Analysis on arithmetic quotients

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Chapter III. Modular forms and representations of GL(2)

This essay will explain the relationship between classical automorphic forms and representations of $\mathrm{GL}_2(\mathbb{R})$.

Contents

1. Introduction

Let $G = \mathrm{GL}_2^+(\mathbb{R})$. I'll define here three equivalent spaces of functions on which G acts.

BASES. Let

$$\mathfrak{H} = \left\{ (z_1, z_2) \in \mathbb{C}^2 \;\middle|\; z_2 \neq 0,\ \mathrm{IM}(z_1/z_2) > 0 \right\}.$$

The group G acts on this through the standard left action:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} : \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \longmapsto \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}.$$

Define

$$\Omega_k^\omega(\mathfrak{H}) = \left\{ F \text{ holomorphic on } \mathfrak{H} \;\middle|\; F(c\zeta) = c^{-k} F(\zeta) \text{ for all } c \in \mathbb{C}^\times \right\}.$$

The group G commutes with scalar multiplication, so acts on Ω_k^ω by the left regular action:

$$L_g F(\zeta) = F(g^{-1}\zeta).$$

UPPER HALF-PLANE. The map $(z_1, z_2) \mapsto z = z_1/z_2$ identifies the quotient $\mathfrak{H}/\mathbb{C}^\times$ with

$$\mathcal{H} = \{ z \in \mathbb{C} \,|\, \mathrm{IM}(z) > 0 \}.$$

The map taking z to $(z, 1)$ is a section of the quotient projection. The image of the section is not G-stable, but instead

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} z \\ 1 \end{bmatrix} = (cz + d) \begin{bmatrix} (az + b)/(cz + d) \\ 1 \end{bmatrix}.$$

The factor $cz + d = j(g, z)$ is called the **automorphy factor**. The equation above implies immediately that

(III.1.1)
$$j(gh, z) = j(g, h(z))j(h, z)\,.$$

and that G acts on $\mathcal{H}$ by fractional linear transformations. The projection is G-equivariant, and the factor $j(g, z)$ measures the extent to which the section is not G-stable. Equation (III.1.1) is a kind of cocycle relation.

Suppose F to lie in $\Omega^\omega_k(\mathfrak{H})$. Define

$$f(z) = f_F(z) = F(z, 1)\,,$$

If $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then

$$F(g(z, 1)) = F(az + b, cz + d) = (cz + d)^{-k} F(az + b/cz + d, 1) = [f\,\big|\,[g]_k](z)$$

if

(III.1.2)
$$\big[f\,\big|\,[g]_k\big](z) = j(g, z)^{-k} f(g(z))\,.$$

This defines a right action of G on $\Omega^\omega_k(\mathcal{H}) = \Omega^\omega(\mathcal{H})$, the space of holomorphic forms on $\mathcal{H}$. Associated to it is a left action

$$L_{k,g} f = f\,\big|\,[g^{-1}]_k\,.$$

I must emphasize that the space doesn't depend on k, but the action of G does.

The action is motivated by the observation that if $\omega = f(z)dz$ is a holomorphic differential form on $\mathcal{H}$, then

$$L^*_g \omega(z) = f(g(z))\,\frac{\det(g)\,dz}{(cz + d)^2}\,.$$

In general, $\mathcal{H}$ is a G-homogeneous space, and the isotropy subgroup of i is the copy of $\mathbb{C}^\times$ obtained from the embedding

$$\iota\colon\ a + ib \longmapsto \begin{bmatrix} a & -b \\ b & a \end{bmatrix}\,.$$

The space $\Omega^\omega_k(\mathcal{H})$ is the space of holomorphic sections of an associated line bundle.

The map from F to f has a simple inverse:

$$F(z_1, z_2) = z_2^{-k} F(z_1/z_2, 1) = z_2^{-k} f(z_1/z_2)\,.$$

THE GROUP. The space $\mathfrak{H}$ is a principal homogeneous G-space. For F in in $\Omega^\omega_k(\mathfrak{H})$, define

$$\Phi(g) = \Phi_F(g) = F(g(i, 1))\,.$$

It is a smooth function on G. Since

$$\iota(a + bi)(i, 1) = (a + bi)(i, 1)\,,$$

(III.1.3)
$$\Phi(g \cdot \iota(c)) = \Phi(g)\, c^{-k}$$

for all c in $\mathbb{C}^\times$.

How does holomorphicity relate to a condition on Φ? The Lie algebra of G acts on the right on $C^\infty(G)$, and contains the element

$$x_\pm = \begin{bmatrix} 1 & \mp i \\ \mp i & \mp 1 \end{bmatrix} .$$

It will turn out, as we shall see soon, that holomorphicity on F translates to the condition

(III.1.4)
$$R_{x_+} \Phi = 0 \,.$$

Here R is the right regular representation. Thus if F is in $\Omega^\omega_k(\mathfrak{H})$ then Φ will satisfy both (III.1.3) and (III.1.4) .

Let $\Omega^\omega_k(G)$ be the space of all such functions.

I have defined maps from Ω^ω_k to $\Omega^\omega(\mathcal{H})$ and to $\Omega^\omega_k(G)$. There is also a map from $\Omega^\omega_k(\mathcal{H})$ to $\Omega^\omega_k(G)$, taking f to

$$\Phi(g) = f(g(i)) j(g, i)^{-k} \,.$$

Conversely, of course:

$$f(x + iy) = \Phi(p) \quad \text{if} \quad p = \begin{bmatrix} y & x \\ 0 & 1 \end{bmatrix} .$$

There are maps between the three spaces I have just defined. We therefore have a diagram

$$\begin{tikzcd} \Omega^\omega_k(\mathfrak{H}) \arrow[rr] \arrow[dr] & & \Omega^\omega_k(G) \\ & \Omega^\omega_k(\mathcal{H}) \arrow[ur] & \end{tikzcd}$$

that is easily verified to be commutative. One of the principal results of this note is:

III.1.5. Theorem. *All maps in this diagram are isomorphisms, equivariant with respect to the left regular representations.*

The only non-trivial part of this the isomorphism of $\Omega^\omega_k(\mathfrak{H})$ with $\Omega^\omega_k(G)$. This requires looking at the Lie algebra of G more closely, which I'll do in the next section.

2. Proof of the Theorem

Next, I'll show how to characterize the image of $C^{\omega}_{m}(\mathfrak{H})$. with respect to Φ. Before I state the main result, I'll recall some elementary facts.

COMPLEX ANALYSIS. A smooth $\mathbb{C}$-valued function $f = u(x, y) + iv(x, y)$ on an open subset of $\mathbb{C}$ is holomorphic if and only if the real Jacobian matrix of f

$$\begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{bmatrix}$$

considered as a map from $\mathbb{R}^2$ to itself lies in the image of $\mathbb{C}$ in $M_2(\mathbb{R})$. (Since this image generically coincides with the group of orientation-preserving similitudes, this means precisely that it is conformal.) This condition is equivalent to the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}\,.$$

Holomorphicity may also be expressed by the single equation

$$\frac{\partial f}{\partial \overline{z}} = 0 \quad \text{where} \quad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\,\frac{\partial}{\partial y}\right)\,.$$

When f is holomorphic, its complex derivative is

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\,\frac{\partial}{\partial y}\right)\,.$$

The notation is designed so that for an arbitrary smooth function

$$df = \frac{\partial f}{\partial z}\,dz + \frac{\partial f}{\partial \overline{z}}\,d\overline{z}$$

where $dz = dx + i\,dy$.

LIE ALGEBRA AND HOLOMORPHY. I recall that

$$x_{\pm} = \begin{bmatrix} 1 & \mp i \\ \mp i & \mp 1 \end{bmatrix}\,.$$

Suppose f holomorphic on $\mathcal{H}$, and set $\Phi = \Phi_f$, so that $\Phi(g) = f(g(i, 1))j(g, i)^{-k}$.

III.2.1. Lemma. *In these circumstances*

$$R_{x_+}\Phi_F(p) = -4iy\,\frac{\partial f(z)}{\partial \overline{z}}$$

if

$$p = \begin{bmatrix} y & x \\ 0 & 1 \end{bmatrix}, \quad z = x + iy\,.$$

The proof will occupy the rest of this section.

The Lie algebra of G has as real basis:

$$\zeta = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$\alpha = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$
$$\kappa = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$
$$\nu_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

In addition, we shall find these useful:

$$\eta = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = (1/2)(\alpha + \zeta)$$
$$x_+ = (2\eta - \zeta) - i(2\nu_+ + \kappa)\,.$$

The significance of x_+ and its conjugate x_- is that

$$[\kappa, x_\pm] = \pm 2i x_\pm\,.$$

Associated to some of these are vector fields on $\mathcal{H}$:

$$\Lambda_{\nu_+} = \frac{\partial}{\partial x}$$
$$\Lambda_\eta = x\,\frac{\partial}{\partial x} + y\,\frac{\partial}{\partial y}\,.$$

HOLOMORPHICITY CHARACTERIZED. Now for F in $C^\infty(\mathfrak{H})$, define a function f_F on the upper half-plane $\mathcal{H}$ to be its restriction to its image in $\mathfrak{H}$:

$$f_F(z) = F(z, 1)\,.$$

I now conclude the proof of the Lemma. Since κ and ζ are in the Lie algebra of $\iota(\mathbb{C}^\times)$:

$$R_\kappa F = -miF, \quad R_\zeta F = -mF\,.$$

But then

$$\begin{aligned} R_{x_+}F(p) &= (R_\alpha - 2iR_{\nu_+} - iR_\kappa)F(p) \\ &= (2R_\eta - R_\zeta - 2iR_{\nu_+} - iR_\kappa)F(p) \\ &= (2R_\eta - 2iR_{\nu_+} + m - i(-mi))F(p) \\ &= (2R_\eta - 2iR_{\nu_+})F(p) \end{aligned}$$

Now I apply the basic formula $R_X f(g) = [L_{gXg^{-1}} f](g)$ to get

$$(2R_\eta - 2iR_{\nu_+})F(p) = (2\Lambda_{p\eta p^{-1}} - 2i\Lambda_{p\nu_+ p^{-1}})F(p)\,.$$

But

$$L_\eta = x\,\frac{\partial}{\partial x} + y\,\frac{\partial}{\partial y}$$

and

$$p\eta p^{-1} = y\eta - x\nu_+$$

so

$$\begin{aligned}(2\Lambda_{p\eta p^{-1}} - 2i\Lambda_{p\nu_+ p^{-1}})F(p) &= 2y\,\frac{\partial f}{\partial y} - 2iy\,\frac{\partial f}{\partial x} \\ &= -2iy\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) \\ &= -4iy\,\frac{\partial f}{\partial \overline{z}}\,.\end{aligned}$$

3. Modular forms

Suppose Γ to be a proper subgroup of $\mathrm{SL}_2(\mathbb{R})$. For convenience, I shall assume it to have exactly one cusp, and that at ∞. I shall also assume $\Gamma \cap N(\mathbb{Z})$ to be $N(\mathbb{Z})$.

A **modular form** of weight $k > 0$ for Γ is a function on $\mathcal{H}$ of moderate growth—i.e. bounded by some norm $\|z\|^N$—such that $f|[\gamma]_k = f$ for all γ in Γ. Invariance with respect to Γ and the requirement of moderate growth imply that

$$f(z) = \sum\nolimits_{n=0}^{\infty} f_n\, e^{2\pi i n z}\,,$$

since $|e^{2\pi i n(x+iy)}| = e^{-2\pi n y}$. It is called a **cusp form** if$f_0 = 0$, in which case f will be exponentially decreasing at ∞.

Following the previous section with a slight twist, define the corresponding function on $G = \mathrm{GL}_2^+(\mathbb{R})$

$$\Phi_f(g) = f\big(g(i)\big)\, j(g,i)^{-m}\, \overset{m/2}{\det}(g)\,.$$

It is actually left-invariant under Γ and again of moderate growth on G. From the results of the previous section:

III.3.1. Theorem. *The space of holomorphic forms of weight m for Γ is isomorphic to the space of smooth (in fact, necessarily real analytic) functions Φ on $\Gamma\backslash G$ of moderate growth such that*

- *(a) $\Phi(gk) = \varepsilon^{-m}(k)\Phi(g)$ for k in SO_2;*
- *(b) $R_{x_+}\Phi = 0$;*
- *(c) $\Phi(gz) = \Phi(g)$ for z in the connected component of the center of G.*

THE PETERSON METRIC. Suppose Φ to satisfy the conditions of the Theorem. Then

$$|\Phi(g)|^2 = |f(g(i))|^2 = |j(g,i)|^{-2k}\, \overset{k}{\det}(g) = |f(g(i))|^2\, \mathrm{IM}(g(i))^k\,,$$

and for k in SO_2

$$|\Phi(gk)|^2 = |\Phi(g)|^2\,.$$

But then

$$\int_{\Gamma\backslash G} |\Phi(g)|^2 \, dg = \int_{\Gamma\backslash\mathcal{H}} |f(z)|^2 \, \mathrm{IM}(z)^m \, \frac{dx\,dy}{y^2}$$

If f is a cusp form, the function $|f|^2$ will be of rapid decrease at ∞. and the integral above finite. Hence I have proved:

III.3.2. Corollary. *If f is a cusp form then Φ_f lies in* $\mathrm{L}^2(\Gamma\backslash G)$.

4. References

1. Adolph Hurwitz, 'Grundlagen einer independenten Theorie der elliptischen Modulfunctionen und Theorie der Multiplicatorgleichungen erster Stufe', *Mathematische Annalen* **18** (1881), 528–592.

2. J-P. Serre, **Cours d'Arithmétique**, Presses Universitaires de France 1970.