The classical groups

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Suppose G to be a split reductive group defined over a field F. It is natural to ask, how does one do computations in it? For arbitrary groups, there are two methods explained in [Cohen-Murray-Taylor:2005]. One of them applies formulas of Chevalley and Tits involving generators and relations, along with the Bruhat decomposition. This is not usually very pleasant and often barely possible to do by hand, although it is quite reasonable to use a computer for the task (and indeed the program MAGMA implements these methods). This is a rather abstract process, and the difficulty of doing it seems to have intimidated many.

However, the so-called **classical** reductive groups are those with simple defining realizations as matrix groups, in which computation is straightforward. Another significant property of the classical groups is that they occur in infinite families: SL_n for $n \ge 2$; Sp_{2n} for $n \ge 2$; SO_n for $n \ge 3$. There are also the spin groups, which one may work with either in terms of the Clifford algebra or as coverings of the orthogonal groups. For each of these families there is an infinite series of inclusions such as

$$\ldots \subset \mathrm{SL}_n \subset \mathrm{SL}_{n+1} \subset \ldots$$

This makes possible proofs and constructions by induction.

I deal with the spin groups elsewhere, because they are somewhat complicated, and require a more extended treatment. Here I mention only that they are responsible for most of the 'accidents':

$$\begin{aligned} \mathrm{Sp}_2 &= \mathrm{SL}_2 \\ \mathrm{Spin}_3 &= \mathrm{SL}_2 \\ \mathrm{Spin}_4 &= \mathrm{SL}_2 \times \mathrm{SL}_2 \\ \mathrm{Spin}_5 &= \mathrm{Sp}_4 \\ \mathrm{Spin}_6 &= \mathrm{SL}_6 \,. \end{aligned}$$

After a brief introduction to root data, I'll discuss the classical groups in terms of these.

I should add, this essay is incomplete. It begain as notes to help me program calculations in classical groups, and is updated often as I find additions useful.

Contents

In all cases, ε_i is the character of the diagonal matrices taking x to $x_{i,i}$ and $\varepsilon_i^\vee$ the corresponding co-character taking x to the diagonal matrix with $x_{i,i} = x$ but $x_{j,j} = 1$ for $i \ne j$. Thus

$$\langle \varepsilon_i, \varepsilon_j^\vee \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

In this note I shall not usually distinguish multiplicative characters from their (additive) differentials.

1. Root data and the structure of reductive groups

A **reductve group** defined over an algebraically closed field is is an algebraic group that does not contain a unipotent normal subgroup. Equivalently, its maximal solvable normal subgroup is an algebraic torus. In characteristic 0, it is one all of whose finite-dimensional algebraic representations are semi-simple. Without further mention, I'll require also that a reductive group be Zariski-connected. Over any field of definition there exists a unique split reductive group defined by descent from its algebraic closure.

ROOT DATA. Split reductive groups are characterized by **root data**.

Suppose given a lattice L, which is to say a free module over $\mathbb{Z}$ of finite rank. Let $L^\vee$ be the dual lattice $\mathrm{Hom}(L,\mathbb{Z})$, $\Delta \subset L$ a finite subset of L, and $\alpha \mapsto \alpha^\vee$ a map from Δ to $L^\vee$. These data define a (based) **root datum** $\mathcal{L} = (L, \Delta, L^\vee, \Delta^\vee)$ if the integral matrix $C = (\langle \alpha, \beta^\vee \rangle)$ indexed by Δ^2 is a positive definite **Cartan matrix**. This means that

(a) $c_{\alpha,\alpha} = 2$ for all α;
(b) $c_{\alpha,\beta} \le 0$ for all $\alpha \ne \beta$;
(c) there exists a diagonal matrix D with positive diagonal entries such that CD is symmetric and positive definite.

Modulo trivial variations, for example, the possible Cartan matrices of dimension 2 are

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \quad \begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix}, \quad \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}.$$

The first is **reducible**, the last three **irreducible**. All possible irreducible Cartan matrices have been classified.

For each α in Δ the linear transformation

$$s_\alpha\colon\ \lambda \longmapsto \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$$

is a reflection in L, fixing vectors in the hyperplane $\alpha^\vee = 0$. The Weyl group of the datum is that generated by the s_α. If d_α is the diagonal entry of the matrix D in condition (c), then

$$\alpha \bullet \beta = \langle \beta, \alpha^\vee \rangle d_\alpha$$

defines a W-invariant metric on the lattice L_Δ spanned by Δ. The group W acts trivially on a complement. Therefore there exists on L a Euclidean metric invariant under the Weyl group. Because the group W takes L into itself, it is finite.

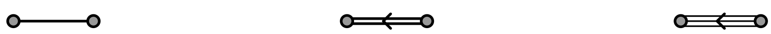
The set Σ of **roots** of the datum is the smallest W-stable subset of L containing Δ, which is therefore also the W-orbit of Δ. The **coroots** are the transforms of $\Delta^\vee$ under the dual transformations of the Weyl group.

The Cartan matrix gives rise to its **Dynkin diagram**. This is a finite graph whose nodes are indexed by Δ, with a directed edge from α to β precisely when $\langle \alpha, \beta^\vee \rangle \ne 0$. In this case the edge is indexed by $\langle \alpha, \beta^\vee \rangle$. For the groups we are looking at

$$0 \le n_{\alpha,\beta} = \langle \alpha, \beta^\vee \rangle \langle \beta, \alpha^\vee \rangle \le 3\,,$$

and in the graphical representation of the diagram this is indicated by an arrow from α to β if $|\langle \alpha, \beta^\vee \rangle| > 1$, as well as the multiplicity of edges.

Here are the Dynkin diagrams of the non-trivial systems of dimension two:



The Cartan matrix is irreducible if and only if the Dynkin diagram is connected. The classification of Cartan matrices reduces to the classification of irreducible ones. The map $\lambda \mapsto (\langle \lambda, \alpha^\vee \rangle$ is a homomorphism from L to $L^\vee_{\Delta^\vee}$ The root datum is said to be semi-simple, for reasons that will be apparent later on, if this is injective. The classification of semi-simple data is well known. But in general the classification of root data involves the K-theory of certain group algebras. For example, there are two inequivalent root data of dimension two. As we shall see later, one corresp[onds to the group GL_2 and the other to $\mathrm{SL}_2 \times \mathbb{G}_m$.

Example. Take $L = \mathbb{Z}^n$ with basis ε_i, dual basis $\varepsilon_i^\vee$. Let Δ be the set of $\varepsilon_i - \varepsilon_{i+1}$ for $0 \le i \le n-2$. The simple reflections swap ε_i with ε_{i+1}, which generate the symmetric group $\mathfrak{S}_n$. The elements of $\Delta^\vee$ are the $\varepsilon_i^\vee - \varepsilon_{i+1}^\vee$.

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ROOT DATA OF REDUCTIVE GROUPS. How, exactly, do root data relate to reductive groups? Suppose G to be reductive defiend over F, which is algebraically closed. Let B be a maximal solvable subgroup of G, and T a maximal torus in B. All choices of B are conjugate in G, and all choices of T are conjugate in B. The group T acts by the adjoint action on the Lie algebra $\mathfrak{g}$. Since T is a torus, $\mathfrak{g}$ decomposes into eigenspaces. That corresponding to the trivial character of T is the Lie algebra $\mathfrak{t}$ of T itself, and the other eigenspaces are one-dimensional. The non-trivial characters λ for which $\mathfrak{g}_\lambda \ne 0$ are the roots of $\mathfrak{g}$ with respect to T. Those λ occurring in the adjoint action on $\mathfrak{b}$ are the **positive roots**. The roots are a subset Σ of $X^*(T)$. Let Σ^+ the subset of positive roots. There exists a subset Δ of Σ^+ with the property that every root in Σ^+ is a non-negative integral combination of elements in Δ. These are the **simple** roots determined by the choice of B.

Example. The simplest examples are the groups GL_n, with T the diagonal matrices, B the upper triangular matrices. (We'll see more about this later.)

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Fix for each α in Δ an element $e_\alpha \ne 0$ in $\mathfrak{g}_\alpha$. All choices are conjugate with respect to the adjoint action of T. This choice now determines a **frame** for $\mathfrak{g}$, the triple $(B, T, \{e_\alpha\})$. The set of frames is a principal homogeneous space under the adjoint action of the quotient of G by its center. There exists in $\mathfrak{g}_{-\alpha}$ a unique $e_{-\alpha}$ such that $h_\alpha = [e_\alpha, e_{-\alpha}]$ satisfies $\alpha(h_\alpha) = 2$. These data determine a unique embedding of $\mathfrak{sl}_2$ into $\mathfrak{g}$, taking

$$\begin{aligned} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} &\mapsto e_\alpha \\ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} &\mapsto h_\alpha \\ \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} &\mapsto e_{-\alpha} \,. \end{aligned}$$

Applying known generators and relations for SL_2, one deduces that this gives rise in turn to a homomorphism φ_α from SL_2 to G. Define the **coroot**

$$\alpha^\vee \colon \mathbb{G}_m \longrightarrow T \subseteq G, \quad x \longmapsto \varphi_\alpha\left(\begin{bmatrix} x & 0 \\ 0 & 1/x \end{bmatrix}\right) .$$

It depends only on the choice of B and T, not on the choice of frame. Let $\Delta^\vee \subset X_*(T)$ be the set of coroots.

There exists a unique involution θ of G acting as $t \mapsto t^{-1}$ on T and inducing the map $e_\alpha \mapsto -e_{-\alpha}$, called the **canonical involution** determined by the frame. There exists for each root $\lambda \notin \Delta$ a choice of $e_\lambda \ne 0$ in $\mathfrak{g}_\lambda$ such that $\theta(e_\lambda) = -e_{-\lambda}$. The e_λ are unique up to sign and, together with a suitable basis of $\mathfrak{t}$, make up a **Chevalley basis** of $\mathfrak{g}$. (There are different definitions of this in the literature, but in all cases $\theta(e_\lambda) = \pm e_{-\lambda}$.) To each e_λ is associated a homomorphism φ_λ from SL_2 to G.

Let t_λ be the image of

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

with respect to φ_λ. Hence $[e_\lambda, e_{-\lambda}] = t_\lambda$. Furthermore

$$[t_\lambda, e_\mu] = \langle \mu, \lambda^\vee \rangle e_\mu \,.$$

If we set $L = X^*(T)$, then $(L, \Delta, L^\vee, \Delta^\vee)$ is a root datum. The group $L^\vee$ may be identified with the cocharacter lattice $X_*(T)$. The group G and frame $(B, T, \{e_\alpha\})$ are said to **realize** the datum.

For every α in Δ, let $\dot{s}_\alpha$ be the image under φ_α of

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

If w is any element of W, it can be expressed as a product of m elementary reflections s_i of least length. The product

$$\dot{w} = \dot{s}_1 \ldots \dot{s}_m$$

depends only on w. The normalizer of T in G fits into an exact sequence

$$1 \longrightarrow T \longrightarrow N_G(T) \longrightarrow W \longrightarrow 1$$

and the map $w \mapsto \dot{w}$ is a section. Since $\dot{s}_\alpha^2 = \alpha^\vee(-1)$, computation in $N_G(T)$ reduces to computation in W and T. The group $G(\mathbb{C})$ is the disjoint union of double cosets $B\dot{w}B$, and every element of $G(\mathbb{C})$ has a canonical representive. Using formulas of Chevalley and Tits, one can multiply explicitly (see [Cohen-Murray-Taylor:2005]).

Chevalley proved in 1955 the astonishing result that *every root datum can be realized*, even in arbitrary characteristic. In effect, root data classify reductive groups up to isomorphism. I say 'astonishing' because, as will become apparent later on, small characteristic—especially characteristic two—causes serious trouble in finding explicit realizations. In any case, if the base field is not assumed to be algebraically closed, root data classify **split** groups.

The rest of this essay will describe the root data of what are called the 'classical' groups—essentially those with explicit realizations as matrix groups in relatively small dimensions. Thus orthogonal groups are called classical (dimension $O(n^2)$ if n is the rank), but the spin groups (dimension 2^n) are not.

DERIVED ROOT DATA. Suppose G to be a reductive group with root datum $(L, \Delta, L^\vee, \Delta^\vee)$. There are several other groups associated to G—its derived group G_{der}, the simply connected covering G_{sc} of G_{der}, the centre Z_G of G, its adjoint quotient G_{adj}, the maximal torus S in Z_G, the maximal torus quotient. *How are all these determined by the root datum of G?*

• The center Z_G is contained in T. It may be disconnected, but it is a **diagonalizable** group, determined by its character group $X^*(Z_G)$. For example, if $G = \mathrm{SL}_n$ then Z_G is isomorphic to the group μ_n of n-th roots of unity, and $X^*(Z_G)$ is isomorphic to Z/n. More generally, the embedding determines a surjective homomorphism from $X^*(T)$ to $X^*(Z_G)$. The roots are trivial on the center, so that the root lattice L_Δ is in the kernel. This gives an isomorphism

$$X^*(Z_G) = X^*(T)/L_\Delta\,.$$

• As a consequence, the center is finite, or equivalently G is semi-simple, if and only if L/L_Δ is finite. In this case,

$$L_\Delta \subseteq L \subseteq \text{ dual of } L^\vee_{\Delta^\vee}\,.$$

In programming, it is useful to distinguish between weights—i.e. elements of L—and root vectors—i.e. weights in the span of Δ. Any weight determines an element of the dual of $\mathrm{L}^\vee_{\Delta^\vee}$, and may be assigned the partial coordinate vector $(\langle v, \alpha^\vee \rangle)$. Any root vector may be expressed as a sum $\sum c_\alpha \alpha$, but also as a weight vector.

• A semi-simple group is in addition simply connected if the embedding of $L^\vee_{\Delta^\vee}$ into $L^\vee$ is an isomorphism. In this case, a weight v is therefore completely specified by the array $(\langle v, \alpha^\vee \rangle)$. The **fundamental weights** are the basis of L dual to $\Delta^\vee$. Row α of the Cartan matrix C holds the coordinates $\langle \alpha, \beta^\vee \rangle$ of the root α with respect to the basis of fundamental weights. These rows span the root lattice. The fundamental weights may be expressed as fractional linear combinations of the simple roots in terms of C^{-1}. The quotient L/L_Δ may be computed by finding the Smith normal form of the transpose of the Cartan matrix. This gives us a basis λ_i of L and an array of $n_i > 0$ such that $(n_i\lambda_i)$ is a basis of L_Δ.

• The derived group G_{der} of G is the maximal semi-simple subgroup of G. What is its root datum? The embedding of G_{der} in G determines an embedding of its maximal torus T_{der} into T, hence a surjection from L to $L_{\mathrm{der}} = X^*(T_{\mathrm{der}})$. This induces an isomorphism

$$L_{\mathrm{der}} = L/\mathrm{Ann}_L(\Delta^\vee)\,.$$

• Let T_{quot} be the maximal torus quotient of G. Then $X^*(T_{\mathrm{quot}})$ embeds into $X^*(T)$. Its image is equal to $\mathrm{Ann}_L(\Delta^\vee)$. This is the same as $X^*(G)$.

• What is the root datum of G_{adj}? Because of the exact sequence

$$1 \longrightarrow Z_G \longrightarrow G \longrightarrow G_{\mathrm{adj}} \longrightarrow 1\,,$$

we have also an exact sequence

$$0 \longrightarrow L_{\mathrm{adj}} \longrightarrow L \longrightarrow X^*(Z_G) \longrightarrow 0\,,$$

so that $L_{\mathrm{adj}} = L_\Delta$.

• The group Z_G contains a maximal torus S. The embedding of S into Z_G is equivalent to a surjective map from $X^*(Z_G)$ to $X^*(S)$. Its kernel is the torsion subgroup of $X^*(Z_G)$.

If G_{sc} is the simply connected covering of G_{der}, the product map from $S \times G_{\mathrm{sc}}$ to G is surjective. Hence:

1.1. Proposition. *Any reductive group can be represented as a quotient of a product of a semi-simple, simply connected one and a torus.*

This enables a classification of root data.

CONSTRUCTION. Root data are the DNA of reductive groups—from a small amount of data an enormous and complicated structure is assembled.

In the previous section I have recalled how a reductive group gives rise to a root datum. Conversely, and perhaps more miraculously, every root datum $\mathcal{L}$ gives rise to a reductive group, which can be constructed through a series of steps.

Step 1. The root datum gives $\Delta \subset L$, as well as the formula for simple reflections. From this one can construct all the roots and coroots, distinguishing the positive ones.

Step 2. The construction of the split semi-simple Lie algebra corresponding to a given root system was done initially for almost all systems case by case, depending on particular representations. This has been done uniformly for all systems by an idea due originally and independently to Chevalley and Harish-Chandra. The precise result is Theorem 1 in §VII.1 of [Jacobson:1962]. A version of this was later applied to the construction of Kac-Moody algebras. If $n = |\Delta|$, the Lie algebra in question is defined as a certain quotient of the free Lie algebra determined by $3n$ generators e_α, h_α, f_α for α in Δ.

Step 3. In general, whenever $\mathfrak{g}$ is a reductive Lie algebra, suppose e_λ to be a generator of $\mathfrak{g}_\lambda$. Then

$$[e_\lambda, e_\mu] = \begin{cases} \text{an element of } \mathfrak{t} & \text{if } \mu = -\lambda \\ N_{\lambda,\mu} e_{\lambda+\mu} & \text{if } \lambda+\mu \text{ is a root} \\ 0 & \text{otherwise.} \end{cases}$$

In particular, if λ is a negative root and μ positive, $[e_\lambda, e_\mu] = 0$ unless $\lambda = -\mu$. Furthermore, if $[e_\lambda, e_\mu] = N_{\lambda,\mu} e_\nu$ then

$$[e^\theta_\lambda, e^\theta_\mu] = N_{\lambda,\mu} e^\theta_\nu\,.$$

In practice, for classical groups it seems easiest to compute Lie brackets by machine as you would by hand. First find a Chevalley basis in the sense I have defined above. Then to each root associate a matrix, implementing it as a list of pasrticular $e_{i,j}$. At the same time as this list is registered, associate also to each (i,j) the corresponding root. This is well defined. Use the rule for products of elementary matrices to compute the brackets of the e_λ.

If the e_λ are part of a Chevalley basis, the structure constants $N_{\lambda,\mu}$ are integers, relatively simple to determine up to sign. Explicit formulas for Lie brackets and group products are due originally to Chevalley. They are explained in [Carter:1972], and in slightly greater generality in [Cohen-Murray-Taylor:2005]. Chevalley discovered how the $e_\lambda \neq 0$ for each root λ can be specified, and then how to find $\varepsilon(\lambda,\mu)$ such that

$$[e_\lambda, e_\mu] = \varepsilon(\lambda,\mu)(p_{\lambda,\mu}+1)e_{-\nu}$$

whenever $\lambda+\mu+\nu=0$. Here $p_{\lambda,\mu}$ is the distance of μ from the left hand end of the λ-string through it. Tits discovered how to describe $N_G(T)$ explicitly. One can also find simple formulas for the action of $N_G(T)$ on $\mathfrak{g}$.

Step 4. The last step is from the Lie algebra to the group determined by the root datum. The final result is of a smooth reductive group scheme over $\mathbb{Z}$. There are in the literature many ways to do this, but as far as I can see none are perfect. It is perhaps best done in a uniform way by ideas explained roughly in [Kostant:1966], but this is not a complete account. Other relevant references are [Demazure-Grothendieck:1962/4], [Conrad-Gabber-Prasad:2015], [Steinberg:1967], and [Lusztig:2009].

In the rest of this essay, I'll define the classical groups, and determine for each one a certain subset of fundamental data. Eventually, these will include

- ∘ a maximal torus T
- ∘ a Borel subgroup B containing T
- ∘ the lattice $X^*(T)$
- ∘ the centre Z_G
- ∘ the Lie algebra
- ∘ a Chevalley basis of $\mathfrak{g}$, as matrices
- ∘ the roots Σ
- ∘ the simple roots Δ
- ∘ the coroots $\Sigma^\vee$
- ∘ the subset $\Delta^\vee$
- ∘ the special embeddings of SL_2
- ∘ the Weyl group W
- ∘ the fundamental weights, as elements of the dual of $L^\vee_{\Delta^\vee}$

But at the moment I am content with less.

Every classical group G comes with a fixed embedding into some $\mathrm{GL}_n(\mathbb{C})$. In all cases, I shall normalize things in such a way that the subgroup of upper triangular matrices will be the Borel subgroup of G, the subgroup of diagonal matrices will be the torus T, and the canonical involution will take X to ${}^tX^{-1}$. Because I am interested in writing programs, I'll label the simple roots for a group of semi-simple rank r by indices in $[0,r-1]$. With a shift of 1, I'll follow the conventions of [Bourbaki:1965].

2. The general linear groups

The simplest reductive groups are the general linear and special linear groups. The group GL_n contains the subgroup SL_n of all $n \times n$ matrices of determinant 1.

The group GL_n contains as maximal torus the subgroup T of diagonal matrices, and as Borel subgroup the group $B = TN$ of upper triangular matrices. The centre Z_G is the group of scalar matrices. The center Z_{SL} of the derived group SL_n is isomorphic to the group μ_n of n-th roots of unity. For general fields of definition, its group of rational points may hence be quite small. The dual $X^*(Z_{\mathrm{SL}})$ is canonically isomorphic to $\mathbb{Z}/n$.

CHARACTER LATTICE. The lattice $X^*(T)$ has as basis the characters ε_i (for $0 \le i < n$), with

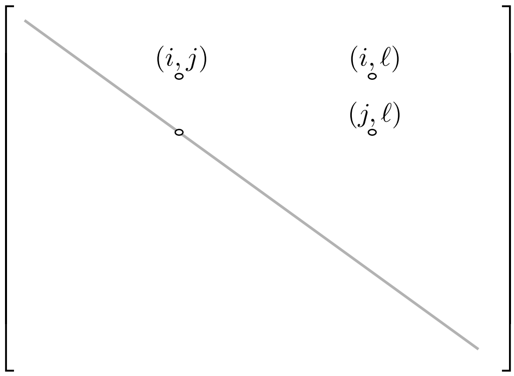
$$\varepsilon_i(t) = t_{i,i} .$$

LIE ALGEBRA. The Lie algebra of GL_n is the vector space of $n \times n$ matrices, with bracket $[X, Y] = XY - YX$. That of SL_n is the subspace of matrices of trace 0.

The elementary matrix $e_{i,j}$ is that with zero entries everywhere except in location (i, j), where it is 1. The matrix product is

(2.1)
$$e_{i,j} \cdot e_{k,\ell} = \begin{cases} e_{i,\ell} & \text{if } j = k \\ 0 & \text{otherwise.} \end{cases}$$

This is one of the basic formulas underlying all computation in matrix groups. It can be roughly visiualized:



The canonical involution takes $e_{i,j}$ to $-e_{j,i}$, and the $e_{i,j}$ make up a Chevalley basis of $\mathfrak{g} = \mathfrak{gl}_n$. The $e_{i,i}$ span the Lie algebra of T. If $t = (t_{i,i})$ is in T then

$$t \cdot e_{i,j} \cdot t^{-1} = (t_{i,i}/t_{j,j}) e_{i,j}$$

so the $e_{i,j}$ with $i \ne j$ are the generators of the root spaces. The roots are parametrized by such pairs.

Lie brackets follow from (2.1) :

$$[e_{i,j}, e_{k,\ell}] = \begin{cases} e_{i,i} - e_{j,j} & \text{if } i = \ell \text{ and } j = k \\ e_{i,\ell} & \text{if } i \ne \ell \text{ and } j = k \\ -e_{k,j} & \text{if } i = \ell \text{ and } j \ne k \\ 0 & \text{otherwise.} \end{cases}$$

Computation in GL_n is the basis of calculations in other classical groups.

The roots are (in additive notation) the $\alpha_{i,j} = \varepsilon_i - \varepsilon_j$ for $i \ne j$. The corresponding root space is spanned by the elementary matrix $e_{i,j}$. The positive roots are those with $i < j$. The simple roots are the $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$. Every positive root can be expressed as a linear combination of simple roots:

$$\alpha_{i,j} = \alpha_i + \cdots + \alpha_{j-1} .$$

COROOTS. Let $\varepsilon_i^\vee$ be the homomorphism from $\mathbb{G}_m$ to GL_n taking t to the diagonal matrix with $s_{i,i} = t$ and $s_{j,j} = 1$ for $j \neq i$.

The coroots are the $\varepsilon_i^\vee - \varepsilon_j^\vee$ with $i \neq j$. The simple ones are those with $j = i + 1$, and these are a basis of $X_*(T_{\mathrm{der}})$ for SL_n, which is simply connected.

Another basis of weights is made up of the fundamental weights

$$\begin{aligned} \varpi_0 &= \varepsilon_0 \\ \varpi_1 &= \varepsilon_0 + \varepsilon_1 \\ &\ldots \\ \varpi_{n-1} &= \varepsilon_0 + \varepsilon_1 + \cdots + \varepsilon_{n-1} \end{aligned}$$

If σ is the standard representation on $\mathbb{C}^n$, then ϖ_i is the highest weight of $\bigwedge^{i+1} \sigma$. The dominant weights are those of the form $\sum_0^{n-1} c_i \varpi_i$ with c_i integral, $c_i \geq 0$ for $i < n$.

For SL_n, the lattice $X^*(T_{\mathrm{der}})$ is the quotient of $X^*(T)$ by ϖ_n. The images of the ε_i with $i < n$ form a basis of $L_{\mathrm{der}} = X^*(T_{\mathrm{der}})$.

The the projection from $X^*(T_{\mathrm{der}})$ onto $X^*(Z_{\mathrm{SL}})$ takes (c_i) to $\sum c_i$ modulo n. The quotient $L_{\mathrm{der}}/L_\Delta$ is isomorphic to $\mathbb{Z}/n$, and we can see this very explicitly. Let

$$\begin{aligned} \psi_0 &= \overline{\varepsilon}_0 \\ \psi_1 &= \overline{\varepsilon}_1 - \overline{\varepsilon}_0 \\ &\ldots \\ \psi_k &= \overline{\varepsilon}_k - \overline{\varepsilon}_{k-1} \quad (2 \leq k < n-1)\,. \end{aligned}$$

Then the ψ_i clearly make up a basis of L_{der}. But since $\overline{\varepsilon}_0 + \cdots + \overline{\varepsilon}_{n-1} = 0$, the weights

$$\begin{aligned} n\psi_0 &= n\overline{\varepsilon}_0 \\ &= n\overline{\varepsilon}_0 - (\overline{\varepsilon}_0 + \cdots + \overline{\varepsilon}_{n-1}) \\ &= (\overline{\varepsilon}_0 - \overline{\varepsilon}_1) + \cdots + (\overline{\varepsilon}_0 - \overline{\varepsilon}_{n-1}) \\ &= (n-1)\alpha_0 + (n-2)\alpha_1 + \cdots + \alpha_{n-2} \\ \psi_k &= -\alpha_{k-1} \quad (k \geq 1) \end{aligned}$$

make up a basis of L_Δ.

SL(2) EMBEDDINGS. For a positive pair (i, j), the embedding of SL_2 into G corresponding to α_i maps a 2×2 matrix into a block in entries $[i, j] \times [i, j]$. For $j = i + 1$ this is a simple 2×2 block. For example, with $n = 4$, $i = 0$, $j = 2$:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto \begin{bmatrix} a & \circ & b & \circ \\ \circ & 1 & \circ & \circ \\ c & \circ & d & \circ \\ \circ & \circ & \circ & 1 \end{bmatrix}.$$

WEYL GROUP. The Weyl group is $\mathfrak{S}_n$, acting by permutation on diagonal entries. There are in this case two natural sections of the sequence

$$1 \longrightarrow T \longrightarrow N_G(T) \longrightarrow W \longrightarrow 1\,.$$

One of them takes place in GL_n, with image the permutation matrices, and is a homomorphism. The other takes place in SL_n, and is Tits' map $w \mapsto \dot{w}$. In general, this sequence does not split.

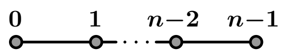
DYNKIN DIAGRAM. The Cartan matrix has

$$\langle \alpha_i, \alpha_j^\vee \rangle = \begin{cases} 2 & \text{if } i = j \\ -1 & \text{if } j = i \pm 1 \\ 0 & \text{otherwise.} \end{cases}$$

For example, that for SL_4 is

$$\begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{bmatrix}$$

The diagram is



3. Preliminaries concerning isometry groups

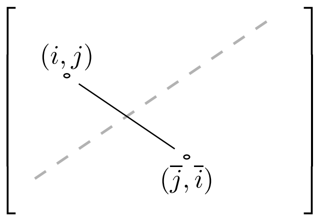
Fix for the rest of this essay the $n \times n$ symmetric matrix

$$\omega_n = \begin{bmatrix} 0 & \ldots & 1 \\ & \ldots & \\ 1 & \ldots & 0 \end{bmatrix}.$$

It is its own inverse. The operator

$$X \longmapsto \omega\, {}^tX\, \omega$$

swaps entries of X along the SW-NE axis, and in particular takes diagonal matrices to diagonal matrices. Explicitly, it takes the elementary matrix $e_{i,j}$ to $e_{n-1-j,n-1-i}$. For convenience I set $\overline{m} = n - 1 - m$, and write this as $e_{\overline{j},\overline{i}}$.



Suppose Ω to be either

$$\omega_n \quad \text{or} \quad \begin{bmatrix} 0 & -\omega_m \\ \omega_m & 0 \end{bmatrix},$$

a non-degenerate symmetric or anti-symmetric matrix of size $n \times n$. It satisfies the equation $\Omega^2 = \pm I$. Define G_Ω to be the group of all $m \times m$ matrices such that

$${}^tX \Omega X = c\,\Omega$$

in which $c = \mu(X)$ is a scalar. We can rewrite this equation as

(3.1) $$X\,{}^*X = {}^*X X = c\,I \quad \text{with} \quad {}^*X = \Omega^{-1}\,{}^tX\,\Omega\,.$$

It is a Zariski-closed algebraic subgroup of GL_n, since the condition to be a scalar amounts to a set of polynomial identities. The map $X \mapsto \mu(X)$ is then a character of G. Let $\mathbf{S}_\Omega$ be the subgroup in which $\mu(X) = 1$. Its Lie algebra is the space of all X such that

(3.2)
$$ {}^tX\Omega + \Omega X = 0 \ \text{ or } \ X = -{}^*X \, . $$

Because $\Omega^{-1} = \pm\Omega$, the transpose adjoint ${}^tX^{-1}$ will lie in G when X does. This will be the canonical involution of G. It is also true that Ω itself lies in S.

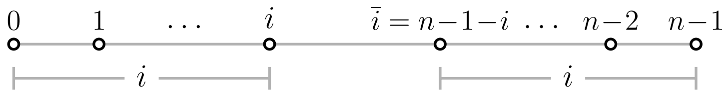
The diagonal matrices in G will form a maximal torus T. The subgroup of upper triangular matrices will be a Borel subgroup (i.e. maximal solvable), because the involution $X \mapsto \Omega\,{}^tX\Omega$ takes an upper triangular matrix to another one. Every t in T will satisfy the condition $t_{i,i}t_{\overline{i},\overline{i}} = \mu(t)$ for all i. I recall that ε_i is the character of the diagonal matrices taking t to $t_{i,i}$. Let $\overline{\varepsilon}_i$ be its restriction to T.

The group S_Ω will also contain elements

$$ \begin{bmatrix} X & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \omega\,{}^tX^{-1}\omega \end{bmatrix} \qquad (X \in \mathrm{GL}_k) $$
$$ \begin{bmatrix} I & 0 & 0 \\ 0 & X & 0 \\ 0 & 0 & I \end{bmatrix} \qquad (X \in \mathbf{S}_\ell) $$

for every m such that $2k \le n$ and $\ell \le n$. Here S_ℓ is the group defined by the central $\ell \times \ell$ minor of Ω.

To help mental calculation later on, I offer the diagram



4. The symplectic groups

Now let $G = \mathrm{GSp}_{2n}$, the group of symplectic similitudes of a non-degenerate alternating form in $2n$ dimensions. Here, I take

$$ \Omega = \Omega_{2n} = \begin{bmatrix} 0 & \omega_n \\ -\omega_n & 0 \end{bmatrix} , $$

and then

$$ \mathrm{GSp}_{2n} = \{ X \mid {}^tX\,\Omega X = c\,\Omega \} $$

for some non-zero scalar $c = \mu(X)$.

The associated alternating form is

$$ \sum_0^{n-1} (x_i y_{\overline{i}} - x_{\overline{i}} y_i) \, . $$

There is an exact sequence

$$ 1 \longrightarrow \mathrm{Sp}_{2n} \longrightarrow \mathrm{GSp}_{2n} \xrightarrow{\ \mu\ } \mathbb{G}_m \longrightarrow 1 \, . $$

One section of the map μ takes c to

$$ \mu^\vee(c) = \begin{bmatrix} I_n & 0 \\ 0 & cI_n \end{bmatrix} \, . $$

Remark. Other choices of alternating matrix are common. They are all equivalent, since any two non-degenerate symplectic forms of the same dimension are equivalent. One common choice is

$$\begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix} .$$

This would be convenient in many circumstances, but the choice I have made has the great virtue that the upper triangular matrices in G form a Borel subgroup.

∘ ———— ∘

The group Sp_{2n} contains the matrices

$$\begin{bmatrix} X & 0 \\ 0 & \omega\, {}^tX^{-1}\omega^{-1} \end{bmatrix} \qquad (X \in \mathrm{GL}_n)$$

$$\begin{bmatrix} I & 0 & 0 \\ 0 & X & 0 \\ 0 & 0 & I \end{bmatrix} \qquad (X \in \mathrm{Sp}_{2m})$$

$$\begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \qquad (\omega\, {}^tX\omega^{-1} = X) \ .$$

This last condition means that X is symmetric with respect to the NE–SW axis.

LIE ALGEBRA. The Lie algebra of Sp_{2n} is the vector space of $2n \times 2n$ such that ${}^tX\Omega + \Omega X = 0$. There are four distinct types making up a basis. Two of these come from the embedding of GL_m, and two from he skew-symmetric matrix S. I list those in $\mathfrak{b}$:

(a) the semisimple elements

$$(e_{i,i} - e_{i+1,i+1}) + (e_{\overline{i}-1,\overline{i}-1} - e_{\overline{i},\overline{i}}) \quad (0 \le i < n) ;$$

(bi) the elements in the copy of $\mathfrak{gl}_n$:

$$e_{i,j} - e_{\overline{j},\overline{i}} \quad (0 \le i < j < n) ;$$

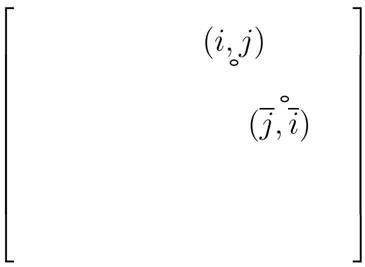
(bii) the cross-diagonal elements:

$$e_{i,\overline{i}} \quad (0 \le i < n) ;$$

(biii) the remainder in the upper right block:

$$e_{i,j} + e_{\overline{j},\overline{i}} \quad (0 \le i < n \le j < 2n - 1) ;$$

(c) the transposes of elements in (b).



Each of these elements contains in its expression some unique $e_{i,j}$ in the region

$$\{0 \le i < n, \ 0 \le j < \overline{i}\} .$$

In computation it is useful to index it by (i, j).

MAXIMAL TORUS. The maximal torus is the group of diagonal matrices

$$a = \begin{bmatrix} a_0 & \circ & \ldots & \circ & \circ \\ \circ & a_1 & \ldots & \circ & \circ \\ & & \ldots & & \\ \circ & \circ & \ldots & a_{2n-2} & \circ \\ \circ & \circ & \ldots & \circ & a_{2n-1} \end{bmatrix}$$

with $a_0 a_{2n-1} = a_1 a_{2n-2} = \ldots = a_{n-1} a_n = \mu(a)$. If $\overline{\varepsilon}_i$ is the restriction to T of ε_i, then $X^*(T)$ has as basis the $\overline{\varepsilon}_i$, for $0 \le i < n$ together with μ, which is in additive notation the common sum $\overline{\varepsilon}_i + \overline{\varepsilon}_{2n-1-i}$.

ROOTS. For $0 \le i < n$ let

$$\alpha_i = \overline{\varepsilon}_i - \overline{\varepsilon}_{i+1} \,.$$

The α_i are the simple roots. The last one is $\alpha_{n-1} = \overline{\varepsilon}_{n-1} - \overline{\varepsilon}_n$, but in terms of the chosen basis it is $2\overline{\varepsilon}_{n-1} - \mu$.

The positive roots on T_{der} may be expressed in two ways.

$$\begin{aligned} \overline{\varepsilon}_i - \overline{\varepsilon}_j &= \textstyle\sum_{i \le k < j} \alpha_k & (0 \le i < j < n) \\ 2\overline{\varepsilon}_i &= 2 \textstyle\sum_{i \le k < n-1} \alpha_k + \alpha_{n-1} & (0 \le i < n-1) \\ \overline{\varepsilon}_i + \overline{\varepsilon}_j &= \textstyle\sum_{i \le k < j} \alpha_k + 2 \textstyle\sum_{j \le k < n-1} \alpha_k + \alpha_{n-1} & (0 \le i < j < n) \end{aligned}$$

There are $n(n-1)/2 + n(n-1)/2 + n = n^2$ roots in all.

WEYL GROUP. The Weyl group is the semi-direct product $(\pm 1)^\Delta \rtimes \mathfrak{S}_n$, acting by signed permutations $(\overline{\varepsilon}_i) \mapsto (\overline{\varepsilon}^{\pm 1}_{\sigma(i)})$.

SL(2) EMBEDDINGS. For $0 \le i < n-1$, the embedding of SL_2 into G corresponding to α_i is that for the copy of GL_n. For α_{n-1}, it is the central embedding of SL_2. For example, when $n = 2$:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto \begin{bmatrix} 1 & \circ & \circ & \circ \\ \circ & a & b & \circ \\ \circ & c & d & \circ \\ \circ & \circ & \circ & 1 \end{bmatrix}$$

This is consistent with an earlier remark, since $\mathrm{SL}_2 = \mathrm{Sp}_2$.

LIE BRACKETS. This subsection is meant to help in detailed calculations. It will give very explicit formulas for elements of the Lie algebra of upper triangular matrices in $\mathfrak{sp}_{2n}$. The non-trivial cases come in two flavours: (1) those pairs lying the embedded copy of $\mathfrak{gl}_n$. This we already know; (2) those pairs with the first in the embedded $\mathfrak{gl}_n$ and the second associated to the upper right square.

(a) The site (k, ℓ) lies on the cross-diagonal. We are looking at

$$\begin{aligned} [e_{i,j} - e_{\overline{j},\overline{i}}, e_{k,\overline{k}}] &= [e_{i,j}, e_{k,\overline{k}}] - [e_{\overline{j},\overline{i}}, e_{k,\overline{k}}] \\ &= e_{i,j} \cdot e_{k,\overline{k}} - e_{k,\overline{k}} \cdot e_{j,k} - e_{\overline{j},\overline{i}} \cdot e_{k,\overline{k}} + e_{k,\overline{k}} \cdot e_{\overline{j},\overline{i}} \end{aligned}$$

This vanishes unless $j = k$, and is in that case equal to $e_{i,\overline{k}} + e_{k,\overline{i}}$. Furthermore, $i < j = k$, so that these are two distinct terms.

(b) It does not, so that we are therefore looking at

$$[e_{i,j} - e_{\overline{j},\overline{i}}, e_{k,\ell} + e_{\overline{\ell},\overline{k}}]$$

with $0 \le i < j < n$, $0 \le k < n-1 < n \le \ell < 2n-1$. This comes in two pieces. The first is

$$\begin{aligned}[e_{i,j}, e_{k,\ell} + e_{\overline{\ell},\overline{k}}] &= [e_{i,j}, e_{k,\ell}] + [e_{i,j}, e_{\overline{\ell},\overline{k}}] \\ &= e_{i,j}\cdot e_{k,\ell} - e_{k,\ell}\cdot e_{i,j} + e_{i,j}\cdot e_{\overline{\ell},\overline{k}} - e_{\overline{\ell},\overline{k}}\cdot e_{i,j} \\ &= e_{i,j}\cdot e_{k,\ell} + e_{i,j}\cdot e_{\overline{\ell},\overline{k}}\,.\end{aligned}$$

Keep in mind that $\overline{m}$ lies in $[0, n)$ if and only if m lies in $[n, 2n)$.

The second is

$$\begin{aligned}[-e_{\overline{j},\overline{i}}, e_{k,\ell} + e_{\overline{\ell},\overline{k}}] &= -[e_{\overline{j},\overline{i}}, e_{k,\ell}] - [e_{\overline{j},\overline{i}}, e_{\overline{\ell},\overline{k}}] \\ &= e_{k,\ell}\cdot e_{\overline{j},\overline{i}} - e_{\overline{j},\overline{i}}\cdot e_{k,\ell} + e_{\overline{\ell},\overline{k}}\cdot e_{\overline{j},\overline{i}} - e_{\overline{j},\overline{i}}\cdot e_{\overline{\ell},\overline{k}} \\ &= e_{k,\ell}\cdot e_{\overline{j},\overline{i}} + e_{\overline{\ell},\overline{k}}\cdot e_{\overline{j},\overline{i}}\,.\end{aligned}$$

Comparing these expressions, we see that they vanish unless $j = k$ or $j = \overline{\ell}$. If $j = k$, two terms sum to $e_{i,\ell} + e_{\overline{\ell},\overline{i}}$. If $j = \overline{\ell}$, we get $e_{i,\overline{k}} + e_{k,\overline{i}}$. These two cases do not overlap. But it can happen that $i = k$, in which case we get $2\,e_{k,\overline{k}}$.

COROOTS. The torus T is identified with a subtorus of the diagonal torus in GL_n, and hence the lattice $L^\vee = X_*(T)$ with a subtorus of the lattice spanned by the $\varepsilon^\vee$. Cocharacters in $X_*(T)$ may therefore be identified with their images in $\mathbb{Z}^{2n}$. As a basis:

$$\begin{aligned}\overline{\varepsilon}_i^\vee &= \varepsilon_i^\vee - \varepsilon_{2n-1-i}^\vee \quad (0 \le i < n) \\ \mu^\vee &= \sum\nolimits_n^{2n-1} \varepsilon_i^\vee\,.\end{aligned}$$

The simple coroots are

$$\alpha_i^\vee = \begin{cases} \overline{\varepsilon}_i^\vee - \overline{\varepsilon}_{i+1}^\vee & 0 \le i < n \\ \overline{\varepsilon}_{n-1}^\vee & \text{otherwise.}\end{cases}$$

FUNDAMENTAL WEIGHTS.

$$\begin{aligned}\varpi_0 &= \varepsilon_0 \\ \varpi_1 &= \varepsilon_0 + \varepsilon_1 \\ &\dots \\ \varpi_{n-1} &= \varepsilon_0 + \varepsilon_1 + \cdots + \varepsilon_{n-1}\end{aligned}$$

together with μ.

THE CENTRE. The weights $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $0 \le i \le n-2$ together with ε_{n-1} make up a basis of weights, while the α_i for $0 \le i \le n-2$ together with $\alpha_{n-1} = 2\varepsilon_{n-1}$ make up a basis of roots. So $L_{\mathrm{der}}/L_\Delta = \mathbb{Z}/2$.

DOMINANT WEIGHTS. The dominant weights are integral linear combinations

$$\sum_0^{n-1} c_k \varpi_k + c_n \mu$$

with $c_k \ge 0$ for $0 \le k < n$.

The weight $\overline{\varepsilon}_0$ is the highest weight of the defining representation of G, its embedding into GL_{2n}. The other weights of this representation are the $\overline{\varepsilon}_m$ with $1 \le m < 2n$ (recall that for $m < n \le 2m$ we have $\overline{\varepsilon}_m = \mu - \varepsilon_{2n-1-m}$).

The highest weight $\overline{\varepsilon}_0 = \varpi_0$ is **minuscule**—i.e. all weights are in a single Weyl orbit.

DYNKIN DIAGRAM.

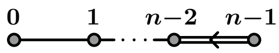
The Cartan matrix has

$$\langle \alpha_i, \alpha_j^\vee \rangle = \begin{cases} 2 & \text{if } i = j \\ -1 & \text{if } i < n-1 \text{ and } j = i \pm 1 \\ -2 & \text{if } i = n-1, j = n-2 \\ 0 & \text{otherwise.} \end{cases}$$

For example, when $n = 2$:

$$\begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -2 & 2 \end{bmatrix}$$

The Dynkin diagram is



5. Even orthogonal groups

In this section and the next I'll look at orthogonal groups—in this section for quadratic forms of even dimension, and in the next those of odd. But there are certain features in common.

ALL DIMENSIONS. Let $\Omega = \omega_N$, with $u \circ v$ the associated inner product. Let (u_i) be the standard basis, so that

(5.1) $$u_i \circ u_j = \begin{cases} 1 & \text{if } j = N-1-i \\ 0 & \text{otherwise.} \end{cases}$$

The group G_Ω is that of all X such that

$${}^tX\Omega X = c\Omega$$

for some $c = \mu(X) \neq 0$, O_Ω is the subgroup with $\mu = 1$, and SO_Ω to be subgroup of O_Ω with $\det = 1$. The maximal torus of SO is made up of the diagonal matrices

$$\begin{bmatrix} x_0 & & & & \\ & x_1 & & & \\ & & \ldots & & \\ & & & 1/x_1 & \\ & & & & 1/x_0 \end{bmatrix}.$$

The Lie algebra is the set of $N \times N$ matrices such that

$${}^tX\Omega + \Omega X = 0 \quad \text{or} \quad X = -\Omega\,{}^tX\Omega\,.$$

This translates to the condition $x_{i,j} = -x_{N-1-j,N-1-i}$. In other words, the matrix is hence skew-symmetric around the NE-SW axis where $i+j = n-1$. Its entries are determined by the ones in the region $i+j < n-1$, and the dimension of the Lie algebra is hence $N(N-1)/2$. It has as basis the matrices

$$e_{i,N-1-j} - e_{j,N-1-i}$$

for $0 \le i < j < N$. The corresponding linear transformation

$$u_k \longmapsto \begin{cases} u_i & \text{if } k = N-1-j \\ -u_j & \text{if } k = N-1-i \\ 0 & \text{otherwise.} \end{cases}$$

otherwise expressed as

$$v \longmapsto (v \circ u_j)\,u_i - (v \circ u_i)\,u_j\,,$$

because of (5.1) .

There are some types of these in common to both N odd and N even. Let $n = \lfloor N/2 \rfloor$.

(a) the semisimple elements

$$(e_{i,i} - e_{i+1,i+1}) + (e_{\overline{i}-1,\overline{i}-1} - e_{\overline{i},\overline{i}}) \quad (0 \le i < n)\,;$$

(bi) from the copy of GL_n:

$$e_{i,j} - e_{\overline{j},\overline{i}} \quad (0 \le i < j < n)\,;$$

(bii) from the upper right block

$$e_{k,\ell} - e_{\overline{\ell},\overline{k}} \quad (0 \le k < n-1 < n \le \ell < 2n-1)\,;$$

(c) transposes of elements in (b).

The group O_N contains the matrices

$$\begin{bmatrix} I_k & 0 & 0 \\ 0 & X & 0 \\ 0 & 0 & I_k \end{bmatrix} \quad (X \in \mathrm{O}_{N-2k})\,,$$

which allows induction.

EVEN DIMENSIONS. Now assume $N = 2n$. Let $\omega = \omega_n$, so that

$$\Omega = \begin{bmatrix} 0 & \omega \\ \omega & 0 \end{bmatrix}.$$

It corresponds to the quadratic form

$$2x_0x_{2n-1} + \cdots + 2x_{n-1}x_n\,.$$

The group SO_Ω contains the matrices

$$\begin{bmatrix} X & 0 \\ 0 & \omega\,{}^tX^{-1}\omega^{-1} \end{bmatrix} \qquad (X \in \mathrm{GL}_n)$$

$$\begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \qquad (\omega\,{}^tX\omega^{-1} = -X)\,.$$

This last condition means that X is skew-symmetric with respect to the NE–SW axis.

The simple roots are the $\alpha_i = \overline{\varepsilon}_i - \overline{\varepsilon}_{i+1}$ for $0 \le i < n-1$ and $\overline{\varepsilon}_{n-2} + \overline{\varepsilon}_{n-1}$. The first few come about from GL_n. The last from the nilpotent matrix

$$\begin{bmatrix} \circ & \circ & 1 & \circ \\ \circ & \circ & \circ & -1 \\ \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ \end{bmatrix}$$

in SO_4.

The positive roots of SO_Ω are the

$$\begin{aligned} &\overline{\varepsilon}_i - \overline{\varepsilon}_j \quad (0 \le i < j < n-1) \\ &\overline{\varepsilon}_i + \overline{\varepsilon}_j \quad (0 \le i < j < n-1) \end{aligned}$$

SO(4). What happens in general is best understood if one looks first at the case $n = 2$. Here, there is a homomorphism from $\mathrm{SL}_2 \times \mathrm{SL}_2$ to $\mathrm{SO}(4)$ that accounts for, among other things, the basic embeddings of SL_2 in general. In effect, although I won't elaborate, this product is $\mathrm{Spin}(4)$, and the map to be described is the canonical quotient.

This is clearest of one uses a different realization of Ω. Identify the vector space with M_2 and

$$\Omega(x) = \det(x)\colon \begin{bmatrix} x_0 & x_1 \\ x_2 & x_3 \end{bmatrix} \longmapsto x_0x_3 - x_1x_2 \,.$$

The sign is not quite right, but I'll adjust that in a moment.

The group $\mathrm{SL}_2 \times \mathrm{SL}_2$ maps to $\mathrm{SO}(\Omega)$ through the action

$$X \mapsto A\cdot X\cdot B^{-1} \,.$$

At first, I take as my basis of M_2 the matrices

$$m_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad m_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad m_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad m_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \,.$$

With respect to this basis, left multiplication takes

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{bmatrix},$$

and right multiplication takes

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} d & -c & 0 & 0 \\ -b & a & 0 & 0 \\ 0 & 0 & d & -c \\ 0 & 0 & -a & a \end{bmatrix} \,.$$

Note that

$$\begin{bmatrix} d & -c \\ -b & a \end{bmatrix} = \omega \begin{bmatrix} a & -b \\ -c & d \end{bmatrix} \omega^{-1}$$

so that if I change the right action to

$$\Omega \longmapsto \Omega\cdot\omega B^{-1}\omega$$

the matrix of the right action becomes

$$\begin{bmatrix} a & -b & 0 & 0 \\ -c & d & 0 & 0 \\ 0 & 0 & a & -b \\ 0 & 0 & -c & d \end{bmatrix} \,.$$

Now I finally adjust the sign of Ω. Instead of

$$x_0x_3 - x_1x_2$$

I want

$$x_0x_3 - x_1x_2 \,.$$

so I change the sign of my second basis matrix. The left and right action matrices now become

$$\begin{bmatrix} a & 0 & b & 0 \\ 0 & a & 0 & -b \\ c & 0 & d & 0 \\ 0 & -c & 0 & d \end{bmatrix}, \quad \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & -b \\ 0 & 0 & -c & d \end{bmatrix} \,.$$

This can be seen more intelligently by identifying $M_2 \oplus M_2$ with the four-dimensional Clifford algebra of Ω and $\mathrm{SL}_2 \times \mathrm{SL}_2$ with the spin group.

SL(2) EMBEDDINGS. The first $n-1$ embeddings from SL_2 are the ones into GL_n. That for α_n embeds the Lie algebra as it does for SO_4, which itself is embedded into every SO_{2n}.

LIE BRACKETS. These are much like the ones for $\mathfrak{sp}_{2n}$, and I won't give details.

DYNKIN DIAGRAM.

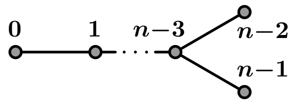
The Cartan matrix has

$$\langle \alpha_i, \alpha_j^\vee \rangle = \begin{cases} 2 & \text{if } i = j \\ -1 & \text{if } i, j < n-1 \text{ and } j = i \pm 1 \\ -1 & \text{if } i = n-3,\, j = n-1 \\ 0 & \text{otherwise.} \end{cases}$$

For $n = 2$:

$$\begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & -1 \\ & -1 & 2 & 0 \\ & -1 & 0 & 2 \end{bmatrix}$$

The Dynkin diagram is



There is an automorphism swapping ε_{n-1} and $-\varepsilon_n$, hence α_{n-1} and α_n, which comes from conjugation by the matrix

$$\begin{bmatrix} I & \circ & \circ & \circ \\ \circ & \circ & 1 & \circ \\ \circ & 1 & \circ & \circ \\ \circ & \circ & \circ & I \end{bmatrix}$$

in O_{2n}, which swaps the two embeddings of SL_2 exhibited above.

The fundamental weights of the covering spin groups:

$$\varpi_i = \begin{cases} \overline{\varepsilon}_1 + \cdots + \overline{\varepsilon}_i & \text{if } 0 \le i \le n-3 \\ (1/2)(\overline{\varepsilon}_1 + \cdots + \overline{\varepsilon}_{n-3} + \overline{\varepsilon}_{n-2} - \overline{\varepsilon}_{n-1}) & \text{if } i = n-2 \\ (1/2)(\overline{\varepsilon}_1 + \cdots + \overline{\varepsilon}_{n-3} + \overline{\varepsilon}_{n-2} + \overline{\varepsilon}_{n-1}) & \text{if } i = n-1 \end{cases}$$

CENTER. The center of GO is made up of the scalar matrices. That of SO_{2n} is made up of $\{\pm I\}$.

6. Odd orthogonal groups

Let $\omega = \omega_n$ and

$$\Omega = \begin{bmatrix} \circ & \circ & \omega \\ \circ & 1 & \circ \\ \omega & \circ & \circ \end{bmatrix}$$

It is the matrix of the quadratic form

$$2x_0x_{2n} + \cdots + 2x_{n-1}x_{n+1} + x_n^2\,.$$

The elements in the Lie algebra are those X in M_{2n+1} such that ${}^tX\Omega+\Omega X=0$. In addition to types occurring in both even and odd dimensions, we have here the matrices

$$\begin{bmatrix} 0 & x & 0 \\ 0 & 0 & -{}^tx\,\omega \\ 0 & 0 & 0 \end{bmatrix}.$$

The group GO contains matrices

$$\begin{bmatrix} I & x & -x\cdot{}^tx\,\omega/2 \\ 0 & 1 & -{}^tx\,\omega \\ 0 & 0 & I \end{bmatrix}$$

$$\begin{bmatrix} A & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega\,{}^tA^{-1}\omega \end{bmatrix} \qquad (A\in \backslash GL_n)$$

$$\begin{bmatrix} I & 0 & X \\ 0 & 1 & 0 \\ 0 & 0 & I \end{bmatrix} \qquad (\omega\,{}^tX\omega=-X)$$

The simple roots are $\alpha_i=\varepsilon_i-\varepsilon_{i+1}$ for $0\le i<n-1$ and ε_{n-1}. The first several come about from GL_n, the last from the nilpotent matrix

$$\begin{bmatrix} \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & 1 & \circ & \circ \\ \circ & \circ & \circ & -1 & \circ \\ \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & \circ \end{bmatrix}$$

on which the torus acts as x_{n-1}.

SO(3). I want to explain here the map from SL_2 to $\mathrm{SO}(3)$ that identifies SL_2 with $\mathrm{Spin}(\Omega)$ and PGL_2 with what is sometimes called $\mathrm{Pin}(\Omega)$.

6.1. Proposition. *There exists a unique homomorphism from* SL_2 *to* $\mathrm{SO}(3)$ *taking*

$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & x & -x^2/2 \\ 0 & 1 & -x \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} t & 0 \\ 0 & 1/t \end{bmatrix} \longmapsto \begin{bmatrix} t^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/t^2 \end{bmatrix}$$

$$w=\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \longmapsto \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Proof. The standard three-dimensional representation of SL_2 acts on symmetric vectors e_0^2, e_0e_1, e_1^2. With respect to the basis with $-e_2^1/2$ the matrices of this representation are as above.

We have already seen that the image is contained in $\mathrm{SO}(3)$. Uniqueness is because these matrices generate SL_2. ∎

Explicit formulas for an arbitrary matrix in SL_2 are messy, but for any fixed matrix they can be found from the Bruhat decomposition $\mathrm{SL}_2=B\sqcup BwB$.

SL(2) HOMOMORPHISMS. The first $n-1$ maps from SL_2 arise through embeddings into GL_n, while the last through the map into $\mathrm{SO}(3)$.

Why does the copy of SL_2 corresponding to α_{n-1} not embed, while those for the others do? For $n = 5$ this is OK because the standard copy of SL_2 contains the centre, while that of the central SL_2 does not. So it is the first that collapses. This agrees with the fact that although the root systems for B_2 and C_2 are the same, the indexing of simple roots is different.

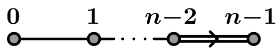
DYNKIN DIAGRAM. The Cartan matrix has

$$\langle \alpha_i, \alpha_j^\vee \rangle = \begin{cases} 2 & \text{if } i = j \\ -1 & \text{if } j < n-1 \text{ and } j = i \pm 1 \\ -2 & \text{if } j = n-1, i = n-2 \\ 0 & \text{otherwise.} \end{cases}$$

For example when $n = 2$:

$$\begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -2 \\ & & -1 & 2 \end{bmatrix}$$

The diagram is



FUNDAMENTAL WEIGHTS. Of the covering spin group.

$$\begin{aligned} \varpi_i &= \varepsilon_0 + \cdots + \varepsilon_i \quad (0 \le i < n-1) \\ \varpi_{n-1} &= (1/2)(\varepsilon_0 + \cdots + \varepsilon_{n-1}) \end{aligned}$$

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