Clifford algebras and spinors

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This essay will present a brief outline of the theory of Clifford algebras, together with a small amount of material about quadratic forms. I follow loosely the well known book **Geometric algebra** by Emil Artin, but with some simplifying modifications that I saw originally in lecture notes by Raoul Bott (dating from 1962/3), subsequently included in [Atiyah-Bott-Shapiro:1964].

In the first section, I'll recall a few facts about quadratic forms and orthogonal groups. In the second, I'll discuss quaternion algebras, which will play a major role later on. In the third, I'll look at Clifford algebras and spin groups. In the next, I'll look at a number of examples. Finally, in the last section I'll make some observations about spin groups and root systems.

Throughout, unless specified otherwise, F will be an arbitrary field of characteristic other than two, and Q will be a non-degenerate quadratic form on the F-vector space V.

Contents

1. Quadratic spaces

A quadratic form on a vector space V over F is a homogeneous function Q of degree 2 such that the inner product

$$u \circ v = (1/2)\,(Q(u+v) - Q(u) - Q(v))$$

is bilinear. Thus

$$Q(u+v) = Q(u) + 2(u \circ v) + Q(v), \quad u \circ u = Q(u)\,.$$

The bilinear form $u \circ v$ is associated to a linear map β from V to its dual $\widehat{V} = \mathrm{Hom}_F(V, F)$, defined by the condition

$$\langle \beta(u), v \rangle = u \circ v\,.$$

Because Q is assumed to be non-degenerate, this is an isomorphism.

A vector v is called **isotropic** of $Q(v) = 0$, otherwise **anisotropic**.

If V is given a basis $\{e_i\}$, the form Q corresponds to a symmetric matrix $M = M_Q$ such that

$$u \circ v = {}^t u M v\,.$$

The (i, j) entry in M is $e_i \circ e_j$. Non-degeneracy means that M is non-singular. A change of basis changes M to some ${}^t X M X$, which implies that the determinant of M modulo $(F^\times)^2$ is a well defined invariant of the form. It is usually called its discriminant, but this terminology conflicts with usage in number theory, and I'll just call it the **determinant** of the form. (Perhaps 'reduced determinant' would be better.)

If X is any subset of V, let

$$X^\perp = \{v \in V \mid v \circ x = 0 \text{ for all } x \in X\}\,.$$

It is always a vector subspace. The intersection $X \cap X^\perp$ may be non-trivial, in which case it is made up of isotropic vectors. The assumption that Q is non-degenerate means that $V^\perp = \{0\}$.

1.1. Lemma. *If V has dimension n and U is a linear subspace of V of dimension d, then $U^\perp$ has dimension $n-d$.*

Proof. It is the kernel of the composition of β with restriction to U.

1.2. Lemma. *If U is a subspace of V on which the restriction of Q is non-degenerate, then $V = U \oplus U^\perp$.*

Proof. Because $U \cap U^\perp = \{0\}$.

As a special case:

1.3. Lemma. *If $Q(v) \neq 0$ then every vector w in V can be expressed as $c \cdot v + u$ with u in $v^\perp$.*

Proof. Familiar. Explicitly

$$c = \frac{w \circ v}{Q(v)}\,.$$

With our assumption that the characteristic of F is not two, one can always choose a basis that makes M_Q diagonal:

1.4. Proposition. *There exists in V a basis (v_i) such that $v_i \circ v_j = 0$ if $i \neq j$.*

Proof. By induction on the dimension n of V. If $n = 1$, the result is trivial. Otherwise, choose v_n such that $Q(v_n) \neq 0$, and applying the previous Lemma write $V = F \cdot v_n \oplus v_n^\perp$. Apply induction.

Remark. In the arithmetic theory of quadratic forms, the bilinear form associated to a quadratic form Q is

$$Q(x+y) - Q(x) - Q(y)\,,$$

and hence is lacking the factor $1/2$. For example, the quadratic form $x^2 + y^2$ is associated in this way to the inner product $2x_1x_2 + 2y_1y_2$ rather than $x_1x_2 + y_1y_2$. This is natural—modulo 2 this form factors as $(x+y)^2$ and is rather degenerate, and there should be some way to take this into account. In this theory, a quadratic form is not defined by an integral matrix. For example, the form $x^2 + xy + y^2$, which is the norm form on the algebraic integers in $\mathbb{Q}(\zeta_3)$, is associated to the matrix

$$\begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}.$$

If the characteristic is not 2, there is an equivalence between quadratic forms and symmetric bilinear forms. Without this equivalence the theory of Clifford algebras would look quite different.

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THE ORTHOGONAL GROUP. The orthogonal group $\mathrm{O}(Q)$ is that of all X in $\mathrm{GL}(V)$ preserving Q, and $\mathrm{SO}(Q)$ is the intersection $\mathrm{O}(Q) \cap \mathrm{SL}(V)$. If V is given a coordinate system, the group $\mathrm{O}(Q)$ is that of all matrices X such that

$${}^tX\,MX = M \quad (M = M_Q)\,.$$

It follows from this formula that $\det(X) = \pm 1$, so that $\mathrm{O}(Q)/\mathrm{SO}(Q)$ has at most two elements. If $Q(v) \neq 0$ the map

$$r_v\colon\ u \longmapsto u - 2\left(\frac{u \circ v}{Q(v)}\right) \cdot v$$

is an orthogonal reflection in the hyperplane $v^\perp$. The determinant of a reflection is -1, so $\mathrm{O}(Q)/\mathrm{SO}(Q) = \{\pm 1\}$.

The Lie algebra $\mathfrak{so} = \mathfrak{so}_Q$ if SO_Q is the space of all matrices X such that $I + \varepsilon X$ lies in SO_Q for an infinitesimal ε. This translates to the condition

$${}^tXM + MX = 0\,.$$

The Lie bracket on $\mathfrak{so}$ is that inherited from $\mathrm{End}(V)$:

$$[x, y] = xy - yx\,.$$

The exterior product $\bigwedge^2 V$ is canonically defined to be the quotient of $\bigotimes^\bullet V$ by the two-sided ideal generated by tensors $u \otimes v - v \otimes u$. The exterior product $u \wedge v$ is the image of $u \otimes v$. Since 2 is invertible, there is an $\mathfrak{S}_2$-equivariant section of this quotient, taking $u \wedge v$ to

$$(1/2)(u \otimes v - v \otimes u)\,,$$

which I'll identify with $u \wedge v$ from now on. There is a canonical isomorphism of $V \otimes \widehat{V}$ with $\mathrm{End}(V)$, taking $v \otimes \widehat{v}$ to the endomorphism

$$u \longmapsto \langle \widehat{v}, u \rangle v\,.$$

The linear isomorphism of V with $\widehat{V}$, composed with this, gives a linear map from $\bigwedge^2 V$ to $\mathrm{End}(V)$:

(1.5) $$\tau_{u \wedge v}(w) = (1/2)\big((v \circ w)u - (u \circ w)v\big)\,.$$

The group SO_Q acts in the natural way on $\bigwedge^2 V$, and by the adjoint action on $\mathfrak{so}_Q \subset \mathrm{End}(V)$.

1.6. Proposition. *The map taking $u \wedge v$ to $\tau_{u\wedge v}$ is an SO_Q-equivariant isomorphism of $\bigwedge^2 V$ with $\mathfrak{so}_Q$.*

In other words, there is an avatar of $\mathfrak{so}_Q$ which in some sense is independent of Q.

Proof. First, to verify that the image is contained in $\mathfrak{so}$. This requires that

$$(Tu) \circ v + u \circ (Tv) = 0\,,$$

if $T = \tau_{a \wedge b}$. For this, compare:

(1.7) $$\begin{aligned} T(u) \circ v &= (1/2)(b \circ u)(a \circ v) - (a \circ u)(b \circ v) \\ u \circ T(v) &= (1/2)(b \circ v)(a \circ u) - (a \circ v)(b \circ u)\,. \end{aligned}$$

Equivariance is immediate, since the map from $V \otimes V$ to $V \otimes \widehat{V}$ is $\mathfrak{so}$-equivariant and the map from $V \otimes \widehat{V}$ to $\mathrm{End}(V)$ is equivariant with respect to the diagonal action of GL_n.

Why an isomorphism? We can construct an inverse. If T lies in $\mathfrak{so}_Q$ then

$$A(u, v) = T(u) \circ v = -u \circ T(v) = -A(v, u)\,.$$

This means that the bilinear form $A(u, v)$ is alternating, hence in $\bigwedge^2 \widehat{V}$. But β identifies V with $\widehat{V}$, and $\beta \otimes \beta$ identifies $\bigwedge^2 V$ with $\bigwedge^2 \widehat{V}$, and (1.7) shows that $\beta^{-1} \otimes \beta^{-1}$ applied to $A(T_{a\wedge b})$ gives $a \wedge b$.

The following gives a formula for the Lie bracket on $\bigwedge^2 V$, but also just expresses the equivariance of the map from $\bigwedge^2 V$ to $\mathfrak{so}_Q$.

1.8. Proposition. *We have*

$$[\tau_{a\wedge b}, \tau_{c \wedge d}] = \tau_{u,d} + \tau_{c,v}) \quad (u = \tau_{a\wedge b}(c),\ v = \tau_{a \wedge b}(d)\,.$$

HYPERBOLIC SPACES. In general, V will contain isotropic vectors. The basic example is the **hyperbolic plane** H_2, with $V = F^2$ and $Q(x, y) = xy$. With respect to a hyperbolic basis e, f its matrix is

$$\begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}.$$

Its orthogonal group consists of all matrices

$$\begin{bmatrix} t & 0 \\ 0 & 1/t \end{bmatrix}, \quad \begin{bmatrix} 0 & t \\ 1/t & 0 \end{bmatrix}.$$

The basis $(e+f)/2$, $(e-f)/2$ is often more convenient to work with. The matrix of the form is then

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} .$$

Hyperbolic planes occur whenever there are isotropic vectors. More precisely:

1.9. Lemma. *If v is an isotropic vector than there exists an isotropic u in V such that $u \circ v = 1$.*

In other words, the pair u, v span a hyperbolic plane.

Proof. Because of non-degeneracy, there exists w in V such that $v \circ w = 1$. But then $(cv + w) \circ v = 1$ for all c, and the equation

$$Q(cv + w) = cv \circ w + Q(w) = 0$$

may be solved for c.

The space H_{2m} is the orthogonal sum of m copies of H_2. There are several convenient choices of a coordinate system on a hyperbolic space. I'll choose one in which the form is

$$x_1 x_{n+1} + \cdots + x_n x_{2n} ,$$

whose matrix is

$$(1/2) \cdot \begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix} .$$

A subspace of V is called **isotropic** if Q vanishes identically on it. So in this example the subspace of vectors whose last m coordinates vanish is isotropic, as is that of vectors whose first ones do.

A repeated application of this and Lemma 1.2 proves:

1.10. Lemma. *If $\{u_i\}$ is the basis of an isotropic subspace U of V, there exist vectors w_i in V spanning an isotropic subspace W such that*

$$u_i \circ w_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

In these circumstances,

$$V = U \oplus W \oplus (U \oplus W)^\perp ,$$

and the subspace $U \oplus W$ is the direct sum of hyperbolic planes.

This leads to:

1.11. Proposition. *Every quadratic space may be represented as a direct sum of a hyperbolic space H^{2m} and an anisotropic subspace.*

Because of a theorem of Ernst Witt, these summands are unique up to isomorphism, but I shan't need that.

In H_{2m} a matrix

$$\sigma = \begin{bmatrix} I & S \\ 0 & I \end{bmatrix} \qquad \textbf{(1.12)}$$

lies in $O(Q)$ if and only if S is skew-symmetric. If $1 \le i \le n$ then $\sigma(e_i) = e_i$, while otherwise $\sigma(e_i) - e_i$ lies in the space spanned by the e_i with $i \le m$. Thus in this case (i) all vectors fixed by σ are isotropic and (ii) for any vector v, $\sigma(v) - v$ is isotropic. Such a linear transformation is very special.

1.13. Lemma. *Conversely, if σ lies in $\mathrm{O}(Q)$ and satisfies conditions (i) and (ii), then the quadratic space is hyperbolic and in a suitable coordinate system the formula (1.12) holds.*

Proof. In a few steps.

Step 1. Let W be the image of the linear map $v \mapsto \sigma(v) - v$. The form Q vanishes identically on W, so it has dimension $d \le n/2$. I claim now that σ *fixes every v in $W^\perp$*, which is of dimension $n - d$.

Suppose v to lie in $W^\perp$. Then for every u in V

$$\begin{aligned}(\sigma(v) - v)\circ u &= \sigma(v)\circ u - v\circ u \\ &= v\circ\sigma^{-1}(u) - v\circ u \\ &= v\circ(\sigma^{-1}(u) - u) \\ &= v\circ(x - \sigma(x)) \\ &= 0\end{aligned}$$

where $x = \sigma^{-1}(u)$. But since Q is non-degenerate, $v = 0$.

Step 2. Since σ fixes no anisotropic vector, all vectors in $W^\perp$ are isotropic, and must have dimension at most $\lfloor n/2 \rfloor$. Therefore $n - d = d$, $n = 2d$, and $W = W^\perp$ is a maximal isotropic subspace of V. *The space V itself must be a hyperbolic space of dimension $2d$*, a direct sum of hyperbolic planes.

Step 3. Now choose a basis $\{w_i\}$ of W and a dual basis $\widehat{w}_i$ of an isotropic complement U. The matrix of σ must be of the form

$$\begin{bmatrix} I & S \\ 0 & I \end{bmatrix}.$$

This result is complemented by:

1.14. Lemma. *Suppose σ to be any element of $\mathrm{O}(Q)$ such that $\sigma(v) - v$ is isotropic whenever v is anisotropic. The $\sigma(v) - v$ is always isotropic.*

This should be intuitive. For example, suppose $F = \mathbb{R}$. The set of all $\sigma(v) - v$ is a linear subspace W of V. The anisotropic vectors in V are dense, and therefore their image in W is also dense. But Q is continuous, so it vanishes in all of W.

If a subset Ω of $\mathbb{R}^n$ is open and v is an arbitrary vector in $\mathbb{R}^n$, then perturbations of v will generically lie in Ω. This suggests the following proof.

Proof. Suppose ν to be isotropic. Then for any anisotropic v and ε in F

$$Q(\nu + \varepsilon v) = \varepsilon\, \nu\circ v + \varepsilon^2 Q(v)\,.$$

But since V is non-degenerate the vector space $\nu^\perp$ has dimension $n - 1 > 1$, so we may find an anisotropic v in it, and then

$$Q(\nu + \varepsilon v) = \varepsilon^2 Q(v)\,.$$

Thus $Q(\nu + \varepsilon v)$ is anisotropic for all $\varepsilon \neq 0$. For $\varepsilon \neq 0$

$$\begin{aligned}0 &= Q(\sigma(\nu + \varepsilon v) - (\nu + \varepsilon v)) \\ &= Q((\sigma(\nu) - \nu) + \varepsilon\,(\sigma(v) - v)) \\ &= Q(\sigma(\nu) - \nu) + \varepsilon\,(\sigma(\nu) - \nu)\circ(\sigma(v) - v)\,.\end{aligned}$$

But $|F| > 2$, so each term in this linear function of ε must vanish, which proves the claim.

As a consequence of these:

1.15. Lemma. *If σ is in $\mathrm{O}(Q)$ and $\det(\sigma) = -1$, then either (i) σ fixes an anisotropic vector or (ii) there exists an anisotropic vector v such that $\sigma(v) - v$ is anisotropic.*

This allows us to prove the following, which is Theorem 3.20 of [Artin:1966]. It is crucial in understanding spinors.

1.16. Theorem. *In a non-degenerate quadratic space of dimension n, every transformation σ in $\mathrm{O}(Q)$ can be expressed as a product of at most n reflections.*

Proof. The proof will be by induction on n, and will take a while.

If σ fixes an anisotropic vector v (that is to say, such that $Q(v) \neq 0$) then it is essentially an isometry in the space $v^\perp$ of dimension $n-1$, so we may apply induction to see that it is a product of at most $n-1$ reflections.

If there exists an anisotropic vector v such that $u = \sigma(v) - v$ is anisotropic, let ρ be the reflection r_u, which will take v to $\sigma(v)$, and $\rho\sigma$ will be an isometry that fixes v, so we may apply the earlier case.

According to Lemma 1.15, if neither of these conditions holds, then σ lies in SO_Q. Suppose now that ρ is aany reflection at all. Then $\det(\rho\sigma) = -1$, so according to what we have seen, we may write

$$\rho\sigma = r_1 \ldots\, r_m$$

with $m \leq n$. Since $\det(\rho\sigma) = -1$, m must be odd, hence $m \leq n-1$. Then $\sigma = \rho r_1 \ldots\, r_m$.

Remark. The expression for σ as a product of reflections is certainly not unique. The proof depends on finding certain anisotropic vectors, and any proposed algorithm for finding an expression would seem to have a probabilistic flavour. Is there something deterministic? Some way to classify all reflection products? There is one situation in which a related question is interesting—in which an element of a Coxeter group is expressed as a product of a small numer of root reflections.

2. Quaternion algebras

Suppose E/F to be a (separable) quadratic extension. It might even be the algebra $F \oplus F$, with a suitable embedding of F. Let $x \mapsto \overline{x}$ be an involutory automorphism fixing elements in the copy of F. For α in $F^\times$, let $B = B_{E,\alpha}$ be the algebra generated over F by E and an element σ, with relations

$$x\sigma = \sigma\overline{x}, \quad \sigma^2 = \alpha\,.$$

The field F embeds into it, and its image is the centre of B.

If c lies in $E^\times$, then changing σ to $c\sigma$ determines an isomorphic algebra, and α changes to $c\overline{c}\alpha$, so the isomorphism class of $B_{E,\alpha}$ depends only on the image of α in $F^\times/NE^\times$.

The field E acts on the right on B, so the identification with E^2 is the map

$$(x, y) \longmapsto x + \sigma y\,.$$

Acting by multiplication on the left, B commutes with this right action of E. This gives us an embedding of B into $M_2(E)$. Explicitly, $x + \sigma y$ takes

$$\begin{aligned} 1 &\longmapsto x + \sigma y \\ \sigma &\longmapsto x + \sigma y \sigma \\ &\qquad = x\sigma + \sigma^2\overline{y} \\ &\qquad = \sigma\overline{x} + \alpha\overline{y}\,. \end{aligned}$$

In other words, it corresponds to the matrix

$$\mu(x + \sigma y) = \begin{bmatrix} x & \alpha\overline{y} \\ y & \overline{x} \end{bmatrix}.$$

In conformity with the theory of Galois descent, these are the matrices such that $\gamma^{-1}\overline{X}\gamma = X$, with

$$\gamma = \begin{bmatrix} 0 & \alpha \\ 1 & 0 \end{bmatrix}.$$

The embedding of B into $M_2(E)$ determines by restriction the trace operator $X \mapsto \mathrm{trace}\;(X)$. The determinant of $\mu(x + \sigma y)$ is

$$N_{B/F}(x) = x\overline{x} - \alpha y\overline{y}\,.$$

Both the trace and the norm lie in F, and the norm is a multiplicative homomorphism from $B^\times$ to $F^\times$. Considering B as a vector space over F, this gives us a non-degenerate quadratic form of dimension four.

The field E is equal to $F(\sqrt{\beta})$ for some β. Say $\lambda = \sqrt{\beta}$. Thus B has as basis

$$1,\ \lambda,\ \sigma,\ \sigma\lambda$$

whose squares are

$$1,\ \beta,\ \alpha,\ -\alpha\beta\,.$$

The norm form is

$$N(x,y,z,w) = x^2 - \beta y^2 - \alpha z^2 + \alpha\beta w^2\,.$$

This form represents 1—i.e. has 1 in its image—and has determinant 1, and in fact these two properties characterize the four-dimensional forms defined by quaternion algebras—every non-degenerate quadratic form with determinant 1 is equal to a multiple of the norm form of some quaternion algebra. What is not quite transparent, as far as I can see, is that the quaternion algebra is uniquely determined by the quadratioc form. Later on, we shall see this to be true.

If $F = \mathbb{R}$, $E = \mathbb{C}$, and $\alpha = -1$ then B is Hamilton's quaternions $\mathbb{H}$.

Define the **conjugate** of $z = x + \sigma y$ to be $\overline{\overline{z}} = \overline{x} - \overline{y}\sigma$.

2.1. Lemma. *Conjugation is an involutory anti-automorphism.*

Proof. Let $z = x + \sigma y$, $X = \mu(z)$. Then

$$\mu(X) = \begin{bmatrix} x & \alpha\overline{y} \\ y & \overline{x} \end{bmatrix}, \quad \mu(\overline{\overline{X}}) = \begin{bmatrix} \overline{x} & -\alpha y \\ -\overline{y} & x \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1/\alpha \end{bmatrix} {}^t\overline{\overline{X}} \begin{bmatrix} 1 & 0 \\ 0 & -\alpha \end{bmatrix} \mu(\overline{z})$$

We have

$$(x+\sigma y)(\overline{x} - \overline{y}\sigma) = x\overline{x} - \sigma y\overline{y}\sigma + \sigma y\overline{x} - x\overline{y}\sigma\,.$$

The norm map can then therefore be expressed as $N(z) = z\overline{\overline{z}}$.

2.2. Proposition. *If α lies in $NE^\times$ then $B_{E,\alpha}$ is isomorphic to $M_2(F)$, and otherwise it is a division algebra.*

In particular, if $E = F \oplus F$ then B is isomorphic to $M_2(F)$. If E is a field, then $E \otimes E$ is isomorphic to $E \oplus E$, so that $B \otimes K$ is isomorphic to $M_2(K)$ for any extension field K/F into which E embeds.

Proof. Suppose λ to be a generator of E/F, satisfying the quadratic equation

$$\lambda^2 - a\lambda + b = 0\,.$$

Take $1, \lambda$ as a basis of E/F. Since

$$\begin{aligned} \lambda\cdot 1 &= \lambda \\ \lambda\cdot\lambda &= a\lambda - b \end{aligned}$$

we get an embedding of E into $M_2(F)$ taking

$$\lambda \longmapsto \begin{bmatrix} 0 & -b \\ 1 & a \end{bmatrix}.$$

But then

$$\overline{\lambda} \longmapsto \begin{bmatrix} a & b \\ -1 & 0 \end{bmatrix}$$

If

$$\sigma = \begin{bmatrix} 1 & a \\ 0 & -1 \end{bmatrix}$$

then $\sigma^2 = I$ and

$$\sigma \begin{bmatrix} a & b \\ -1 & 0 \end{bmatrix} \sigma^{-1} = \begin{bmatrix} 0 & -b \\ 1 & a \end{bmatrix}$$

so that $\sigma\lambda\sigma^{-1} = \overline{\lambda}$, and $B_{E,1}$ is isomorphic to $M_2(F)$.

Now suppose α does not lie in $NE^\times$. Then from one can see that $N(z) \neq 0$ unless $z = 0$, and $z \neq 0$ has as inverse $\overline{z}/N(z)$.

The data (E, α) determining B are not at all unique. If K/F is any quadratic extension embedded in B there exists some β in B such that (K, β) also defines B. It commonly happens that B contains lots of quadratic extensions. For $\mathfrak{p}$-adic fields there is exactly one quaternion division algebra, into which all quadratic field extensions embed. Only $M_2(F)$ contains a copy of $F \oplus F$.

What is the orthogonal group of $N_{B/F}$? First of all, if ν lies in $N^1_{E/F}$ then multiplication

$$(\boldsymbol{a}) \qquad X \longmapsto \nu\, X$$

preserves the norm. So does conjugation by any element of $B^\times$:

$$(\boldsymbol{b}) \qquad X \longmapsto \mu X \mu^{-1}\,.$$

Since

$$\mu \cdot \nu X \cdot \mu^{-1} = \mu\nu\mu^{-1} \cdot \mu X \mu^{-1}$$

the two types of orthogonal transformations in (a) and (b) give rise to a homomorphism from the semi-direct product $N^1_{B/F} \rtimes (B^\times/F^\times)$ to the special orthogonal group of $N_{B/F}$.

2.3. Proposition. *This homomorphism from $N^1_{B/F} \rtimes (B^\times/F^\times)$ to $\mathrm{SO}(N_{B/F})$ is an isomorphism.*

The orthogonal group contains in addition the conjugation operator.

Any subspace of B inherits a quadratic form, of course. This applies to possibly several quadratic field extensions, but there is also a canonical one of dimension three, on the kernel B_0 of the trace from B to F. Let $N_{B,0}$ be the restriction of $N_{B/F}$ to B_0. Conjugations by elements of $B^\times$ take this space to itself and preserve $N_{B,0}$.

2.4. Proposition. *The canonical homomorphism from $B^\times/F^\times$ to $\mathrm{SO}(N_{B,0})$ is an isomorphism.*

Remarks. The group $F^\times$ may be considered as the F-rational points of a one-dimensional algebraic subvariety

$$\{(x, y) \,|\, xy = 1\}$$

of F^2. Similarly $E^\times$ may be regarded as the F-rational points of a two-dimensional variety, and $B^\times$ as those on an algebraic variety of dimension four. Related algebraic groups defined over F include $B^\times/F^\times$ and the norm-kernel $N^1_{B/F}$. The embedding of $N^1_{B/F}$ into $B^\times$ induces a homomorphism from $N^1_{B/F}$ to $B^\times/F^\times$, whose kernel is $\{\pm 1\}$. As a homomorphism of algebraic groups it is a surjection, which means that the associated map of points rational over an algebraic closure is surjective, but it is not generally a surjection of F-rational points.

This is already apparent if $B = M_2(\mathbb{R})$, since the image of $\mathrm{SL}_2(\mathbb{R})$ in $\mathrm{PGL}_2(\mathbb{R})$ has index two. Of course the homomorphism of $\mathbb{C}$-rational points, from $\mathrm{SL}_2(\mathbb{C})$ to $\mathrm{GL}_2(\mathbb{C})/\mathbb{C}^\times$, is surjective.

3. Spinors

The **Clifford algebra** $C = C(V, Q)$ is the quotient of $\bigotimes^{\bullet} V$ by the two-sided ideal generated by tensors of the form $v \otimes v - Q(v)$. It inherits a multiplication from the tensor algebra. Now

$$Q(u+v) - Q(u) - Q(v) = 2(u \circ v) = (u+v)^2 - u^2 - v^2 = u \cdot v + v \cdot u ,$$

so that if $u \circ v = 0$ then $u \cdot v = -v \cdot u$. Thus u and v do not generally commute.

3.1. Lemma. *Suppose V to have dimension n, and let (e_i) be an orthogonal basis. The algebra C has dimension 2^n, and a basis is made up of the images e_S of the tensors*

$$e_{i_1} \otimes \ldots \otimes e_{i_k}$$

as S runs through ordered subsets $\{i_1 < \ldots < i_k\}$ of $[1, n]$.

Proof. This is because C may be filtered by order, and the graded module is the exterior algebra.

There is a simple formula for the product of two basis elements. Define an operation on ordered subsets of $[1, n]$:

$$S \dot{+} T = (S \cup T) - (S \cap T) .$$

In effect, this is bit-wise addition modulo 2. Define also a function on pairs from $[1, n]$:

$$(s,t) = \begin{cases} 1 & \text{if } s \le t \\ -1 & \text{if } s > t. \end{cases}$$

3.2. Lemma. *For S, T ordered subsets of $[1, n]$*

$$e_S \cdot e_T = \prod_{S \times T} (s,t) \cdot \prod_{S \cap T} Q(e_u) \cdot e_{S \dot{+} T} .$$

Proof. By induction on $|S|$, since $e_s \cdot e_t = -e_t \cdot e_s$ if $s \ne t$.

3.3. Corollary. *For S, T*

$$e_S \, e_T \, e_S^{-1} = (-1)^{|S|\,|T| - |S \cap T|} e_T .$$

3.4. Corollary. *If n is even, the center of $C(V, Q)$ is the embedded copy of F. Otherwise, it has as basis 1 and*

$$e_{[1,n]} = \prod\nolimits_1^n e_i .$$

The ring C is itself graded by parity of degree, defining C^0 and C^1. There exists a canonical embedding of V into C^1, and I shall identify V with its image. It generates C. The space C^0 is a ring, and C^1 is a free module over C^0 of rank one.

Example. Say $V = F \cdot e$, $Q(x \cdot e) = ax^2$. Then

$$\begin{aligned} C^0 &= F \\ C^1 &- F \cdot e \\ e^2 &= -a . \end{aligned}$$

If $-a = b^2$ is a square in F, let $u = (1 + e/c)/2$, $v = (1 - e/b)/2$. Then $u^2 = u$, $v^2 = v$, $u + v = 1$. The map

$$x \longmapsto (x \cdot u, x \cdot v)$$

will then be an isomorphism of C with the split ring $F \oplus F$. If $-a$ is not a square then C will be a quadratic field extension of F.

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The Clifford algebra possesses a natural universal property, which is immediate from its definition:

3.5. Lemma. *Any map φ from V to an F-algebra R such that $\varphi(v)^2 = Q(v)$ induces a unique homomorphism from $C(V, Q)$ to R.*

• Because of this, since $Q(-v) = Q(v)$, there exists a unique involutory automorphism

$$\alpha\colon\ C \longrightarrow C, \quad v \longmapsto -v\,.$$

The parity grading of $C(V)$ is the eigenspace decomposition of α.

(3.6)
$$\alpha(e_I) = (-1)^n e_I \quad (n = |I|)\ .$$

• There is also the **transpose**

$$x \mapsto {}^t x, \quad e_1 \otimes \ldots \otimes e_k \mapsto e_k \otimes \ldots \otimes e_1\,,$$

which is an anti-automorphism. How it acts can also be explicitly determined:

(3.7)
$${}^t e_I = (-1)^{n(n-1)/2} e_I\,.$$

• And finally, there is the **conjugation** anti-automorphism

$$x \mapsto \overline{x} = \alpha({}^t x) = {}^t\alpha(x)\,.$$

(3.8)
$$\overline{e}_I = (-1)^{n(n+1)/2} e_I\,.$$

For every x in $C(Q)$, define its **norm**

$$N(x) = x \cdot \overline{x}\,.$$

Example. Suppose V to be $\mathbb{R}^2$ with basis i, j and $Q(xi + yj) = -(x^2 + y^2)$. If $k = ij$ in $C(Q)$, then $k^2 = -1$ and the algebra $C(Q)$ may then be identified with Hamiltonian's quaternions $\mathbb{H}$. The conjugate of $w + xi + yj + zk$ is $w - xi - yj - zk$,

$$N(w + xi + yj + zk) = w^2 + x^2 + y^2 + z^2\,,$$

and the norm map is a homomorphism.

∘————- ∘

In general, the norm map does not behave in such a simple fashion as it does in this example. The **Clifford group** of $C = C(V, Q)$ is the subgroup Γ of x in $C^\times$ such that the map

$$\rho(x)\colon\ v \mapsto \alpha(x) v x^{-1}$$

takes V to itself. It is a group, stable under transpose and conjugation, and ρ is a group homomorphism into $\mathrm{GL}(V)$.

Remark. Here I am following [Bott:1962-3]. [Artin:1966] defines a version of ρ by ordinary conjugation, not this twisted conjugation. This causes him a fair amount of difficulty with signs. (There is an illuminating discussion of this point in §3 of Brian Conrad's Math 210 lecture notes.) The basic problem with Artin's definition is that, when n is odd, the center of the Clifford algebra—and hence the kernel of his ρ—is made up of linear combinations of elements in C^0 and C^1. Lack of homogeneity makes many formulations and proofs rather awkward.

What is the image of ρ? What is its kernel? We shall work slowly towards proving:

3.9. Theorem. *The homomorphism ρ is a surjection from Γ onto $\mathrm{O}(Q)$. The kernel of the map consists of the scalars $F^\times$.*

There are three things to be shown: (a) that the image contains $\mathrm{O}(Q)$, (b) that its kernel is the scalars, and (c) that the image is contained in $\mathrm{O}(Q)$. I show these in several steps.

Step 1. The vector v is a unit in $C(V,Q)$ if and only if $Q(v) \neq 0$, in which case

$$v^{-1} = v/Q(v)\,.$$

Then $\alpha(v) = -v$ and

$$\begin{aligned}\alpha(v)uv^{-1} &= -vuv/Q(v)\\ &= -v(2u\circ v - vu)/Q(v)\\ &= -2\cdot\frac{u\circ v}{v\circ v}\cdot v + u\\ &= r_v(u)\,.\end{aligned}$$

Hence:

3.10. Lemma. *If $Q(v) \neq 0$ then v lies in Γ and the corresponding linear transformation is the orthogonal reflection r_v.*

With Artin's definition the image is $-r_v$. This causes trouble because whether or not this lies in $\mathrm{SO}(Q)$ depends on the dimension of V.

By Theorem 1.16, this implies claim (a).

Step 2. Next is (b):*the kernel of ρ is the group of scalars $F^\times$.*

This is Proposition 3.2 in [Bott:1962]. The proof is straightforward if a bit long. Suppose

$$\alpha(x)v = vx$$

for all v in V. If $x = x_0 + x_1$ with $x_i \in C^i$, then upon matching parities this translates to

$$x_0 v = v x_0, \quad -x_1 v = v x_1\,.$$

Suppose $e = e_i$ to be a basis element of V. We can write x_0 uniquely as $a + eb$ in terms of the basis e_S, with a in C^0 and b in C^1, neither of them involving any terms with a factor e. Setting $v = e$ we get

$$ae + ebe = ea + Q(e)b\,.$$

Since neither a nor b contains any factor e, Lemma 3.2 tells us that $ae = ea$ and $be = -eb$. We can hence cancel $ea = ae$ from both sides. We then deduce that $Q(e)b = -Q(e)b$, which implies that $b = 0$. Hence the expansion of x_0 cannot involve any e_S with i in S. Since e_i was arbitrary, x_0 must be a scalar.

Suppose similarly that $x_1 = a + eb$ with a in C^1, b in C^0. Now

$$av + ebv = -va - veb$$

for all v. Setting $v = e$ we conclude immediately this time that $b = 0$. Now set $a = ec$ with c in C^0. We have

$$vec = -ecv$$

for all v in V. Setting $v = e$ we deduce that $c = 0$, hence $x_1 = 0$.

Step 3. Recall the **norm** map

$$N(x) = x\cdot\overline{x} = x\cdot\alpha({}^t x)\,.$$

I claim that $N(x) \in F^\times$ *for x in* Γ.

Proof. It suffices to show that it is in the kernel of ρ. Let u in V be given, and let $v = \rho(x)u$. Then

$$\alpha(x)ux^{-1} = v\,.$$

Apply the transpose map to get

$${}^t x^{-1}\, u\, {}^t\alpha(x) = v = \alpha(x)ux^{-1}$$

so that

$$\alpha(\alpha({}^t x)x)u(\alpha({}^t x)x)^{-1} = u\,.$$

Therefore $\overline{x}x = c$ lies in $F^\times$ and $x^{-1} = \overline{x}/c$ so $x\overline{x} = c$ as well.

Step 4. *The map N is a homomorphism from Γ to $F^\times$.*

Proof. By the previous step

$$N(xy) = xy \cdot \overline{yx} = x \cdot N(y) \cdot \overline{x} = N(x)N(y)\,.$$

Step 5. *The image of ρ is contained in* $\mathrm{O}(Q)$.

Proof. By the previous step

$$N(\rho(x)v) = N(\alpha(x)yx^{-1}) = N(\alpha(x))N(v)N(x)^{-1} = N(v)\,.$$

This concludes the proof of the Theorem.

3.11. Proposition. *The group Γ is the union of $\Gamma \cap C^0$ and $\Gamma \cap C^1$.*

Proof. Follows from Theorem 1.16 and Theorem 3.9.

Define

$$\mathrm{GSpin}(Q) = \Gamma \cap C^0\,.$$

The image of $\mathrm{GSpin}(Q)$ with respect to ρ is equal to $\mathrm{SO}(Q)$. More precisely:

3.12. Theorem. *This sequence is exact:*

$$1 \longrightarrow F^\times \longrightarrow \mathrm{GSpin}(Q) \longrightarrow \mathrm{SO}(Q) \longrightarrow 1\,.$$

Let $\mathrm{Spin}(Q)$ be the subgroup of $\mathrm{GSpin}(Q)$ on which $N = 1$. The kernel of the restriction of ρ to Spin os $\{\pm 1\}$, so the map

$$\mathrm{Spin}(Q) \longrightarrow \mathrm{SO}(Q)$$

is a double covering. If F is separably closed it is surjective, but in general not. The long exact sequence in Galois cohomology gives us a map, called the spinor norm, from the cokernel to $H^1(\mathcal{G}(F_s/F), \{\pm 1\})$. For $\mathfrak{p}$-adic fields, this is a bijection.

4. Examples

I'll begin with a general result about direct sums of quadratic spaces. Suppose (V, Q) to be any quadratic space with basis (e_i) and R to be a form on F^2, with basis $\{f_1, f_2\}$. Suppose $f_i^2 = c_i$, and let $c = c_1 c_2$. Then the e_i together with the f_i make up a basis for $V \oplus F^2$. Let φ take

(4.1) .
$$\begin{aligned} e_i &\longmapsto e_i \otimes f_1 f_2 \\ f_i &\longmapsto 1 \otimes f_i \end{aligned}$$

Since
$$\varphi(e_i)^2 = (e_i \otimes -f_1^2 f_2^2 = -cQ(e_i)\,,$$
it determines a ring homomorphism from $C(Q \oplus R)$ to $C(-cQ) \otimes C(R)$.

4.2. Proposition. *The map (4.1) is an isomorphism.*

Proof. It takes a basis to a basis, and the two spaces have the same dimension.

Let (V, Q) be the hyperbolic plane. Suppose u, v to be a hyperbolic basis, with $Q(u) = Q(v) = 0$, $u \circ v = 1$. Then
$$f_1 = (u + v)/2, \quad f_2 = (u - v)/2$$
satisfy
$$f_1^2 = 1, \quad f_2^2 = -1, \quad f_1 \circ f_2 = 0\,.$$
The map
$$f_1 \longmapsto \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad f_2 \longmapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
determines an isomorphism of $C(V, Q)$ with $M_2(F)$, which takes
$$f_1 f_2 \longmapsto \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The ring C^0 is the ring of diagonal matrices, the group GSpin is the group of invertible diagonal matrices
$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = a \cdot \frac{(1 - f_1 f_2)}{2} + b \cdot \frac{(1 + f_1 f_2)}{2}\,,$$
and the group Spin is that of diagonal matrices
$$\begin{bmatrix} t & 0 \\ 0 & 1/t \end{bmatrix},$$
which act as hyperbolic rotations $(x, y) \mapsto (t^2 x, y/t^2)$.

As a corollary of this example and Proposition 4.2:

4.3. Proposition. *If (V, Q) is an arbitrary quadratic space and $C = C(V, Q)$ then*
$$C(V \oplus H_2) = M_2(C)$$
and C^0 is the subring of diagonal matrices, isomorphic to $C \oplus C$.

4.4. Corollary. *We have*
$$C(H_{2m}) = M_{2^m}(F)\,.$$

Example. Let $V = F \cdot e_1$, $Q(x \cdot e_1) = ax^2$. Then $e^2 = -a$, $C(V) = F(\sqrt{-a})$, and $C^0 = F$.

4.5. Proposition. *If* $V = H_{2m} + ax^2$ *and* $E = F(\sqrt{-a})$ *then*

$$C(V) = M_{2^m}(E)$$

These results deal completely with the case F is algebraically closed, since in that case every quadratic space of even dimension $2m$ is isomorphic to H_{2m}, and every one of odd dimension $2m+1$ equal to $H_{2m} \oplus x^2$.

In a convenient table:

n	$C^0(H_{2n})$	$C(H_{2n})$
0	F	F
1	F	$F \oplus F$
2	$F \oplus F$	$M_2(F)$
3	$M_2(F)$	$M_2(F) \oplus M_2(F)$
$\ldots$		
$2m$	$M_m \oplus M_m(F)$	$M_m(F)$
$2m+1$	$M_{m-1}(F)$	$M_{m-1}(F) \oplus M_m(F)$

We shall see later how to describe the Lie algebras of GSpin and Spin.

• Now let V be the space of 2×2 matrices of trace 0, $Q = -\det$. If we take as coordinates

$$\begin{bmatrix} z & x \\ y & -z \end{bmatrix}$$

the form Q becomes $xy + z^2$. The group $\mathrm{GL}_2(F)$ acts on this by conjugation, preserving Q. The group $\mathrm{PGL}_2(F)$ therefore embeds into $\mathrm{O}(Q)$. This is the unique irreducible representation of $\mathrm{PGL}_2(F)$. The image of $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ (suitably rearranged) is

$$\begin{bmatrix} a/b & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & b/a \end{bmatrix} .$$

and the image of PGL_2 therefore lies in $\mathrm{SO}(Q)$, and in fact is isomorphic to it. The group $\mathrm{O}(Q)$ is generated by its image and any one reflection. The group GSpin is equal to $\mathrm{GL}_2(F)$.

Example. Suppose e_1, e_2 to be a basis of V with

$$Q(x, y) = ax^2 + by^2$$

Then

$$\begin{aligned} e_1^2 &= -a \\ e_2^2 &= -b \\ (e_1 e_2)^2 &= -ab \,. \end{aligned}$$

The associated norm is

$$x^2 - ay^2 - bz^2 + abw^2 = (x^2 - ay^2) - b(z^2 - aw^2) \,,$$

with determinant 1. Thus $C(V, Q)$ is a quaternion algebra.

Example. Suppose Q to be a form of dimension 4 and determinant 1 that represents 1. If $Q(v) = 1$, then the restriction of Q to $v^\perp$ is a quadratic form of dimension 3 and also of determinant 1. According to a theorem of Witt it is, up to equivalence, independent of the choice of v. Choose coordinates so that it becomes

$$Q(x, y, z) = -ax^2 - by^2 + abz^2 \,.$$

The original form may be expressed as

$$w^2 - ax^2 - by^2 + abz^2 \,. \tag{4.6}$$

What is the Clifford algebra of the three-dimensional form Q? It contains elements e_i for $1 \le i \le 3$ with

$$e_1^2 = a, \quad e_2^2 = b \quad e_3^2 = -ab \,.$$

The algebra $C^0(V, Q)$ has as basis the elements $e_i e_j$ with

$$(e_2 e_3/b)^2 = a \quad (e_3 e_1/a)^2 = b \quad (e_1 e_2)^2 = -ab \,.$$

This implies that $C^0(V, Q)$ is a quaternion algebra with norm form (4.6) , and proves:

4.7. Proposition. *The map associating $N_{B/F}$ to B induces an equivalence between isomorphism classes of quaternion algebras and equivalence classes of quadratic forms of dimension 4 and determinant 1 that represent 1.*

Example. Let F be arbitrary, E/F quadratic. Let V be the vector space of all 2×2 Hermitian matrices

$$h = \begin{bmatrix} y + x & z \\ \overline{z} & y - x \end{bmatrix}$$

Let Q be the quadratic form

$$(-1/2)\det(h) = z\overline{z} + x^2 - y^2 \,,$$

which takes values in F. The group $\mathrm{SL}_2(E)$ acts on this:

$$h \mapsto g h^t \overline{g} \,,$$

and we get therefore an embedding of $\mathrm{SL}_2(E)$ into $\mathrm{SO}(Q)$. In fact, this identifies $\mathrm{SL}_2(E)$ with $\mathrm{Spin}(Q)$.

How to see this? Let $E = F(\sqrt{\alpha})$,

$$\begin{aligned} v_1^2 &= 1 \\ v_2^2 &= -1 \\ v_3^2 &= 1 \\ v_4^2 &= -\alpha \end{aligned}$$

The ring $C^0(Q)$ therefore contains the Clifford algebra associated to the quadratic space $H \oplus F$, which we know to be $M_2(F)$. If

$$\gamma = v_1 v_2 v_3 v_4 \,.$$

then

$$\gamma^2 = \alpha \,.$$

This means that $F \oplus F \cdot \gamma$ generates a ring isomorphic to E. Furthermore, γ generates the center of C^0. This gives an embedding of $M_2(E)$ into C^0, which turns out to be an isomorphism. The group Γ is that of all x in $\mathrm{GL}_2(K)$ such that $\det(x)$ lies in $F^\times$, and $\mathrm{Spin}(Q)$ must be $\mathrm{SL}_2(K)$.

Example. Let $F = \mathbb{R}$. Every quadratic form over F is a direct sum of three types: (i) hyperbolic planes; (ii) x^2; (iii) $-x^2$. Proposition 4.2 reduces the computation of $C(V)$ to that of the last two cases. For these we have a kind of periodicity.

Let $(\mathbb{R}^m, Q^\pm)$ be the form

$$\pm x_1^2 + \cdots + x_m^\pm \,.$$

As an immediate consequence of Proposition 4.2:

4.8. Proposition. *For* $m \geq 1$

$$C^+_{m+2} = C^-_m \otimes C^+_2$$
$$C^-_{m+2} = C^+_m \otimes C^-_2$$

In order to apply this, we need to understand the cases $m = 1, 2$.

Take $F = \mathbb{R}$, $V = \mathbb{R} \cdot j + \mathbb{R} \cdot k$, with $Q(xj + yk) = -(x^2 + y^2)$. If $i = jk$, then also $i^2 = -1$. Embed $\mathbb{C}$ into C^0, taking $\sqrt{-1}$ to i. Then C is the space $\mathbb{C} + \mathbb{C}j$ and

$$jz = \overline{z} j\,.$$

We may identify C^1 with $\mathbb{C}j$. We have relations

$$i^2 = j^2 = k^2 = -1$$
$$ij = k,\ jk = i,\ ki = j\,,$$

so C is isomorphic to the quaternion algebra $\mathbb{H}$. The group GSpin is isomorphic to $\mathbb{C}^\times$ and if V is identified with $\mathbb{C}\,j$, z acts as rotation by $z/\overline{z}$.

Example. Take $F = \mathbb{R}$, $V = \mathbb{R}^3$ with basis e_i, $Q(x, y, z) = x^2 + y^2 + z^2$. Set

$$i = e_2 e_3$$
$$j = e_3 e_1$$
$$k = e_1 e_2$$

Then C^0 is isomorphic to $\mathbb{H}$, GSpin is $\mathbb{H}^\times$ and Spin is the kernel of $N_{\mathbb{H}/\mathbb{R}}$.

This gives us another table, illustrating something called **Bott periodicity**.

n	C^+_n	C^-_n
1	$\mathbb{C}$	$\mathbb{R} \oplus \mathbb{R}$
2	$\mathbb{H}$	$M_2(\mathbb{R})$
3	$\mathbb{H} \oplus \mathbb{H}$	$M_2(\mathbb{C})$
4	$M_2(\mathbb{H})$	$M_2(\mathbb{H})$
5	$M_4(\mathbb{C})$	$M_2(\mathbb{H}) \oplus M_2(\mathbb{H})$
6	$M_8(\mathbb{R})$	$M_4(\mathbb{H})$
7	$M_8(\mathbb{R}) \oplus M_8(\mathbb{R})$	$M_8(\mathbb{C})$
8	$M_{16}(\mathbb{R})$	$M_{16}(\mathbb{R})$

5. Root systems

The group $\mathrm{Spin}(Q)$ is semi-simple. In case the form Q is split, what is its root system? It has to be the same as taht of SO_Q, but an explicit determination woudl be nice.

By definition of Spin, the Lie algebar is the space of all X in $C(V, Q)$ such that

$$Xv - vX$$

lies in V for all v in V.

5.1. Lemma. *For every* u, v *in* V *the element* suv *in* $C(V, Q)$ *lies in the Lie algebra of* Spin.

Proof. Choose a hyperbolic basis (u_i) of V, and let $u_{\overline{i}}$ be the dual element of u_i. Thus

$$u_i \circ u_j = \begin{cases} 1 & \text{if } j = \overline{i} \\ 0 & \text{otherwise.} \end{cases}$$

Therefore

$$u_i u_j + u_j u_i = \begin{cases} 2 & \text{if } j = \overline{i} \\ 0 & \text{otherwise.} \end{cases}$$

The effect of $u_i u_j/2$ on v is the same as

$$v \longmapsto (v \circ u_j)u_i - (v \circ u_i)u_j \,.$$

The dimensions match, so these are the root spaces.

LANGLANDS DUALITY.

5.2. Proposition. *The groups* GSp_{2n} *and* GSpin_{2n+1} *are Langlands duals.*

This is Proposition 2.4 of [Asgari:2002].

There are a few more low-dimensional accidents worth noting. One is when $(V, Q) = H_6$, for which the spin group is $\mathrm{SL}_4(F)$. As a another example:

5.3. Proposition. *The group* GSpin_5 *is isomorphic to* GSp_4*.*

Proof. This is not quite immediate. We need a suitable isomorphism τ of $X^*(T)$ with $X_*(T)$. It must take Δ to $\Delta^\vee$, as must also its transpose ${}^tT$. The transpose ${}^tT$ is defined by the equation

$$\langle {}^t\tau(u), v\rangle = \langle u, \tau(v)\rangle \,.$$

But if $\gamma = \varepsilon_1$ the pairing matrix is

$$\begin{bmatrix} \langle\alpha,\alpha^\vee\rangle & \langle\alpha,\beta^\vee\rangle & \langle\alpha,\gamma^\vee\rangle \\ \langle\beta,\alpha^\vee\rangle & \langle\beta,\beta^\vee\rangle & \langle\beta,\gamma^\vee\rangle \\ \langle\gamma,\alpha^\vee\rangle & \langle\gamma,\beta^\vee\rangle & \langle\gamma,\gamma^\vee\rangle \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ -2 & 2 & -1 \\ -1 & 0 & 0 \end{bmatrix} \,,$$

so the map

$$\begin{aligned} \tau\colon \alpha &\longmapsto \beta^\vee \\ \beta &\longmapsto \alpha^\vee \\ \gamma &\longmapsto \gamma^\vee \,. \end{aligned}$$

is the one we are looking for.

The isomorphism is not unique, since there exists an involution of GSp_{2n} acting as I in Sp_{2n} but taking μ to $-\mu$:

$$X \longmapsto {}^*X^{-1} \,.$$

Therefore there are two possible identifications of the dual of GSp_4 with GSp_4, both taking B to B, T to T. (This caused me some confusion in verifying the Proposition.)

6. References

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