Compact groups as algebraic groups

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A classic result of [Tannaka:1939], in the formulation of [Chevalley:1946], asserts that every compact subgroup of $\mathrm{GL}_n(\mathbb{C})$ is the group of real points on an algebraic group defined over $\mathbb{R}$. Chevalley introduced a more algebraic approach to the topic, but his underlying argument is not so different from Tannaka's. The literature since then has followed one of two threads. One is algebraic (as in Chapter 9 of [Robert:1983]), inspired by Grothendieck's notion of Tannakian categories (as in [Grothendieck:1970]). The other is analytic (as in §XI.11 of [Yoshida:1965] or §30 of [Hewitt-Ross:1970]), and follows Tannaka's original exposition more closely. There is surprisingly little overlap between the two, and in fact neither group seems to be very aware of the other. There is an account somewhere in between in [Bröcker-tom Dieck:1985], but nonetheless it looks to me as if there were room for clarification. In this essay I shall offer a brief argument that takes advantage of terminology and concepts that have been introduced since Chevalley's book was written. The basic ideas are still due to Chevalley.

I shall also explain several variants of this theorem, including the refinement of Chevalley that characterizes precisely which real algebraic structure is constructed.

Throughout this essay, K will be a compact Lie group. All representations of K will be assumed to be continuous homomorphisms into some $\mathrm{GL}(V)$, with V finite-dimensional and complex. If V is a vector space, $\widehat{V}$ will be its linear dual. If X is any topological sp[ace, $\mathbb{C}(X)$ will be the space of continuous complex-valued functions on X.

Contents

1. Preliminaries

Suppose H to be a closed subgroup of K. Suppose (π, V) to be an irreducible representation of K. If $\widehat{v}$ lies in $\widehat{V}^H$ and v in V, the function

$$F_{\hat{v}\otimes v}(k) = \langle \widehat{\pi}(k)\widehat{v}, v\rangle$$

is a continuous function on K/H. According to Frobenius reciprocity, the map

(1.1) $$\widehat{V}^H \otimes V \longrightarrow \mathrm{L}^2(K/H), \quad \widehat{v}\otimes v \longmapsto F_{\hat{v}\otimes v}$$

is a K-equivariant embedding (with K acting on the right factor). The following is well known.

1.2. Proposition. (Peter-Weyl) *The space $\mathrm{L}^2(K/H)$ is isomorphic to the Hilbert direct sum of the images of these embeddings.*

1.3. Corollary. *The subgroup H is equal to K if and only if the only irreducible representation (π, V) of K such that $V^H \neq \{0\}$ is the trivial one.*

Proof. If $H \neq K$, Urysohn's Lemma asserts the existence of non-constant continuous functions on K/H. Therefore $H = K$ if and only if $\mathrm{L}^2(K/H) = \mathbb{C}$. ▯

Let $\mathfrak{C}(K/H)$ be the subspace of $\mathbb{C}(K/H)$ spanned by the images of the maps (1.1). According to Proposition 1.2, it is dense in $\mathbb{C}(K)$.

A K-finite function on K/H is one contained in a finite-dimensional K-stable subspace of $\mathbb{C}(K/H)$. All functions in $\mathfrak{C}(K/H)$ are clearly K-finite. In fact:

1.4. Lemma. *The space $\mathfrak{C}(K/H)$ is the space of all K-finite functions on K/H.*

Take H to be $\{1\}$. If (π, V) is an irreducible representation of K, the map defined by (1.1) embeds $\widehat{V} \otimes V$ equivariantly into $\mathrm{L}^2(K)$ as a representation of $K \times K$. Then Proposition 1.2 says that $\mathrm{L}^2(K)$ is the orthogonal direct sum of such representations.

Any K-finite function is certainly H-finite. Restriction hence defines an H-equivariant linear map from $\mathfrak{C}(K)$ to $\mathfrak{C}(H)$.

1.5. Proposition. *The canonical map from $\mathfrak{C}(K)$ to $\mathfrak{C}(H)$ is surjective, and an injection if and only if $H = K$.*

Proof. The question comes down to: *If (π, V) is an irreducible representation of K, what is the restriction of $\widehat{V} \otimes V$ to H?* Suppose the restriction of π to H is $\oplus n_i \rho_i$, with each (ρ_i, U_i) an irreducible representation of H. Then

$$\widehat{V} \otimes V = \oplus_{i,j} \widehat{U}_i \otimes U_j \, .$$

The map of matrix coefficients is compatible. So surjectivity follows from the fact that each (ρ, U) occurs in the restriction to H of some representation of K. What about injectivity? Suppose the map is injective. If U_i is not isomorphic to U_j, then the matrix coefficients of $\widehat{U}_i \otimes U_j$ vanish. Therefore all ρ_i must be isomorphic. There will be a kernel if its multiplicity is greater than one. Therefore under the assumption of injectivity, every irreducible representation of K remains irreducible when restricted to H. However, according to Corollary 1.3 if $H \neq K$ there will exist non-trivial irreducible representations of K whose restriction to H contains the trivial representation. ▯

The representation of K on $\mathrm{L}^2(K)$ is faithful, so the intersection of all the kernels of irreducible representations of K is $\{1\}$.

On the other hand, since K is a Lie group there exists a neighbourhood of the identity in K in which $\{1\}$ is the only subgroup. Therefore a decreasing sequence of closed subgroups in K is stationary. (This is not true of all compact groups, for example the p-adic integers $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n$ and other p-adic analytic groups.) Combining this observation with that in the previous paragraph:

1.6. Lemma. *There exists a faithful finite-dimensional representation of K.*

The group K may, and subsequently will, be thus identified with a closed subgroup of some $\mathrm{GL}(V)$, in which V is a complex vector space of finite dimension. Choosing a suitable coordinate system, I shall assume that

$$K \subseteq \mathrm{U}(n) \subset \mathrm{GL}_n(\mathbb{C}) \, .$$

Remark. In this discussion, it need only be assumed that K does not possess arbitrarily small subgroups. It will follow from subsequent discussion that it is then necessarily a Lie group.

The space $\mathfrak{C}(K)$ is a ring because the product of matrix coefficients is the matrix coefficient of a tensor product. The Peter-Weyl theorem implies that it is the same as the K-finite functions in $\mathrm{L}^2(K)$, and also the same as the ring referred to the .

The space $\mathfrak{C}(K)$ is stable under conjugation, and there is an interesting observation in connection with this. If V is any vector space over $\mathbb{C}$, its conjugate vector space $\overline{V}$ is the same space, but with a conjugate scalar multiplication. If $v \mapsto c \cdot v$ is that on V, that on $\overline{V}$ is

$$v \mapsto c \circ v = \overline{c} \cdot v \, .$$

Hermitian forms $u \bullet v$ on V are equivalent to linear maps $V \times \overline{V} \to \mathbb{C}$.

Suppose (π, V) to be a representation of K over $\mathbb{C}$. Since

$$[\pi(k)](c \circ v) = [\pi(k)](\overline{c} \cdot v) = \overline{c} \cdot [\pi(k)](v) = c \circ [\pi(k)](v) \, ,$$

we obtain thus a conjugate representation $\overline{\pi}$. Given a basis of V, if $\pi(k)$ is represented by the matrix $(x_{i,j})$ then $\overline{\pi}(k)$ is represented by the matrix $(\overline{x}_{i,j})$.

2. Defining the algebraic group

The group $\mathrm{GL}_n(\mathbb{C})$ may be identified with the algebraic variety

$$\{(x, y) \in M_n(\mathbb{C}) \times M_n(\mathbb{C}) \,|\, x \cdot {}^t y = 1\} \,.$$

The Stone-Weierstrass approximation theorem asserts that the polynomials in the matrix entries $x_{i,j}$, $y_{i,j}$ and their conjugates are dense in $\mathbb{C}(K)$. They clearly lie in $\mathfrak{C}(K)$, and therefore span $\mathfrak{C}(K)$.

If S in any set in $\mathbb{C}^n$, let $\mathcal{I}_S$ be the ideal of all polynomials in $\mathbb{C}[x_1, \ldots, x_n]$ vanishing on S. Let $\mathfrak{A}(K)$ be the quotient of $\mathbb{C}[x, y]$ by the ideal $\mathcal{I}_K$ of polynomials that vanish on K. Since $K \subseteq \mathrm{U}(n)$ and $y = {}^t x^{-1}$, the restrictions to K of conjugates of the $x_{i,j}$ are the same as the $y_{i,j}$. Therefore:

2.1. Proposition. *The canonical homomorphism from $\mathfrak{A}(K)$ to $\mathfrak{C}(K)$ is an isomorphism.*

If S is any subset of $\mathbb{C}^n$, let S_* of S is the set of all x in $\mathbb{C}^n$ such that $P(x) = 0$ for all P in $\mathcal{I}_S$. It is the closure of S in the Zariski topology.

2.2. Lemma. *The Zariski closure of a group is also a group.*

Proof. This is elementary, but I copy here an elegant note by Matt Samuel on `math.stackexchange.com`. Let G be the original group. If g, h lie in G_*, let U be a neighborhood of gh. Since multiplication is continuous, there exist neighborhoods A of g and B of h such that $AB \subseteq U$. Since both A and B contain elements of G, so must U. Therefore gh lies in G_*. Since inversion is a homeomorphism, if U is a neighborhood of g then U^{-1} is a neighborhood of g^{-1}, and since the former neighborhood contains an element of G, so does the latter. Then g^{-1} lies in G_*.

Now let K_* be the Zariski closure of K in $\mathrm{GL}_n(\mathbb{C})$. The previous result implies immediately:

2.3. Proposition. *The Zariski closure K_* is an algebraic subgroup of $\mathrm{GL}_n(\mathbb{C})$.*

2.4. Lemma. *The group K_* is stable under the anti-holomorphic involution $x \mapsto {}^t\overline{x}^{-1}$.*

Proof. By assumption, $x = {}^t\overline{x}^{-1}$ for x in K. Therefore if $P(x, y) = 0$ for all (x, y) in K, so is $P({}^t\overline{x}^{-1}, {}^t\overline{y}^{-1})$.

According to the theory of Galois descent, this involution defines a unique algebraic group $\boldsymbol{K}$ over $\mathbb{R}$ such that $K_* = \boldsymbol{K}(\mathbb{C})$. The following is Chevalley's formulation of 'Tannaka duality':

2.5. Theorem. *The embedding of K into $\boldsymbol{K}(\mathbb{R})$ is a bijection.*

Proof. The group $\boldsymbol{K}(\mathbb{R})$ is the subgroup of $\boldsymbol{K}(\mathbb{C})$ on which $x = {}^t\overline{x}^{-1}$. It is therefore contained in $\mathrm{U}(n)$, and must be compact. The Theorem now follows from this:

2.6. Lemma. *The group K is maximal compact in $\boldsymbol{K}(\mathbb{C})$.*

Proof. Suppose G to be compact in $\boldsymbol{K}(\mathbb{C})$, containing K. The earlier discussion applies to G as well as K, so that $\mathfrak{A}(K)$ is isomorphic to $\mathfrak{C}(G)$ as well as $\mathfrak{C}(K)$. Apply Proposition 1.5.

Remark. Tannaka duality is just the observation that evaluating the matrix coefficients of K at points of K establishes a bijection between K and the conjugation-invariant homomorphisms from the affine ring to $\mathbb{C}$—i.e. that K is the real spectrum of the affine ring. Analysts seem to find this a sort of miracle.

3. Chevalley's characterization

The immediate consequence of Theorem 2.5 is that K is the group of $\mathbb{R}$-rational points on a real algebraic variety. In general there will exist many real algebraic varieties with this property. For example, if $K = \{1\}$ in $\mathbb{R}^\times$ it is the group of rational points on all the varieties $x^n = 1$ with n odd. But the variety $\boldsymbol{K}$ is rather special. Here is one simple characterization due to Chevalley:

3.1. Theorem. *The group $K = \boldsymbol{K}(\mathbb{R})$ meets all the connected components of / $\boldsymbol{K}(\mathbb{C})$, and the product map is a bijection of $\boldsymbol{K}(\mathbb{R}) \times \exp(i\mathfrak{k})$ with $\boldsymbol{K}(\mathbb{C})$.*

This is from §IX of Chapter VII of [Chevalley:1946]. It has consequences I'll specify later.

I'll deduce this theorem from the more precise result that Chevalley proved. I start with a special case:

3.2. Lemma. *Every g in GL_n can be factored uniquely as $k \cdot s$ with k in $\mathrm{U}(n)$ and s a positive definite Hermitian matrix.*

This is compatible with the Theorem since every positive definite Hermitian matrix is of the form $\exp(y)$ with y Hermitian, or $y = ix$ with x skew-Hermitian, hence in the Lie algebra of $\mathrm{U}(n)$.

Proof. This is a well known exercise in many undergraduate classes. For every matrix x let $x^* = {}^t\overline{x}$. If $g = k \cdot s$, then $g^* = s^* \cdot k^* = s \cdot k^{-1}$ and $g^*g = s^2$. This suggests the reverse step. The matrix g^*g will be positive definite Hermitian, so we may write it as s^2 with s also positive definite Hermitian. The matrix $k = gs^{-1}$ will then be unitary.

According to Lemma 2.4 the group $\boldsymbol{K}(\mathbb{C})$ is stable under the involution $x \mapsto x^*$. The following is therefore applicable:

3.3. Proposition. *Suppose G to be an algebraic subgroup of $\mathrm{GL}_n(\mathbb{C})$ stable under the involution $x \mapsto x^*$. If / $g = k \cdot s$ according to the previous Lemma, then both k and s lie in G.*

Proof. As in an earlier remark, $g^*g = s^2$, hence all s^{2k} lie in G. We can express $s = x\rho x^{-1}$ with x in $\mathrm{U}(n)$ and ρ positive diagonal. The same hypotheses apply to $x^{-1}Gx$, so we may assume that all ρ^{2k} lie in G. Write $\rho = \exp(r)$. If P is a polynomial vanishing on G then $P(2kr) = 0$ for all k. But this means that the polynomial $P(tr)$ in the variable t has an infinite number of roots, hence must vanish identically. Thus $\rho = \exp(r)$ also lies in G.

Now to finish the proof of Chevalley's theorem. This proposition says that the map from $\boldsymbol{K}(\mathbb{R}) \times \exp(i\mathfrak{k})$ to $\boldsymbol{K}(\mathbb{C})$ is surjective. But because of Lemma 3.2 it is also injective.

Remark. Suppose $\boldsymbol{G}$ to be an algebraic group defined over $\mathbb{R}$, and let $\mathfrak{g}_{\mathbb{R}}$ be its real Lie algebra. Following loosely a suggestion of [Deligne:1972], I'll call a real algebraic group **$\boldsymbol{G}$ strictly compact** if (a) $\boldsymbol{G}(\mathbb{R})$ is compact and (b) the product map from $\boldsymbol{G}(\mathbb{R}) \times \exp(i\mathfrak{g}_{\mathbb{R}})$ to $\boldsymbol{G}(\mathbb{C})$ is a bijection. Theorem 3.1 thus asserts that $\boldsymbol{K}$ is strictly compact. [Mostow:1955] proves that if $\boldsymbol{G}(\mathbb{R})$ meets every component of $\boldsymbol{G}(\mathbb{C})$ then it is strictly compact. [Deligne:1972] proves that any algebraic subgroup of a strictly compact group is also strictly compact.

Remark. [Robert:1983] characterizes a compact group in terms of the tensor category of all of its finite-dimensional representations. This is in fact very close to Tannaka's original result! The account of Tannaka's theorem in Chapter IX of [Chevalley:1946] is very clear.

Remark. The algebraic group $\boldsymbol{K}$ is completely defined by its affine ring $\mathfrak{A}(K)$, which is the same as $\mathfrak{C}(K)$. This depend only on the compact group. Therefore homomorphisms of algebraic groups are the same as those of the underlying compact groups:

$$\mathrm{Hom}(K, L) = \mathrm{Hom}(\boldsymbol{K}, \boldsymbol{L})\,.$$

Remark. There is a well known converse to the main result of this essays—every complex reductive group possesses a compact form defined over $\mathbb{R}$. If G is connected this can be constructed explicitly from its root datum, and [Mostow:1955] reduces the general case to this one.

4. References

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