Essays in analysis

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Compact operators

This is a sequel to an introductory essay on Hilbert spaces. It deals specifically with compact operators and, for the most part, follows [Reed-Simon:1972]. The only unusual feature is the exposition of the theorem in [Duflo:1972] concerning the trace of integral operators as an integral over the diagonal. This is a basic result, frequently referred to without clear justification.

Duflo's own account, as pointed out to me by Chris Brislawn, contains a few small but confusing errors. In fact, much of the literature on the subject is either inadequate to cover cases one needs or in error. The most satisfactory current account of this theorem is probably that in [Brislawn:1991] (which extends earlier work of his dealing with $\mathbb{R}^n$, but it involves a number of techniques I am not familiar with (primarily, the application of martingales to analysis). This is a project for the future.

Contents

All Hilbert spaces in this essay are assumed to be separable, which means they possess countable orthonormal bases.

1. The singular value decomposition

For any bounded operator T on a Hilbert space, define its conjugate transpose:

$$T^* = {}^t\overline{T}\,.$$

Here ${}^tT$ is the transpose with respect to linear duality, so T^* is that with respect to the Hermitian inner product:

$$T(u) \bullet v = u \bullet T^*(v)\,.$$

The operator T is unitary if and only if this is the same as T^{-1}.

The principal result of this section is a generalization of a familiar result in linear algebra, which I recall here:

1.1. Proposition. *If T is a non-singular finite-dimensional complex matrix, there exist unitary matrices U_1 and U_2 and a diagonal matrix D with positive entries such that $T = U_1 D U_2$*

This is called the **singular value decomposition** of T, and the eigenvalues of D (its entries along the diagonal) are called the **singular values** of T. Roughly speaking, the singular values of a matrix measure its size. In particular, if $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_n$ are the entries in D then $\lambda_n \le \|T\| \le \lambda_1$.

An equivalent formulation is that $T = US$ with U unitary and S positive definite Hermitian, since $U_1 D U_2 = U_1 U_2 \cdot U_2^{-1} D U_2$. It is not really necessary to restrict to non-singular T, as we shall see later, but it simplifies the discussion.

Something similar is true for real matrices. In dimension 2, this says that if we define an ellipse to be a circle scaled along perpendicular axes then any linear transformation takes a circle to an ellipse, and tells you that to find the axes of the ellipse you should expect to find eigenvalues and eigenvectors.

Proof. If $T = U_1 D U_2$ then since $U^{-1} = U^*$

$$T^* \cdot T = U_2^{-1} D\, U_1^{-1} \cdot U_1 D U_2 = U_2^{-1} D^2 U_2 \,.$$

Conversely, since the product $T^* \cdot T$ is positive definite and Hermitian, we can find a unitary matrix U such that

$$T^* \cdot T = U^{-1} D^2 U \,,$$

where D is real, diagonal, with non-negative entries. Set $S = U^{-1}\, DU$. Then S is positive definite Hermitian and:

$$\begin{aligned} T^* \cdot T &= S^2 \\ S^{-1} T^* \cdot T S^{-1} &= I \,, \end{aligned}$$

which means that $TS^{-1} = U_\circ$ is unitary, and then $T = U_\circ S = U_\circ U^{-1} DU$. ∎

This proof does not remain valid for infinite-dimensional vector spaces, since the eigenvalue decomposition fails. That part of the argument is replaced by a suitable square root construction.

POSITIVE OPERATORS. Recall that an operator T is called **positive** (more properly non-negative) if $Tu \bullet u \ge 0$ for all u in its domain.

1.2. Lemma. *A positive bounded operator on a complex Hilbert space is self adjoint.*

Proof. We want to show that $Tu \bullet v = u \bullet Tv = \overline{Tv \bullet u}$, or in other words

$$\mathrm{RE}(Tu \bullet v) = \mathrm{RE}(Tv \bullet u), \quad \mathrm{IM}(Tu \bullet v) = -\mathrm{IM}(Tv \bullet u)$$

for all u, v. Since

$$T(u+v) \bullet (u+v) - Tu \bullet u - Tv \bullet v = Tu \bullet v + Tv \bullet u \,,$$

we know that $Tu \bullet v + Tv \bullet u$ is real, which implies that $\mathrm{IM}(Tu \bullet v) = -\mathrm{IM}(Tv \bullet u)$. But if we substitute iv for v in this equation, we see that $\mathrm{RE}(Tu \bullet v) = \mathrm{RE}(Tv \bullet u)$. ∎

> *From now on, I'll not distinguish real from complex, and assume without further mention that any positive operator is bounded and self-adjoint as well as non-negative.*

A BIT OF CALCULUS. I begin the construction of the singular value factorization of an arbitrary bounded operator $T\colon H_1 \to H_2$ by associating to it a positive operator $|T|$ from H_1 to itself through which it factors. In the previous discussion, this is S. We can't diagonalize an arbitrary positive operator, so we have to modify the proof of the previous result—instead of diagonalizing $T^* \cdot T$ and then taking the square root of its diagonalization, we shall derive directly the square root of a positive operator. Towards this goal:

1.3. Lemma. *The Taylor series of $(1-x)^{1/2}$ converges absolutely for all $|x| \le 1$.*

Proof. Let $\alpha = 1/2$. The series is

$$(1-x)^\alpha = 1 + \sum_{m=1}^{\infty} c_m x^m = 1 - \alpha z + \frac{\alpha(\alpha-1)}{2} x^2 - \frac{\alpha(\alpha-1)(\alpha-2)}{3!} x^3 + \cdots \,.$$

The series converges for $|x| < 1$. For $|x| = 1$, all coefficients c_m but $c_0 = 1$ are negative, so

$$\begin{aligned}\sum_1^n |c_m| &= -\sum_1^n c_m \\ &= \lim_{x\to 1-} -\sum_1^n c_m x^m \\ &\le \lim_{x\to 1-} -\sum_1^\infty c_m x^m \\ &= \lim_{x\to 1-} 1-(1-x)^\alpha \\ &= 1\,. \end{aligned}$$

This will be complemented by:

1.4. Lemma. *Suppose $\sum a_i$ and $\sum_i b_i$ to converge absolutely, with*

$$A_m = \sum_{i\le m} |a_i|, \quad B_m = \sum_{i\le m} |b_i|\,.$$

If

$$c_m = \sum_{k=0}^n a_k b_{n-k}$$

then

$$\left|\Big(\sum_{i\le m} a_i\Big)\Big(\sum_{j\le m} b_j\Big) - \Big(\sum_{k\le 2m} c_k\Big)\right| \le A_m(B_\infty - B_m) + (A_\infty - A_m)B_m + (A_\infty - A_m)(B_\infty - B_m)\,.$$

Note that, at least formally,

$$\sum c_k = \Big(\sum a_k\Big)\Big(\sum b_k\Big)\,.$$

I leave this as a straightforward exercise.

SQUARE ROOTS. The elementary calculus exercises we have just seen have an important consequence for operators:

1.5. Lemma. *If T is any positive operator, there exists a unique positive operator S such that $S^2 = T$. Its kernel is the same as that of T, and it commutes with any operator that commutes with T.*

Proof. It suffices to prove this for any multiple λT with $\lambda > 0$, so we may assume $\|T\| \le 1$. But then for $\|u\| = 1$ we have by Cauchy-Schwartz

$$\begin{aligned} 0 \le T(u)\bullet u \le \|T(u)\|\,\|u\| \le u\bullet u = 1 \\ u\bullet u \ge (u - T(u))\bullet u \ge 0 \end{aligned}$$

so that $I - T$ is a positive operator and $\|I - T\| \le 1$. According to Lemma 1.3, the series S for $S = \big(I - (I-T)\big)^{1/2}$ converges absolutely. Lemma 1.4 implies that $S^2 = T$. It clearly commutes with anything that commutes with T. Furthermore, if $\|u\| = 1$ then

$$Su\bullet u = 1 + \sum_1^\infty c_m (I-T)^m(u)\bullet u \ge 0\,,$$

so S is positive and therefore self-adjoint. This concludes the proof of existence.

I make the second claim formally:

1.6. Lemma. *If S is a bounded operator with $S^2 = T$, the kernel of S is the same as the kernel of T.*

Proof of the Lemma. If $S(v) = 0$ then of course $T(v) = 0$. If $T(v) = S^2(v) = 0$ then $S^2(v) \bullet v = S(v) \bullet S(v) = 0$ so $S(v) = 0$.

Now for uniqueness. Since the convergence of the series for $\sqrt{T} = \big(I - (I - T)\big)^{1/2}$ is absolute, S commutes with any operator that commutes with T. Suppose now that $R^2 = T$ for some possibly different positive operator R. Then R commutes with T since

$$RT = R^3 = TR\,,$$

so R and S commute. According to the previous Lemma, the kernel of R is also the kernel of T, so R, S, T all induce operators on $\mathrm{Ker}(T)^\perp$, and we may assume $\mathrm{Ker}(T) = 0$.

We want to show $R = S$, or equivalently $R - S = 0$. Since $R^2 - S^2 = (R - S)(R + S) = 0$, we know that $R - S = 0$ on the range of $R + S$, so it suffices to show that the range of $R + S$ is dense. But since $R + S$ is self-adjoint, the complement of its range is $\mathrm{Ker}(R + S)$. Since R and S are both positive and $\mathrm{Ker}(R) = \mathrm{Ker}(S) = 0$, so is $\mathrm{Ker}(R + S) = 0$.

If T is a bounded operator from H_1 to H_2, its adjoint T^* is a map from H_2 to H_1, and $T^* \cdot T$ is a positive operator from H_1 to itself. Define

$$|T| = \sqrt{T^* \cdot T}\,.$$

It is positive, and if $T\colon H \to H$ is positive then $|T| = T$. It is an immediate consequence of the definition that

$$T(u) \bullet T(u) = |T|(u) \bullet |T|(u)\,.$$

From this follows directly:

1.7. Lemma. *The kernel of $|T|$ is the same as the kernel of T.*

THE MAIN RESULT. We now construct the singular value decomposition of an arbitrary bounded operator. A **partial isometry** is a bounded operator U satisfying the condition that $\|U(v)\| = \|v\|$ on the perpendicular complement of its kernel.

The factorization $T = U\,|T|$ can also be written as $T = U|T|U^{-1} \cdot U$, and $U|T|U^{-1}$ is also positive, so there is a certain left-right symmetry.

1.8. Theorem. (Singular value decomposition) *If T is an arbitrary bounded operator $T\colon H_1 \to H_2$, there exists a partial isometry $U\colon H_1 \to H_2$ such that $T = U \cdot |T|$. Then*

- *(a) the partial isometry is unique if its kernel is required to be the kernel of T;*
- *(b) if $H_1 = H_2$, there exists a unique unitary U which restricts to the identity operator on the kernel of T.*

Proof. Since $|T|$ is self-adjoint, the closure of the range of $|T|$ is the orthogonal complement of $\mathrm{Ker}(T)$:

$$H_1 = \mathrm{Ker}(T) \oplus \text{ closure of the range of } |T|\,.$$

In order to define U, it suffices to define it on each summand. On $\mathrm{Ker}\,|T| = \mathrm{Ker}\,T$ we choose it arbitrarily. If $T = U \cdot |T|$ then $U\big(|T|(v)\big) = T(v)$. This defines U on the range of $|T|$. Since $T(v) \bullet T(v) = |T|(v) \bullet |T|(v)$, the map U is an isometry on the range of $|T|$. It may hence be extended as one to its closure.

2. Compact operators

There is one case in which the polar decomposition can be made rather explicit.

An operator $T\colon H_1 \to H_2$ is said to be of **finite rank** if its image has finite dimension. The simplest such map has rank one, and is of the form

$$u \otimes v\colon\ w \longmapsto (w \bullet v)u \quad (v \in H_1, u \in H_2)\,.$$

Any map of finite rank is a sum of maps of rank one, and if we apply the singular value decomposition we may put it in a special form. Suppose $T\colon H_1 \to H_2$ to be a continuous linear map of finite rank, with $T = U\,|T|$ (U a partial isometry). Here $|T|$ is a positive, bounded, self-adjoint operator of finite rank. Its kernel is the same as $K = \mathrm{Ker}(T)$, and T takes the orthogonal complement $K^\perp$ of K to itself. This space is finite-dimensional, and hence there exists an orthonormal basis $\{v_i\}$ of $K^\perp$ and eigenvalues $\lambda_i > 0$ such that $|T|(v_i) = \lambda_i v_i$. If $u_i = U(v_i)$, the u_i form an orthonormal set in H_2 and

$$T\colon\ w \longmapsto \sum \lambda_i (w \bullet v_i) u_i\,.$$

Continuing this idea, suppose $\{\lambda_i\}$ to be any bounded sequence of non-zero complex numbers, $\{u_i\}$ to be an orthonormal subset of H_1, $\{v_i\}$ to be one of H_1. Then the formula

$$T\colon\ w \longmapsto \sum \lambda_i (w \bullet v_i) u_i$$

defines a bounded operator from H_1 to H_2, with bound equal to $\limsup |\lambda_i|$. We can read off the singular value decomposition easily:

$$|T|\colon\ w \longmapsto \sum |\lambda_i| (w \bullet v_i) v_i, \quad U\colon v_i \longmapsto (\lambda_i/|\lambda_i|) u_i\,.$$

.

There is an important difference between sequences λ_i that converge to 0 and those that do not. It is those in the first group that this essay is concerned with. What is the difference, exactly? Simplify things slightly by assuming the λ_i to be a decreasing positive real sequence. If T_n is the operator

$$T_n\colon\ v = \sum c_i v_i \longmapsto \sum_{i \le n} c_i \lambda_i v_i$$

then

$$\big\|T(v) - T_n(v)\big\|^2 \le \lambda_{n+1}^2 \Big(\sum_{i>n} |c_i|^2\Big) \le \lambda_{n+1}^2 \|v\|^2$$

so that $\|T - T_n\| \le \lambda_{n+1}$. Therefore if $\lambda_n \to 0$ the operator T is the limit in the norm topology of the operators T_n, all of which have finite rank.

The operators that possess this property are quite special. A bounded operator T from one Hilbert space to another is called **compact** (for reasons that will become apparent in a moment) if for every $\varepsilon > 0$ there exists an operator F of finite rank such that $\|T - F\| < \varepsilon$. The following is immediate:

2.1. Proposition. *The subspace of compact operators is closed in the space of all bounded operators.*

Here is another general fact about compact operators:

2.2. Proposition. *If S is bounded and T compact then ST and TS are also compact.*

Here, too, the proof is immediate.

WHY ARE THEY CALLED COMPACT?. There is another useful way to characterize such operators, which explains the terminology.

2.3. Theorem. *A bounded linear operator is compact if and only if it takes bounded subsets into relatively compact ones.*

I recall that a set is called **relatively compact** if its closure is compact.

The proof requires a preliminary discussion of compactness. Let X be an arbitrary complete separable metric space. (Separability means there exists a countable dense subset.) The classic theorem of Heine-Borel asserts that there are two equivalent definitions of compactness of a subset K of X: (1) every covering of K by open sets possesses a finite sub-covering; (2) every sequence of points in K contains a subsequence of points converging to a point in K. Compact sets are closed.

But there is a third criterion. A set is called **totally bounded** if for any $\varepsilon > 0$ it may be covered by a finite set of ε-balls.

2.4. Proposition. *If X is a complete separable metric space, the following are equivalent conditions on a subset K of X:*

(a) every sequence of points in K contains a subsequence that converges to a point of X;
(b) the subset K is relatively compact;
(c) the subset K is totally bounded.

Proof. (a) implies (b): Let y_i be a sequence of points in $\overline{K}$. For each of these, let x_i be a point of K such that $|x_i - y_i| \le 1/i$. By assumption, there exists a subsequence x_{i_j} converging to some y in X. The subsequence y_{i_j} converges to the same point.

(b) implies (c): Immediate.

(c) implies (a): Let x_i be any sequence of points in K. The set K can be covered by a finite number of balls of radius 1, so one of them must contain an infinite subsequence of them. And so on for balls of radius $1/n$ for all $n > 1$. In this way we get a Cauchy subsequence in K.

The last part of the proof uses the Axiom of Choice.

Now for the proof of Theorem 2.3. Suppose first that $T\colon H_1 \to H_2$ is an operator of finite rank. Then the image of any bounded subset of H_1 is a bounded subset of the image of T, hence compact. Now suppose that T is an arbitrary compact operator. For each $\varepsilon > 0$ we can find an operator F of finite rank such that $\|T - F\| \le \varepsilon/2$. It suffices to show that T takes B_1 into a relatively compact subset. If $X = B_1$ then in H_2 the set $F(X)$ may be covered by a finite number of balls $B_{\varepsilon/2}(x_i)$ of radius $\varepsilon/2$. But then for any v in X we know that

$$\|T(v) - F(v)\| \le \varepsilon/2$$

and we also know that there exists x_i with

$$\|F(v) - x_i\| \le \varepsilon/2 \, .$$

But then

$$\|T(v) - x_i\| \le \varepsilon$$

and hence $T(X)$ is covered by the balls $B_\varepsilon(x_i)$.

Conversely, suppose that the image of B_1 is relatively compact in H_2. Given $\varepsilon > 0$, the image $T(B_1)$ may be covered by a finite collection of balls $B_{\varepsilon/2}(x_i)$ for $i = 0, \ldots, n$. Let Π be orthogonal projection onto the space spanned by the x_i, and F the composition of T followed by Π. If v lies in B_1 then there exists x_i such that $\|T(v) - x_i\| \le \varepsilon/2$. Since orthogonal projection does not increase lengths

$$\begin{aligned}\|T(v) - x_i\| &\le \varepsilon/2 \\ \|\Pi(T(v)) - x_i)\| &= \|\Pi(T(v)) - \Pi(x_i)\| \\ &\le \varepsilon/2 \\ \|\Pi(T(v)) - T(v)\| &\le \|\Pi(T(v)) - x_i\| + \|x_i - T(v)\| \\ &\le \varepsilon\,.\end{aligned}$$

FREDHOLM THEORY. Let H be a Hibert space, $\mathcal{L}(H)$ the ring of bounded operators in H. Suppose D to be an open subset of $\mathbb{C}$, $F(z)$ a function on D with values in the space $\mathcal{L}(H)$. It is called analytic if it may be locally expanded in power series converging in the norm topology.

The following is a classic result.

2.5. Proposition. *If T is a compact operator from a Hilbert space H to itself, the operator $(I - zT)^{-1}$ is a meromorphic function of z whose poles are the inverses of the non-zero eigenvalues of T. For each $\lambda \ne 0$ the subspace H_λ of vectors annihilated by some power of $(T - \lambda I)$ has finite dimension.*

This last means that the filtration

$$\mathrm{Ker}(T - \lambda I) \subseteq \mathrm{Ker}(T - \lambda I)^2 \subseteq \mathrm{Ker}(T - \lambda I)^3 \subseteq \ldots$$

is stable. One can deduce the nature of this filtration from the nature of the pole of $(I - zT)^{-1}$ at $1/\lambda$, but all I'll mention here is that T acts as a scalar on this space if and only if the pole is simple. You can get a good idea of what happens by looking at finite matrices in Jordan form.

One important thing about the first assertion is that the poles of $(I - zT)^{-1}$ have no accumulation point in $\mathbb{C}$.

Proof of the Proposition 2.5. Suppose T to be an arbitrary compact operator, and let $C(z) = zT$. Suppose for the moment z_0 to be any point of $\mathbb{C}$. Choose r such that $\|C(z) - C(z_0)\| < 1/2$ if $|z - z_0| < r$, and choose an operator F_0 of finite rank such that $\big\|C(z_0) - F_0\big\| < 1/2$. Set $\Delta(z) = C(z) - F_0$. Then $\big\|\Delta(z)\big\| < 1$ for $|z - z_0| < r$ and in that disc the operator $I - \Delta(z)$ is invertible since the series

$$I + \Delta(z) + \Delta(z)^2 + \cdots$$

converges.

Now

$$\begin{aligned}I - C(z) &= I - (C(z) - F_0) - F_0 \\ &= I - \Delta(z) - F_0 \\ &= \big(I - F_0\big(I - \Delta(z)\big)^{-1}\big)\big(I - \Delta(z)\big) \\ &= \big(I - G(z)\big)\big(I - \Delta(z)\big)\end{aligned}$$

where

$$G(z) = F_0\big(I - \Delta(z)\big)^{-1}\,.$$

We now require a Lemma.

2.6. Lemma. *Suppose $\varphi(z)$ to be a holomorphic family of bounded operators defined on the open region $D \subseteq \mathbb{C}$. Suppose F to be an operator of finite rank, and set*

$$E(z) = F(\varphi(z))\,.$$

Suppose that there does not exist a vector $v \neq 0$ fixed by all $E(z)$. Then the operator $(I - E(z))^{-1}$ is meromorphic on D and its poles are at the values of z for which $E(z)$ has a non-trivial fixed vector. For any z, the dimension of the space of vectors annihilated by some power of $I - E(z)$ is finite.

Proof. Suppose that the image of F is contained in the finite dimensional space U. Let $u_1, \ldots, u_n$ be an orthonormal basis of U, and extend it to an orthonormal basis (u_i) of H. Since F has finite rank, we have

$$F = \sum u_i \otimes v_i$$

with

$$[u \otimes v](w) = (w \bullet v)\, u\,.$$

Then

$$[E(z)](w) = F\big([\varphi(z)](w)\big) = \sum \big([\varphi(z)](w) \bullet v_i\big)\, u_i = \sum \big(w \bullet [\varphi(z)]^*(v_i)\big)\, u_i\,.$$

Therefore if we set

$$\varphi_{i,j}(z) = u_j \bullet [\varphi(z)]^*(v_i) = [\varphi(z)](u_j) \bullet v_i$$

the matrix of $I - E(z)$ with respect to the basis u_i is

$$\begin{bmatrix} 1 - \varphi_{1,1}(z) & -\varphi_{1,2}(z) & \ldots & -\varphi_{1,n}(z) & -\varphi_{1,n+1}(z) & \ldots & \\ -\varphi_{2,1}(z) & 1 - \varphi_{2,2}(z) & \ldots & -\varphi_{2,n}(z) & -\varphi_{2,n+1}(z) & \ldots & \\ & & \ldots & & & & \\ -\varphi_{n,1}(z) & -\varphi_{n,2}(z) & \ldots & 1 - \varphi_{n,n}(z) & -\varphi_{n,n+1}(z) & \ldots & \\ 0 & 0 & \ldots & 0 & 1 & 0 & \ldots \\ 0 & 0 & \ldots & 0 & 0 & 1 & \ldots \\ & & \ldots & & & & \end{bmatrix}$$

This matrix has the form

$$I - E(z) = \begin{bmatrix} I_n - \Phi(z) & N(z) \\ 0 & I \end{bmatrix},$$

in which $\Phi(z)$ a holomorphic function taking values in the space of $n \times n$ complex matrices. Either $\det\big(I_n - \Phi(z)\big)$ is identically 0, or not. In the second case, one can solve explicitly for $(I_n - E(z))^{-1}$ in terms of the cofactor matrix of $I - \Phi(z)$. Its poles are where $\Phi(z)$ has eigenvalue 1. Also in this case we may write for each $n > 0$

$$(I_n - E(z))^n = \begin{bmatrix} \big(I_n - \Phi(z)\big)^n & N_n(z) \\ 0 & I \end{bmatrix}$$

which implies that the vectors annihilated by some power of $(I_n - E(z))$ are the same as those annihilated by some power of $I_n - \Phi(z)$, which has finite dimension. This concludes the proof of the Lemma. ▯

The operator $G(z)$ is of finite rank, since its image is contained in the image of F_0. Thus in the neighbourhood of z_0 the operator $I - C(z)$ is invertible if and only if $I - G(z)$ is. As for $G(z)$, Lemma 2.6 shows that the set of points where it is invertible is either discrete, or empty. Now since $C(0) = I$, $I - C(z)$ is certainly invertible in the neighbourhood of the origin, and hence by analytic continuation the operator $\big(I - C(z)\big)^{-1}$ is meromorphic with a discrete set of poles. This concludes the proof of the Theorem. ▯

2.7. Corollary. *If T is a compact operator from a Hilbert space to itself, then its spectrum $\sigma(T)$ is a discrete set having no limit points except possibly* 0.

2.8. Corollary. *If T is a self-adjoint compact operator, then for any $\lambda \neq 0$ the eigenspace H_λ for T has finite dimension.*

2.9. Corollary. *A self-adjoint operator T is compact if and only if there exists a complete orthonormal basis v_i of the subspace complementary to the kernel of T, and a sequence of real numbers $\lambda_i \neq 0$ with limit* 0 *such that $Tv_i = \lambda_i v_i$.*

A CONVERSE. An argument in a previous subsection can be reversed:

2.10. Theorem. *The operator $T\colon H_1 \to H_2$ is compact if and only if there exist orthonormal sets $\{v_i\}$ in H_1 and $\{u_i\}$ in H_2 and a set of positive $\{\lambda_i\}$ with $\lambda_i \to 0$ such that*

$$T(x) = \sum \lambda_i (x \bullet v_i)\, u_i \,.$$

Proof. We have seen at the beginning of this section that T is compact if such a sequence exists.

So now assume T compact. The proof of the formula for T is motivated by the singular value decomposition in finite dimensions. If $T = US$ with U unitary and S positive Hermitian with $\{v_i\}$ an orthonormal basis of eigenvectors, then

$$\begin{aligned} S(u) &= \sum \lambda_i (u \bullet v_i)\, v_i \\ T(u) = US(u) &= \sum \lambda_i (u \bullet v_i)\, U(v_i) \\ &= \sum \lambda_i (u \bullet v_i)\, u_i \quad \big(u_i = U(v_i)\big)\,. \end{aligned}$$

So now we continue. Since T is compact, so is $T^* \cdot T$. It is also positive. Therefore by Corollary 2.9 there exists an eigenpair sequence $\{v_i, \mu_i\}$ $(i \ge 1)$ with

$$T^* \cdot T(v_i) = \mu_i\, v_i$$

and $T^* \cdot T = 0$ on the complement of the v_i. Since $\mu_i > 0$, we may define $\lambda_i = \sqrt{\mu_i} > 0$. We have

$$w = w_0 + \sum_{i \ge 1} (w \bullet v_i)\, v_i$$

for every w, where w_0 lies in the kernel of T, hence

$$T(w) = \sum_{i \ge 1} (w \bullet v_i) T(v_i)\,.$$

Set $u_i = T(v_i)/\sqrt{\lambda_i}$. Then

$$u_i \bullet u_j = \frac{T(v_i) \bullet T(v_j)}{\lambda_i \lambda_j} = \frac{T^* T(v_i) \bullet v_j}{\lambda_i \lambda_j} = \frac{\lambda_i}{\lambda_j}\,(v_i \bullet v_j)$$

so $\{v_i\}$ is an orthonormal basis for the complement of the kernel of T, and

$$T(w) = \sum \lambda_i (w \bullet v_i)\, u_i \,.$$

As for uniqueness, it is easy to see that any representation of this kind has to arise from the singular value factorization. ▯

The λ_i in this result are called (what else?) the **singular values** of T.

3. Hilbert-Schmidt operators

3.1. Lemma. *If T is a positive operator on the Hilbert space H the sum $\sum T(u_i) \bullet u_i$ is independent of the choice of orthonormal basis $\{u_i\}$.*

The terms are all non-negative. The sum might be infinite. Whether finite or infinite, it is called the **trace** of T.

Proof. Let $S = T^{1/2}$. Suppose $\{u_i\}$ and $\{v_i\}$ to be two orthonormal bases.

$$\begin{aligned}\sum_i T(u_i) \bullet u_i &= \sum_i \|S(u_i)\|^2 \\ &= \sum_i \Big(\sum_j |S(u_i) \bullet v_j|^2 \Big) \\ &= \sum_{i,j} |S(v_j) \bullet u_i|^2 \\ &= \sum_i \|S(v_i)\|^2 \\ &= \sum_j T(v_j) \bullet v_j \,. \end{aligned}$$

By analogy with what happens in finite dimensions, this sum is called $\mathrm{trace}(T)$.

Each composite $T^* \cdot T$ is positive. An operator $T\colon H_1 \to H_2$ is called a **Hilbert-Schmidt** operator if

$$\mathrm{trace}\ T^* \cdot T = \sum \|T(u_i)\|^2 < \infty \,.$$

for some—hence by the Lemma any—orthonormal basis $\{u_i\}$.

Let $\mathcal{I}_2 = \mathcal{I}_2(H_1, H_2)$ be the set of Hilbert-Schmidt operators from H_1 to H_2. Define on $\mathcal{I}_2$ the norm

$$\|T\|_2^2 = \mathrm{trace}\ T^* \cdot T$$

3.2. Proposition. *If T is any bounded operator then $\|T\| \le \|T\|_2$.*

Proof. If u is a unit vector we may choose it to be the first element of a basis, so that $\|Tu\|^2 \le \|T\|_2^2$. Thus $\|T\| \le \|T\|_2$.

3.3. Proposition. *The space $\mathcal{I}_2$ with the norm $\|T\|_2$ is a Hilbert space, and if T is in $\mathcal{I}_2$ then $\|T\|_2 = \|T^*\|_2$.*

Proof. It is immediate that $\|\lambda T\|_2 = |\lambda|\,\|T\|_2$ and

$$\|S + T\|_2 \le \|S\|_2 + \|T\|_2 \,,$$

so the set $\mathcal{I}_2$ is a vector space.

Suppose $\{u_i\}$ and $\{v_j\}$ to be orthonormal bases of H_1 and H_2. If T is any bounded operator from H_1 to H_2

$$T(u_i) = \sum t_{i,j} v_j, \quad \|T(u_i)\|^2 = \sum_j |t_{i,j}|^2$$

so that T is a Hilbert-Schmidt operator if and only if $\|T\|_2^2 = \sum |t_{i,j}|^2 < \infty$. Conversely, every such infinite matrix $(t_{i,j})$ corresponds to a unique Hilbert-Schmidt operator, and $\mathcal{I}_2$ is in fact isomorphic to the Hilbert space of all such infinite matrices. The adjoint T^* corresponds to the matrix which is the conjugate transpose of that of T.

3.4. Theorem. *A Hilbert-Schmidt operator may be approximated in the $\mathcal{I}_2$ norm by operators of finite rank.*

Proof. Let $(t_{i,j})$ be the matrix corresponding to the Hilbert-Schmidt operator T. Thus $\sum |t_{i,j}|^2 < \infty$. Choose n so $\sum_{\inf(i,j)>n} |t_{i,j}|^2 < \varepsilon^2$. If F is the operator of finite rank whose matrix is the first n rows of the matrix then $\|T - F\|_2 < \varepsilon$.

3.5. Corollary. *Hilbert-Schmidt operators are compact.*

Proof. Because $\|T\| \le \|T\|_2$.

3.6. Proposition. *If S is an arbitrary bounded operator and T is Hilbert-Schmidt, then ST and TS are both Hilbert-Schmidt.*

In other words, the linear space of Hilbert-Schmidt operators is an ideal in the ring of bounded operators.

Proof. There are two situations to investigate:

$$H_1 \xrightarrow{S} H_2 \xrightarrow{T} H_3$$
$$H_1 \xrightarrow{T} H_2 \xrightarrow{S} H_3\,.$$

On the one hand

$$\sum \|ST(u_i)\|^2 \le \|S\|^2 \sum \|T(u_i)\|^2\,.$$

On the other, $TS = (S^*T^*)^*$. Apply Proposition 3.3.

3.7. Proposition. *The operator T is Hilbert-Schmidt if and only if there exist orthonormal sets $\{u_i\}$ and $\{v_i\}$ and a sequence $\{\lambda_i\}$ with $\sum |\lambda_i|^2 < \infty$ such that*

$$Tw = \sum \lambda_i (w \cdot v_i) u_i\,.$$

The λ_i are the singular values of T.

Proof. Since

$$T(u_i) = \sum_j \lambda_j (u_i \bullet u_j) v_j. = \lambda_i v_i \quad \|T(u_i)\|^2 = \lambda_i^2\,.$$

4. Example: differential operators on the circle

Now let $H = \mathrm{L}^2(\mathbb{S})$, and identify $\mathbb{S}$ with $\mathbb{R}/\mathbb{Z}$. Set $Dy = y''$. The eigenvalues of D are the μ with periodic solutions $y(x)$ to the equation $y'' = \mu y$. The solutions of this equation on $\mathbb{R}$ are the linear combinations of $e^{\lambda x}$ and $e^{-\lambda x}$ where $\lambda^2 = \mu$. These solutions will be periodic if and only if $\lambda = 2\pi i n$, in which case $\mu = -4\pi^2 n^2$.

4.1. Proposition. *Suppose $\mu \ne -4\pi^2 n^2$ for n in $\mathbb{Z}$. Then $D - \mu I$ is an isomorphism of the domain of D with $\mathrm{L}^2(\mathbb{S})$.*

Proof. Suppose μ not to be one of these eigenvalues.

There are two ways to prove the Proposition.

The first in terms of Fourier series. If F is in $\mathrm{L}^2(\mathbb{S})$ then

$$F(x) = \sum\nolimits_{\mathbb{Z}} F_n e^{2\pi i n x}, \quad F_n = F \bullet e^{2\pi i n x}\,.$$

The map taking F to (F_n) is an isomorphism of $\mathrm{L}^2(\mathbb{S})$ with $\mathrm{L}^2(\mathbb{Z})$. Since $(DF)_n = -4\pi^2 n^2 F_n$, the domain of D is the subspace of those F such that $n^2 F_n$ is square-integrable. The distribution $(D - \mu I)F$ has coefficients $(-4\pi^2 n^2 - \mu)F_n$, and under the assumption on μ none of the factors vanishes. The inverse of $D - \mu I$ therefore takes the function with Fourier coefficients F_n to that with coefficients $F_n/(-4\pi^2 n^2 - \mu)$.

The second way to understand the inverse of $D - \mu I$ is in terms of an integral operator. If Φ is a distribution on $\mathbb{S}$, its derivative is defined by the equation

$$\langle \Phi', f \rangle = -\langle \Phi, f' \rangle \,,$$

leading to a formula for the second derivative

$$\langle \Phi'', f \rangle = \langle \Phi, f'' \rangle \,,$$

where in both cases f is an arbitrary smooth function on $\mathbb{S}$. A fundamental solution of $D - \mu I$ is a distribution F_y on $\mathbb{S}$ depending on the parameter y such that

$$(D - \mu I)F_y = \delta_y \,.$$

This may be constructed explicitly, since the distribution equation amounts to the conditions that (a) $F = F_y$ is smooth and periodic on $\mathbb{R}$ of period 1 except at the points $y + n$, and (b) F is continuous at these points, but $F'(y+) - F'(y-) = 1$. For any x in $\mathbb{R}$ let $\langle x \rangle = x - \lfloor x \rfloor$ be the fractional part of x. If we define

$$f(x) = \frac{1}{\lambda} \cdot \frac{e^{\lambda\langle x \rangle} + e^{-\lambda\langle x \rangle}}{e^{\lambda/2} - e^{-\lambda/2}} \,,$$

then this function satisfies these conditions at the points in $1/2 + \mathbb{Z}$. Since D commutes with translations, we set

$$F_y = f(y - 1/2) \,.$$

The formula

$$\varphi(x) = \int_0^1 \varphi(y) F_y(x) \, dy$$

defines the inverse to $D - \mu I$.

Now I want to look at a much more general situation, one in which nothing explicit can de done, but much can be said in a general way. I'll leave out details, since the subject of Sobolev spaces deserves, and gets, a longer treatment elsewhere. Let

$$Lf = -d^2 f/dx^2 + a(x) f$$

where the real function $a(x)$ is smooth and periodic of order 2π, and hence L may be considered as an operator on $C_c^\infty(\mathbb{S})$, where $\mathbb{S}$ is the unit circle. This operator is symmetric and essentially self-adjoint. We have

$$\int_{\mathbb{S}} Lf(x)\overline{f(x)} \, dx = \int_{\mathbb{S}} |f'(x)|^2 + a(x)|f(x)|^2 \, dx \,.$$

Replacing L by $L + pI$ if necessary, we may assume that L is a positive operator.

4.2. Proposition. *The set of eigenvalues of L is infinite and discrete in $\mathbb{R}$.*

For $m \ge 0$ the **Sobolev space H**^{m} is that of f such that every $d^k f/dx^k$ lies in $\mathrm{L}^2(\mathbb{S})$ for all $k \le m$. Fourier analysis tells us that this is the same as the distributions whose Fourier coefficients c_ℓ satisfy

$$\sum \left(1 + |\ell|^2\right)^{m/2} |c_\ell|^2 < \infty \,.$$

This condition defines $\mathbf{H}^m$ for $m < 0$, too. Every $\mathbf{H}^m$ is contained in $\mathbf{H}^{m-1}$, and this embedding is Hilbert-Schmidt, since it is the composite of the operators

$$(c_n) \longrightarrow \big(1 + |n|\big)c_n, \quad (c_n) \longrightarrow c_n/\big(1 + |n|\big)\,,$$

The second is Hilbert-Schmidt, and we know that the composite of a Hilbert-Schmidt operator and a bounded operator is Hilbert-Schmidt.

Functions in $\mathbf{H}^m$ are in C^{m-1}, so the intersection of all $\mathbf{H}^m$ is $C^\infty(\mathbb{S})$. Furthermore, $\mathbf{H}^2$ is the domain of both $y \mapsto y''$ and L. Each of these takes $\mathbf{H}^m$ to $\mathbf{H}^{m-2}$.

The operator $I + L$ is in fact an isomorphism of $\mathbf{H}^k$ with $\mathbf{H}^{k-2}$ for all k. Therefore $(I + L)^{-1}$ is a compact operator, and from this we deduce an orthogonal decomposition of L^2 into finite-dimensional eigenspaces of L. If φ is smooth and $Lf = \varphi$ then f is also smooth. Hence all eigenfunctions are smooth.

5. Nuclear operators

An arbitrary bounded operator T on the Hilbert space H is said to be **nuclear**, or of **trace class**, if the trace of $|T|$ is finite—if

$$\sum |T|(u_i) \bullet u_i = \sum |T|^{1/2}(u_i) \bullet |T|^{1/2}(u_i) < \infty$$

for one, hence by Lemma 3.1 all, orthonormal bases $\{u_i\}$.

Let $\mathcal{I}_1(H)$ be the set of nuclear operators from H to itself. We shall see later how to extend the trace function to all of $\mathcal{I}_1(H)$ (instead of just positive operators), but that will take some preparation.

5.1. Lemma. *If T is a positive operator and U is a partial isometry then* $\mathrm{trace}(U^* TU) \le \mathrm{trace}(T)$. *Equality holds if U is unitary.*

Proof. Choose a basis $\{u_i\}$ such that each u_i is in either the kernel of U or its perpendicular complement. The vectors $U(u_i)$ for u_i in the complement may be extended to a full orthonormal basis $\{v_i\}$. Then

$$\sum U^* TU(u_i) \bullet u_i = \sum TU(u_i) \bullet U(u_i) \le \sum T(v_i) \bullet v_i = \mathrm{trace}\, T\,.$$

If U is unitary, equality holds. ▯

5.2. Proposition. *The set of nuclear operators is a vector space. More precisely:*

(a) if T is nuclear, so is λT;
(b) if S and T are nuclear, so is $S + T$, and

$$\mathrm{trace}\,|S + T| \le \mathrm{trace}\,|S| + \mathrm{trace}\,|T|\,.$$

Proof. Claim (a) is immediate, but (b) is a bit tricky (even in finite dimensions).

Suppose S and T to be in $\mathcal{I}_1$. Start with unitary singular-value decompositions:

$$\begin{aligned} S &= U\,|S| \\ T &= V\,|T| \\ S + T &= W\,|S + T|\,, \end{aligned}$$

which are equivalent to

$$\begin{aligned} |S| &= U^*\,S \\ |T| &= V^*\,T \\ |S + T| &= W^*\,(S + T)\,. \end{aligned}$$

Then for any u

$$\begin{aligned} u \bullet |S+T|u &= u \bullet W^*(S+T)(u) \\ &\le \big|u \bullet W^* \, U|S|(u)\big| + \big|u \bullet W^* \, V|T|(u)\big| \\ &= \big|\,|S|^{1/2}U^* \, W(u) \bullet |S|^{1/2}(u)\big| + \big|\,|T|^{1/2}V^* \, W(u) \bullet |T|^{1/2}(u)\big| \\ &\le \big\|\,|S|^{1/2}U^* \, W(u)\big\| \cdot \big\|\,|S|^{1/2}(u)\big\| + \big\|\,|T|^{1/2}V^* \, W(u)\big\| \cdot \big\|\,|T|^{1/2}(u)\big\| \end{aligned}$$

and thus

$$\begin{aligned} \sum_1^\infty u_i \bullet |S+T|(u_i) &\le \sum_1^n \big\|\,|S|^{1/2}U^* \, W(u_i)\big\| \cdot \big\|\,|S|^{1/2}(u_i)\big\| \\ &\qquad + \sum_1^\infty \big\|\,|T|^{1/2}V^* \, W(u_i)\big\| \cdot \big\|\,|T|^{1/2}(u_i)\big\| \\ &\le \Big(\sum_1^\infty \big\|\,|S|^{1/2}U^* \, W(u_i)\big\|^2\Big)^{1/2} \Big(\sum_1^\infty \big\|\,|S|^{1/2}(u_i)\big\|^2\Big)^{1/2} \\ &\qquad + \Big(\sum_1^\infty \big\|\,|T|^{1/2}V^* \, W(u_i)\big\|^2\Big)^{1/2} \Big(\sum_1^\infty \big\|\,|T|^{1/2}(u_i)\big\|^2\Big)^{1/2} \\ &= \Big(\sum_1^\infty \big\|\,|S|^{1/2}(u_i)\big\|^2\Big)^{1/2} \Big(\sum_1^\infty \big\|\,|S|^{1/2}(u_i)\big\|^2\Big)^{1/2} \\ &\qquad + \Big(\sum_1^\infty \big\|\,|T|^{1/2}(u_i)\big\|^2\Big)^{1/2} \Big(\sum_1^\infty \big\|\,|T|^{1/2}(u_i)\big\|^2\Big)^{1/2} \\ &= \Big(\sum_1^\infty \big\|\,|S|^{1/2}(u_i)\big\|^2\Big) \qquad\qquad + \Big(\sum_1^\infty \big\|\,|T|^{1/2}(u_i)\big\|^2 \\ &= \operatorname{trace}|S| + \operatorname{trace}|T| \,. \end{aligned}$$

5.3. Lemma. *Every bounded linear operator from a Hilbert space to itself is a linear combination of four unitary operators.*

Proof. If T is any bounded operator, The operators $A = T + T^*$ and $B = i(T - T^*)$ are self-adjoint, and

$$T = (1/2)A - (i/2)B \,.$$

So to prove the theorem, we may suppose T to be self-adjoint and $\|T\| \le 1$. Then

$$T = (1/2)\Big(T + i\sqrt{I - T^2}\,\Big) + (1/2)\Big(T - i\sqrt{I - T^2}\,\Big) \,,$$

and each of these terms is unitary since

$$\Big(T \pm i\sqrt{I - T^2}\,\Big)^* = T \mp i\sqrt{I - T^2}, \quad \Big(T \pm i\sqrt{I - T^2}\,\Big)\Big(T \mp i\sqrt{I - T^2}\,\Big) = I \,.$$

5.4. Proposition. *If S is bounded and T in $\mathcal{I}_1$ then ST and TS are also in $\mathcal{I}_1$, and* $\operatorname{trace} ST = \operatorname{trace} TS$.

Proof. By Lemma 5.3 we may assume T unitary. Let $\{u_i\}$ be an orthonormal basis $\{v_i = T(u_i)\}$ another.

$$\begin{aligned}\operatorname{trace} ST &= \sum ST(u_i) \bullet u_i \\ &= \sum ST(u_i) \bullet T^* T(u_i) \\ &= \sum S(v_i) \bullet T^*(v_i) \\ &= \sum TS(v_i) \bullet v_i \\ &= \operatorname{trace} TS\,.\end{aligned}$$

5.5. Proposition. *Any operator of trace class is Hilbert-Schmidt.*

Proof. If T is of trace class then so are $|T|$ and, by Proposition 5.4, $|T|^2$. But $\operatorname{trace}|T|^2 = \sum \|T(u_i)\|^2$.

5.6. Corollary. *Every operator of trace class is compact. A compact operator is in $\mathcal{I}_1$ if and only if $\sum \lambda_i < \infty$, where the λ_i are its singular values.*

Proof. The first part follows from the previous two results. For the last part, if T then so is $|T| = U^* T$, and its canonical expansion is

$$|T|u = \sum \lambda_i (u \bullet u_i) u_i\,.$$

But then $\sum |T|u_i \bullet u_i = \sum \lambda_i$.

Define the norm on $\mathcal{I}_1$:

$$\|T\|_1 = \sum \lambda_i$$

where the λ_i are the singular values of T.

5.7. Proposition. *The space $\mathcal{I}_1$ together with the norm $\|T\|_1$ is a Banach space, and $\|T\| \le \|T\|_1$. The operators of finite rank are dense in this Banach space.*

Proof. Exercise.

5.8. Proposition. *If T is in $\mathcal{I}_1$, the sum*

$$\operatorname{trace} T = \sum T(u_i) \bullet u_i$$

converges absolutely and is independent of the orthonormal basis $\{u_i\}$.

Proof. Write the unitary singular value decomposition $T = U|T|U^* \cdot U$. Then $S = U|T|U^*$ is positive and self-adjoint, and also in $\mathcal{I}_2$. Also, $S^{1/2}$ and $S^{1/2}U$ are in $\mathcal{I}_2$. Hence

$$|T(u_i) \bullet u_i| = \left||S|^{1/2}U(u_i) \bullet S^{1/2}(u_i)\right| \le \|S^{1/2}U(u_i)\|\,\|S^{1/2}(u_i)\|$$

and

$$\sum |T(u_i) \bullet u_i| \le \sum \|S^{1/2}U(u_i)\|\,\|S^{1/2}(u_i)\| \le \Big(\sum \|S^{1/2}U(u_i)\|^2\Big)^{1/2}\Big(\sum \|S^{1/2}(u_i)\|^2\Big)^{1/2}$$

so the sum converges absolutely.

Independence is formal:

$$\begin{aligned}\sum_i T(u_i) \bullet u_i &= \sum_i T\Big(\sum_j (u_i \bullet v_j) v_j\Big) \bullet u_i \\ &= \sum_{i,j} (v_j \bullet u_i)(T(v_j) \bullet u_i) \\ &= \sum_j T(v_j) \bullet \Big(\sum_i (v_j \bullet u_i) u_i\Big) \\ &= \sum_j T(v_j) \bullet v_j \,.\end{aligned}$$

This concludes the proof of the Proposition.

5.9. Theorem. *An operator is in $\mathcal{I}_1$ if and only if it factors as the composite of two Hilbert-Schmidt operators.*

Proof. From Proposition 5.4 and the singular value decomposition.

5.10. Corollary. *For a bounded operator S, $\operatorname{trace} S^* = \overline{\operatorname{trace} S}$.*

Proof. This is immediate.

The following is also a corollary of the previous result.

5.11. Theorem. *The operator T is of trace class if and only if there exist orthonormal sets u_i, v_i with*

$$T(x) = \sum_i \lambda_i (x \bullet v_i) u_i \,,$$

in which $\sum |\lambda_i| < \infty$. In this case its trace is

$$\sum \lambda_i (u_i \bullet v_i) \,.$$

6. Integral operators and traces

One commonly encountered example of a Hilbert-Schmidt operator is an integral operator defined by a kernel function.

Suppose (M, dx) to be a measure space such that $\mathrm{L}^2(M)$ is separable. For example, M could be a locally compact space with countable basis, dx a Baire measure, which I'll eventually assume to be the case. Let $K(x, y)$ be an L^2 function on $M \times M$. Then the integral formally defined as

$$[T_K f](x) = \int_M K(x, y) f(y) \, dy$$

determines a bounded operator T_K from $\mathrm{L}^2(M)$ to itself. More precisely, it is defined by Riesz' Lemma (identifying a Hilbert space with its conjugate dual) and the equation

$$[T_K f] \bullet g = \int_M K(x, y) f(y) \overline{g(x)} \, dx dy$$

for every g in $\mathrm{L}^2(M)$.

6.1. Proposition. *A bounded linear operator T on $\mathrm{L}^2(M)$ is Hilbert-Schmidt if and only if $T = T_K$ for some K in $\mathrm{L}^2(M \times M)$. Furthermore,*

$$\|T_K\|_2^2 = \int_{M\times M} \big|K(x,y)\big|^2 \, dx\, dy \, .$$

Proof. Let $\{u_i\}$ be an orthonormal basis of $\mathrm{L}^2(M)$. Then the products $u_{i,j}(x,y) = u_i(x)\overline{u}_j(y)$ are an orthonormal basis of $\mathrm{L}^2(M \times M)$ (§II.4 of [Reed-Simon:1973]). We may therefore express

$$K = \sum\nolimits_{i,j} c_{i,j} u_{i,j} \, .$$

We have

$$\operatorname{trace} T_K^* T_K = \sum |c_{i,j}|^2 = \|K\|_2^2 \, .$$

Thus $K \mapsto T_K$ is an isometric embedding of $\mathrm{L}^2(M \times M)$ into $\mathcal{I}_2$. It has closed range. But the finite rank operators are contained in it and are dense in $\mathcal{I}_2$.

If K is a function on the product of a finite set S with itself, T_K may be identified with a finite matrix, and the trace of T_K is the sum $\sum K(s,s)$ of its diagonal entries. There are many generalizations of this in the literature. Most have rather restrictive hypotheses, and and not all are correct. One whose hypothesis is fairly simple and whose proof is not too complicated can be found in [Duflo:1972]:

6.2. Theorem. *Suppose $K(x,y)$ to be a continuous square-integrable kernel function on some σ-compact space M with a Baire measure dx whose support is all of X. If T_K is of trace class, then the restriction of K to the diagonal is integrable and the trace of T_K is*

$$\int_M K(x,x)\, dx \, .$$

In practice, the hypothesis comes easily—for example, when T_K is the product of two Hilbert-Schmidt operators. In certain circumstances the hypothesis can be simplified—if K is continuous and T_K is a positive operator, it will be automatically of trace class if its integral over the diagonal is finite. This does not seem to be all that useful.

The proof can be motivated by an argument that will play an important role later on. Suppose that (a) the functions u_i making up an orthonormal basis of $\mathrm{L}^2(M)$ are continuous on M, and (b) K is defined by an abolutely and uniformly converging sum

(6.3)
$$K(x,y) = \sum \lambda_i u_i(x) \overline{v}_i(y) \, .$$

Thsi kernel is therefore manifestly continuous. The operator T_K takes

$$w \longrightarrow \sum \lambda_i (w \bullet v_i) u_i$$

and its trace is

$$\sum \lambda_i (u_i \bullet v_i) = \sum \lambda_i \int_M u_i(x) \overline{v}_i(x)\, dx \, .$$

But the assumption about convergence allows us to interchange sum and integral to get this equal to

$$\int_M \sum \lambda_i \cdot u_i(x) \overline{v}_i(x)\, dx = \int_M K(x,x)\, dx \, .$$

The point of the proof is to make this argument valid. The problem is that neither (a) nor (b) is valid. The convergence in (6.3) is only in the L^2-norm, and in addition the u_i, v_i are not necessarily continuous.

The proof of the theorem that I'll present will follow Duflo's argument closely, and will be in several steps. In this first version of this essay I'll quote without proof a couple of basic results in measure theory that are needed. In a later version I hope to include them in an appendix.

6.4. Lemma. (Riesz) *If the sequence f_n converges in L^p-norm ($p \ge 1$) to f then there exists a subsequence which converges pointwise to f almost everywhere.*

This result is well known, but my source for it is the undated article by Péter Medvegyev in the reference list (Propositions 4 and 6).

6.5. Corollary. *In the same circumstances, if the sequence f_n is monotonic then it converges almost everywhere.*

6.6. Lemma. (Egoroff) *Suppose the sequence of measurable functions f_n to converge almost everywhere locally to f. Then for every compact subset Ω of M and $\varepsilon > 0$ there exists a compact set $X \subseteq \Omega$ such that (a) $\mathrm{meas}(\Omega - X) < \varepsilon$; (b) the restriction of each f_n to X is continuous; (c) the sequence f_n converges uniformly on X to f (which is consequently continuous).*

This form of Egoroff's theorem is to be found in §IV.5.4 of [Bourbaki:2007].

So now I begin the proof proper of Theorem 6.2.

Step 1. The starting point is the defining formula

(6.7)
$$[T_K f] \bullet g = \int_M K(x,y) f(y) \overline{g(x)}\, dx dy \,.$$

By assumption and Theorem 5.11, there exist orthonormal bases $\{u_i\}$ and $\{v_i\}$ and a sequence λ_i such that

$$\begin{aligned} \sum |\lambda_i| &< \infty \\ T_K f &= \sum_i \lambda_i (f \bullet v_i)\, u_i \\ \operatorname{trace} K &= \sum \lambda_i (u_i \bullet v_i) \,. \end{aligned}$$

Step 2. Since

$$\begin{aligned} \sum \int_M |\lambda_i| |u_i(x)|^2 \, dx &= \sum |\lambda_i| < \infty \\ \sum \int_M |\lambda_i| |v_i(x)|^2 \, dx &= \sum |\lambda_i| < \infty \,, \end{aligned}$$

the two series

(6.8)
$$\sum |\lambda_i| |u_i(x)|^2, \quad \sum |\lambda_i| |v_i(x)|^2$$

converge in L^1 norm, say to $U(x)$, $V(x)$.

Corollary 6.5 implies that the sequences

$$U_n(x) = \sum_1^n |\lambda_i| |u_i(x)|^2, \quad V_n(x) = \sum_1^n |\lambda_i| |v_i(x)|^2$$

converge pointwise to $U(x)$, $V(x)$ a. e..

Lemma 6.6 implies that for every compact $\Omega \subseteq M$ and $\varepsilon > 0$ there exists a compact $X \subset \Omega$ such that (a) $\mathrm{meas}(\Omega - X) < \varepsilon$; (b) all u_i, v_i are continuous on X; (c) both the sequences $U_n(x)$, $V_n(x)$ converge uniformly there.

Step 3. Now choose an increasing sequence of compact subsets Ω_n whose union is M, and for each n a sequence of compact subsets $X_n^m \subseteq \Omega_n$ verifying the properties above with $\varepsilon = 1/m$.

For each n let

$$X_n = \bigcup_{k=1}^{n} X_k^n \subseteq \Omega_n, \quad X = \bigcup X_n .$$

Each X_n is compact; $X_n \subseteq X_{n+1}$; the restrictions of u_i, v_i to X_n are continuous; the series (6.8) converge uniformly on X_n; and $M - X$ has measure 0. Because of the assumption on the support of dx, we may also assume that the support of dx on X_n is X_n.

By Cauchy-Schwarz

$$\Big(\sum |\lambda_i||u_i(x)|\,|v_i(y)|\Big)^2 \le \Big(\sum |\lambda_i||u_i(x)|^2\Big)\Big(\sum |\lambda_i||v_i(y)|^2\Big) .$$

The series on the left therefore converges uniformly on $X_n \times X_n$. For (x, y) in $X \times X$, define the continuous kernel

$$K_\bullet(x, y) = \sum \lambda_i u_i(x)\overline{v}_i(y) .$$

The formal argument presented earlier may now be applied to $K_\bullet$, whose trace may now be evaluated as

$$\sum \lambda_i (u_i \bullet v_i) = \sum \lambda_i \int u_i(x)\overline{v}_i(x)\, dx ,$$

since

$$\sum |\lambda_i| \int |u_i(x)||v_i(x)|\, dx < \infty .$$

Step 4. Fubini's Theorem allows us to interchange sum and integral to deduce that

$$\int |K_\bullet(x, x)|\, dx \le \sum |\lambda_i| \int |u_i(x)||v_i(x)|\, dx < \infty$$

and

$$\mathrm{trace}\, K_\bullet = \int K_\bullet(x, x)\, dx .$$

On the other hand

$$\int K_\bullet(x, y) f(x) g(y)\, dx\, dy = [T_K f] \bullet g . \tag{6.9}$$

Step 5. Comparing (6.7) with (6.9) , we deduce that the functions

$$K(x, y), \quad K_\bullet(x, y)$$

are almost everywhere equal on $X \times X$. They are also continuous, so that the set on which they differ is both open and of measure 0. By assumption on the support of dx on X_n, this open set must be empty.

Remark. The most interesting applications of Duflo's theorem these days are probably to automorphic forms, which is what Duflo had in mind. An early example can be found in [Duflo-Labesse:1971]. At the bottom of p. 225 in this paper the authors simply refer to a manuscript of Bourbaki—unpublished then and still unpublished now—for the result above. This is presumably the same thing that Duflo refers to as the origin of his theorem.

More examples are in Arthur's development of the Selberg trace formula. There, one wants an expression for the trace of convolution operator R_f (f in $C_c^\infty(G)$) on $\mathrm{L}^2_{\mathrm{cusp}}(\Gamma\backslash G)$. It is relatively easy to verify that it is the composite of two Hilbert-Schmidt operators, and that its kernel is continuous. Arthur's proof (Theorem 3.9 in [Arthur:1970]) demonstrates these points, but then to apply his version of an integral formula requires some extra work. His argument has something in common with that of Duflo, but is more elementary because his hypotheses are stronger. In subsequent accounts of the Arthur-Selberg trace formula the formula for the trace as an integral over the diagonal is always, as far as I know, passed over in an almost inaudible mumble.

Remark. There is a satisfying—close to definitive—generalization of Duflo's (or Bourbaki's) theorem to be found in the paper [Brislawn:1991]. This paper is a sequel to [Brislawn:1988] and [Brislawn:1990]. In the earlier papers he applies the maximal functions of Hardy-Littlewood to obtain new and strong results for $\mathbb{R}^n$. But the natural development of the results of Hardy and Littlewood incorporates a regularization process involving martingales to make sense of the diagonal integral for *any* trace class kernel. In this last paper he does not even assume that the space on which measures exist is locally compact, but implies Duflo's result easily if it is. Martingales are a powerful tool, and I wish that there existed a self-contained account sufficient to justify Brislawn's argument.

I should mention that Brislawn helped me to decipher Duflo's paper, which is somewhat elliptic and in addition contains a few confusing typographical slips.

7. References

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