Analysis on SL(2)

Growth conditions and the constant term

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Let

$$\begin{aligned} G &= \mathrm{SL}_2(\mathbb{R}) \\ K &= \mathrm{SO}(2) \\ \mathcal{H} &= \{z \,|\, \mathrm{IM}(z) > 0\} \\ \Gamma &= \text{ a proper discrete subgroup of } G\,. \end{aligned}$$

The group G acts by fractional linear transformations on $\mathcal{H}$. The isotropy subgroup of i is K, so that $\mathcal{H}$ may be identified with G/K. The condition on Γ means that $\Gamma\backslash\mathcal{H}$ is the union of a compact subset and a finite number of parabolic domains.

What is a parabolic domain? Suppose q to be either ∞ or a real number, and let $P = P_q$ be its stabilizer in $\mathrm{SL}_2(\mathbb{R})$. It will be a conjugate of P_∞, the subgroup of upper triangular matrices. Let N_q be its unipotent radical, which will be a conjugate of $N = N_\infty$, the subgroup of upper triangular unipotent matrices

$$\nu(x) = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}\,.$$

Let $N(\mathbb{Z})$ be the subgroup of integral matrices.

The point q is called a **cusp** if $\Gamma \cap N_q$ is an infinite cyclic group, and in this case its stabilizer P_q is called a cuspidal parabolic subgroup. For example, ∞ is a cusp of the group $\mathrm{SL}_2(\mathbb{Z})$. Some conjugate of $\Gamma \cap N_q$ in $\mathrm{SL}_2(\mathbb{R})$ will be exactly $N(\mathbb{Z})$, and the pull backs x_q, y_q of the functions x, y I call **parabolic coordinates** on $\mathcal{H}$ associated to q. A parabolic domain associated to q is one of the regions $\mathcal{H}_{q,Y}$ where $y_q \geq Y$, or its image in $\Gamma\backslash\mathcal{H}$. If $Y \gg 0$ the projection from $(\Gamma \cap P_q)\backslash\mathcal{H}_{q,Y}$ to $\Gamma\backslash\mathcal{H}$ is an embedding.

The standard example is $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. In this case the region

$$\{z \,|\, \mathrm{IM}(z) > 0, |z| \geq 1, \mathrm{RE}(z) \leq 1/2\}$$

is a fundamental domain for Γ. It is the union of the parabolic domains where $\mathrm{IM}(z) \geq 1$ and a small compact subset. Suppose F to be a holomorphic automorphic form of weight k with respect to Γ. Then it is invariant under translation by $N(\mathbb{Z})$, so may be expressed as a convergent series

$$F(z) = \sum_{n \geq 0} F_n e^{2\pi i n z} = \sum_{n \geq 0} F_n e^{-2\pi n y}\, e^{2\pi i n x}\,.$$

The difference between $F(z)$ and its constant term F_0 is exponentially decreasing as a function of y.

This essay will be concerned with analogous properties for other smooth functions on arithmetic quotients $\Gamma\backslash\mathcal{H}$, for an arbitrary proper subgroup Γ. The rough idea is that on a parabolic domain $F(z)$ is asymptotic to its constant term at the corresponding cusp. In analyzing this behaviour, one may as well assume that the cusp is ∞ with stabilizer $P = AN$, and that $\Gamma \cap N = N(\mathbb{Z})$. The function $F(z)$ may be expanded in a Fourier series

$$F(x + iy) = \sum\nolimits_{\mathbb{Z}} F_n(y) e^{2\pi i n x}$$

with Fourier coefficients

$$F_n(y) = \int_0^1 F(x+iy)e^{-2\pi inx}\,dx$$

that depend on y. There are several variations on the theme that $F(z) \sim F_0(y)$ as $y \to \infty$. One is that in which F is an eigenfunction of the Laplacian Δ of moderate growth as $y \to \infty$ (it is a **Maass form**), another in which F satisfies a certain somewhat technical condition of *uniform* moderate growth. Yet another concerns the Laplacian as an unbounded operator on $\mathrm{L}^2(\Gamma\backslash\mathcal{H})$.

The relationship between constant terms and asymptotic behaviour is fundamental in the theory of automorphic forms.

Contents

1. Maass forms

The Laplacian on $\mathcal{H}$ has the formula

$$\Delta = \Delta_{\mathcal{H}} = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

and is invariant under G. A **Maass form** on $\Gamma\backslash\mathcal{H}$ is a smooth function F on $\Gamma\backslash\mathcal{H}$ such that

- $\Delta F = \gamma F$ for some scalar γ;
- F is of moderate growth at every cusp q:

$$F(x_q + iy_q) = O(y_q^M)$$

  for some M, as $y \to \infty$.

The non-Euclidean Laplacian Δ is an elliptic differential operator. Solutions of an elliptic differential equation with locally smooth coefficients are smooth, so F is necessarily a smooth function of z. Solutions of an elliptic differential equation with analytic coefficients are real analytic, so that F is in fact real analytic. It is also true, if not immediately apparent, that an equivalent definition of a Maass form is as an eigendistribution of Δ that is tempered in some sense.

What is the asymptotic behaviour of F near the cusps of Γ? As I have already mentioned in the Introduction, conjugating Γ in $\mathrm{SL}_2(\mathbb{R})$ if necessary, I may assume the cusp at hand is ∞ and that $\Gamma \cap N_\infty = N(\mathbb{Z})$. In addition, that F is invariant under all of Γ will play no role in the discussion to come. Therefore:

> *From now on, I assume only that F is an eigenfunction of Δ on the quotient $(\Gamma \cap P)\backslash\mathcal{H}$ with $F(x+iy) = O(y^M)$ for some M, as $y \to \infty$.*

This hypothesis, for example, applies to holomorphic forms of even weight $k > 0$. What I am going to say is trivial in this case, but it can serve as a simple model.

Since F is smooth, it may be expanded in a Fourier series

$$F(x+iy) = \sum_{-\infty}^{\infty} F_n(y)e^{2\pi inx}$$

with smooth Fourier coefficients

$$F_n(y) = \int_0^1 F(x+iy)e^{-2\pi i n x}\,dx\,.$$

If F is holomorphic, the condition of moderate growth implies that the expansion has no terms of negative index and is therefore

$$F(z) = \sum_{n\ge 0} F_n e^{2\pi i n z}\,.$$

Since $e^{2\pi i n z} = e^{2\pi i n x}e^{-2\pi n y}$ we have

$$\big|F(z) - F_0\big| = O(e^{-2\pi y}) \quad (y\to\infty)\,.$$

In this section and the next I'll explain how this conclusion remains valid in general.

Since the Laplacian and N commute, the Fourier terms $F_n(y)$ are also eigenfunctions of Δ. This means that the coefficients $F_n(y)$ satisfy an ordinary differential equation. The two cases in which $n=0$ and $n\ne 0$ are very different.

• ***n* = 0**. For the constant term F_0 we get the differential equation

$$y^2F_0'' = \gamma F_0\,.$$

It is an Euler equation

$$D^2F_0 - DF_0 - \gamma F_0 = 0$$

in which D is the multiplicatively invariant derivative $y\,d/dy$. This differential equation has a **regular singularity** at ∞. The operator $y\,d/dy$ is invariant on the multiplicative group of real numbers, which is isomorphic to the additive group of real numbers via the exponential map $y=e^x$. If I set $\Phi(x) = F(e^x)$ then Φ now satisfies the equation

$$\Phi'' - \Phi' - \gamma\Phi = 0\,.$$

This equation has constant coefficients, and for all but one value of γ it will have as basis of solutions e^{s_1x} and e^{s_2x} where the s_i are solutions of the equation

$$s^2 - s - \gamma = 0, \qquad \text{hence } \; s = \frac{1\pm\sqrt{1+4\gamma}}{2}\,.$$

The exception is when $\gamma = -1/4$, when a basis of solutions is made up of $e^{x/2}$ and $xe^{x/2}$. Thus, the solutions of the original equations are the linear combinations of y^{s_1} and y^{s_2} as long as $\gamma\ne -1/4$. If $\gamma=-1/4$, on the other hand, the solutions are linear combinations of $y^{1/2}$ and $y^{1/2}\log y$.

• ***n* ≠ 0**. For the Fourier coefficient $F_n(y)$ we get the differential equation

$$y^2\big(F_n'' - 4\pi^2n^2F_n\big) = \gamma F_n, \quad F_n'' - \big(4\pi^2n^2 + \gamma/y^2\big)F_n = 0\,,$$

which has an **irregular singularity** at ∞. As $y\to\infty$ this differential equation has as limit the constant coefficient equation

$$F'' - 4\pi^2n^2F = 0\,.$$

with solutions $F(y) = e^{\pm 2\pi n y}$, so one might expect some similarity between the behaviour of F_n and of the functions $e^{\pm 2\pi n y}$. Since one of these grows exponentially and the other decreases, the following is plausible:

1.1. Proposition. *The space of solutions of the equation*

$$F'' - 4\pi^2n^2F = (\gamma/y^2)F$$

that are of moderate growth on $(1,\infty)$ *has dimension one, and it has as basis the unique solution which is asymptotic to* $e^{-2\pi ny}$ *as* $y\to\infty$.

This will take some work to explain. First we simplify by a scale change. Let $\lambda = 2\pi|n|$, and then set $F(y) = \mathcal{W}(\lambda y) = \mathcal{W}(2\pi|n|y)$. This gives us

$$F''(y) - 4\pi^2n^2F(y) = \lambda^2\mathcal{W}''(\lambda y) - \lambda^2\mathcal{W}(2\lambda y)\,,$$

so that if we set $x = \lambda y$ we see that $\mathcal{W}$ satisfies the differential equation

$$\mathcal{W}''(x) - \mathcal{W}(x) = (\gamma/x^2)\mathcal{W}\,,$$

which is now independent of λ. According to the standard formula found in Theorem 4.7 of [Brauer-Nohel:1967], there exist solutions of this equation with **asymptotic series expansions**

$$\mathcal{W}(x) = e^{\pm x}\left(1 + \frac{c_1}{x} + \frac{c_2}{x^2} + \cdots\right)\,,$$

which means that

$$\mathcal{W}(x)/e^{\pm x} - \left(1 + \frac{c_1}{x} + \frac{c_2}{x^2} + \cdots + \frac{c_n}{x^n}\right) = O(x^{-n-1})$$

for each n as $x\to\infty$.

Because of the growth condition on F as $y\to\infty$, only the solution with leading term e^{-x} is relevant here. The coefficients c_i of the formal series can be calculated by a recursion, but before doing that it is probably easiest to make a slight change, setting $\mathcal{W} = e^{-x}G$. Then

$$\begin{aligned}
\mathcal{W} &= e^{-x}G \\
\mathcal{W}' &= -e^{-x}G + e^{-x}G' \\
\mathcal{W}'' &= G - e^{-x}G' + e^{-x}G'' \\
\mathcal{W}'' - \mathcal{W} &= e^{-x}G'' - e^{-x}G'\,,
\end{aligned}$$

thus getting for G the differential equation

$$G'' - G' = \frac{\gamma}{x^2}G\,.$$

Setting formally

$$G = 1 + \frac{c_1}{x} + \frac{c_2}{x^2} + \frac{c_3}{x^3} + \cdots + \frac{c_n}{x^n} + \cdots$$

leads us to expansions

$$\begin{aligned}
\frac{\gamma G}{x^2} &= \quad \frac{\gamma}{x^2} + \frac{\gamma c_1}{x^3} + \frac{\gamma c_2}{x^4} + \cdots + \frac{\gamma c_{n-1}}{x^{n+1}} + \cdots \\
G' &= -\frac{c_1}{x^2} - \frac{2c_2}{x^3} - \frac{3c_3}{x^4} - \cdots - \frac{nc_n}{x^{n+1}} + \cdots \\
G'' &= \qquad\qquad \frac{2c_1}{x^3} + \frac{3\cdot 2c_2}{x^4} + \cdots + \frac{n(n-1)c_{n-1}}{x^{n+1}} + \cdots\,.
\end{aligned}$$

and recursion formulas

$$\begin{aligned}
c_1 &= \gamma \\
n(n-1)c_{n-1} + nc_n &= \gamma \\
c_n &= \frac{\gamma - n(n-1)}{n}\,c_{n-1} \quad (n\ge 2)\,.
\end{aligned}$$

The numerator here grows more rapidly than the denominator, so the series certainly does not converge. The functions $\mathcal{W}$ are a variant of the Bessel functions called **Whittaker functions**.

In the next section I'll follow Chapter 5 of [Coddington-Levinson:1955] in sketching the proof of the asymptotic expansion. In the rest of this one I'll just assume this to be so and prove:

1.2. Proposition. *Suppose F to be an eigenfunction of Δ on $(\Gamma \cap P)\backslash\mathcal{H}$ of moderate growth. Then as $y \to \infty$*

$$\big|F(x+iy) - F_0(y)\big| \ll_F e^{-2\pi y}\,.$$

Here I use Serge Lang's generalization of O-notation—$\ll_F X$ means $\le CX$ where C depends on F.

Proof. We start with

$$\begin{aligned} F(x+iy) - F_0(y) &= \sum_{n\ne 0} e^{2\pi nix} F_n(y) \\ &= \sum_{n\ne 0} c_n e^{2\pi inx}\mathcal{W}(2\pi|n|y) \\ \big|F(x+iy) - F_0(y)\big| &\le \sum_{n\ne 0} |c_n|\big|\mathcal{W}(2\pi|n|y)\big|\,. \end{aligned}$$

Since for $k > 0$

$$\frac{\partial^k F(x+iy)}{\partial x^k} = \sum_{n\ne 0}(2\pi in)^k c_n \mathcal{W}(2\pi|n|y)e^{2\pi inx}$$

we have

$$\begin{aligned} c_n(2\pi in)^k\mathcal{W}(2\pi|n|y) &= \int_0^1 \frac{\partial^k F(x+iy)}{\partial x^k}\, e^{-2\pi nix}\,dx \\ c_n &= \frac{1}{(2\pi in)^k}\frac{1}{\mathcal{W}(2\pi|n|y)}\int_0^1 \frac{\partial^k F(x+iy)}{\partial x^k}\, e^{-2\pi nix}\,dx \end{aligned}$$

for every k and y, as long as $\mathcal{W}(2\pi|n|y) \ne 0$. But we also know that $\mathcal{W}(t) \sim e^{-t}$. Choose t_0 large enough so $1/2 < \mathcal{W}(t)/e^{-t} < 2$ for $t > t_0$. Let $y_0 = t_0/2\pi$. Thus for $y \ge y_0$

$$\begin{aligned} c_n &= \frac{1}{(2\pi in)^k}\frac{1}{\mathcal{W}(2\pi|n|y_0)}\int_0^1 \frac{\partial^k F(x+iy_0)}{\partial x^k}\, e^{-2\pi nix}\,dx \\ |c_n| &\le \frac{2e^{2\pi|n|y_0}}{|2\pi n|^k}\int_0^1 \left|\frac{\partial^k F(x+iy_0)}{\partial x^k}\right| dx \\ \sum_{n\ne 0}|c_n|\big|\mathcal{W}(2\pi|n|y)\big| &\le C_k\sum_{n\ne 0}\frac{e^{-2\pi|n|(y-y_0)}}{|n|^k} \\ &\le C_k^*\, e^{-2\pi y}\Big(\sum_{n>0}\frac{1}{n^k}\Big)\,. \ \blacksquare \end{aligned}$$

2. Asymptotic expansions of Whittaker functions

If $\mathcal{W}'' - \mathcal{W} = (\gamma/x^2)\mathcal{W}$ and $W(x) = \mathcal{W}(x/2)$ then

$$W'' - \frac{W}{4} = \frac{\gamma}{x^2} W \,.$$

The differential equation

$$W'' + \left(-\frac{1}{4} + \frac{k}{x} + \frac{1/4 - m^2}{x^2} \right) W = 0 \,.$$

is called the **Whittaker equation** with parameters k, m. The equation for W in the previous section is a special case where $k = 0$. In general, there is exactly one solution with an asymptotic expansion

$$W \sim x^k e^{-x/2} \left(1 + \frac{c_1}{x} + \frac{c_2}{x^2} + \cdots \right) \,.$$

This is designated $W_{k,m}$, the **Whittaker function** with parameters k, m. As explained in §16.2 of [Whittaker-Watson:1952], Whittaker functions occur frequently—the error function, the incomplete Gamma function, the logarithmic integral, and Bessel functions all have simple expressions in terms of certain Whittaker functions. What is important for our purposes is that they also occur as Fourier coefficients of automorphic forms on arithmetic quotients and in the **Whittaker models** of representations of real groups of rank one. Because of this, they play an important role in the zeta functions of automorphic representations.

The Whittaker equation may be transformed into a system of first order equations by the usual trick of introducing a new dependent variable $V = W'$. The system we get is

$$\begin{bmatrix} W \\ V \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ 1/4 - k/x - (1/4 - m^2)/x^2 & 0 \end{bmatrix} \begin{bmatrix} W \\ V \end{bmatrix} \,.$$

it is therefore a special case of a system

$$y'' = A(z)y$$

in which $A(z)$ has a convergent expansion

$$A(z) = A_0 + \frac{A_1}{z} + \frac{A_2}{z^2} + \cdots$$

satisfying the condition that the eigenvalues of A_0 are distinct. We shall now look at this more general situation.

2.1. Proposition. *Suppose*

$$A(x) = A_0 + \frac{A_1}{x} + \frac{A_2}{x^2} + \cdots$$

is a convergent expansion near ∞, with the eigenvalues of A_0 distinct. Then there exists a matrix solution of

$$F' = A(x)F$$

which has an asymptotic expansion of the form

$$F(x) \sim \widehat{F}(x) = P(x) x^R e^{\Lambda x} \,,$$

where R and Λ are diagonal complex matrices, and P an asymptotic series in non-negative powers of $1/x$.

Proof. I'll follow Chapter 5 of [Coddington-Levinson:1955], in which a more general result about systems

$$F' = x^r A(x) \quad (r \in \mathbb{N})$$

is treated. The proof comes in two steps, the first explaining how to find the components $P(x)$, R, and Λ of the formal solution, and the second explaining how to relate the formal solution to an asymptotic expansion.

First of all, we may reduce to the case where A_0 is diagonal, replacing $F(x)$ by $EF(x)$ if EA_0E^{-1} is diagonal, since from $F' = AF$ we deduce $EF' = EAE^{-1}\,EF$. If $F(x) = P(x)x^R e^{\Lambda x}$ then

$$F'(x) = P'(x)x^R e^{\Lambda x} + (1/x)P(x)Rx^R e^{\Lambda x} + P(x)x^R \Lambda\, e^{\Lambda x}$$

so we must solve

$$P'(z)z^R e^{\Lambda x} + (1/x)P(x)Rz^R e^{\Lambda x} + P(x)z^R \Lambda e^{\Lambda x} = AP(x)z^R e^{\Lambda x}\,.$$

Since R, z^R, and Λ are all diagonal, they commute, so we may cancel $z^R e^{\Lambda x}$, leading to the equation

$$P'(x) + (1/x)P(x)R + P(x)\Lambda = A(x)P(x)\,.$$

We now equate coefficients of the powers of $1/x$. The constant term gives us

$$P_0\Lambda = A_0$$

and we can set $P_0 = I$, $\Lambda = A_0$. Equating coefficients of $1/x$ gives us

$$R + P_1A_0 - A_0P_1 = A_1\,.$$

Now if B is any matrix and the diagonal entries of A_0 are a_i, then $BA_0 - A_0B$ is a matrix with entries

$$B_{i,j}(a_j - a_i)\,.$$

In particular, its diagonal vanishes. I introduce notation—for any matrix M let $M = D(M) + M^*$, where $D(M)$ is the diagonal of M and M^* is off-diagonal. So the equation above requires that $R = D(A_1)$ and that

$$(P_1)_{i.j} = \frac{(A_1)_{i,j}}{a_i - a_j}$$

for $i \neq j$. It says nothing, however, about the diagonal $D_1 = D(P_1)$, which will be determined only in the next stage. Suppose that we are given inductively the off-diagonal P^*_{n-1} of P_{n-1}. Equating coefficients of $1/x^n$ we get

$$\begin{aligned}
-(n-1)P_{n-1} + P_{n-1}R + P_n\Lambda - \Lambda P_n &= A_1P_{n-1} + \cdots + A_n \\
&= (R + A_1^*)P_{n-1} + \cdots + A_n \\
&= RP_{n-1} + A_1^*P_{n-1} + \cdots + A_n\,.
\end{aligned}$$

This equation determines at once the diagonal of P_{n-1} and the off-diagonal of P_n. This concludes the construction of a formal solution.

I'll not include here the proof that $\widehat{F}(x)$ is an asymptotic approximation to a fundamental solution to the differential equation, except in the special case of Whittaker's equation with $k = 0$. Details are to be found in §5.4 of [Coddington-Levinson:1955]. The example I shall look at closely has a few of the features to be found in the general case.

2.2. Proposition. *The differential equation*

$$W'' - (1/4)W = \frac{\gamma}{x^2}W$$

has solutions with asymptotic expansions

$$W \sim e^{\pm x/2}\left(1 + \frac{c_1}{x} + \frac{c_2}{x^2} + \cdots\right)$$

The solution asymptotic to $e^{-x/2}$ is unique, but that asymptotic to $e^{x/2}$ is not, since the sum of it and any exponentially decreasing solution will have the same asymptotic behaviour.

Proof. The proof will make a couple of applications of the technique called 'variation of constants' in elementary courses on differential equations. Suppose φ to be a function which (a) is on each of the positive and negative axes a solution to the differential equation

$$W'' - (1/4)W = 0\,;$$

(b) is continuous at 0, but at 0 its first derivative jumps by 1. The first example we shall use is

$$\varphi(x) = \begin{cases} 0 & \text{if } x < 0; \\ 2\sinh(x/2) & \text{otherwise.} \end{cases}$$

Such a function is a solution of the distributional equation

$$\varphi'' - \varphi/4 = \delta_0\,.$$

Hence the function

$$F(x) = Ae^{-x/2} + Be^{x/2} + \int_a^b \varphi(x-s)G(s)\,ds = Ae^{-x/2} + Be^{x/2} + 2\int_a^x \sinh\left(\frac{x-s}{2}\right)G(s)\,ds$$

is a solution of $F'' - F/4 = G$ in the interval $[a,b]$. This can also be seen by applying the formula

$$H'(x) = h(x,x) + \int_a^x \frac{\partial h}{\partial x}(x,s)\,ds$$

if

$$H(x) = \int_a^x h(x,s)\,ds\,.$$

The first application of this idea will be to construct a solution asymptotic to $e^{-x/2}$. More explicitly, it will construct a solution of the integral equation

$$F(x) = e^{-x/2} - 2\int_x^\infty \sinh\left(\frac{x-s}{2}\right)\frac{F(s)}{s^2}\,ds$$

for $x > 0$ by a sequence of approximate solutions. We set $F_0(x) = 0$, and then in succession

$$F_{n+1}(x) = e^{-x/2} - 2\int_x^\infty \sinh\left(\frac{x-s}{2}\right)\frac{F_n(s)}{s^2}\,ds\,.$$

Thus $F_1(x) = e^{-x/2}$. I shall now show that the sequence $F_n(x)$ converges to a solution of the integral equation $F(x)\,e^{-x/2}$. By induction, I'll assume $F_n(x) \le e^{-x/2}$. This is an argument standard at the very beginning of the theory of differential equations. We have

$$F_{n+1}(x) - F_n(x) = -2\int_x^\infty \sinh\left(\frac{x-s}{2}\right)\frac{F_n(s) - F_{n-1}(s)}{s^2}\,ds$$

which implies that

$$\big|F_{n+1}(x) - F_n(x)\big| \le \int_x^\infty e^{\frac{s-x}{2}} \cdot \frac{|F_n(s) - F_{n-1}(s)|}{s^2}\, ds\,.$$

Thus

$$|F_2(x) - F_1(x)| \le \int_x^\infty e^{\frac{s-x}{2}} \cdot \frac{e^{-s/2}}{s^2}\, ds = e^{-x/2} \int_x^\infty \frac{1}{s^2}\, ds = \frac{e^{-x/2}}{x}$$

and by induction one can prove that

$$\big|F_{n+1} - F_n(x)\big| \le \frac{e^{-x/2}}{n!\, x^n}\,,$$

leading to

$$\big|F(x)\big| \le e^{-x/2} e^{1/x}, \quad F(x) \sim e^{-x/2}\,.$$

Integration by parts will lead to a proof that the entire asymptotic series is valid.

The second application of the idea will use

$$\varphi(x) = \begin{cases} -e^{x/2} & \text{if } x < 0; \\ -e^{-x/2} & \text{otherwise.} \end{cases}$$

Again we start off with $F_0(x) = 0$ and set

$$F_{n+1}(x) = e^{x/2} + \int_1^\infty \varphi(x-s) \frac{F_n(s)}{s^2}\, ds\,.$$

Thus $F_1(x) = e^{x/2}$, and $F_n(x) = O(e^{-x/2})$, as a similar argument will show. The sequence converges to a solution of the integral equation

$$F(x) = e^{x/2} + \int_1^\infty \varphi(x-s) \frac{F(s)}{s^2}\, ds$$

which is asymptotic to $e^{x/2}$.

We shall see later that all partial derivatives of F also decrease exponentially.

Remark. Nicolas Templier has explained to me that the graph of the Whittaker function is really remarkable. For reasons related to quantum mechanics, the exponential drop-off as $y \to \infty$ is extremely sudden. It is not easy to draw this graph, although I believe standard computer packages do it.

3. Uniform moderate growth

Continue to assume that $\Gamma \cap P = N(\mathbb{Z})$ is made up of integral horizontal translations $z \mapsto z + n$ or, equivalently, that we have chosen parabolic coordinates (x, y) in the neighbourhood of the cusp fixed by P. If F is a smooth function on $\mathcal{H}$ invariant under the subgroup $\Gamma \cap N$, it can be expressed in a Fourier series

$$F(x+iy) = \sum_{-\infty}^{\infty} F_n(y) e^{2\pi i n x} \,.$$

The main result of the previous sections is that if F is a Maass form then $F(z) - F_0(y)$ is exponentially decreasing as $y \to \infty$. This a special case of a more fundamental if also somewhat weaker result, which is one of the basic tools in analysis on arithmetic quotients.

On $G = \mathrm{SL}_2(\mathbb{R})$ define the norm

$$\|g\| = \sup_{\|v\|=1} \|g(v)\|$$

(with v in $\mathbb{R}^2$). Thus

$$\|g\| = \sup |x|, 1/|x| \ \text{ if } \ g = \begin{bmatrix} x & 0 \\ 0 & 1/x \end{bmatrix} .$$

Define

$$\|g\|_{\Gamma\backslash G} = \inf_\gamma \|\gamma g\|$$

on $\Gamma\backslash G$. It is easy to see (and is a well known result in reduction theory) that on any parabolic domain associated to the cusp q we have

$$\|g\| \asymp \|g\|_{\Gamma\backslash G} \asymp \sqrt{y_q} \,.$$

A smooth function F on $\Gamma\backslash G$ is said to be of **moderate growth** if there exists $N > 0$ such that $F(g) = O(\|g\|^N)$, and of **uniform moderate growth** if F is smooth and there exists $N > 0$ with $R_X F = O(\|g\|^N)$ for all X in $U(\mathfrak{g})$. (The uniformity is that N does not depend on X.) Let $A_{\mathrm{umg}}(\Gamma\backslash G)$ be the space of functions of uniform moderate growth on $\Gamma\backslash G$. It is stable under the right regular representation of G. Since functions on $\Gamma\backslash\mathcal{H}$ may be identified with functions on $\Gamma\backslash G$, we may also speak of $A_{\mathrm{umg}}(\Gamma\backslash\mathcal{H})$.

The constant term at the cusp q of a function F on $\Gamma\backslash G$ is the function

$$F_q(g) = \int_{\Gamma\cap N_q\backslash N_q} F(ng)\, dn \,.$$

It is a function on $(\Gamma \cap P_q)N_q\backslash G$.

3.1. Proposition. *Suppose F to lie in $A_{\mathrm{umg}}(\Gamma\backslash G)$, and assume that ∞ is a cusp of Γ. The difference $F(g) - F_0(g)$ is $O(y^{-M})$ for all M, as $y \to \infty$.*

Here the equation $g(i) = x + iy$ defines y.

Proof. As I have already mentioned, there is a technical difficulty. A function F on $\mathcal{H}$ may be identified with functions on G/K, but $R_X F$ may not be K-invariant. However, since $\mathcal{H}$ may also be identified with $P/(K\cap P)$ and $P \hookrightarrow G$, we may obtain functions on $\mathcal{H}$ by restriction to P of functions on G.

I combine this with a simple observation about the action of R_X in terms of operators L_X: *For F on G and X in $U(\mathfrak{g})$*

$$R_X F(g) = L_{\mathrm{Ad}(g)X} F(g) \,.$$

In particular, if

$$g = p = \begin{bmatrix} t & x \\ 0 & 1/t \end{bmatrix}$$

and

$$X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

then $R_X F(p) = L_{t^2 X}(p)$, which may be identified with $y\partial F/\partial x$ if F is a function on $\mathcal{H}$ and $p(i) = x + iy$. By Proposition 1.9, $\|g\|_{\Gamma\backslash G} \asymp y$ on the parabolic domain. So we suppose now that $R_{X^k} F(p) \le C_k \delta_P(p)^N$ for all k. From

$$F(z) - F_0(y) = \sum_{n \ne 0} F_n(y) e^{2\pi i n x}$$

we obtain for every $k > 0$

$$\begin{aligned} R_{X^k} F &= \sum_{n \ne 0} (2\pi i y n)^k F_n(y) e^{2\pi i n x} \\ F_n(y) &= \frac{1}{(2\pi i n y)^k} \int_0^1 R_{X^k} F(x + iy)\, dx \\ \big|F_n(y)\big| &\le \frac{1}{|2\pi n y|^k} \int_0^1 C_k |y|^N \, dx \\ \big|F(z) - F_0(y)\big| &\le \frac{C_k^*}{y^{k-N}} \Big(\sum_{n > 0} \frac{1}{n^k} \Big) . \end{aligned}$$

Because we can transform any cusp to ∞, this result implies:

3.2. Proposition. *If F lies in $A_{\mathrm{umg}}(\Gamma\backslash\mathcal{H})$, the difference between F and its constant term at any cusp is rapidly decreasing in the neighbourhood of that cusp.*

4. The Hecke algebra

The relationship between the result in the previous section and Proposition 1.2 may not be apparent, since it is not obvious that a Maass form lies in $A_{\mathrm{umg}}(\Gamma\backslash\mathcal{H})$. In this section, I'll sketch the proof that it does, but postpone details.

The most important point can be formulated roughly by saying that all Maass forms with the same eigenvalue are in some sense all incarnations of the same one.

Suppose for the moment that (π, V) is any continuous representation of G on a topological vector space V. I'll not spell out precisely what this means, but under a weak assumption on V (local convexity, quasi-completeness) one can then define for every f in $C_c^\infty(G)$ the operator

$$\pi(f) = \int_G f(g)\pi(g)\, dg$$

if one is given a Haar measure on G. It is characterized by the condition that if $\widehat{v}$ is a continuous linear function on V then

$$\langle \widehat{v}, \pi(f)v \rangle = \int_G f(g) \langle \widehat{v}, \pi(g)v \rangle \, dg$$

which makes sense because the integrand is a continuous function on G of compact support. In many cases, including the one we are about to see, the integral may be defined directly. The **Hecke algebra** $H(G/\!/K)$ of G with respect to K is that of all smooth compactly supported functions on G that are right- and left-invariant with respect to K. Operators $\pi(f)$ for f in the Hecke algebra take V^K into itself.

A function on $\mathcal{H}$ may be identified with one on G/K, that is to say a function on G fixed by K. Right convolution by functions in the Hecke algebra

$$R_f F(g) = \int_G f(x)F(gx)\,dx$$

may therefore be identified with operators on the space of functions on $\mathcal{H}$. The reason I am introducing these notions is:

4.1. Proposition. *If f is in $H(G//K)$ and F a continuous function of moderate growth on $\Gamma\backslash\mathcal{H}$ then $R_f F$ lies in $A_{\mathrm{umg}}(\Gamma\backslash\mathcal{H})$.*

This is complemented by:

4.2. Proposition. *Every Maass form on Γ can be expressed as $R_f F$ for some f in $H(G//K)$ and F a Maass form on $\Gamma\backslash\mathcal{H}$.*

As a consequence, we get a weak version of Proposition 1.2 in which exponential is replaced by polynomial decay.

I'l try to sketch why these are true, but in the opposite order.

Proof of Proposition 4.2. This is a basic result of the representation theory of G. I'll explain later on what is going on, but let me try to give a rough idea here.

For each s in $\mathbb{C}$ define the character χ_s of P by the formula

$$\chi_s\colon \begin{bmatrix} t & x \\ 0 & 1/t \end{bmatrix} \longmapsto |t|^s\,.$$

and

$$\mathrm{Ind}_s = \left\{ f \in C^\infty(G,\mathbb{C}) \,\middle|\, f(pg) = \chi_{s+1}(p)f(g) \text{ for all } p \in P, g \in G \right\}.$$

Because $G = PK$ and $P \cap P = \pm I$, its restriction to K is $C^\infty(K/\{\pm I\})$, and hence the subspace Ind_s^K has dimension 1. Also, the subspace of K-finite functions in I_s, on which the Lie algebra $\mathfrak{g}$ acts, is a direct sum of one-dimensional eigenspaces on which K acts by an evenpower of the character

$$\varepsilon\colon \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \longmapsto c + is\,.$$

The Hecke algebra $H(G//K)$ acts by scalars on this, giving rise to a homomorphism φ_s from $H(G//K)$ to $\mathbb{C}$. The Casimir element of $U(\mathfrak{g})$ acts on this by a scalar $\lambda_s = s^2 - 1$. The main theorem of the subject is that if (π, V) is any smooth representation of G with V^K finite-dimensional, and on which C acts by λ_s, then the Hecke algebra acts by φ_s. We can find functions in $H(G//K)$ approximating the Dirac δ at i, and therefore for any s we can find f in the Hecke algebra for which $\varphi_s(f) \neq 0$. Since the space of Maass forms for a given eigenvalue qualifies, we can find a function in the Hecke algebra that acts as the identity on it.

Proof of Proposition 4.1. This is more straightforward. For any smooth representation π of G

$$\pi(g)\pi(f)v = \pi(g)\int_G f(x)\pi(x)v\,dx = \pi(L_g f)v$$

so $\pi(X)\pi(f) = \pi(L_X f)$ and

$$R_X R_f F(g) = \int_G [L_X f](x)F(xg)\,dx$$

But If $F(x) = O(\|x\|_{\Gamma\backslash G}$ then

$$
\begin{aligned}
R_X R_f F(g) &= \int_G [L_X f](x) F(gx)\, dx \\
\big|R_X R_f F(g)\big| &\le \int_G \big|[L_X f](x)\big|\, \|g\|^M \|x\|^M\, dx \\
&= \|g\|^M \int_G \big|L_X f(x)\big|\, \|x\|^M\, dx\,.
\end{aligned}
$$

4.3. Corollary. *Maass forms lie in* $A_{\mathrm{umg}}(\Gamma\backslash G)$.

Remark. A second proof is possible. If $\Delta F = \lambda F$ and F is of moderate growth, then for some fixed M all $\Delta^n F$ are $O(y^M)$. The space of all functions that are $O(\|g\|^M_{\Gamma\backslash G})$ is a Banach space, so by a little known result of Langlands' Ph. D. thesis (explained relatively well in §4 of Chapter 1 of [Robinson:1991]), we also have all $R_X F$ in this space.

5. References

1. F. Brauer and J. Nohel, **Ordinary differential equations**, Benjamin, 1967.
2. E. A. Coddington and N. Levinson, **Theory of ordinary differential equations**, McGraw-Hill, 1955.
3. Derek Robinson, **Elliptic operators and Lie groups**, Clarendon Press, Oxford, 1991.
4. E. T. Whittaker and G. N. Watson, **Modern Analysis**, Cambridge, 1952.