

Essays on convexity

Remarks on Weyl's convexity paper

Bill Casselman
University of British Columbia
cass@math.ubc.ca

This is an elementary discussion of the geometry of convex polyhedra. I follow a classic paper of Hermann Weyl.

Contents

1. Convex cones
2. Consequences
3. References

1. Convex cones

Following Weyl, I begin the discussion with convex cones.

The **homogeneous convex hull** of $X \subset \mathbb{R}^n$ is the set of all points $\sum_X c_x x$ with all $c_x \geq 0$, the convex homogeneous cone spanned by X . The points of X span a linear subspace whose dimension I call the **essential dimension** of X . The set X will be called **linearly degenerate** if its essential dimension is less than n , otherwise linearly non-degenerate. If it is linearly non-degenerate, then it contains at least one subset of n linearly independent points. A closed linear half-space containing X will be called **extremal** (for X) if its boundary contains $n - 1$ linearly independent points of X . There are only a finite number of these subsets, hence at most a finite number of extremal linear half-spaces. For example, if X is a collection of n linearly independent points its homogeneous convex hull is called a **simplicial cone**. In this case, every subset of $n - 1$ points determines an extremal half-space.

In general, there are two ways to describe any convex body—as the convex hull of a set of points, or as the intersection of half-spaces. Being able to pass back and forth between the two is the fundamental procedure in the theory. For homogeneous cones, the main result takes this form:

1.1. Theorem. *If X is a non-degenerate finite set of points in \mathbb{R}^n , then its homogeneous convex hull is the intersection of all linear half-spaces extremal for X .*

It may very well happen that the homogeneous convex hull is all of \mathbb{R}^n , in which case the only linear function f with $f(X) \geq 0$ is the constant 0. In this case the set of extremal half-spaces is empty. In general, the only problem is to show that there are enough extremal half-spaces to determine X .

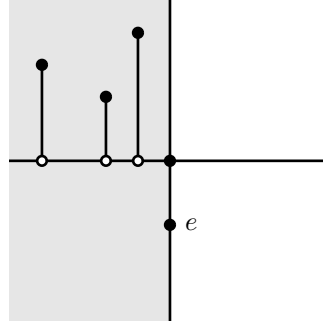
Proof. One half of the Theorem is trivial, since the homogeneous convex hull is certainly contained in the intersection of all extremal half-spaces. It remains to be shown that if a point lies in the intersection of extremal half-spaces then it lies in the homogeneous convex hull.

The proof proceeds by induction on n . If $n = 0$ there is nothing to be proven. If $n = 1$ then the points of X span a non-trivial segment. If this segment contains 0 in its interior, then the homogeneous convex hull is all of \mathbb{R}^1 , and there are no linear functions cx taking a non-zero value on all of X except

0 itself. Otherwise, the homogeneous hull is a half-line with 0 on its boundary, and again the result is straightforward.

Assume the Theorem to be true for dimension up to $n - 1$, and suppose X to have essential dimension n . There are two parts to the argument, which is constructive. The first either produces an extremal half-space or demonstrates that the homogeneous convex hull of X is all of \mathbb{R}^n . The second proves the theorem when it is known that extremal half-spaces exist.

Part 1. Suppose $\alpha = 0$ to be any hyperplane containing $n - 1$ linearly independent points of X . Either all points of X lie on one side of this hyperplane or they do not. If they do, then this hyperplane is extremal, and we can pass on to part 2. Suppose they do not. Let e be a point with $\langle \alpha, e \rangle < 0$. We may as well assume α to be the coordinate function x_n , and $e = (0, 0, \dots, -1)$. Let Y be the projection onto the plane $x_n = 0$ of the points X^+ of X with $x_n \geq 0$. The new system Y is non-degenerate of dimension $n - 1$, by assumption on α . Either it possesses an extremal half-space $\langle \beta, x \rangle \geq 0$ with $\langle \beta, x \rangle = b_1 x_1 + \dots + b_{n-1} x_{n-1}$ or it does not. If it does, the half-space $\langle \beta, x \rangle + x_n \geq 0$ is extremal and contains, in addition to the points in X^+ the point e .



So we can now loop in our argument. In the second case, when Y does not possess an extremal half-space, we can apply the induction assumption. Every point in the hyperplane $\langle \alpha, x \rangle = 0$ is in the convex hull of Y . Adding multiples of e , we see that every point where $x_n \leq 0$ is in the convex hull. Now choose a point f with $x_n > 0$; a similar argument implies that every point where $x_n \geq 0$ is also there.

Part 2. At this point we know there exist extremal half-spaces. Let Δ be a set of affine functions α such that the inequalities $\alpha > 0$ define them. We know that Δ is not empty.

Choose a point e in the interior of the homogeneous convex hull of X . Thus $\langle \alpha, e \rangle > 0$ for every α in Δ . Let p be an arbitrary point in the intersection of extremal half-spaces, so that also $\langle \alpha, p \rangle \geq 0$ for all α in Δ . Then for $\lambda > 0$ and α in Δ

$$\langle \alpha, p - \lambda e \rangle = \langle \alpha, p \rangle - \lambda \langle \alpha, e \rangle < \langle \alpha, p \rangle$$

Let λ be the least among the numbers $\langle \alpha, p \rangle / \langle \alpha, e \rangle$ for α in Δ . Then $q = p - \lambda e$ has these properties:

- (1) $p = q + \lambda e$ where $\lambda \geq 0$ and e is in the homogeneous convex hull of X ;
- (2) $\langle \alpha, q \rangle \geq 0$ for all α in Δ ;
- (3) $\langle \alpha, q \rangle = 0$ for some α in Δ .

In order to prove that p is in the homogeneous convex hull of X , it suffices to show that q is in it. (Note that the construction of q is reminiscent of a step in the simplex algorithm, which implements a practical version of the proof.)

We want to apply the induction hypothesis to q . Let Y be the set of all x in X on the hyperplane $\langle \alpha, x \rangle = 0$, which contains q . Because α is an extremal support, there exist at least $n - 1$ linearly independent points in Y , so that it, too, is non-degenerate in $n - 1$ dimensions. Choose coordinates so

that $\alpha = x_n$. It must be shown that any extremal function of the first $n - 1$ coordinates, with respect to the set Y , is non-negative on q . So suppose

$$\beta = \beta_1 x_1 + \cdots + \beta_{n-1} x_{n-1}$$

to be extremal for Y , and more particularly suppose $\langle \beta, y \rangle \geq 0$ for y in Y and that it vanishes on a subset $\Upsilon \subseteq Y$ of dimension $n - 2$. Consider the extension

$$b_1 x_1 + \cdots + b_{n-1} x_{n-1} - \mu x_n$$

I claim that $\mu \geq 0$ can be chosen so that it defines an extremal support for X . To arrange this, let μ be the minimum value of

$$\frac{b_1 x_1 + \cdots + b_{n-1} x_{n-1}}{x_n}$$

as x ranges over the points of $X - Y$. ◻

2. Consequences

Suppose X to be a finite set of points in a real affine space of dimension n . The **convex hull** of X is the set of all points of the form $\sum_X c_x x$ with all $c_x \geq 0$, $\sum c_x = 1$. It is closed and bounded. The **affine dimension** of X is the dimension of the smallest affine subspace containing it. If X has affine dimension n , a hyperplane is called **extremal** for X if it contains a subset of X of affine dimension $n - 1$ and all of X is contained either in that hyperplane or on one side of it. The interior of the intersection of the convex hull of X with an extremal hyperplane is called a **face** of the hull, and the hyperplane is called its support. The supports of extremal hyperplanes are often called **walls**.

2.1. Proposition. *If the affine support of X is all of \mathbb{R}^n then the convex hull of X is the intersection of all its extremal half-spaces.*

Proof. Embed \mathbb{R}^n into \mathbb{R}^{n+1} by tacking on an extra coordinate equal to 1. Apply to the new X . ◻

Now assume \mathfrak{h} to be a finite set of hyperplanes, and in fact that the hyperplanes in \mathfrak{h} are the supports of the walls of a chamber bounded convex set C . For each H in \mathfrak{h} let α_H be an affine function such that $\alpha_H(x) = 0$ defines H and $\alpha_H(C) > 0$. Let Δ be the set of all the α_H . Thus

$$C = \{x \mid \alpha(x) > 0 \text{ for all } \alpha \in \Delta\}.$$

2.2. Proposition. *An affine function f satisfies $f(C) > 0$ if and only if*

$$f = \sum_{\alpha \in \Delta} c_\alpha \alpha$$

with all $c_\alpha \geq 0$ and at least one $c_\alpha > 0$.

Proof. First of all, homogenize the problem by adding on an extra dimension. Now it may be assumed that all hyperplanes are linear. Then dualize—apply to the subset Δ in the dual V^* of V instead of to a point set in V . The set \overline{C} is that of all linear functions x on V^* such that $\langle \alpha, x \rangle \geq 0$. If f is any ‘point’ in V^* with $\langle f, x \rangle \geq 0$ for all x in \overline{C} then implies it is a non-negative linear combination of elements of Δ . ◻

3. References

1. H. Weyl, ‘The elementary theory of convex polyhedra’, pp. 3–18 in volume I of **Contributions to the theory of games**, *Annals of Mathematics Studies* **21**, Princeton Press, 1950. This is a translation by H. Kuhn of the original ‘Elementare Theorie der konvexen Polyeder’, *Commentarii Mathematici Helvetici* **7**, 1935, pp. 290–306.