Introduction to ordinary differential equations

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This essay will summarize what is needed about ordinary differential equations in representation theory, particularly that of $\mathrm{SL}_2(\mathbb{R})$, as well as a bit more involving fundamental solutions of ordinary differential equations. All differential equations will be linear.

Contents

Part I. Equations on R

1. Linear differential equations

This section will be concerned with solving a system of differential equations with initial condition:

$$y'(x) = A(x)y(x), \quad y(0) = y_0$$

where $A(x)$ is a smooth matrix-valued function defined on a possibly infinite interval $(-C, C)$ and $y = (y_i(x))$ is a smooth vector-valued function.

What's special about linear equatiosn is that solutions of this equation form a vector space.

If $A = (a_{i,j})$ is an $r \times c$ complex matrix define its norm:

$$|A| = c \sup |a_{i,j}|$$

where $c = \mathrm{col} A$ is the number of columns in A.

1.1. Lemma. *For any matrices A and B*

$$\begin{aligned} \sup |a_{i,j}| &\le |A| \\ |A+B| &\le |A| + |B| \\ |AB| &\le |A|\,|B|\,. \end{aligned}$$

Proof. The first and second are immediate. As for the last:

$$\begin{aligned}
|AB| &= \mathrm{col}AB \sup_{i,k} \Big| \sum_j a_{i,j} b_{j,k} \Big| \\
&= \mathrm{col}B \sup_{i,k} \Big| \sum_j a_{i,j} b_{j,k} \Big| \\
&\le \mathrm{col}B \sup_{i,k} \sum_j |a_{i,j}|\,|b_{j,k}| \\
&\le \mathrm{col}A \mathrm{col}B \sup_{i,j} |a_{i,j}| \sup_{j,k} |b_{j,k}| \\
&= |A|\,|B| \,.
\end{aligned}$$

1.2. Lemma. *The function y satisfies the differential equation $y' = A(x)y$ if and only if it satisfies the integral equation*

$$y(x) = y_0 + \int_0^x A(s)y(s)\,ds \,.$$

This is the simplest example we'll see of an intimate relationship between differential and integral equations. This relationship is essentially just an elaboration of the fundamental theorem of calculus.

Proof of the Lemma. The fundamental theorem asserts that if y satisfies this integral equation then $y' = Ay$, and clearly $y(0) = y_0$ also. On the other hand, if $y' = Ay$ then

$$y(x) - y(0) = \int_0^x y'(s)\,ds = \int_0^x A(s)y(s)\,ds \,.$$

1.3. Lemma. *A solution of the system $y' = Ay$ is uniquely determined by its initial value.*

Apply the following Lemma to the difference of two solutions.

1.4. Lemma. *Suppose I to be a finite closed interval in $(-C, C)$ containing 0 and $M = \sup_I |A(x)|$. If y is a solution of $y' = Ay$ then for x in I*

$$|y(x)| \le |y(0)| \left(1 + \frac{e^{M|x|} - 1}{M}\right) \,.$$

Proof. If y is a solution then for $x > 0$ in I

$$\begin{aligned}
y(x) &= y(0) + \int_0^x A(s)y(s)\,ds \\
|y(x)| &\le |y(0)| + \int_0^x |A(s)|\,|y(s)|\,ds \\
&\le |y(0)| + M \int_0^x |y(s)|\,ds \,.
\end{aligned}$$

Set

$$\delta = |y(0)|, \quad Y(x) = \int_0^x |y(s)|\,ds \,.$$

Then $Y'(x) = |y(x)|$, $Y(0) = 0$, $Y(x) \geq 0$, and hence

$$\begin{aligned}
Y'(x) - MY(x) &\leq \delta \\
e^{-Mx}\big(Y'(x) - MY(x)\big) &\leq \delta e^{-Mx} \\
\frac{d\,Y(x)e^{-Mx}}{dx} &\leq \delta e^{-Mx} \\
e^{-Mx}Y(x) &\leq \delta \int_0^x e^{-Ms}\,ds \\
&= \delta\left(\frac{1-e^{-Mx}}{M}\right) \\
Y(x) &\leq \delta\left(\frac{e^{Mx}-1}{M}\right) \\
|y(x)| &\leq \delta\left(1 + \frac{e^{Mx}-1}{M}\right).
\end{aligned}$$

The argument for $x < 0$ is almost the same.

1.5. Theorem. *Suppose $A(x)$ to be a smooth $M_n(\mathbb{C})$-valued function on the interval $(-C, C)$ (with C possibly ∞). For every y_0 in $\mathbb{C}^n$ the linear system of differential equations*

$$y' = A(x)y$$

has a unique smooth solution on $(-C, C)$ satisfying $y(0) = y_0$.

In other words, the vector space of solutions of the differential equation has dimension n, and the map taking y to $y(0)$ is an isomorphism with $\mathbb{C}^n$.

Proof. Uniqueness follows from the previous Lemma. For existence, we define a sequence of functions

$$\begin{aligned}
y_0(x) &= 0 \\
y_{k+1}(x) &= y_0 + \int_0^x A(s)y_k(s)\,ds\,.
\end{aligned}$$

and a sequence of differences

$$\Delta_k(x) = y_{k+1}(x) - y_k(x)\,.$$

For the Δ_k we have

$$\begin{aligned}
\Delta_{k+1}(x) &= \int_0^x A(s)\,\Delta_k(s)\,ds \\
\big|\Delta_{k+1}(x)\big| &\leq \int_0^x \big|A(s)\big|\big|\Delta_k(s)\big|\,ds \\
&\leq M_x \int_0^x \big|\Delta_k(s)\big|\,ds \qquad \left(M_x = \sup_{0\leq s\leq x} |A(s)|\right)
\end{aligned}$$

and therefore by induction

$$\big|\Delta_k(x)\big| \leq M_x^k\,|y(0)|\,\frac{|x|^k}{k!}\,,$$

which means that the sequence

$$y_k = y_0 + \sum_1^k \Delta_i$$

converges uniformly on any bounded subinterval of $(-C, C)$ to a solution of the integral equation.

CONSTANT COEFFICIENTS. There are very few cases where a simple formula for solutions can be found. One of them is that where A is constant. For X a square matrix define

$$e^X = I + X + \frac{X^2}{2} + \frac{X^3}{3!} + \cdots$$

The following is immediate:

1.6. Proposition. *If A is a constant matrix then the solution of*

$$y' = Ay, \quad y(0) = y_0$$

is

$$y = e^{Ax} y_0 \, .$$

SCALAR FUNCTIONS. Another is that where y is a scalar-valued function.

1.7. Proposition. *If y and A are scalar-valued functions, the solution of*

$$y' = A(s)y, \quad y(0) = y_0$$

is

$$y = e^{\alpha(x)} y_0, \quad \alpha(x) = \int_0^x A(s)\, ds \, .$$

WRONSKIANS. Suppose now that $\Phi(x)$ is a square matrix whose columns are solutions. The determinant of $\Phi(x)$ is called a **Wronskian** of the system.

1.8. Proposition. *Any Wronskian $W(x)$ of the system $y' = A(x)y$ is of the form $e^{\tau(x)} W(0)$ where*

$$\tau(x) = \int_0^x \operatorname{trace}\big(A(s)\big)\, ds \, .$$

Proof. We start with:

1.9. Lemma. *If Φ^* is the transposed adjoint of Φ, then the derivative of* $\det$ *at Φ is the linear map*

$$X \longmapsto \operatorname{trace}(X\Phi^*) \, .$$

Proof. What this means is that

$$\left[\frac{d \det(\Phi + tX)}{dt}\right]_{t=0} = \operatorname{trace}(X\Phi^*) \, .$$

Now $\det(\Phi + tX)$ is a polynomial in the coefficients of Φ and X, as well as t. Its constant term is $\det(\Phi)$, and what is to be shown is that the coefficient of t in this polynomial is $\operatorname{trace}(X\Phi^*)$.

The basic property of the transposed adjoint is that $\Phi\Phi^* = \det(\Phi) I$. It is easy to see that

$$\det(I + ty) = 1 + t \operatorname{trace}(y) + \cdots$$

and if we multiply by $\det(\Phi)$ we get

$$\begin{aligned}
\det(\Phi + t\Phi y) &= \det(\Phi) + t \det(\Phi) \operatorname{trace}(y) + \ldots \\
\det(\Phi + t\Phi y) &= \det(\Phi) + t \operatorname{trace}\big(\det(\Phi) y\big) + \ldots \\
&= \det(\Phi) + t \operatorname{trace}(\Phi^* \, \Phi y) + \ldots
\end{aligned}$$

so that $\det(\Phi + tX) = \det(\Phi) + t\,\mathrm{trace}(X\Phi^*) + \cdots$ if $X = \Phi y$. If Φ is non-singular, any X can be expressed as Φy, but since the invertible Φ are an open set in the space of all square matrices, the equation remains true for all X.

Now to prove the Proposition. The equation $\Phi\Phi^* = \det(\Phi)I$ and the chain rule imply that the derivative of $W = \det\big(\Phi(x)\big)$ is

$$\begin{aligned}\mathrm{trace}\big(\Phi'\Phi^*\big) &= \mathrm{trace}\big(A\cdot\Phi\Phi^*\big)\\ &= \mathrm{trace}\big(A\cdot\det(\Phi)I\big)\\ &= \det(\Phi)\mathrm{trace}\big(A(x)\big)\end{aligned}$$

so that W satisfies the differential equation $W'(x) = \mathrm{trace}\,\big(A(x)\big)\,W(x)$.

If $\Phi(x)$ is a matrix whose columns are solutions of the system, it is called a matrix solution.

1.10. Corollary. *If $\Phi(x)$ is a square matrix solution such that the columns of $\Phi(0)$ are linearly independent, then the columns of $\Phi(x)$ are linearly independent for all x.*

A matrix whose columns are linearly independent functions forming a basis of solutions of the equation $y' = A(x)y$ is called a **fundamental matrix** for the system. If $\Phi(x)$ is a fundamental matrix then $\Phi(x)$ is invertible for all x. If A is constant, then e^{Ax} is a fundamental matrix.

N-TH ORDER EQUATIONS. An n-th order differential equations is that specifies relation between its first n-derivatives:

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_0(x)y = 0\,.$$

It is equivalent to a system of equations for y, where Y is a vector function whose entries are $y(x)$, $y'(x)$, $y''(x)$, ...

2. Change of independent variable

Very often we are given a function $y(x)$ of one variable x satisfying a differential equation

$$y''(x) + a(x)y'(x) + b(x)y(x) = 0$$

and a change of independent variable from x to t, and wish to discover the differential equation satisfied by the new function $Y(t) = y(x) = y(x(t))$.

The coefficients $a(x)$, $b(x)$, $c(x)$ are replaced by $A(t) = a(x(t))$, $B(t) = b(x(t))$, $C(t) = c(x(t))$, and $y(x)$ is replaced by $Y(t)$. As for the derivatives, the basic identity is one of operators:

$$\frac{d}{dx} = \frac{\dfrac{d}{dt}}{\dfrac{dx}{dt}}$$

which is a variant of the chain rule. Applying this rule just once we see that

$$\frac{dy}{dx} = \frac{dY/dt}{dx/dt}$$

which allows us to subsitute

$$y'(x) = dy/dx = \frac{dY/dt}{dx/dt} = \frac{Y'(t)}{x'(t)}$$

in the original equation, which is what we want to do because the right hand side will be an expression in t. Applying the rule to dy/dx instead of $y(t)$ we see that

$$\begin{aligned} y''(x) &= \frac{d}{dx}\frac{dy}{dx} \\ &= \frac{d(dy/dx)/dt}{dx/dt} \\ &= \frac{(d/dt)(Y'(t)/x'(t))}{x'(t)} \\ &= \frac{Y''(t)x'(t) - Y'(t)x''(t)}{x'(t)^3} \end{aligned}$$

Example. As a first example, suppose we start with the constant coefficient equation

$$y''(x) + ay'(x) + by(x) = 0, \quad x = \log t.$$

Then $dx/dt = 1/t$ so we derive the translations

$$\begin{aligned} y'(x) &= tY'(t) \\ y''(x) &= \frac{d}{dx}\frac{dy}{dx} \\ &= \frac{\dfrac{d}{dt}(dy/dx)}{\dfrac{dx}{dt}} \\ &= \frac{\dfrac{d}{dt}(tY'(t))}{1/t} \\ &= t^2Y''(t) + tY'(t) \end{aligned}$$

When we substitute these into the original differential equation we get the **Euler's equation**

$$t^2Y''(t) + tY'(t) + atY'(t) + bY(t) = t^2Y''(t) + (a+1)tY'(t) + bY(t) = 0\ .$$

This tells us is a general way how to solve Euler's equations. The constant coefficient equation has a solution $y(x) = e^{\lambda x}$ where λ is a root of the characteristic equation. But this means that the Euler's equation has a solution of the form $Y(t) = t^\lambda$. If the characteristic equation has a double root, then the original differential equation has a solution $y = xe^{\lambda x}$, which translates to $t^\lambda \log t$.

Example. As a second example start with the equation

$$f''(x) + a(x)f'(x) + b(x)f(x) = 0$$

and make a change of variable $x = 1/\xi$, setting $F(\xi) = f(x) = f(1/\xi)$. Here $dx/d\xi = -1/\xi^2$, so

$$\begin{aligned} f'(x) &= \frac{F'(\xi)}{(-1/\xi^2)} \\ &= -\xi^2F'(\xi) \\ f''(x) &= -\xi^2\frac{d}{d\xi}(-\xi^2F'(\xi)) \\ &= \quad \xi^2\frac{d}{d\xi}(\quad \xi^2F'(\xi)) \\ &= \xi^4F''(\xi) + 2\xi^3F'(\xi). \end{aligned}$$

and we get the new equation

$$\xi^4 F''(\xi) + (2\xi^3 - \xi^2 A(\xi))F'(\xi) + B(\xi)F(\xi) = 0.$$

where

$$A(\xi) = a(1/\xi), \quad B(\xi) = b(1/\xi)\ .$$

Example. Now let $x = t^n$. Then

$$\begin{aligned} x'(t) &= nt^{n-1} \\ x''(t) &= n(n-1)t^{n-2} \end{aligned}$$

so

$$\begin{aligned} y'(x) &= \frac{Y'(t)}{nt^{n-1}} \\ y''(x) &= \frac{nY''(t)t^{n-1} - n(n-1)Y'(t)t^{n-2}}{n^3 t^{3(n-1)}} \\ &= \frac{Y''(t) - (n-1)tY'(t)}{n^2 t^{2(n-1)}}\ . \end{aligned}$$

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4. Inhomogeneous equations

An inhomogeneous differential equation is one of the form $Ly(x) = \varphi(x)$, where Ly is a linear expression in y. A simple example is $y'(x) = e^x$. The solutions of this equation are of the form $y(x) + a(x)$ where y is an arbitrary solution of $Ly = 0$, and a is some particular solution of $La = \varphi$. There are several ways to express solutions to inhomogeneous differential equations, but the basic fact is that solving an inhomogeneous equation reduces to an integration if you know enough about the homogeneous equation.

4.1. Theorem. *If $\Phi(x)$ is a fundamental matrix for the differential equation $y' = A(x)y$ then*

$$y(x) = y_0 + \Phi(x) \int_{x_0}^{x} \Phi^{-1}(s)\varphi(s)\, ds$$

is the solution of $y' = A(x)y + \varphi(x)$ with $y(x_0) = y_0$.

Proof. A straightforward exercise in the fundamental theorem of calculus.

The real point is that the function $\Phi_s = \Phi(x)\Phi^{-1}(s)$ is the **fundamental solution** of the system satisfying the initial condition $\Phi_s(s) = I$. This will motivate what we are about to do, which is to show how to solve similarly a higher order differential equation

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_0(x)y = f(x)\,.$$

The corresponding system is of the form

$$Y' = A(x)Y + F(x)$$

where

$$A(x) = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & & \dots & & \\ -a_0(x) & -a_1(x) & \dots & & -a_{n-1}(x) \end{bmatrix}, \quad F = \begin{bmatrix} 0 \\ 0 \\ \dots \\ f(x) \end{bmatrix}.$$

Let's look first at a familiar case. The formula derived by the technique called 'variation of parameters' solves the differential equation and initial conditions

$$y'' + a_1(x)y' + a_0(x)y = f(x) \quad \begin{cases} y(x_0) = 0 \\ y'(x_0) = 0 \end{cases}$$

by the formula

$$\begin{aligned} y &= -\varphi_1(x) \int_0^x \frac{f(s)\varphi_2(s)}{W(\varphi_1,\varphi_2)(s)}\, ds + \varphi_2(x) \int_0^x \frac{f(s)\varphi_1(s)}{W(\varphi_1,\varphi_2)(s)}\, ds \\ &= \int_{x_0}^x f(s) \cdot \frac{\varphi_2(x)\varphi_1(s) - \varphi_1(x)\varphi_2(s)}{W(\varphi_1,\varphi_2)(s)} ds\,, \end{aligned}$$

where W is the Wronskian determinant

$$W(\varphi_1,\varphi_2) = \begin{vmatrix} \varphi_1 & \varphi_2 \\ \varphi_1' & \varphi_2' \end{vmatrix}.$$

With the usual derivation, this formula seems to come from nowhere. It is in fact a special case of a formula valid for any higher order equation, one which fits into the general theory of partial as well as ordinary differential equations. Let

$$Ly = y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_0(x)y\,,$$

and for each s let φ_s be the solution of

$$L\varphi_s = 0, \quad \varphi_s^{(i)}(s) = \begin{cases} 0 & \text{if } i < n-1 \\ 1 & \text{if } i = n-1 \,. \end{cases}$$

4.2. Proposition. *The solution of*

$$Ly = y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_0(x)y = f(x), \quad y^{(i)}(x_0) = 0 \text{ for } 0 \le i < n$$

is

$$F(x) = \int_{x_0}^{x} \varphi_s(x) f(s)\, ds \,.$$

Proof. This is a special case of the formula in Theorem 4.1, but I'll derive it directly. I recall first a Lemma from calculus:

4.3. Lemma. *If*

$$F(x) = \int_0^x f(x,s)\, ds$$

then

$$F'(x) = f(x,x) + \int_0^x \frac{\partial f(x,s)}{\partial x}\, ds \,.$$

Proof. More gneerally, set

$$F(x) = \int_a^b f(x,s)\, ds = \Phi(a,b,x)\,,$$

and then let a and b be functions of x. By the chain rule

$$F'(x) = \frac{\partial \Phi}{x} + \frac{\partial \Phi}{\partial a}\frac{\partial a}{\partial x} + + \frac{\partial \Phi}{\partial b}\frac{\partial b}{\partial x} \,.$$

Remark. One can also prove this in a way that suggests generalizations. The basic result in calculus is that if $f(x,s) = f(s)$ (no dependence on s) then $F'(x) = f(x)$. This can be interpreted as saying that if $\chi_{[0,x]}$ is the characteristic function of $[0,x]$ then if

$$F(x) = \int_{-\infty}^{\infty} \chi_{[0,x]} f(s)\, ds$$

then

$$F'(x) = \int_{-\infty}^{\infty} \frac{d}{dx}\chi_{[0,x]} f(s)\, ds = f(x) = \int_{\infty}^{\infty} \delta_x(s) f(s)\, ds\,,$$

which can in turn be interpreted as saying that

$$\frac{d\chi_{[0,x]}}{dx} = \delta_x \,.$$

But then if

$$F(x) = \int_0^x f(x,s)\, ds = \int_{-\infty}^{\infty} \chi_{[0,x]}(s) f(x,s)\, ds$$

we have

$$\begin{aligned} F'(x) &= \int_{-\infty}^{\infty} \left[\frac{d\chi_{[0,x]}(s)}{dx} \cdot f(x,s) + \chi_{[0,x]}(s) \cdot \frac{\partial f(x,s)}{\partial x} \right] ds \\ &= \int_{-\infty}^{\infty} \left[\delta_x(s) \cdot f(x,s) + \chi_{[0,x]}(s) \cdot \frac{\partial f(x,s)}{\partial x} \right] ds \\ &= f(x,x) + \int_0^x \frac{\partial f(x,s)}{\partial x}\, ds \,. \end{aligned}$$

Given this Lemma, take up again the proof of Proposition 4.2. We have

$$
\begin{aligned}
F(x) &= \int_{x_0}^{x} \varphi_s(x) f(s)\, ds \\
F'(x) &= f(x)\varphi_x(x) + \int_{x_0}^{x} \frac{d\varphi_s(x)}{dx} f(s)\, ds \\
&= \int_{x_0}^{x} \frac{d\varphi_s(x)}{dx} f(s)\, ds \\
&\ldots \\
F^{(n)}(x) &= f(x)\varphi^{(n-1)}(x) + \int_{x_0}^{x} \frac{d^n\varphi_s(x)}{dx^n} f(s)\, ds \\
&= f(x) + \int_{x_0}^{x} \frac{d^{n-1}\varphi_s(x)}{dx^n} f(s)\, ds \\
LF &= f(x) + \int_{x_0}^{x} L\varphi_s(x) f(s)\, ds \\
&= f(x)\,.
\end{aligned}
$$

There are useful variants of the same idea. Here is one that occurs in representation theory.

4.4. Proposition. *Suppose $f(x)$ to be a smooth function on $(0,\infty)$, such that for some $M > 0$*

$$f(x) = O(x^M)$$

as $x \to \infty$. Suppose $a > 0$. Then the function y satisfies the equation

$$y'' - a^2 y = f(x)$$

on $(0,\infty)$ if and only if

$$y = c_+ e^{ax} + c_- e^{-ax} + \frac{1}{2a}\int_1^\infty e^{-a|x-s|} f(s)\, ds$$

for suitable $c_\pm$.

Proof. The integrand $\varphi_s = e^{-a|x-s|}$ satisfies

$$\varphi_s(s) = 0, \quad \varphi_s'(s) = 1$$

so the calculation is the same as that of Proposition 4.2.

There remain a few fine points that I take up in detail elsewhere. We know that there exist smooth solutions and that they are uniquely determined by initial conditions. But what about solutions that are not smooth? Say, with only continuous first derivatives? This question can be handled directly, but it turns out that all solutions of the differential equation $y' = A(x)y$ are smooth everywhere:

4.5. Theorem. *If $f(x)$ is a smooth function on an interval and Φ is any distributions such that $L\Phi = f$ in the interval, then Φ is a smooth function.*

This is a special case of a general theorem about **elliptic** differential operators, and I do not know any simple proof.

Another point concerns a more systematic approach to Proposition 4.2. What's really going on is that this is an example of Laurent Schwartz's theory of a fundamental solution to a differential equation. Continue to let

$$Ly = y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_0(x)y$$

Recall that $\varphi_s(x)$ is the solution of $L\varphi_s = 0$ such that

$$\varphi_s^{(i)}(s) = \begin{cases} 0 & \text{if } i < n-1 \\ 1 & \text{if } i = n-1\,. \end{cases}$$

Now define a variant of this. Suppose $\varphi(x)$ to be any solution of $L\varphi = 0$ and define Φ_s by the formula

$$\Phi_s(x) = \begin{cases} \varphi(x) & \text{if } x < s \\ \varphi(x) + \varphi_s(x) & \text{if } x \geq s. \end{cases}$$

If $n = 1$ this will have a jump discontinuity from 0 to 1 at $x = s$, but if $n > 1$ it will be continuous at $x = s$, with value $\varphi(s)$.

4.6. Proposition. *The function $\varphi_s(x)$ is a solution of the distribution equation*

$$L\Phi_s = \delta_s\,.$$

Proof. I recall that a distribution is defined to be a continuous linear functional on the space of smooth functions with compact support. The derivative of a distribution Φ is defined by the characterization

$$\langle \Phi', f\rangle = -\langle \Phi, f'\rangle\,,$$

which by the formula for integration by parts is an extension of the definition of the derivative of functions. What we want to show is that $\langle L\Phi_s(x), f(x)\rangle = f(s)$, which we at least now know how to interpret.

4.7. Lemma. *If the function F is smooth on each interval $(-\infty, s]$ and $[s, \infty)$ then considering F as a distribution*

$$\langle F', f\rangle = f(s)\big(F_+(s) - F_-(s)\big) + \int_{-\infty}^{\infty} F'(x)f(x)\,dx\,.$$

Proof. Applying integration by parts we have

$$\begin{aligned}
\langle F', f\rangle &= -\langle F, f'\rangle \\
&= -\int_{-\infty}^{\infty} F(x)f'(x)\,dx \\
&= -\int_{-\infty}^{s} F(x)f'(x)\,dx + -\int_{s}^{\infty} F(x)f'(x)\,dx \\
&= -[F(x)f(x)]_s^{\infty} + \int_{-\infty}^{x} F'(x)f(x)\,dx - [F(x)f(x)]_{-\infty}^{s} + \int_{x}^{\infty} F'(x)f(x)\,dx \\
&= f(s)\big(F_+(s) - F_-(s)\big) + \int_{-\infty}^{\infty} F'(x)f(x)\,dx\,.
\end{aligned}$$

The Proposition now follows if we apply this to $F = \Phi_s, \Phi_s', \ldots, \Phi^{(n)}$.

We now have two formulas involving $\varphi_s(x)$, one involving an integral against ds, the other against dx. The first formula implies that (if we choose x_0 suitably) that if f has compact support in $\mathbb{R}$ then

$$LF = f(x), \quad \text{where } F(x) = \int_{-\infty}^{\infty} \Phi_s(x)f(x)$$

is a solution of $LF = f(x)$, and the second asserts that

$$L\Phi_s = \delta(s)\,.$$

which means that

$$\int_{-\infty}^{\infty} \Phi_s(x) L^* f(x)\, dx = f(s)$$

where L^* is the differential operator adjoint to L. The proofs of these look quite different. But there is in fact a formulation essentially equivalent to both. Define on $\mathbb{R}^2$ the function

$$\Phi(x, y) = \Phi_y(x)\,.$$

4.8. Proposition. *We have an equation of distributions in two variables:*

$$L_x \Phi(x, y) = \delta(x - y)\,.$$

I'll not say more about this here, except to remark that both of the earlier formulas are consequences.

Part II. Equations on C

5. Differential equations with analytic coefficients

We now look at differential equations in which the independent variable is complex. Assume the functions making up the entries of the matrix $A(z)$ to be analytic in a neighbourhood of 0.

5.1. Theorem. *Given a linear system of differential equations*

$$y' = A(z)y$$

with coefficients analytic for $|z| < r$ there exists a unique analytic function $y(z)$ in the complex disk $|z| < r$ with $y(0) = y_0$.

The connection between this result and previous ones on differential equations on subsets of $\mathbb{R}$ is that if $z(t)$ is a path in the disk with $z(0) = 0$ then the composite $y\big(z(t)\big)$ to $\mathbb{R}$ solves the lifted equation $y'(t) = A\big(z(t)\big)y(t)$.

Proof. I'll do here just the case where $A(z) = a(z)$ is a scalar-valued function. We may as well assume $y(0) = 1$. We express

$$\begin{aligned} a &= a_0 + a_1 z + a_2 z^2 + \cdots \\ y &= 1 + c_1 z + c_2 z^2 + \cdots \\ ay &= a_0 + (a_0 c_1 + a_1)z + (a_0 c_2 + a_1 c_1 + a_2)z^2 + \cdots \\ y' &= c_1 + 2c_2 z + 3c_3 z^2 + \cdots \end{aligned}$$

leading to

$$\begin{aligned} c_1 &= a_0 \\ c_2 &= \frac{1}{2}(a_0 c_1 + a_1) \\ c_3 &= \frac{1}{3}(a_0 c_2 + a_1 c_1 + a_2) \\ &\cdots \\ c_k &= \frac{1}{k}(a_0 c_{k-1} + \cdots + a_{k-1}) \end{aligned}$$

I now follow [Brauer-Nohel:1967]. The method is a variant of that of **majorants** invented by Cauchy, which compares the solution of the given differential equation to one that can be solved explicitly.

We want to show that the series $\sum c_i z^i$ converges for $|z| < r$. It suffices to show that for every $R < r$ we can find a sequence C_k with (a) $|c_k| \le C_k$ and (b) $\lim_{k\to\infty} C_{k+1}/C_k = 1/R$. We know that since $\sum a_k z^k$ converges for $|z| < r$ that for some $M > 0$ we have $|a_k| \le MR^{-k}$ for all k.

Now consider the differential equation

$$y'(z) = \frac{My(z)}{1-(z/R)}, \quad y(0) = M\,.$$

Its solution is $\sum C_k (z/R)^k$ with

$$\begin{aligned} C_0 &= M \\ C_{k+1} &= \frac{1}{k+1}\cdot(MC_k + MR^{-1}C_{k-1} + \cdots + MR^{-k})\,. \end{aligned}$$

The previous inequalities allow us to see that $|c_k| \le C_k$ for all k. But

$$\begin{aligned} (k+1)C_{k+1} &= MC_k + R^{-1}kC_k \\ &= C_k(M + kR^{-1}) \\ \frac{C_{k+1}}{C_k} &= \frac{M+k/R}{k+1} \\ &\longrightarrow 1/R \quad \text{as } k \longrightarrow \infty\,. \end{aligned}$$

This result applies only to disks in the complex numbers, but it can be extended to more general regions. There is an important feature of solving differential equations in regions in $\mathbb{C}$, however. Consider the differential equation $y' = y/2z$, which is analytic in the complement of 0. Formally, its solution is $y = z^{1/2}$, but this does not make sense in that region. In general, there exists solutions to differential equations in a region of $\mathbb{C}$ only if that region is simply connected.

5.2. Corollary. *If $A(z)$ is analytic in a simply connected open subset U of $\mathbb{C}$, then for each u in U and each vector y_u there exists a unique solution $y(z)$ of the system $y' = A(z)y$ with $y(u) = y_u$, defined and analytic throughout U.*

In general, suppose u and v to be any points of a region U that is not necessarily simply connected and $\gamma(t)$ to be a path in U with $\gamma(0) = u$, $\gamma(1) = v$. Let $\widetilde{U}$ be the covering space of U. The differential equation lifts to one on $\widetilde{U}$, and if $\widetilde{u}$ is any point in $\widetilde{U}$ over u the path γ lifts to a unique path starting out at $\widetilde{u}$. Since $\widetilde{U}$ is simply connected, there exists a unique solution on $\widetilde{U}$ agreeing with some initial condition at $\widetilde{u}$. If $u = v$ then the endpoint of the lifted path will lie over v, and we shall get a vlaue of the solutiuon at v. The map from $\mathbb{C}^n$ to itself is called the monodromy of the path. It depends only on the homotopy class of γ. For the equation $y' = \lambda y/z$ it is multiplication by $e^{2\pi i\lambda}$, so for the equation above the monodromy is multiplication by -1. F or the equation $y'' + y'/z = 0$ a basis of solutions is $1, \log(z)$. The monodromy associated to one positive revolution is

$$\begin{bmatrix} 1 & 2\pi i \\ 0 & 1 \end{bmatrix}.$$

6. Regular singularities

Now suppose that $A(z)$ is analytic in $0 < |z| < r$. The system $y' = Ay$ has a possibly multi-valued basis of solutions in $0 < |z| < r$, in the sense that it has a fundamental matrix analytic in all of the region $-\infty < \mathrm{RE}(z) < \log r$ for the lifted system arising through the the exponential map e^z.

The prototypical example is one of Euler's equations

$$y' = \frac{A}{z}\,y \quad \text{or } zy' = Ay$$

where A is a constant matrix. If we make a change of variables $z = e^w$, $y(w) = y(z)$, the operator $z\,d/dz$ becomes d/dw so the system becomes

$$y' = Ay$$

which has constant coefficients and the solutions $y = e^{Aw}y_0$ and the original system has the multi-valued solutions $z^A z_0$.

In general, a system is said to have a **simple singularity** at 0 if it is of the form

$$zy' = A(z)y$$

where $A(z)$ is analytic in the neighbourhood of 0. Associated to it is the Euler system

$$zy' = A(0)y\,.$$

The relationship between the two is close—the Euler system picks out the **leading terms** of the solution to the original system. I'll explain in detail for single higher order equations.

Let D be the differential operator xd/dx. If we interpret higher order equations in terms of systems, we see that a higher order equation

$$D^n y + a_{n-1}(x)D^{n-1}y + \cdots + a_0(x) = 0$$

has a regular singularity at 0 if the $a_k(x)$ are analytic at 0. In that case there exist solutions

$$x^r(1 + c_1x + c_2x^2 + \cdots)$$

where r is a root of the **indicial equation**

$$r^n + a_1(0)r^{n-1} + \cdots a_n(0) = 0$$

Equivalently, an equation

$$x^n y^{(n)} + x^{n-1}a_{n-1}(x)y^{(n)} + \cdots + a_0 y = 0$$

has regular singularities at 0 if each $a_k(x)$ is analytic at 0, and and the indicial equation is then

$$r^{[n]} + a_{n-1}(0)r^{[n-1]} + \cdots a_0(0) = 0$$

where

$$r^{[k]} = (r-1)(r-2)\ldots\big(r-(k-1)\big)\,.$$

I'll say more only for equations of second order.

6.1. Theorem. *Suppose that r_1 and r_2 are the roots of the indicial equation of the differential equation*

$$x^2y'' + xa_1(x)y' + a_0(x) = 0\,.$$

(a) If $r_1 - r_2$ is not an integer, then for either root r the equation has a solution of the form

$$x^r(1 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots);$$

(b) if $r_1 = r_2 =$ say r then there exists a solution of the form

$$y_1 = x^r(1 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots)$$

and another of the form

$$y_2 = x^r(d_1 x + d_2 x^2 + d_3 x^3 + \cdots) + y_1 \log x;$$

(b) if $r_1 = r_2 + n$ with $n > 0$ then there exists a solution of the form

$$y_1 = x^{r_1}(1 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots)$$

and another of the form

$$y_2 = x^{r_2}(1 + d_1 x + d_2 x^2 + d_3 x^3 + \cdots) + \mu\, y_1 \log x$$

where μ might be 0.

The coefficients of the series may be found by recursion. Any series found will automatically converge, because of this:

6.2. Theorem. *Any formal series solving a differential equation with a regular singularity at 0 will automatically converge.*

It is instructive to compute the monodromy of the third type of solution. Write

$$\begin{aligned} y_1 &= z^{r_1} f_1 \\ y_2 &= z^{r_2} f_2 + \mu y_1 \log(z). \end{aligned}$$

Then since $e^{2\pi i r_1} = e^{2\pi i r_2}$

$$\begin{aligned} y_1 &\mapsto e^{2\pi i r_1} y_1 \\ y_2 &\mapsto e^{2\pi i r_2} z^{r_2} f_2 + \mu e^{2\pi i r_1} y_1(\log(z) + 2\pi i) \\ &= e^{2\pi r_2} y_2 + 2\pi i \mu e^{2\pi i r_1} y_1 . \end{aligned}$$

To every differential equation with singularities among a discrete subset of $\mathbb{C}$ one gets a representation of the fundamental group of the region into $\mathbb{C}^n$. The **Riemann-Hilbert problem**, posed most notably by Hilbert as the twenty-first among his famous list of 1906, is to ascertain whether every such representation arises from a differential equation with regular singularities. Many variations on this have been shown to be true.

7. Irregular singularities

A differential equation may also have a regular singularity at ∞. This may be tested, and the solution found, by a change of variables $w = 1/z$. Consider the equation

$$y'' + a(x)y' + b(x)y = 0$$

where each of a and b is analytic at ∞:

$$a(x) = \sum_0 a_i x^{-i}, \quad b(x) = \sum_0 b_i x^{-i} .$$

Setting $t = 1/x$, $y(t) = y(x)$ this becomes

$$t^4 y'' + \left[2t^3 - t^2 \sum_0 a_i w^i \right] y' + \left[\sum_0 b_i t^i \right] = 0 ,$$

which has a regular singularity at 0, and the original equation at ∞, only if $a_0 = b_0 = b_1 = 0$.

If the equation does not have a regular singularity at ∞ it is said to have an **irregular singularity** there. The prototypical case here is the equation with constant coefficients, and it turns out that the general equation with an irregular singularity at ∞ is modeled to some extent on that one.

Differential equations with irregular singularities also have series solutions, but they are not convergent—they are, instead, **asymptotic series**.

7.1. Theorem. *Suppose that the indicial equation*

$$\lambda^2 + a_0 \lambda + b_0 = 0 .$$

has two distinct roots λ_1, λ_2. Then for each i there exist formal solutions of the differential equation

$$y'' + a(x)y' + b(x)y = 0$$

of the form

$$e^{\lambda_i x} x^{r_i} (1 + c_1/x + c_2/x^2 + \cdots)$$

where

$$(2\lambda_i + a_0) r_i = -\lambda_i a_1 - b_1 .$$

Each of these formal solutions describes the asymptotic behaviour of a true solution.

Part III. Applications

8. Separation of variables I. The Laplacian in the plane

The Laplacian in the Euclidean plane is

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} .$$

It commutes with rotations, hence preserves the eigenspaces of the rotation group. If f is an eigenfunction of Δ and also one of rotations corresponding to the character $e^{in\theta}$, it is determined by its restriction to the x-axis. By the traditional method of separation of variables, it will satisfy an ordinary differential equation there. This will be determined by expressing the Laplacian in polar coordinates.

I recall now how coordinate changes affect partial differential operators. There is one small trick to take into account, that occurs already in dimension one. In changing from the independent variable x to the new variable t, we first wrote down the chain rule

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

and then divided to get

$$\frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt}$$

which I wrote as the operator equation

$$\frac{d}{dx} = \frac{d}{dt}\Big/\frac{dx}{dt}$$

The point of this was that these two steps were necessary in order to get on the left an expression all in terms of x, and on the right one all in terms of t.

Something similar happens for several variables. We apply the chain rule for two variables

$$\begin{aligned}\frac{\partial f}{\partial r} &= \frac{\partial x}{\partial r}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial f}{\partial y}\\ \frac{\partial f}{\partial \theta} &= \frac{\partial x}{\partial \theta}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial f}{\partial y}\end{aligned}$$

which we can write as a matrix equation

$$\begin{bmatrix}\partial f/\partial r\\ \\ \partial f/\partial \theta\end{bmatrix} = \begin{bmatrix}\partial x/\partial r & \partial y/\partial r\\ \\ \partial x/\partial \theta & \partial y/\partial \theta\end{bmatrix}\begin{bmatrix}\partial f/\partial x\\ \\ \partial f/\partial y\end{bmatrix}$$

and the way to get the equation we want is to solve this equation by multiplying by the inverse of the 2×2 matrix of partial derivatives, the *Jacobian matrix* of the coordinate change. We then get

$$\begin{bmatrix}\partial f/\partial x\\ \\ \partial f/\partial y\end{bmatrix} = \begin{bmatrix}\partial x/\partial r & \partial y/\partial r\\ \\ \partial x/\partial \theta & \partial y/\partial \theta\end{bmatrix}^{-1}\begin{bmatrix}\partial f/\partial r\\ \\ \partial f/\partial \theta\end{bmatrix}$$

which I write as an operator equation

$$\begin{bmatrix}\partial/\partial x\\ \\ \partial/\partial y\end{bmatrix} = \begin{bmatrix}\partial x/\partial r & \partial y/\partial r\\ \\ \partial x/\partial \theta & \partial y/\partial \theta\end{bmatrix}^{-1}\begin{bmatrix}\partial/\partial r\\ \\ \partial/\partial \theta\end{bmatrix}$$

For polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$, and the Jacobian matrix is

$$\begin{bmatrix}\cos\theta & \sin\theta\\ -r\sin\theta & r\cos\theta\end{bmatrix}.$$

There are many ways to see what its inverse is, but one illuminating way to start is to notice that this matrix may be factored as a product of matrices

$$\begin{bmatrix}1 & 0\\ 0 & r\end{bmatrix}\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}.$$

If $A = BC$ (matrices A, B, C) then $A^{-1} = C^{-1}B^{-1}$. Each of these factors has a simple inverse, the first since it is a *diagonal* matrix, the second because it is *orthogonal*—i.e. each row is a vector of length 1 and the two rows are orthogonal. The inverse of an orthogonal matrix is simply its *transpose* (its flip about the diagonal). Thus the inverse of our Jacobian matrix is

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/r \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta/r \\ \sin\theta & \cos\theta/r \end{bmatrix}$$

and so

$$\begin{aligned} \frac{\partial}{\partial x} &= \cos\theta\,\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\,\frac{\partial}{\partial\theta} \\ \frac{\partial}{\partial y} &= \sin\theta\,\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\,\frac{\partial}{\partial\theta} \end{aligned}$$

gives the basic operator equation we are looking for. To get second order partial derivatives, such as $\partial^2 f/\partial x^2$, we apply these formulas twice. First setting for example $F(x) = \partial f/\partial x$, we get

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2} &= \frac{\partial F}{\partial x} = \cos\theta\,\frac{\partial F}{\partial r} - \frac{\sin\theta}{r}\,\frac{\partial F}{\partial\theta} \\ &= \cos^2\theta\,\frac{\partial^2 f}{\partial r^2} - \frac{2\sin\theta\cos\theta}{r}\,\frac{\partial^2 f}{\partial r\partial\theta} + \frac{\sin^2\theta}{r^2}\,\frac{\partial^2 f}{\partial\theta^2} + \frac{\sin^2\theta}{r}\,\frac{\partial f}{\partial r} \\ \frac{\partial^2 f}{\partial y^2} &= \sin^2\theta\,\frac{\partial^2 f}{\partial r^2} + \frac{2\sin\theta\cos\theta}{r}\,\frac{\partial^2 f}{\partial r\partial\theta} + \frac{\cos^2\theta}{r^2}\,\frac{\partial^2 f}{\partial\theta^2} + \frac{\cos^2\theta}{r}\,\frac{\partial f}{\partial r} \\ \Delta f &= \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\,\frac{\partial f}{\partial r} + \frac{1}{r^2}\,\frac{\partial^2}{\partial\theta^2}. \end{aligned}$$

If a function f is an eigenfunction of Δ with eigencharacter $e^{in\theta}$ with respect to rotation, it is of the form $e^{in\theta}F(r)$ in polar coordinates with

$$F''(r) + \frac{1}{r}F'(r) - \frac{\lambda + n^2}{r^2}F(r) = 0\,.$$

It is singular at 0 and ∞. At 0 it is regular, and the indicial equation depends on n but not on λ. Ar ∞ the opposite is true, the behaviour does not depend on n. If $\lambda = 0$ the solutions are polynomials in $x + iy$ or $x - iy$, but the functions one obtains in general are **Bessel functions** of order n. Their graphs are the wave patterns you see in a vibrating cup of coffee.

9. Separation of variables II. The Laplacian in non-Euclidean geometry

In non-Euclidean geometry, the Laplacian in polar coordinates is

$$\frac{\partial^2}{\partial r^2} + \frac{1}{\tanh r}\frac{\partial}{\partial r} + \frac{1}{\sinh^2 r}\frac{\partial^2}{\partial\theta^2}\,.$$

An eigenfunction of Δ and the rotation group satisfies

$$f''(r) + \frac{1}{\tanh r}\,f'(r) - \frac{n^2}{\sinh^2 r}\,f(r) = \lambda f(r)\,.$$

This has a regular singularity at 0 and an irregular one at ∞. But there is a major difference between this case and the Euclidean one—if we change variables $y = e^r$ we get

$$D^2 f - \left(\frac{1+y^2}{1-y^2}\right) Dpt f - \frac{n^2 y^2}{(1-y^2)^2}\,f = \lambda f\,,$$

where $D = y\,d/dy$. This has a regular singularity at both 1 and ∞.

10. Whittaker functions

Let's look at an example that arises in the theory of automorphic forms. Suppose F to be an eigenfunction of the (non-Euclidean) Laplacian on the quotient $\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}$ (where $\mathcal{H}$ is the upper half plane). We assume it further to be an automorphic form, which means that it is of moderate growth on the fundamental domain $-1/2 \le x < 1/2$, $|y| \ge 1$. Thus

$$y^2 \left(\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial x^2} \right) = -\lambda^2 y \,.$$

and

$$|F| \le C y^M$$

for some $C > 0$ on the strip $|x| \le 1/2$, $1 \le y$. Since $F(z+1) = F(z)$, we can express

$$F = \sum_n F_n(y) e^{2\pi i n x}$$

with

$$F_n(y) = \int_0^1 F(x+iy) e^{-2\pi i n x} \, dx \,.$$

The function $F_n(y)$ also satisfies $|F_n| \le C y^M$ for $1 \le y$. Because $\Delta F = -\lambda^2 F$ and Δ commutes with horizontal translation, the function $F_n(y)e^{2\pi i n x}$ satisfies the same eigen-equation. Therefore

$$y^2 \big(F_n''(y) - 4\pi^2 n^2 F_n \big) = -\lambda^2 F_n \,.$$

Thus we are led to consider very generally solutions $F(x)$ of the differential equation

$$F'' - \left(a^2 + \frac{b}{x^2} \right) F = 0$$

where $|F| \le C x^M$ on $[1, \infty)$ and $a \ne 0$.

This equation has two singularities, at 0 and ∞. The singularity at 0 is regular; that at ∞ is irregular.

We can write the equation as

$$F'' - a^2 F = -\left(\frac{bF}{x^2} \right)$$

and according to the previous section we can find formal series solutions

$$e^{\pm a x} (1 + c_1/x + c_2/x^2 + \cdots)$$

that are in fact asymptotic. For automorphic forms, the condition of moderate growth forces us to discard the solution of exponential growth, and leaves us with one of exponential decrease. I do not see an elementary way to derive this asymptotic expansion, but Proposition 4.4 allows us to see that it is rapidly decreasing.

The functions F_n are special cases of the **Whittaker functions** $W_{k,m}(x)$, which satisfy the differential equation

$$W'' + \left(-\frac{1}{4} + \frac{k}{z} + \frac{1/4 - m^2}{x^2} \right) W = 0 \,.$$

They possess the asymptotic expansion

$$W_{k,m}(x) \sim e^{-x/2} x^k \left(1 + \frac{(m^2 - (k-1/2)^2)}{1! x} + \frac{(m^2 - (k-1/2)^2)(m^2 - (k-3/2)^2)}{2! x^2} + \cdots \right) .$$

Part IV. References

1. Fred Brauer and John A. Nohel, **Ordinary differential equations - a first course**, W. A. Benjamin, 1967.

For rigour and details, the classic text by Coddington and Levinson is best, but for a discursive account that includes nearly everything needed for representation theory, this book is very good. The discussion of irregular singular points and asymptotic expansions is particularly clear, although with few proofs.

2. Earl A. Coddington and Norman Levinson, **Theory of ordinary differential equations**, McGraw-Hill, 1955.

Clear and thorough if somewhat dense. The (difficult) proof that formal solutions correspond to asymptotic series is in §4 of Chapter 5.

3. E. T. Whittaker and G. N. Watson, **A course of modern analysis**, Cambridge University Press, 4th edition, 1952.

The functions $W_{k,m}$ are discussed in Chapter XVI. The asymptotic expansion is derived there from an expression for $W_{k,m}$ as a contour integral.