Essays on the structure of reductive groups

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Determinants

Suppose D to be a field, possibly non-commutative. The space D^n becomes a module over D through multiplication on the right. Left multiplication by matrices identifies $M_n(D)$ with the ring of endomorphisms of D^n commuting with this right action. Let $\overline{D}^\times$ be the maximal abelian quotient of the multiplicative group $D^\times$, the quotient of $D^\times$ by its commutator subgroup.

The aim of this note is to demonstrate and comment on:

Theorem. *There exists a unique homomomorphism*

$$\det\colon\ \mathrm{GL}_n(D) \longrightarrow \overline{D}^\times$$

such that if X is a diagonal matrix then $\det(X)$ is the image in $\overline{D}^\times$ of $\prod x_{i,i}$.

Of course this is well known if D is commutative, so the problem is how to deal with non-commutativity.

The principal references in the literature are [Dieudonné:1943] and §IV.1 of the book [Artin:1955]. Proofs involve a lot of explicit matrix manipulations. It would be nice to have a treatment for the non-commutative case like that for the commutative case, in which determinants are defined in terms of exterior products. Some version of this is carried out in Appendice 2 of [Bourbaki:1981], but the result is not lucid.

All that is new here is brevity.

Contents

1. Introduction

For $1 \le i, j \le n$ let $e_{i,j}$ be the elementary matrix with $e_{i,j} = 1$, other entries 0, and for $i \ne j$, x in D let

$$u_{i,j}(x) = I + xe_{i,j}\ .$$

It is unipotent. For $1 \le i \le n$ let $d = \varepsilon_i(x)$ be the diagonal matrix with $d_{i,i} = x$, $d_{j,j} = 1$ for $j \ne i$. For $1 \le i < n$ let $\delta_i(x) = \varepsilon_i(x)/\varepsilon_{i+1}(x)$.

Let E_n be the subgroup of GL_n generated by the $u_{i,j}(x)$.

Gauss elimination (aka Bruhat factorization) expresses every matrix as a product n_1wan_2 with the n_i upper triangular, a diagonal, and w one of the twisted permutation matrices with integral entries and determinant 1. It does this by applying row and column operations, which are effected through left and right multiplication by one of the $u_{i,j}(x)$.

Since

$$\begin{bmatrix} 1 & 0 \\ 1/x & 1 \end{bmatrix} \begin{bmatrix} 1 & -x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1/x & 1 \end{bmatrix} = \begin{bmatrix} 0 & -x \\ 1/x & 0 \end{bmatrix}\ .$$

$$\begin{bmatrix} 0 & -x \\ 1/x & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} x & 0 \\ 0 & 1/x \end{bmatrix}\ ,$$

the elements $\delta_i(x)$ lie in E_n. Hence:

1.1. Proposition. *The group* $\mathrm{GL}_n(D)$ *is generated by the matrices* $u_{i,j}(x)$ *and* $\varepsilon_n(x)$.

Therefore every g in G conjugates an element of E_n to another element of E_n:

1.2. Corollary. *The group* E_n *is a normal subgroup.*

1.3. Proposition. *If* c *is a commutator in* $D^\times$ *then* $\delta_n(c)$ *is in* E_n.

Proof. For a, b in $D^\times$

$$\begin{aligned}\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} &= \begin{bmatrix} a & 0 \\ 0 & 1/a \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & ab \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & ba \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & 1/a \end{bmatrix} .\end{aligned}$$

We can therefore find a matrix m in E_n such that

$$m\begin{bmatrix} 1 & 0 \\ 0 & ba \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & ab \end{bmatrix}, \quad m = \begin{bmatrix} 1 & 0 \\ 0 & aba^{-1}b^{-1} \end{bmatrix} .$$

A right-linear map f from D^n to another right-D-module satisfies the condition $f(v\cdot c) = f(v)\cdot c$. Given a coordinate system, a right-linear map may be identified with multiplication on the left by a row vector with coefficients in D.

A **shear** in $\mathrm{GL}_n(D)$ is a linear transformation of D^n of the form

$$v \longmapsto v + h\cdot\langle f, v\rangle ,$$

in which $f \neq 0$ is a right-linear map from D^n to D, and $h \neq 0$ in D^n satisfies the condition $\langle f, h\rangle = 0$. It translates vectors parallel to the hyperplane $f = 0$, in the direction of h. Every $u_{i,j}(x)$ is a shear.

The group GL_n acts transitively on pairs $f \neq 0$, $h \neq 0$ with $\langle f, h\rangle 0$, and

$$\begin{bmatrix} x & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1/x & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & xc \\ 0 & 1 \end{bmatrix} .$$

Conversely, therefore, choosing coordinates suitably one sees that every shear is hence conjugate to some $u_{1,2}(x)$.

Some elementary reasoning then leads to a proof of:

1.4. Proposition. *All shears are conjugate in* $\mathrm{GL}_n(D)$.

1.5. Proposition. *Except when* $D = \mathbb{F}_2$ *and* $n = 2$*, every shear is a commutator.*

In the exceptional case, $\mathrm{GL}_n(D)$ is isomorphic to $\mathfrak{S}_3$, and the proposition is false.

Proof. Except when $D = \mathbb{F}_2$ and $n = 2$, every shear n can be expressed as $n_1 n_2$ with the n_i also shears. All three of these lie in one conjugacy class. With respct to any homomorphism to an abelian group, they all have the same image, say α. Then $\alpha + \alpha = \alpha$, and $\alpha = 0$. ▯

In particular, E_n is contained in the commutator subgroup. According to Proposition 1.1, every g can be written as $u\varepsilon_n(x)$ for u in E_n and x in $D^\times$. Therefore if the homomorphism $\det$ exists and satisfies the hypothesis of the original Theorem, $\det(g)$ is the image of x in $\overline{D}^\times$. This assures the uniqueness of the map $\det$. It remains to define it.

2. The determinant

We now have a candidate for $\det(g)$, found by applying Gauss elimination. But we don't have any idea as to why the calculation always produces the same result. That's what I'll explain here.

The construction of $\det$ goes by induction on n. For $n = 1$ it takes x in $D^\times$ to its image in $\overline{D}^\times$.

Suppose now that A is an $n \times n$ matrix with $n \ge 2$. Choose a non-zero element $a_{i,1}$ in the first column. Subtract off from the other rows a multiple of the i-th row to make the rest of the first column vanish. Let this new matrix be B. Let $\widehat{B}_{i,1}$ be the $(n-1) \times (n-1)$ matrix obtained from B by eliminating the i-th row and first column. The usual formula that is valid in the case of commutative fields suggests that we set

$$\det(A) = (-1)^{i-1} a_{i,1} \det(\widehat{B}_{i,1})\,. \tag{2.1}$$

In order to know that $\det$ is well defined, it is necessary to show it is independent of the choice of i. This will be done by an induction argument I'll explain in a moment.

But first I'll look at $n = 2$. Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Ambiguity arises only if both $a \ne 0$, $c \ne 0$. On the one hand, we go

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longrightarrow \begin{bmatrix} a & b \\ 0 & d - ca^{-1}b \end{bmatrix}, \quad \det = \text{ image of } ad - aca^{-1}b\,.$$

On the other, we go

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & b - ac^{-1}d \\ c & d \end{bmatrix}, \quad \det = \text{ image of } cac^{-1}d - cb\,.$$

So we want to prove that

$$ad - aca^{-1}b \equiv cac^{-1}d - cb$$

modulo the commutator subgroup of $D^\times$. The left hand side is $ad - aca^{-1}c^{-1} \cdot cb$ and the right hand side is $cac^{-1}a^{-1} \cdot ad - cb$. The first is $ad - x \cdot cb$ if $x = aca^{-1}c^{-1}$, while second is $x^{-1} \cdot ad - cb$. But

$$x^{-1} \cdot ad - bc = x^{-1}(ad - x \cdot cb)\,.$$

Since x is a commutator, this proves the equivalence, and concludes also the verification that the determinant of a 2×2 matrix is at least well defined.

I leave as exercise the verification of these properties:

(a) $\det(I_n) = 1$;
(b) if r and s are distinct rows, replacing r by $r + xs$ does not change $\det$;
(c) replacing a row r by xr changes $\det$ to $\overline{x}\det$;

We now come to a major result:

2.2. Proposition. *For all $n \ge 3$, (2.1) is a valid definition, and the determinant so defined satisfies these properties (a)–(c).*

We know it to be true for $n \le 2$. Following Dieudonné's admirable suggestion, I look just at $n = 3$, where all important phenomena appear, in order not to burden notation. So consider

$$A = \begin{bmatrix} a_{1,1} & A_1 \\ a_{2,1} & A_2 \\ a_{3,1} & A_3 \end{bmatrix}.$$

Again for simplicity, I'll look only at the case $a_{i,1} \neq 0$ for $i = 1, 2$. One reduction goes

$$A \longrightarrow \begin{bmatrix} a_{1,1} & A_1 \\ 0 & A_2 - a_{2,1}a_{1,1}^{-1}A_1 \\ 0 & A_3 - a_{3,1}a_{1,1}^{-1}A_1 \end{bmatrix}$$

giving as candidate determinant the image of

$$a_{1,1} \det\left(\begin{bmatrix} A_2 - a_{2,1}a_{1,1}^{-1}A_1 \\ A_3 - a_{3,1}a_{1,1}^{-1}A_1 \end{bmatrix}\right) .$$

The other goes

$$A \longrightarrow \begin{bmatrix} 0 & A_1 - a_{1,1}a_{2,1}^{-1}A_2 \\ a_{2,1} & A_2 \\ 0 & A_3 - a_{3,1}a_{2,1}^{-1}A_2 \end{bmatrix}$$

giving determinant the image of

$$-a_{2,1} \det\left(\begin{bmatrix} A_1 - a_{1,1}a_{2,1}^{-1}A_2 \\ A_3 - a_{3,1}a_{2,1}^{-1}A_2 \end{bmatrix}\right) .$$

So now I must show that

$$a_{1,1} \det\left(\begin{bmatrix} A_2 - a_{2,1}a_{1,1}^{-1}A_1 \\ A_3 - a_{3,1}a_{1,1}^{-1}A_1 \end{bmatrix}\right) \equiv -a_{2,1} \det\left(\begin{bmatrix} A_1 - a_{1,1}a_{2,1}^{-1}A_2 \\ A_3 - a_{3,1}a_{2,1}^{-1}A_2 \end{bmatrix}\right) .$$

We can write these as

$$a_{1,1}a_{2,1} \det\left(\begin{bmatrix} a_{2,1}^{-1}A_2 - a_{1,1}^{-1}A_1 \\ A_3 - a_{3,1}a_{1,1}^{-1}A_1 \end{bmatrix}\right), \quad a_{2,1}a_{1,1} \det\left(\begin{bmatrix} a_{2,1}^{-1}A_2 - a_{1,1}^{-1}A_1 \\ A_3 - a_{3,1}a_{2,1}^{-1}A_2 \end{bmatrix}\right) ,$$

then express the bottom row of one as a row subtraction. I leave it as exercise to finish this, and also to finish the proofs of Theorem 1.

2.3. Proposition. *The determinant is a homomorphism whose kernel is E_n.*

Proof. Straightforward, from what has been proved. ∎

It is conventionally called $\mathrm{SL}_n(D)$.

3. The norms on D

In general, very little is known about $\overline{D}^{\times}$, but there are important cases where it is known completely.

If F is a (commutative) field, a **central simple algebra** over F is an algebra A whose center is isomorphic to F such that $E \otimes_F A \cong M_n(E)$ for some field extension E/F. In these circumstances, the determinant on $M_n(E)$ defines a multiplicative norm on A, which takes its values in F. It induces a homomorphism $\mathrm{NM}\colon A^{\times} \to F^{\times}$, which of course must contain the commutators of $A^{\times}$. It is natural to ask, *when is the kernel of* NM *exactly equal to the commutator subgroup?* This is now part of the subject of algebraic K-theory (see, for example, [Milnor:1971]).

The following is a basic fact in local class field theory. It is due to [Matsushima-Nakayama:1943] for $\mathfrak{p}$-adic groups. The case of the real quaternions is mentioned in [Dieudonné:1943], and presumably well known at that time.

3.1. Theorem. *If F is a local field and A a central simple algebra over F, then the commutator subgroup of $A^{\times}$ is equal to the kernel of* NM.

4. References

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