The Schwartz space of Ehrenpreis and Mautner

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Let

$$\begin{aligned} G &= \mathrm{SL}_2(\mathbb{R}) \\ \Gamma &= \mathrm{SL}_2(\mathbb{Z})\,. \end{aligned}$$

By definition, a smooth function F belongs to $A_{\mathrm{umg}}(\Gamma\backslash G)$, the space of functions of uniform moderate growth on $\Gamma\backslash G$, if and only if there exists a fixed M such that $R_X F(g) = O(\|g\|^M)$ for all X in $U(\mathfrak{g})$. It belongs to the Schwartz space $\mathcal{S}(\Gamma\backslash G)$ if $R_X F$ decreases rapidly at all cusps of Γ for all X. If F is a function on $\Gamma\backslash\mathcal{H}$, this definition requires lifting F to a function on $\Gamma\backslash G$ by using the identification of $\mathcal{H}$ with $G/\mathrm{SO}(2)$. It is a matter of curiousity to have a more intrinsic characterization of such functions on $\Gamma\backslash\mathcal{H}$. It seems to me possible that this question and similar ones for other arithmetic quotients are important, although I'll offer no evidence for that here.

1. Proposition. *A smooth function F on $\Gamma\backslash\mathcal{H}$ lies in $A_{\mathrm{umg}}(\Gamma\backslash\mathcal{H})$ if and only if there exists a fixed M such that $\Delta^k F = O(y^M)$ for all k.*

2. Proposition. *A smooth function F on $\Gamma\backslash\mathcal{H}$ lies in $\mathcal{S}(\Gamma\backslash\mathcal{H})$ if and only if $\Delta^k F$ decreases rapidly in the neighbourhood of all cusps of Γ.*

These are certainly necessary conditions, since Δ is the restriction to functions on $\mathcal{H}$ of the Casimir operator in $U(\mathfrak{g})$. The criterion in the second of these is in fact the definition of the Schwartz space in [Ehrenpreis-Mautner:1962], which I believe to have been the first to introduce the notion.

By truncating F smoothly, we may assume that in the fundamental domain F has support in the region $y > 1$. In that case, we may identify it with a function on $(\Gamma\cap N)\backslash\mathcal{H}$. Resolve F into its Fourier series

$$F(x+iy) = \sum F_n(y)e^{2\pi inx} \quad \left(F_n(y) = \int_0^1 F(x+iy)e^{-2\pi iinx}\,dx\right)\,.$$

Then F satisfies the hypotheses of the theorem if and only if each of F_0 and $F - F_0$ separately do, and similarly for the conclusion.

Let $D = y\partial/\partial y$. It is straightforward to see that the function F_0 lies in A_{umg} if and only if there exists M such that all $D^k F_0 = O(y^M)$.

It is also easy to see that F lies in $\mathcal{S}$ if and only if all $\partial^{k+\ell}F/\partial x^k\partial y^\ell = O(y^{-N})$ for all N. (Remember, F has support in $y > 1$.)

The proof of the Proposition therefore comes in two halves.

Contents

1. The constant term

I'll deal with the constant term first.

1.1. Proposition. *Suppose that F is a smooth function on $(0,\infty)$ with support in $y>1$ such that for some fixed M all $\Delta^k F = O(y^M)$. Then all $D^k F = O(y^M)$; as well.*

Proof. Suppose that $F = F_0$.

$$|F(y)| \le C_0 y^M, \quad (\Delta F) = y^2 F''(y) \le C_2 y^M \ .$$

1.2. Lemma. *Suppose $F(y)$ and $y^2F''(y)$ are both $O(y^M)$, and of support near $y=\infty$. Then $yF'(y) = O(y^M)$ as well.*

Proof. The idea is to reduce it to a variant of a well known lemma due originally to Landau:

1.3. Lemma. *If*

$$g(t) = O(e^{\mu t}), \quad g''(t) = O(e^{\mu t})$$

then so also is

$$g'(t) = O(e^{\mu t}) \ .$$

Proof of Landau's Lemma.

$$\begin{aligned}
g(t) - g(t-\varepsilon) &= \varepsilon g'(t) + (\varepsilon^2/2) g''(\theta_1) \quad (t-\varepsilon < \theta_1 < t) \\
g(t+\varepsilon) - g(t) &= \varepsilon f'(t) + (\varepsilon^2/2) g''(\theta_2) \quad (t < \theta_1 < t+\varepsilon) \\
g(t+\varepsilon) - g(t-\varepsilon) &= 2\varepsilon f'(t) + (\varepsilon^2/2)\big(g''(\theta_1) + g''(\theta_2)\big) \\
2\varepsilon f'(t) &= g(t+\varepsilon) - g(t-\varepsilon) - (\varepsilon^2/2)\big(g''(\theta_1) + g''(\theta_2)\big) \\
2\varepsilon |f'(t)| &\le P\big(e^{\mu(t+\varepsilon)} + e^{\mu(t-\varepsilon)}\big) + Q(\varepsilon^2/2)\big(e^{\mu\theta_1} + e^{\mu\theta_2}\big) \\
&\le P e^{\mu t}\big(e^{\mu\varepsilon} + e^{-\mu\varepsilon}\big) + (\varepsilon^2/2) Q e^{\mu t}\big(e^{\mu\varepsilon} + 1\big) \\
&\le P e^{\mu t}\big(e^{\mu\varepsilon} + 1\big) + (\varepsilon^2/2) Q e^{\mu t}\big(e^{\mu\varepsilon} + 1\big) \\
|f'(t)/e^{\mu t}| &\le \left(\frac{e^{\mu\varepsilon}+1}{2}\right)\left(\frac{P}{\varepsilon} + \frac{Q\varepsilon}{2}\right)
\end{aligned}$$

for all $\varepsilon > 0$. The function of ε

$$E(\varepsilon) = \left(\frac{e^{\mu\varepsilon}+1}{2}\right)\left(\frac{P}{\varepsilon} + \frac{Q\varepsilon}{2}\right)$$

approaches ∞ near 0 and also as $\varepsilon \to \infty$, and somewhere in between takes a minimum positive value $E_{\min}$. Thus for all t

$$|f'(t)| \le E_{\min} e^{\mu t} \ .$$

This concludes the proof of the two lemmas. Applied to all the $\Delta^k F$ in turn these imply that all $D^k F_0 = O(y^M)$, and this in turn implies that F_0 is of uniform moderate growth or rapid decrease, depending on the assumption on the $\Delta^k F$. Q.E.D. We start with

$$F(y) = O(y^m), \quad y^2 F''(y) = O(y^M) \,.$$

We can write

$$y^2 F'' = (D^2 - D)F = (D - 1/2)^2 F - 1/4F$$

if $D = y\partial/\partial y$, and then deduce

$$(D - 1/2)^2 F = O(y^M) \ .$$

If we set $G(y) = y^{\lambda} F(y)$, then

$$\begin{aligned} DG &= y^{\lambda} DF + \lambda y^{\lambda} F \\ &= y^{\lambda}(D+\lambda)G \\ D^2G &= \lambda y^{\lambda}(D+\lambda)F + y^{\lambda} D(D+\lambda)F \\ &= y^{\lambda}(D+\lambda)^2 F \ . \end{aligned}$$

Set $\lambda = -1/2$, so

$$\begin{aligned} G &= y^{-1/2}F(y) \\ &= O(y^{M-1/2}) \\ D^2G &= y^{-1/2}(D-1/2)^2F(y) \\ &= O(y^{M-1/2}) \end{aligned}$$

If we change the independent variable to $y = e^t$ we get equations

$$\begin{aligned} G(t) &= O(e^{(M-1/2)t}) \\ G''(t) &= O(e^{(M-1/2)t}) \end{aligned}$$

2. The rest

Now I'll deal with the rest of F.

2.1. Proposition. *Suppose F to be a smooth function on $\Gamma\cap N\backslash\mathcal{H}$ with support in $y > 1$ and all $\Delta^k F = O(y^M)$. If $F_0 = 0$ then all the $\partial^{k+\ell}\Phi/\partial x^k\partial y^\ell = O(y^{-N})$ for all N.*

Proof. I'll start off by looking at the the individual terms in the Fourier expansion. We have

$$(\Delta F)_n(y) = y^2 F_n''(y) - 4\pi^2 n^2 F_n(y)$$

and we may assume $n \neq 0$. If $\Delta^k F = O(y^M)$ for all k then so is $\Delta^k F_n(y) = O(y^M)$ for all k.

2.2. Proposition. *Let $F(y) = O(y^M)$ be a smooth function on $\mathbb{R}$ with support on $(1,\infty)$, and suppose that for some $\lambda \geq 1$*

$$y^2(F'' - \lambda^2 F) \leq Cy^M \ .$$

Then $F \leq (CC_M/\lambda)y^{M-2}$ for some constant depending only on M.

Since the λ we are concerned with are the $4\pi^2n^2$ and the series $\sum_{n>0} 1/n^2 < \infty$, this imples the bound we want on F itself.

Proof. Let $G(y) = y^2(F'' - \lambda^2 F)$. Since F has support on $(1,\infty)$ so does G. Since $E(x) = -e^{-\lambda|x|}/2\lambda$ satisfies the distributional differential equation

$$E'' - \lambda^2 E = \delta_0$$

an easy calculation tells us that

$$F(y) = c_+e^{\lambda y} + c_-e^{-\lambda y} - \frac{1}{2\lambda}\int_1^\infty e^{-\lambda|x-y|}G(y)\,dy \ .$$

Since $G(y) = O(y^M)$ an easy calculation tells us that the integral is of moderate growth, and therefore since F is of moderate growth the coefficient c_+ has to vanish. Thus

$$F(y) = c_-e^{-\lambda y} - \frac{1}{2\lambda}\int_1^\infty e^{-\lambda|x-y|}G(y)\,dy \ .$$

Since $F(0) = 0$

$$c_{-} = \frac{1}{2\lambda} \int_1^\infty e^{-\lambda x} G(x)\,dx$$

and hence

$$F(y) = \frac{e^{-\lambda y}}{2\lambda} \int_1^\infty e^{-\lambda x} G(x)\,dx - \frac{1}{2\lambda} \int_1^\infty e^{-\lambda |x-y|} G(x)\,dx \ .$$

For the moment, let

$$I_N = \int_1^\infty e^{-\lambda x} x^N\,dx \ .$$

Integration by parts and an easy estimate gives

$$\begin{aligned}
I_N &\le 1 \quad (N \le -2) \\
I_N &= \frac{e^{-\lambda}}{\lambda} + \frac{N}{\lambda}\, I_{N-1} \\
&= \frac{e^{-\lambda}}{\lambda} \left(1 + \frac{N}{\lambda} + \frac{N(N-1)}{\lambda^2} + + \frac{N(N-1)\cdots(N-n)}{\lambda^{n+1}} \right) \\
&= \frac{e^{-\lambda}}{\lambda} \left(1 + N + N(N-1) + \cdots + N(N-1)\cdots(N-n) \right) \\
&= \frac{e^{-\lambda}}{\lambda}\, E_N \\
&\le E_N/e
\end{aligned}$$

if $n = \lceil N+1 \rceil$, keeping in mind that $\lambda \ge 1$. Thus the first term above is bounded by

$$\frac{C E_{M-2}\, e^{-\lambda y}}{2\lambda e} \ .$$

The second term can be broken up into two parts:

$$\begin{aligned}
\int_1^\infty e^{-\lambda |x-y|} G(x)\,dx &= \int_1^y e^{-\lambda (y-x)} G(x)\,dx + \int_y^\infty e^{-\lambda (x-y)} G(x)\,dx \\
&\le C \int_1^y e^{-\lambda (y-x)} x^{M-2}\,dx + C \int_y^\infty e^{-\lambda (x-y)} x^{M-2}\,dx \ .
\end{aligned}$$

For the first of these integrals:

$$\begin{aligned}
\int_1^y e^{-\lambda (y-x)} x^{M-2}\,dx &= \left[\frac{e^{-\lambda (y-x)} x^{M-2}}{\lambda} \right]_1^y - \frac{M-2}{\lambda} \int_1^y e^{-\lambda (y-x)} x^{M-3}\,dx \\
&= \frac{y^{M-2} - e^{-\lambda (y-1)}}{\lambda} - \frac{M-2}{\lambda} \int_1^y e^{-\lambda (y-x)} x^{M-3}\,dx \\
&\le \frac{y^{M-2} - e^{-\lambda (y-1)}}{\lambda} + \frac{y^{M-2} - 1}{\lambda} \\
&\le \frac{2y^{M-2}}{\lambda} \ ,
\end{aligned}$$

which implies that the term is bounded by

$$\frac{C y^{M-2}}{\lambda^2} \le \frac{y^{M-2}}{\lambda} \ .$$

For the second:

$$\int_y^\infty e^{-\lambda(x-y)} x^{M-2}\, dx = \int_0^\infty e^{-\lambda s}(s+y)^{M-2}\, ds$$
$$= \int_0^\infty e^{-\lambda s}(s+y)^{m-2}\, ds \quad (m = \lceil M \rceil) \ .$$

To this last expression we can apply the binomial theorem and the estimate of the first integral to see that this second term is at most $CC^*_M y^{M-2}/\lambda$.

So now we know that F itself is rapidly decreasing. It remains to show that all its partial derivatives are also rapidly decreasing. I leave this as an exercise! Q.E.D.

3. References

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