Variations on a theorem of Émile Borel

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This short note is meant to be an introduction to a class of results that arise frequently in analysis on real manifolds. But in its present form it was written mainly for my own use. In distributing it, I apologize for its roughness, but hope that even as it is it might prove useful.

I am not a true expert in this difficult subject, and shall be glad to receive notice of errors.

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Part I. Borel's Theorem

1. Borel's Theorem—the elementary proof

If f is a smooth function in $\mathbb{R}^n$, its Taylor series at 0 is the power series

$$\tau(f) = \sum \frac{1}{k!} f^{(k)}(0)\, x^k$$

where the k are *multi-indices* (k_i) in $\mathbb{N}^n$ so that

$$k! = \prod k_i!, \quad f^{(k)} = \frac{\partial^k f}{\partial x^k} = \Big(\prod \frac{\partial^{k_i}}{\partial x_i^{k_i}}\Big) f, \quad x^k = \prod x_i^{k_i}\,.$$

This defines a map τ from $C^\infty(\mathbb{R}^n)$ to $\mathbb{C}[[x]]$. It is continuous in the natural topologies on each of these vector spaces.

The following result is found in [Borel:1895] (bottom of p. 44).

1.1. Theorem. (Borel's Lemma) *The map τ is surjective.*

I shall offer an elementary proof in this section, one only slightly different from that in [Narasimhan:1968], and sketch a much less elementary but conceptually simpler one in the next.

Proof. Let $C_b^\infty(\mathbb{R}^n)$ be the subspace of smooth functions on $\mathbb{R}^n$ all of whose derivatives are bounded. On this space we have a semi-norm defined by

$$\|f\|_m = \sup_{\substack{x\in\mathbb{R}^n \\ |k|<m}} \left|f^{(k)}(x)\right|$$

Here $|k| = \sum k_i$. We begin with:

1.2. Lemma. *Suppose $P = P_m$ to be a homogeneous polynomial of degree m on $\mathbb{R}^n$. Given $\varepsilon > 0$ there exists $f(x) = f_P(x)$ in $C_b^\infty(\mathbb{R}^n)$ which is identically P near 0 and such that*

$$\|f\|_m \le \varepsilon\,.$$

This is at least plausible, since the k-th derivatives of P vanish at 0 for $|k| < m$ and we don't care a lot how f behaves away from 0 as long as it is small. The main ideas of the proof of the Lemma appear already for $n = 1$, which I'll assume in order to obtain a slight simplification in notation.

Proof of the Lemma. In order to exhibit the basic idea, I'll look first at a few small values of m.

Say $m = 1$, so $P = c_1x$ for some constant c_1. Given $\varepsilon > 0$, I have to find f in $C^\infty(\mathbb{R})$ identically equal to c_1x near 0, but with $|f(x)| \le \varepsilon$ everywhere. I perform a common trick. First of all, I choose once and for all a smooth function φ identically 1 near 0 and vanishing for $|x| \ge 1$. Let M_0 be a bound for $|\varphi(x)|$. For $c > 0$ the function

$$\varphi_c(x) = \varphi(cx)$$

has support on $|x| \le 1/c$, is still equal to 1 near 0, and is still bounded everywhere by M_0. Thus the product φ_cP will have support on $|x| \le 1/c$ and will be bounded by $M_0|c_1|/c$ in that region since $|P(x)| \le c_1/c$ in the region $|x| \le c$. If we therefore set

$$f = \varphi_cP$$

with $c \ge M_0|c_1|/\varepsilon$, we deduce

$$\left|f(x)\right| \le \varepsilon\,.$$

everywhere.

Now say $m = 1$, so $P = c_2x^2$. We again set

$$f = \varphi_cP$$

and hope to guarantee both $\left|f(x)\right| \le \varepsilon$ and $\left|f'(x)\right| \le \varepsilon$ everywhere if we just choose c large enough. As before f will have support on $|x| \le 1/c$. If M_i is a bound for $\varphi^{(i)}(x)$, then c^iM_i is a bound for $\varphi_c^{(i)}$ since $\varphi_c^{(i)}(x) = c^i\varphi^{(i)}(cx)$. For $|x| \le 1/c$ we have:

$$\begin{aligned}
|f(x)| &= |\varphi_c(x)P(x)| \\
&\le M_0|c_2|/c^2 \\
\left|f'(x)\right| &= \left|\varphi_c'(x)P(x) + \varphi_c(x)P'(x)\right| \\
&\le cM_1\,|c_2|\,|x|^2 + 2M_0\,|c_2|\,|x| \\
&\le cM_1\,|c_2|\,c^{-2} + 2M_0\,|c_2|\,c^{-1} \\
&\le |c_2|\,(M_1/c + 2M_0/c) \\
&= \frac{|c_2|}{c}\,(M_1 + 2M_0)
\end{aligned}$$

so we must choose c large enough so that

$$\begin{aligned} c^2 &\geq M_0|c_2|/\varepsilon \\ c &\geq |c_2|\,(M_1 + 2M_0)/\varepsilon\,, \end{aligned}$$

which is certainly possible.

You can anticipate how this is going to proceed for an arbitrary m. We set $P = c_m x^m$ and $f = \varphi_c P$. We then have by Leibniz' formula, with $k < m$:

$$\begin{aligned} f^{(k)}(x) &= \sum_0^k \binom{k}{i} c^{k-i} \varphi^{(k-i)}(cx) P^{(i)}(x) \\ &= c_m \left[\sum_0^k \binom{k}{i} c^{k-i} \varphi^{(k-i)}(cx) \frac{m!}{(m-i)!}\, x^{m-i} \right] \end{aligned}$$

and in the range $|x| \leq 1/c$ this is bounded by

$$\begin{aligned} &|c_m| \left[\sum_0^k \binom{k}{i} \frac{(m+1)!}{(m-i)!}\, M_{k-i} c^{k-i} c^{i-m-1} \right] \\ &= \frac{|c_m|}{c^{m-k}} \left[\sum_0^k \binom{k}{i} \frac{m!}{(m-i)!}\, M_i \right] \end{aligned}$$

so again we can make $\|f\|_m$ small by choosing c large enough.

The conclusion of the proof of Theorem 1.1 is now straightforward. Given a formal power series $\widehat{f} = \sum c_k x^k$, choose for each $m \geq 1$ a smooth function $f_m(x)$ which is identical to the homogeneous part of the series of degree m near 0 and satisfies $\|f_m\|_m \leq 1/2^m$. Then

$$c_0 + \sum_{m \geq 1} f_m(x)$$

defines a smooth function on all of $\mathbb{R}^n$ whose Taylor series at 0 is $\widehat{f}$.

Very roughly speaking, this proof works because although the derivatives of φ_c increase as c does, the functions x^m with $m \geq 1$ vanish more rapidly near 0 in the range $[-1/c, 1/c]$ as c increases. These two effects cancel each other out in Leibniz' formula.

A very simple application of Borel's Theorem to local number theory occusr in analyzing Tate's local zeta functions for $\mathbb{R}$ and $\mathbb{C}$.

Define the Schwartz space $\mathcal{S}[0,\infty)$ of the closed half-line to be the space of restrictions of functions in $\mathcal{S}(\mathbb{R})$ to this half-line. A simple variant of Borel's Theorem says taht is the same as the space of smooth functions on $(0,\infty)$ that are smoothly asymptotic to a power series at 0. If $\mathcal{S}(0,\infty)$ is the space of functions in $\mathcal{S}(\mathbb{R})$ vanishing identically on $(-\infty, 0]0$, it also says that the following sequence of modules of the multiplicative group of positive real numbers is exact:

$$0 \to \mathcal{S}(0,\infty) \to \mathcal{S}[0,\infty) \to \mathbb{C}[[x]] \to 0\,.$$

Similarly, for $\mathbb{C}$ we get an exact sequence of $\mathbb{C}^\times$ modules

$$0 \to \mathcal{S}(\mathbb{C}^\times) \to \mathcal{S}(\mathbb{C}) \to \mathbb{C}[[z, \overline{z}]] \to 0\,.$$

These are useful in understanding the uniqueness of certain distributions occurring in local functional equations.

2. The proof by functional analysis

The proof I sketch in this section can be found as Theorem 37.2 of [Treves:1967], and has the virtue of being easily applicable in more general circumstances. It is based on the following criterion, which I shall not prove here.

2.1. Lemma. *Let U and V be Fréchet spaces with duals $\widehat{U}$ and $\widehat{V}$. A continuous linear map $f\colon U \to V$ is surjective if and only if the dual map $\widehat{f}\colon \widehat{V} \to \widehat{U}$ is injective with weakly closed image.*

How does this help to prove Borel's Theorem? Both $C^\infty(\mathbb{R}^n)$ and $\mathbb{C}[[x]]$ are Fréchet spaces. The dual of $\mathbb{C}[[x]]$ is the space generated by coefficient evaluations or, equivalently, the evaluation of partial derivatives. The dual of $C^\infty(\mathbb{R}^n)$ is the space of distributions of compact support. The dual map is certainly injective, because the polynomials are in $C^\infty(\mathbb{R})$. Closure is implied by the well known fact, which again I won't prove here, that the image of the dual map is the space of distributions with support at 0. This is the same as the space spanned by the Dirac δ_0 and its derivatives. ∎

The advantage of this proof is that it handles generalizations easily. It is not much more difficult to show, for example, that the map taking f to its family of Taylor series transverse to and along a linear subspace is surjective.

3. Why the proof can't be too elementary

If we are concerned only with finding a function with a given finite Taylor series, we can just choose the Taylor series itself to be the function whose Taylor series it is. In other words, the map from $C^\infty(\mathbb{R}^n)$ to $\mathbb{C}[[x]]/(x^{m+1})$ is not only surjective, but can be split. This is no longer true for the map τ itself. John Mather provided me with the following observation and its proof.

3.1. Proposition. *The surjective map*

$$\tau\colon\ C^\infty(\mathbb{R}) \longrightarrow \mathbb{C}[[x]]$$

does not possess a continuous splitting.

Proof. Suppose we have a splitting Φ. Let U be the open set of f in $C^\infty(\mathbb{R}^n)$ such that $|f(x)| < 1$ for $\|x\| \le 1$. If f lies in U and f does not vanish on $\|x\| \le 1$ then some multiple of f will not lie in U.

Because Φ is continuous, $\Phi^{-1}(U)$ must be an open set in $\mathbb{C}[[x]]$, hence contain some neighbourhood of 0. A basis of neighbourhoods of 0 in $\mathbb{C}[[x]]$ is made up by the sets $U = U(\varepsilon, m)$ of power series $\sum c_k x^k$ with $|c_k| < \varepsilon$ for $|k| < m$. In particular, $\Phi^{-1}(U)$ must contain some space T_m of all series with $c_k = 0$ for $|k| \le m$. The set T_m is a linear subspace of $\mathbb{C}[[x]]$, and hence so is its image under Φ.

Now let $f = \Phi(x^m)$. Since its Taylor series at 0 is x^m the function f does not vanish on $\|x\| \le 1$, so some multiple of f does not lie in U. This is a contradiction, since all multiples of f also lie in T_m. ∎

4. A simple application to representation theory

In this section, let $\mathbb{P} = \mathbb{P}^1(\mathbb{R})$. The points of $\mathbb{P}$ are the lines in $\mathbb{R}^2$, and may be expressed in homogeneous coordinates as $(\!(x, y)\!)$ where at least one of x, y does not vanish. The space $\mathbb{P}$ is the union of points where $y \ne 0$ and where $x \ne 0$. Each of these may be identified with $\mathbb{R}$, the first via $x \mapsto (\!(x, 1)\!)$, the second via $y \mapsto (\!(1, y)\!)$. On the overlap, where neither x nor y vanish, the coordinate change is $y = 1/x$. Let I_w be the subspace of $C^\infty(\mathbb{P})$ consisting of those functions that vanish of infinite order at ∞, and let I_1 be the space of formal series in $y = 1/x$. As an immediate consequence of Borel's theorem:

4.1. Proposition. *The sequence*

$$0 \to I_w \to C^\infty(\mathbb{P}) \to I_1 \to 0$$

is exact.

This is what I call the **Bruhat filtration** of $C^\infty(\mathbb{P})$, which is the space $C^\infty(\mathbb{P})$ of a smooth representation of $\mathrm{SL}_2(\mathbb{R})$. It is the analogue of a much simpler result for representations of p-adic groups induced from parabolic subgroups that gives rise immediately to a filtration of what is called the Jacquet module of such a representation. There are many similarities and differences between the real and p-adic cases—one important and curious one is that in the p-adic case the filtration is by components stable only with respect to parabolic subgroups, whereas here the components are representations of $\mathfrak{sl}_2$.

5. The extension theorems of Seeley and Mather

According to Proposition 3.1, there does not exist a continuous splitting of the map τ. The following result, found in [Seeley:1964], may therefore be a surprise. Define $C^\infty[0,\infty)$ to be the space of functions f on the open interval $(0,\infty)$ all of whose derivatives extend continuously on the closed interval $[0,\infty)$.

It follows easily from Borel's Lemma that the restriction map

$$\mathfrak{Res}\colon\ C^\infty(\mathbb{R}) \longrightarrow C^\infty[0,\infty)$$

is surjective.

5.1. Proposition. *There exists a continuous linear splitting of $\mathfrak{Res}$.*

This is the main result of [Seeley:1964].

The situation is definitely a bit paradoxical. This feeling is not dispelled by a result of [Mather:1966] that I'll now explain. Suppose we are given a continuous linear map φ from $C^\infty(\mathbb{R}^k)$ to $\mathbb{C}[[x_1,\dots,x_n]]$. Each coefficient defines a distribution on $\mathbb{R}^k$. We may therefore speak of the support of the map as the union of the supports of these distributions.

5.2. Proposition. *Suppose we are given a continuous linear map φ from $C^\infty(\mathbb{R}^k)$ to $\mathbb{C}[[x_1,\dots,x_n]]$, and suppose its support is contained in a compact subset of $\mathbb{R}^k$. Then there exists a continuous linear map Φ from $C^\infty(\mathbb{R}^k)$ to $C^\infty(\mathbb{R}^n)$ making the following diagram commute:*

$$\begin{tikzcd} & C^\infty(\mathbb{R}^k) \arrow[dl, dashed] \arrow[d, "\Phi"] \\ C^\infty(\mathbb{R}^n) \arrow[r, "\varphi"] & \mathbb{C}[[x_1,\dots,x_n]] \end{tikzcd}$$

To what extent does there exists a common generalization of Seeley's and Mather's theorems? A theorem of Whitney asserts that the restriction map from $C^\infty(\mathbb{R}^n)$ to $C^\infty(X)$ is surjective, where the second space is defined by local conditions on the arbitrary closed subset $X \subseteq \mathbb{R}^n$. For what sets X is there a continuous linear splitting? This is answered in [Bierstone:1980]:

5.3. Proposition. *If X is a closed subanalytic subset of $\mathbb{R}^n$ whose interior is dense in X, then the surjection $C^\infty(\mathbb{R}^n)$ to $C^\infty(X)$ has a continuous linear splitting.*

6. References

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4. R. Narasimhan, **Analysis on real and complex manifolds**, North-Holland, 1968.

5. Robert Seeley, 'Extensions of C^∞ functions defined in a half space', *Proceedings of the American Mathemetical Society* **15** (1964), 625–626.

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Part II. Whitney's differentiable functions

A result which is nearer to the result we need than Borel's theorem is a mild generalization. If f is in $\mathcal{S}(\mathbb{R}^n)$ then we can map it to its derivatives with respect to the variables $x_{m+1}, \ldots, x_n$ and then restrict to the embedded copy of $\mathbb{R}^m$ in $\mathbb{R}^n$ defined by the equations $x_{m+1} = 0, \ldots, x_n = 0$. This defines a map from $\mathcal{S}(\mathbb{R})^n$ to the space of formal power series with coefficients in $\mathcal{S}(\mathbb{R}^m)$. It is a simple generalization of Borels' result that with C_c^∞ instead of $\mathcal{S}$ this map is surjective. Combining this with the characterization of $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^m)$ in terms of smooth functions on spheres, and using a partition of unity on S^m, we can deduce that *the map on Schwartz spaces is also surjective*. If we define $\mathcal{S}(\mathbb{R}^n - \mathbb{R}^m)$ to be the kernel, in other words, we have an exact sequence

$$0 \longrightarrow \mathcal{S}(\mathbb{R}^n - \mathbb{R}^m) \longrightarrow \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^m)[[x_{m+1}, \ldots, x_n]] \longrightarrow 0 \,.$$

Roughly speaking, I am going to generalize these results to deal with various types of closed sets Y in smooth spaces X, for example closed semi-algebraic subsets of a real algebraic manifold. Eventually, I shall define the Schwartz space of any semi-algebraic manifold, and exhibit an exact sequence of Schwartz spaces and analogues of formal power series spaces when one of these is embedded in another. The key ideas of this material originated with [Whitney:1934], [Łojasiewicz:1959], and [Hironaka:1973]. The recent paper [Gourevitch-Aizenbud:2008] is a useful, but unfortunately not quite complete, summary.

Unfortunately, or perhaps fortunately, although final statements are relatively simple to formulate, I have not been able to resist discussing many technical results along the way. To encourage the reader, I can say that in my opinion the final results are vastly undervalued in representation theory. There have already been many pretty good applications—one of my own favourites is the analysis of Whittaker models for representations of real reductive groups in [Casselman-Hecht-Miličić:2000].

7. Introduction

Define the Schwartz space $\mathcal{S}[0,\infty)$ of the closed half-line $[0,\infty)$ to be the space of restrictions of functions in $\mathcal{S}(\mathbb{R})$ to $[0,\infty)$. This is a simple definition, but it is much better, and quite feasible, to have an equivalent characterization of these restrictions purely in terms of their behaviour on $[0,\infty)$.

Let's look at a somewhat simpler case first. Suppose $\varepsilon > 0$. For $m \ge 0$ one could define the space $C^m[0,\varepsilon)$ to be the space of restrictions of functions in $C^m(-\varepsilon,\varepsilon)$ to $[0,\varepsilon)$. But this is not entirely satisfactory, for many obvious reasons. One part of an intrinsic definition is that the function be in $C^m(0,\varepsilon)$. Making this completely explicit, this means that we are given a polynomial expression

$$\sum_{0 \le k \le m} \frac{f_k(x)}{k!} t^k$$

with the property that

$$f(y) = \sum_{0 \le k \le m} \frac{f_k(x)}{k!} (y-x)^k + o(|y-x|^m)$$

throughout $(0,\varepsilon)$. Further: for each $\ell \le m$ we have

$$f_\ell(y) = \sum_{\ell \le k \le m} \frac{f_k(x)}{k!} (y-x)^{k-\ell} + o(|y-x|^{m-\ell})$$

throughout. But this becomes the natural definition of $C^m[0,\varepsilon)$ if we simply allow x and y to lie in $[0,\varepsilon)$. Justification is that every such function is the restriction to $[0,\varepsilon)$ of a function in $C^m(-\varepsilon,\varepsilon)$. The space $C^\infty[0,\varepsilon)$ is as the intersection of all $C^m[0,\varepsilon)$.

Here is another way to put it. A function $f(x)$ defined in the neighbourhood of a 0 is said to be asymptotic to the power series

$$C(x) = \sum c_k x^k$$

if for all m

$$f(x) - \sum_{k \le m} c_k x^k = o(x^m)$$

locally. If this is so, then f is necessarily differentiable at 0. By induction, it therefore makes sense to call f **smoothly asymptotic** to the series if each derivative $f^{(n)}$ is asymptotic to $C^{(n)}(x)$.

The following is now an easy consequence of Borel's Theorem:

7.1. Proposition. *Suppose f to be a continuous function on $[0,\infty)$. It is the restriction to $[0,\infty)$ of a function in $\mathcal{S}(\mathbb{R})$ if and only if*

(a) it is smooth on the open half-line $(0,\infty)$;
(b) for all n, N

$$f^{(n)}(x) \underset{n,N}{\ll} \frac{1}{(1+x)^N}\,;$$

(c) near 0 the function f is smoothly asymptotic to a power series

$$\sum c_k x^k \,.$$

The multiplicative group open half-line $\mathbb{R}^\times = (0,\infty)$ embeds as a closed algebraic curve in $\mathbb{R}^2$:

$$x \mapsto (x^{-1}, x) \in \{(x,y) \,|\, xy = 1\}\,.$$

It inherits from $\mathbb{R}^2$ the norm

$$\|x\| = \sup |x|, 1/|x|\,.$$

The open half-line $\mathbb{R}^{>0} = (0,\infty)$ is its connected component.

Define the Schwartz space $\mathcal{S}(0,\infty)$ to be the space of all smooth functions f on $(0,\infty)$ such that

$$D^n f(x) \underset{f,n,N}{\ll} \|x\|^{-N}\,.$$

This definition is a special case of the definition of the Schwartz space of any semi-algebraic set. These spaces are an intrinsic part of a sets algebraic structure, as opposed to it analytic structure. For example, this space is ery different from that it inherits from $\mathcal{S}(\mathbb{R})$ via the exponential map.

This Schwartz space may also be identified with the functions in $\mathcal{S}[0,\infty)$ which vanish of infinite order at 0. By Borel's theorem, we have an exact sequence

$$0 \to \mathcal{S}(0,\infty) \to \mathcal{S}[0,\infty) \to \mathbb{C}[[x]] \to 0\,.$$

In other words, we have a natural filtration of $\mathcal{S}[0,\infty)$. It is stable under multiplication by positive real numbers. Similar spaces and filtrations are very useful in analysis on real algebraic varieties.

In higher dimensions, one wants to be able to speak of smooth functions defined on more complicated sets, and define for spaces of such functions certain filtrations, whose graded components are easy to work with. This is exactly what the definitions of [Whitney:1934] will allow you to do.

8. Whitney's extension theorem

In this section I shall sketch a generalization of Borel's theorem due to Hassler Whitney.

Let X be an open set in $\mathbb{R}^n$ and Y a closed subset of X. The space $\mathcal{J}(Y,X)$ of **continuous jets of infinite order** on Y (with respect to its embedding into $\mathbb{R}^n$) may be defined as the ring of formal power series

$$C(Y)[[t]] = C(Y)[[t_1, \ldots t_n]]$$

where $C(Y)$ is the ring of continuous functions on Y. For any jet F and k in $\mathbb{N}^n$ let $F^{[k]}$ be the product of $k!$ and the k-th coefficient of F, so that

$$F = \sum\nolimits_{k \in \mathbb{N}^n} F^{[k]} \, \frac{t^k}{k!}.$$

The space $\mathcal{J}(Y,X)$ may also be identified with that of continuous functions on Y with values in the formal power series ring $\mathbb{C}[[t]]$—to F and y is associated the series

$$\sum F^{[k]}(y) \, \frac{t^k}{k!} \ .$$

The space $\mathcal{J}(Y,X)$ becomes a Fréchet space with the semi-norms

$$\|F\|_{m,\Omega} := \sup\nolimits_{y \in \Omega, |k| \le m} \left| F^{[k]}(y) \right|$$

where m lies in $\mathbb{N}$ and Ω a compact subset of Y.

The motivation for these definitions comes from Taylor series. Let T be the canonical map from $C^\infty(X)$ to $\mathcal{J}(Y,X)$, which to every function f assigns its Taylor's series at y. Explicitly, for any smooth function f and y in Y

$$T_f^{[k]}(y) = f^{(k)}(y) \ .$$

These jets are mutually compatible in some obvious sense. Suppose $F = T_f$ for some smooth function f on X. Then for any x, y in X, hence in particular on Y, and for any $\ell \le m$

(8.1)
$$F^{[\ell]}(y) = \sum_{\ell \le k \le m} \frac{F^{[k]}(x)}{k!} (y-x)^{\ell - k} + o(\|y - x\|^{m-\ell} \ ,$$

uniformly on compact sets.

Conversely, define a jet on Y to be **coherent**, or a **Whitney jet** (also called in the literature a C^∞ function on Y **in the sense of Whitney**), if it satisfies niformly on compact subsets of Y for all m. The peculiarity of this definition is that it does not depend only on the set Y, but very definitely on its embedding in X. For example, if Y is a single point then the corresponding space of functions is isomorphic to a formal power series ring. Define

$$\mathcal{W}(Y,X) := \text{ the space of all coherent jets on } Y.$$

This becomes a Fréchet space if we define on it the semi-norms

$$|||F|||_{m,\Omega} := \|F\|_{m,\Omega} + \sup\nolimits_{x,y \in \Omega, |k| \le m} \left| (R_y^m F)^{[k]}(x) \right| / |x - y|^{m - |k| + 1}.$$

The space $\mathcal{W}(X,X)$ is the same as $C^\infty(X)$, and Whitney jets on $\mathbb{R}^m$ embedded in $\mathbb{R}^n$ may be identified with formal power series in $x_{m+1}, , \ldots, x_n$ whose coefficients lie in $C^\infty(\mathbb{R}^m)$. Borel's Theorem asserts that every formal series at the origin in $\mathbb{R}^n$ is the Taylor series of some smooth function. A result of Whitney, generalizing this, asserts that *every Whitney jet arises as the image under the map J of some smooth function on X*. I will reformulate this slightly.

Define the **Schwartz space** of the open set $X - Y$ (again, with respect to its embedding in X) to be

$$\mathcal{S}(X - Y, X) := \text{ the kernel of the map } J$$

the ideal in $C^\infty(X)$ of those functions whose Taylor's series vanish identically along Y. The terminology I use is not conventional, but justified—I hope—by the emphasis to be placed on these as functions on $X - Y$. Whitney's theorem has then this formulation:

8.2. Proposition. *The sequence of canonical maps*

$$0 \longrightarrow \mathcal{S}(X - Y, X) \longrightarrow C^\infty(X) \xrightarrow{T} \mathcal{W}(Y, X) \longrightarrow 0$$

is exact.

See §I.3 of [Malgrange:1966].

More generally, if Z is in turn a closed subset of Y I define the space of **Schwartz jets** $\mathcal{S}(Y - Z, X)$ on $Y - Z$ to be the subspace of jets in $\mathcal{W}(Y, X)$ vanishing on Z.

8.3. Corollary. *We have the canonical exact sequence*

$$0 \longrightarrow \mathcal{S}(Y - Z, X) \longrightarrow \mathcal{W}(Y, X) \longrightarrow \mathcal{W}(Z, X) \longrightarrow 0 .$$

8.4. Proposition. *The subspace of smooth functions on X which vanish in the neighbourhood of Y is dense in $\mathcal{S}(X - Y, X)$.*

This is the natural generalization of the fact that linear combinations of the Dirac delta δ_0 and its derivatives are the only distributions on $\mathbb{R}^n$ with support at 0. It is demonstrated in the course of the proof of Lemma I.4.3 in [Malgrange:1966].

9. Local growth conditions

I call a smooth function on $X - Y$ **rapidly decreasing along** Y if it vanishes locally more rapidly than any power of $d(x, Y)$, and **smoothly** rapidly decreasing if this is true of all its derivatives as well. I call it of **moderate growth** along Y if it is locally of moderate growth relative to $d(x, Y)$. I call it of smooth moderate growth if this is true for each of its derivatives as well. For the following, I refer to IV.1 of [Malgrange:1966] and IV.4 of [Tougeron:1972].

9.1. Proposition. *The ideal $\mathcal{S}(X - Y, X)$ is the same as the space of all smooth functions on $X - Y$ smoothly rapidly decreasing along Y, and the topology on $\mathcal{S}(X - Y, X)$ induced from $C^\infty(X)$ is the same as that determined by the semi-norms*

$$\sup_{x \in \Omega} |D^k f(x)| / d(x, Y)^N .$$

Proof. Suppose f to lie in $\mathcal{S}(X - Y, X)$. Let Ω be any compact subset of X. If x lies in Ω, then $d(x, Y)$ is bounded by $\delta = d(x, \Omega \cap Y)$. There exists a line segment from x to a point y in $\Omega \cap Y$ of length δ with all but the endpoint in $X - Y$. Since

$$f(x) = \frac{(-1)^k}{(k-1)!} \int_0^1 t^{k-1} (d/dt)^k f(x + t(y - x))\, dt$$

for any positive integer k, there exists C good for all x in Ω such that

$$|f(x)| \le C\, |y - x|^k \sum_{|i| \le k} \sup |D^i f| ,$$

so that f is rapidly decreasing along Y.

Conversely, suppose f to be a function on $X - Y$ smoothly rapidly decreasing along Y. It is then straightforward to see that the extension of f then it is clear that the extension of f to all of X obtained by setting it equal to 0 on Y is smooth and has vanishing Taylor's series along Y. The agreement of topologies is equally straightforward. ▯

It may be of some interest that, at least in some circumstances, if f is of smooth moderate growth along Y and also rapidly decreasing along Y then it is automatically smoothly rapidly decreasing along Y. This follows from an elementary modification of the proof of Landau's Lemma on p. 40 of [Duistermaat:1974].

9.2. Proposition. *If f is of smooth moderate growth and $1/f$ is of moderate growth along Y, then $1/f$ is of smooth moderate growth along Y.*

Proof. Elementary calculus. ▯

As we'll see later, stereographic projection identifies the Schwartz space of $\mathbb{R}^n$ with the smooth functions on $\mathbb{S}^n$ that vanish at a pole. This observation is true because the vector fields $\partial/\partial x_i$, when expressed in coordinates near the pole, are linear expressions of partial derivatives with coefficients of smooth moderate growth. This argument generalizes. Suppose that smooth vector fields $V_1, \ldots, V_n$ are defined on $X - Y$, which at every point of $X - Y$ form a non-degenerate frame. For any $m \in \mathbb{N}^n$ define the differential operator

$$V^m := V^{m_1} \ldots V^{m_n}.$$

9.3. Proposition. *Suppose that the coefficients of the V_i in terms of the fields $\partial/\partial x_i$ are of smooth moderate growth along Y. Then $\mathcal{S}(X - Y, X)$ consists of those smooth functions F on $X - Y$ with all the $V^m F$ rapidly decreasing along Y.*

As a mild generalization of the observation of Schwartz it is not difficult to see that the Schwartz jets along $\mathbb{R}^m$ embedded in $\mathbb{R}^n$ and embedded in turn in the n-sphere may be identified with formal series in $x_{m+1}, \ldots x_n$ with coefficients in $\mathcal{S}(\mathbb{R}^m)$. This generalizes to a result concerning the Schwartz jets $\mathcal{S}(Y, X)$ when Y is singular, but is a bit more difficult to formulate since the notion of transverse derivatives is trickier. In that direction:

The set Y is called **regular** *if (a) Y is locally connected and (b) locally on each component, for some $p > 0$, any two points x, y in Y can be connected by a path in Y of length bounded by $O(d(x, y)^p)$.*

The topology defined above on $\mathcal{W}(Y, X)$ is general distinct from that induced by that on $\mathcal{J}(Y, X)$. However, from IV.3.10–3.12 of [Tougeron:1972]:

9.4. Proposition. *If Y is regular then these topologies agree.*

Two closed subsets Y and Z are said to be **regularly situated** if locally on X for some $p, C > 0$

$$d(x, Y) + d(x, Z) \geq C\, d(x, Y \cap Z)^p,$$

or equivalently

$$d(y, Z) \geq C\, d(y, Y \cap Z)^p$$

on Y. This is elementary:

9.5. Proposition. *If $Y \cup Z$ is regular then Y and Z are regularly situated.*

Two sets that are not regularly situated are the graph of $y = e^{-1/|x|}$ and the x-axis in $\mathbb{R}^2$. They are too close together near the origin.

The following is deeper, and due to [Łojasiewicz:1959] (see I.5.5 of [Malgrange:1966] for a short proof):

9.6. Proposition. *In order for closed subsets Y and Z of X to be regularly situated it is necessary and sufficient that the canonical sequence*

$$0 \longrightarrow \mathcal{W}(Y \cup Z, X) \longrightarrow \mathcal{W}(Y, X) \oplus \mathcal{W}(Z, X) \longrightarrow \mathcal{W}(Y \cap Z, X) \longrightarrow 0$$

be exact.

The point is exactness in the middle.

Suppose now that X is any smooth manifold (automatically assumed Hausdorff and σ-compact). For any point x of X let $\mathfrak{R}_x$ be the ring of germs of smooth functions in the neighbourhood of x, $\mathfrak{M}_x$ the maximal ideal in $\mathfrak{R}_x$ of functions vanishing at x. The space of jets at a point x of X is invariantly characterized as the projective limit of the quotients $\mathfrak{R}_x/\mathfrak{M}_x^n$. The spaces $\mathcal{J}(Y, X)$, $\mathcal{W}(Y, X)$, $\mathcal{S}(X - Y, X)$ as well as the map J from $C^\infty(X)$ to $\mathcal{W}(Y, X)$ may all be defined by local specifications on coordinate patches—in other words, they are spaces of sections of sheaves on X, and even sheaves which are fine. One may define Y to be regular if it is regular locally, and call Y and Z regularly situated in X if they are locally. All of the above results then remain valid even without rewording.

Assume that X is an open subspace of the differentiable manifold M. The Schwartz space $\mathcal{S}(X, M)$ may be identified with the space of all smooth functions on X vanishing locally on $M - X$, along with all derivatives, more rapidly than any power of $d(x, M - X)$. It becomes a nuclear Fréchet space if given the local norms

$$\sup |D^k f(x)|/d(x, M - X)^N .$$

The definition of $\mathcal{S}(X, M)$ certainly depends on the particular embedding of X in M, as opposed to the intrinsic structure of the differentiable manifold X. This may be obvious but as a simple example suppose that $X = \mathbb{R}$. If X is embedded in the circle obtained from stereographic projection in $\mathbb{R}^2$ then the Schwartz space is the usual one, but if mapped *via* the exponential map onto $(0, \infty)$ followed by stereographic projection then the Schwartz space is quite different.

Continue to assume X open in M, and let Y be closed in X. I recall that the Schwartz jets $\mathcal{S}(Y, M)$ are those Whitney jets on $\overline{Y}$ which vanish on $\partial Y = \overline{Y} - Y$. With respect to the map $J\colon C^\infty(M) \to \mathcal{W}(\overline{Y}, M)$ any element in $\mathcal{S}(X, M)$ has as image an element of $\mathcal{S}(Y, M)$. The kernel of this map from $\mathcal{S}(X, M)$ to $\mathcal{S}(Y, M)$ may clearly be identified with $\mathcal{S}(X - Y, M)$.

9.7. Proposition. *If $\overline{Y}$ and $M - X$ are regularly situated then $\mathcal{S}(Y, M)$ may be identified with the space of elements of $\mathcal{W}(\overline{Y} \cup (M - X), M)$ vanishing on $M - X$, and this canonical sequence is exact:*

$$0 \longrightarrow \mathcal{S}(X - Y, M) \longrightarrow \mathcal{S}(X, M) \longrightarrow \mathcal{S}(Y, M) \longrightarrow 0 .$$

Proof. I call temporarily $\mathcal{T}$ the space of elements of $\mathcal{W}(\overline{Y} \cup (M - X), M)$ vanishing on $M - X$. By definition we have an exact sequence

$$0 \longrightarrow \mathcal{T} \longrightarrow \mathcal{W}(\overline{Y} \cup (M - X), M) \longrightarrow \mathcal{W}(M - X, M) \longrightarrow 0$$

and from a previosu result we also have a short exact sequence

$$0 \longrightarrow \mathcal{S}(X - Y, M) \longrightarrow \mathcal{S}(X, M) \longrightarrow \mathcal{T} \longrightarrow 0 .$$

If $\overline{Y}$ and $M - X$ are regularly situated then the identification of $\mathcal{T}$ and $\mathcal{S}(Y, X, M)$ follows. ▮

If X_1 and X_2 are two open subsets of M, then the inclusions induce a map from the direct sum of their Schwartz spaces into the Schwartz space of their union. The following is then simply a reformulation of what we have seen before:

9.8. Proposition. *Let X_1 and X_2 be open subsets of M, $X = X_1 \cup X_2$, and assume that the complements of X_1 and X_2 are regularly situated. Then the space $\mathcal{S}(X)$ is the sum of the images of $\mathcal{S}(X_1)$ and $\mathcal{S}(X_2)$, or equivalently this canonical sequence is exact:*

$$0 \longrightarrow \mathcal{S}(X_1 \cap X_2) \longrightarrow \mathcal{S}(X_1) \oplus \mathcal{S}(X_2) \longrightarrow \mathcal{S}(X) \longrightarrow 0\ .$$

The space of **tempered distributions** $\mathcal{D}(X, M)$ on X (relative to the given embedding into M) is defined to be the strong dual of $\mathcal{S}(X, M)$. By Hahn-Banach, it is the same as the space of distributions on X which may be extended to all of M. For the following, see Prop. IV.1.4 of [Malgrange:1966].

9.9. Proposition. *If f is a smooth function on X of smooth moderate growth along the complement $M - X$ then multiplication by f induces continuous maps from $\mathcal{S}(X, M)$ and $\mathcal{D}(X, M)$ to themselves.*

10. Invariance of Schwartz spaces

There are simple prototypes for the two basic results we shall need. The first is the well known observation found in [Schwartz:1966] (Théorème II, pp. 235–236). The space $\mathcal{S}(\mathbb{R}^n)$ is defined to be that of all smooth functions on $\mathbb{R}^n$ all of whose derivatives decay at infinity more rapidly than any inverse power of $r^2 = x_1^2 + \cdots + x_n^2$. But $\mathbb{R}^n$ may be embedded by means of stereographic projection into the n-sphere S^n in $\mathbb{R}^{n+1}$, and Schwartz' result is this:

10.1. Proposition. *By mean of stereographic projection, the Schwartz space of $\mathbb{R}^n$ may be identified with the space of all smooth functions on $\mathbb{S}^n$ whose Taylor's series vanishes at the north pole.*

The proof is straightforward. We can choose as coordinates at infinity

$$t_i = \frac{x_i}{\sum x_j^2}, \quad x_i = \frac{t_i}{\sum t_j^2}$$

and calculate that

$$\begin{aligned}
\frac{\partial}{\partial t_i} &= \sum_j \frac{\partial x_j}{\partial t_i} \frac{\partial}{\partial x_j} \\
\frac{\partial x_j}{\partial t_i} &= \frac{\left(\sum_k t_k^2\right) - 2t_i^2}{\left(\sum_k t_k^2\right)^2} \quad (i = j) \\
&= \left(\sum_k x_k^2\right) - 2x_i^2 \quad (i = j) \\
&= \frac{-2t_i t_j}{\left(\sum_k t_k^2\right)^2} \quad (i != j) \\
&= -2x_i x_j\ .
\end{aligned}$$

This allows us to prove by inductions that for any multi-index α

$$\partial^\alpha / \partial t^\alpha = \sum_\beta P^\alpha_\beta(x) \frac{\partial^\beta}{\partial x^\beta}$$

where the functions P^α_β are polynomials in x. From this the Proposition follows immediately.

A second prototype is the classical theorem of E. Borel, which in this case asserts that the Taylor's series map at ∞ on S^n, taking smooth functions to formal power series in the t_i, is surjective. Combining these two observations we have an exact sequence

$$0 \longrightarrow \mathcal{S}(\mathbb{R}^n) \longrightarrow C^\infty(\mathbb{S}^n) \longrightarrow \mathbb{C}[[x_1, x_2, \ldots, x_n]] \longrightarrow 0 .$$

Now suppose M to be a real-analytic manifold. A subset of M is called **semi-analytic** if it is defined locally by a finite number of analytic equalities and inequalities. The union and intersection of a finite family of semi-analytic subsets is again semi-analytic, as is the difference of any two, the components, and the boundary of a semi-analytic subset (see [Łojasiewicz:1959] or for some cases [Malgrange:1966]). A subset of M is called **subanalytic** if locally on M it is the image of a semi-analytic subset with respect to a proper analytic map. At the beginning of [Hironaka1973b:] it is shown that such a set need not be semi-analytic. The family of subanalytic subsets is again closed under elementary operations. A third family of subsets may be defined when M is algebraic (i.e. covered by coordinate patches with rational coordinate transformations): a subset in this case is called **semi-algebraic** if defined by polynomial equalities and inequalities. These have the convenient property that the image of a semi-algebraic subset under an algebraic set is again semi-algebraic (see [Łojasiewicz:1959] for a geometric proof of this result due to Seidenberg), so that there is no need to extend this definition in turn. Łojasiewicz proved that every semi-analytic subset is regular, and this has been generalized:

10.2. Proposition. *Every subanalytic subset is regular.*

I do not know to whom this is due, but the outline of a proof can be found in §6 of the article [Hardt:1983], and an almost equivalent result in Theorem 6.17 of [Bierstone:1980] (Bierstone has also explained to me a more elementary line of reasoning than seems to be in the literature).

Along the same lines is Łojasiewicz' Inequality, from Theorem IV.3.1 of [Malgrange:1966]:

10.3. Proposition. *If f is an analytic function then $1/f$ is of moderate growth along the zero set of f.*

10.4. Proposition. *If X is embedded into two manifolds M_1 and M_2 and Φ is a proper analytic map from M_1 to M_2 inducing a diffeomorphism of the images of X, then Φ induces an isomorphism between the Schwartz spaces $\mathcal{S}(X, M_1)$ and $\mathcal{S}(X, M_2)$.*

Proof. [work in progress] .

Now let X be the set of $\mathbb{R}$-valued points on a non-singular algebraic variety defined over $\mathbb{R}$. According to the well known result of [Nagata:1962] it may be embedded into some complete algebraic variety $\overline{X}$ also defined over $\mathbb{R}$. [du Cloux:1990] points out that it may even be embedded into affine space. By [Hironaka:1964] one may assume $\overline{X}$ to be non-singular. Define the Schwartz space $\mathcal{S}(X)$ of X to be $\mathcal{S}(X, \overline{X})$. Since any two choices along the way are dominated by a third, this space is by ctually independent of the choices made, and this space is defined intrinsically by the algebraic structure on X.

If V is an algebraic vector bundle defined over the real algebraic manifold X then one may define as well in an analogous fashion the space of **Schwartz sections** $\Gamma_{\mathcal{S}}(X, V)$ of V over X. If Y is a non-singular closed algebraic submanifold of X then the conormal bundle along Y will be an algebraic vector bundle $\Omega_{Y,X}$. If $\mathcal{I}_Y$ is the ideal sheaf of smooth functions vanishing along Y then locally $\mathcal{I}_Y/\mathcal{I}_Y^2$ may be identified with $\Omega_{Y,X}$, and more generally $\mathcal{I}_Y^m/\mathcal{I}_Y^{m+1}$ may be identified with the symmetric tensor product $S^m(\Omega_{Y,X})$. Define $\mathcal{S}_m(Y, X)$ in two ways: $\mathcal{I}_Y^m\mathcal{S}(Y, X)$ or the functions in $\mathcal{S}(X)$ vanishing of order m along Y—i.e. those belonging everywhere locally to $\mathcal{I}_Y^m$. Then $\mathcal{S}_m(Y, X)$ may be identified also with $\mathcal{I}_Y^m\mathcal{S}(X)$.

10.5. Proposition. *The quotient $\mathcal{I}^m\mathcal{S}(X)/\mathcal{I}^{m+1}\mathcal{S}(X)$ may be identified with the space of Schwartz sections of $S^m(\Omega_{Y,X})$, and $\mathcal{S}(Y, X)$ with the projective limit of the quotients $\mathcal{S}(X)/\mathcal{I}^m\mathcal{S}(X)$.*

Proof. [work in progress].

10.6. Proposition. *If G is a real affine algebraic group then $\mathcal{S}(X)$ is the space of all smooth functions on X all of whose right derivatives $R_X F$ $(X \in U(\mathfrak{g}))$ vanish rapidly at infinity.*

Proof. This is because with respect to any non-singular completion of G the vector fields defined by elements of $\mathfrak{g}$ are meromorphic. ∎

The Schwartz space of $\mathbb{R}$ in this sense is just the usual one. The Schwartz space of the multiplicative group $\mathbb{R}^\times$ consists of all functions in the Schwartz space of $\mathbb{R}$ which vanish of infinite order at 0. The Schwartz space of any unipotent group is, in an obvious way, isomorphic to the Schwartz space of Euclidean space of the same dimension.

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