Symmetric power decompositions for GL(2)

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Let $G = \mathrm{GL}_2(\mathbb{C})$, let σ be the standard representation of G on $\mathbb{C}^2$, and let σ_k be the irreducible representation of G on the space $V_k = S^k(\mathbb{C}^2)$, which is of dimension $k+1$. Thus $\sigma = \sigma_1$. The representation σ_k is the dual of that on polynomials of degree k on $\mathbb{C}^2$.

Let $\boldsymbol{S}_k^m$ be the corresponding representation on the space $S^m(V_k)$ of symmetric tensors of degree m. It has dimension $\binom{m+k}{k}$, since specifying a monomial of degree m in $k+1$ variables amounts to choosing the location of k separators in an array of length $m+k$. It is easy to verify that it is irreducible if and only if $m \le 1$ or $k \le 1$. The following question, in various forms, was asked already in the nineteenth century:

What is the decomposition of $\boldsymbol{S}_k^m$ into irreducible components?

If $k = 1$ these representations are the σ_m. The answer for $k = 2$ is straightforward to come up with, as we shall see momentarily. In general, there are several answers, compatible with each other but each with its own charm. First of all, there is a simple—I should say, deceptively simple—formula for the decomposition that first appeared in [Cauchy:1843]. For him it amounted to an exercise in calculating the Taylor series of a certain rational function that occurred naturally in the theory of partitions. This formula is excellent for computation in any one case, but there is not much structure to it. In this paper, there will be several stages in laying out other solutions. First I'll recall results of [Sturmfels:1995] relevant to decomposing symmetric powers of irreducible representations of arbitrary reductive groups. Next I'll discuss how this applies to the group GL_2, including some examples. Then I'll prove Cauchy's formula and work out some of its consequences. After that, I'll recall some results due to the nineteeth century mathematicians Cayley and Sylvester in the theory of denumerants, and conclude the main part of the paper with a statement and proof of the main theorem and some of its consequences. I have added on an appendix describing the naive way to calculate denumerants, just to demystify the topic somewhat.

My original motive in taking this topic up arose from some remarks in [Langlands:2013], in which he suggests there might be some eventual applications to the theory of automorphic forms. Some information about this can be found in [Casselman:2017], the last sections of which present an initial state of some material found here. Ideally, this would lead to an interpretation of some of the more peculiar features of the decomposition of the $\boldsymbol{S}_k^m$ in terms of modules over Hecke algebras. But I'll not discuss this topic in this paper, because there are still some important unanswered questions.

As far as I know, there is no comprehensive generalization of these formulas to reductive groups other than GL_2, although there is some evidence that one exists. Considering that much of what I'll say in this paper is classical and elementary, I can perhaps justify this note only by expressing my hope that it will lead to a better understanding of the general case. Computer computations regarding this are suggestive, but I'll say nothing about that in this paper.

♡ [sigma3-delta] The decomposition of the representations $\boldsymbol{S}_3^m$, agreeing with Theorem 2.4 in this paper, is given in Theorem 1.3 of [Hahn et al.:2017]. There is also a partial result about $\boldsymbol{S}_4^m$ in their Theorem 3.4. Their methods and even their formulations are very different from mine.

I began writing this paper at an AIM workshop in December, 2015 at which some related questions were raised. More stimulus came at a short workshop at the Institute in Princeton in September, 2016. I am grateful to both institutions for their hospitality.

Contents

1. Symmetric powers [GL2-symm.tex]

The decomposition of symmetric powers of representations is part of the theory of vector partitions. The most relevant result in this subject can be found in [Sturmfels:1995], which I recall.

Suppose A to be a $k \times \ell$ matrix with non-negative integral entries. Let A_i be its i-th column. It will do no harm to assume that the lattice spanned by these columns is the same as $\mathbb{Z}^k$. Then A defines a map from $\mathbb{N}^\ell$ to $\mathbb{N}^k$, taking (x_i) to $Ax = \sum_{i=1}^{\ell} x_i A_i$. I assume also A has no null columns. The associated partition function on $\mathbb{N}^k$ is

$$\varphi_A(u) = \left|\left\{ v \in \mathbb{N}^\ell \,\middle|\, Av = u \right\}\right| .$$

Under my assumptions, this will always be finite. The rational generating function of φ_A can be found easily, and is well known:

[moliens] **1.1. Lemma.** (Molien's formula) *We have*

$$\frac{1}{\prod_{i=1}^{\ell}(1 - q^{A_i})} = \sum\nolimits_{\mathbb{N}^k} \varphi_A(u) q^u .$$

Here I use multi-indices, so if $u = (u_i)$ then $q^u = q_1^{u_1} \dots q_k^{u_k}$.

Example. Suppose

[all-ones] **(1.2)**
$$A = [\, 1 \quad 1 \quad 1 \quad \dots \quad 1 \,] .$$

Then $\varphi_A(m)$ is the number of lattice points in the simplex

$$\left\{ (x_i) \,\middle|\, x_i \geq 0, \ \sum x_i = m \right\} .$$

This is the same as the number of monomials $\prod_{i=1}^{\ell} u_i^{m_i}$ of degree m, which is equal to $\binom{m + \ell - 1}{\ell - 1}$. The generating function is

$$\frac{1}{(1-q)^\ell} \quad (q = x_1) .$$

Example. If

[matA] **(1.3)**
$$A = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 2 & \dots & k \end{bmatrix}$$

the generating function is

$$\frac{1}{(1-t)(1-qt)(1-q^2t)\dots(1-q^kt)} \quad (t = x_1, q = x_2) .$$

We'll see this example again later on.

The matrix A determines a partition of the real cone $\mathcal{C}_A$ spanned by its columns. A subset I of columns of A is called **non-degenerate** if the real cone $\mathcal{C}_I$ spanned by the columns in I has non-empty interior. Call a subset I of $[1, \ell]$ **basic** if it is non-degenerate and of minimal size k. In this case, $\mathcal{C}_I$ is a simplicial cone. A **chamber** in $\mathbb{R}^k$ is the closure of a connected component of the complement of the boundaries of the cones $\mathcal{C}_I$ for I basic. It is a real cone, and the intersection of all $\mathcal{C}_I$ containing it.

♡♡[a][cones] For one example, if A is (1.2) then the only chamber is all of $[0, \infty)$. For another, if A is the matrix in (1.3) then the chambers are the bands

$$C_j = \{(m, i) \,|\, mj \le i \le m(j+1)\} \quad (0 \le j \le k-1).$$

One naturally asks,

what sort of a function is φ_A?

Suppose f to be a function on a lattice $L \subset \mathbb{R}^k$ It is said to be a **quasi-polynomial** function if there exists $N > 0$ with the property that the restriction of f to every congruence class $\{\lambda + NL\}$ is a polynomial function. A function on L is periodic if it factors through some L/NL. If μ_N is the group of N-th roots of unity and ζ is in $\mathrm{Hom}(L, \mu_N)$ then the function $\lambda \mapsto \zeta^\lambda$ is periodic, and Fourier analysis on L/NL tells us that every function of period L/NL is a linear combination of these. This notation is motivated by the observation that if $L = \mathbb{Z}^k$ then $\mathrm{Hom}(L, \mu_N)$ is in bijection with $(\mu_N)^k$, and the corresponding map takes

$$(n_i) \longmapsto \prod \zeta_i^{n_i}.$$

Any function $\zeta^\lambda f(\lambda)$ with f a polynomial is quasi-polynomial.

[quasi-poly-ell] **1.4. Lemma.** *Suppose f to be a function on the lattice L. The following are equivalent:*

(a) the function f is quasi-polynomial;
(b) it is the sum of products of periodic functions and polynomials;
(c) it is the sum of functions $\zeta^\lambda g(\lambda)$ in which g is a polynomial.

Proof. Proof left as exercise.

Also left as exercise:

[qp-props] **1.5. Lemma.** *If f is quasi-polynomial on L then:*

(a) so is $f(\lambda - \mu)$ for any μ in L;
(b) so is the restriction of f to any subgroup of L;
(c) the function f can be expressed uniquely as

$$f(\lambda) = f_1(\lambda) + \sum\nolimits_{\zeta \ne 1} \zeta^\lambda f_\zeta(\lambda), ,$$

with each f_μ a polynomial;
(d) the function f is completely determined by its restriction to the intersection of L with any open cone.

The function f_1 in (c) is called the **polynomial part** of f, the complement the **congruence part**.

The following is a very weak version of the main result of [Sturmfels:1995], but it will suffice for my needs.

[sturmfels] **1.6. Theorem.** (Sturmfels) *The restriction of φ_A to every chamber C is quasi-polynomial.*

The proof applies the algebraic geometry of toric varieties.

There is a simple geometric interpretation of the polynomial part of φ_A—it represents the volume of the inverse images with respect to φ_A in $\mathcal{C}_A$. This is also related to the distribution of lattice points, as is illustrated easily in the first example above, since

$$\binom{m + \ell - 1}{\ell - 1} \sim \frac{m^{\ell-1}}{(\ell-1)!},$$

which is the volume of a real simplex with ℓ vertices.

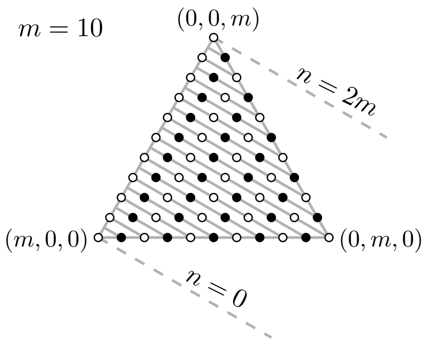
For example, suppose

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}.$$

The map takes

$$(x_1, x_2, x_3) \longmapsto (x_1 + x_2 + x_3, x_2 + 2x_3).$$

In this case there are two chambers in $\mathbb{R}^2$, one where $0 \le n \le m$ and the other where $m \le n \le 2m$. The following figure shows the plane where $x_1 + x_2 + x_3 = m$, and illustrates that the inverse image of (m, n) is a line.



2-images/new-slice-0.eps

It is easy enough to deduce that

$$\varphi_A(m, n) = \begin{cases} (n/2 + 1) - (n/2 - \lfloor n/2 \rfloor) & \text{if } 0 \le n \le m \\ (m - n/2 + 1) - (n/2 - \lfloor n/2 \rfloor) & m \le n \le 2m. \end{cases}$$

Here the polynomial and the congruence parts are indicated by grouping. Note that $n/2 - \lfloor n/2 \rfloor = 0$ if n is even, $1/2$ if n is odd, in accordance with Theorem 1.6.

♡ [sturmfels]

What does this have to do with representation theory? From now on in this section, let

$$\begin{aligned} G &= \text{ arbitrary complex reductive group} \\ T &= \text{ diagonal matrices} \\ N &= \text{ upper trianglar unipotent matrices} \\ B &= \text{ Borel subgroup} \\ &= TN \\ W &= \text{ associated Weyl group.} \end{aligned}$$

Recall that $X_*(T)$ is the lattice of algebraic homomorphisms from $\mathbb{G}_m$ to T, $X^*(T)$ that of algebraic homomorphisms from T to $\mathbb{G}_m$. Let

$$L = \mathrm{Hom}(X_*(T), \mathbb{Z})\,.$$

There is a map from L to $X^*(T)$, taking λ to the multiplicative character e^λ such that

$$e^\lambda(\mu^\vee(x)) = x^{\langle \lambda, \mu^\vee \rangle}$$

for all $\mu^\vee$ in $X_*(T)$. It is an isomorphism, and usually the two are identified, but I wish to use additive notation for L and multiplicative for $X^*(T)$. Thus $e^{\lambda+\mu} = e^\lambda e^\mu$.

Let

$$\begin{aligned} \Delta &= \text{simple roots in } L \\ L_\Delta &= \text{lattice spanned by } \Delta \\ L_\Delta^+ &= \text{cone spanned by } \Delta \\ \Lambda &= \text{dominant weights}\,. \end{aligned}$$

For λ in Λ, let $\pi = \pi_\lambda$ be the irreducible representation of G with highest weight λ, say of dimension d, and let (e_i) be a basis of weight vectors with respect to T, say with weight ε_i. Then ε_i will lie in $\lambda - L_\Delta^+$ and also in the convex hull of the W-orbit of λ. So we may write $\varepsilon_i = \lambda - \lambda_i$, with λ_i in L_Δ^+. Normally, λ_i will be expressed as a linear combination of the e^α for α in Δ.

The weight vectors of the symmetric power $S^m(\pi)$ are then the

$$e_1^{m_1} \dots e_d^{m_d}$$

with $\sum m_i = m$. The corresponding weight will be

[wt-sm] (1.7)

$$m\lambda - \sum_i m_i \lambda_i\,.$$

The multiplicity of the weight $m\lambda - \mu$ in $S^m(\pi)$ is therefore

$$\omega_\lambda(m,\mu) = \left|\left\{ (m_i) \in \mathbb{N}^d \,\middle|\, \sum m_i = m,\ \sum m_i\lambda_i = \mu \right\}\right|$$

In other words, expressing each λ_i as a linear combination of α in Δ:

[sturmfels-wts] 1.8. Lemma. *If $\pi = \pi_\lambda$ and*

$$A = \begin{bmatrix} 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \dots & \lambda_d \end{bmatrix}$$

then $\varphi_A(m,\mu)$ is the multiplicity of $m\lambda - \mu$ in $S^m(\pi)$.

Sturmfels' theorem therefore says that the functions ω_λ will be quasi-polynomial on certain chambers. I have not attempted to figure out what these are for arbitrary reductive groups, but as we shall see it is easy to specify exactly what they are for GL_2.

In possible applications to the representation theory of $\mathfrak{p}$-adic reductive groups and automorphic forms, the partitions of Sturmfels are presumably related to the geometry of certain monoids defined by Vinberg.

The function ω_λ is intimately related to the irreducible decomposition of $\boldsymbol{S}^m_\lambda$. I'll express this in notation convenient later on.

I define the **weight polynomial** of $\sigma = S^m(\pi)$ to be

$$\omega_\sigma = \sum m_{m\lambda-\mu} e^\mu$$

if m_ν is the multiplicity of ν in $S^m(\pi)$. It is of the form $e^{m\lambda}P$ where P is a polynomial in the variables e^α for α in Δ. This polynomial can be immediately derived from Sturmfels' function $\varphi_A(m,\mu)$.

I define the **decomposition polynomial** to be

$$\overline{\delta}_\sigma = \sum\nolimits_\mu c_{m\lambda-\mu} e^\mu$$

if $S^m(\pi) = \sum c_{m\lambda-\mu}\pi_{m\lambda-\mu}$.

Let ρ be one-half the sum of positive roots. Thus $\rho - w\rho$ is always a non-negative sum of simple roots.

[botts-mult] **1.9. Lemma.** *In these circumstances, let*

$$\delta_{S^m(\pi)} = \Big(\sum_W (-1)^{\ell(w)} e^{\rho-w\rho}\Big)\omega_{S^m(\pi)}\,.$$

Then $\overline{\delta}_{S^m(\pi)}$ is the same as $\delta_{S^m(\pi)}$ truncated outside Λ.

Proof. This is an immediate consequence of one of the standard forms of Weyl's character formula.

Example. Let $G = \mathrm{GL}_2$. There is one simple multiplicative root α. If

$$g = \begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix}$$

then its trace on σ_k is

$$p^k + p^{k-1}q + \cdots + q^k = p^k\big(1 + (p/q)^{-1} + \cdots + (p/q)^{-k}\big)$$

so the weight polynomial of σ_k is

$$1 + \alpha^{-1} + \cdots + \alpha^{-k} = \frac{1-\alpha^{-(k+1)}}{1-\alpha^{-1}}\,.$$

From now on G will always be GL_2. What does Sturmfels' result say in this case? Recall that $\boldsymbol{S}^m_k$ is the representation on $S^m(\sigma_k)$. We want to find a formula for the decomposition of $\boldsymbol{S}^m_k$ into irreducible
[botts-mult] [sturmfels] representations, following the path suggested by Theorem 1.6 and Lemma 1.9.

Fix the vectors

$$u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad v = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

and again let

$$g = \begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix}.$$

Thus

$$gu = pu, \quad gv = qv\,.$$

The eigenvectors of $\sigma_k(g)$ are the symmetric tensors $e_{k,i} = u^{k-i}v^i$ with $0 \le i \le k$, with eigenvalues $p^{k-i}q^i$.

[irredecomp] **1.10. Lemma.** *All irreducible constituents of $\mathbf{S}_k^m$ are of the form $\sigma_{km-2i}\cdot\det^i$ with $0 \le i \le km/2$.*

Proof. This is an immediate consequence of an earlier remark about more general G. But the argument is perhaps worth making explicit. The eigenvectors of $\mathbf{S}_k^m$ are the $\mathbf{e}_k^{\boldsymbol{m}} = e_{k,0}^{m_0}\dots e_{k,k}^{m_k}$ with (m_i) in the region

$$\Sigma_m^k = \Big\{(m_i) \in \mathbb{N}^{k+1} \,\Big|\, \sum m_i = m\Big\}\,.$$

This is the set of lattice points in the convex hull of its vertices $(0,\dots,m,\dots,0)$. The eigenvalue of $\mathbf{e}_k^{\boldsymbol{m}}$ is $p^{km-\mu}q^\mu$ with

$$\mu = \sum\nolimits_{i=0}^k i\cdot m_i\,.$$

These eigenvalues are all of the form p^aq^b with (a,b) on the line segment from $(km,0)$ to $(0,km)$, which implies the Lemma.

In particular, the matrix $\begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix}$ with $p = q$ acts on $S^m(\sigma_k)$ as scalar multiplication by p^{km}.

Let

$$\gamma = \begin{bmatrix} 1 & 0 \\ 0 & q \end{bmatrix}\,.$$

Thus

$$\gamma u = u, \quad \gamma v = qv\,.$$

Since

$$\begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix} = \begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & q/p \end{bmatrix},$$

a representation of GL_2 on which scalars act by a single scalar multiplication is determined by the action of γ. In particular, the irreducible representations of GL_2 are of the form $\sigma_k\cdot\det^\ell$, and the representation of γ in this is a direct sum of one-dimensional eigenspaces on which γ acts by $q^{\ell+i}$ for $0 \le i \le k$.

I have already defined the weight polynomial of any finite-dimensional representation π of $\mathrm{GL}_2(\mathbb{C})$, This is the same as the trace of $\pi(\gamma)$. For example, as we have already seen, the weight polynomial of σ_k is

$$1 + q + \dots + q^k = \frac{1-q^{k+1}}{1-q}\,,$$

and more generally that of $\sigma_k\cdot\det^\ell$ is $q^\ell(1-q^{k+1})/(1-q)$.

Let

$$\omega_k^m \;=\; \text{the weight polynomial of } \mathbf{S}_k^m\,.$$

Express

$$\mathbf{S}_k^m = \sum\nolimits_{i=0}^{\lfloor km/2\rfloor} \mu_i\,\sigma_{km-2i}$$

♡ [irredecomp] in accordance with Lemma 1.10. The **decomposition polynomial** $\overline{\delta}_k^m$ may be identified with $\sum \mu_i q^i$, which has degree at most $\lfloor km/2\rfloor$. It might not be clear at first why this should be a polynomial in the same variable ♡ [botts-mult] as that in ω_k^m. However, as Lemma 1.9 tells us, there is a very simple relationship between the two. For GL_2, this is simply expressed:

[gl2-decomp] **1.11. Lemma.** *The decomposition polynomial of $\mathbf{S}_k^m$ is the truncation of the polynomial $\delta_k^m = (1-q)\omega_k^m$ at terms beyond those of degree $\lfloor km/2\rfloor$.*

If $S = \sum c_iq^i$ then $(1-q)S = \sum(c_i - c_{i-1})q^i$ (taking $c_{-1} = 0$), so that in effect the sequence defining δ is the first difference of that defining ω.

Proof. If

$$\mathbf{S}_k^m = \sum_{i=0}^{\lfloor km/2 \rfloor} \mu_i \, \sigma_{km-2i} \, ,$$

then

$$\omega_k^m = \frac{1}{1-q} \cdot \sum_{i=0}^{\lfloor km/2 \rfloor} \mu_i (q^i - q^{km+1-i}) \, ,$$

and

$$(1-q)\omega_k^m = \sum_{i=0}^{\lfloor km/2 \rfloor} \mu_i q^i - \sum_{i=0}^{\lfloor km/2 \rfloor} \mu_i q^{km+1-i} \, .$$

But if $i \le km/2$ then $km+1-i \ge km/2+1$, so all the negative terms are of degree more then $\lfloor km/2 \rfloor$. ▯

Therefore, computing the weight polynomial and the decomposition polynomial are practically equivalent.

♠ [sm-wts] All these things fit in nicely with Sturmfels' result. Because of the function ω_k^m may be identified with φ_A if we take

[matA-bis] **(1.12)**

$$A = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 2 & \dots & k \end{bmatrix}$$

The chambers in this case are the bands

$$mi \le n \le m(i+1) \quad (0 \le i \le k-1)$$

and in any one of these the function $\omega_k(m,n)$ must be a quasi-polynomial. The main theorem will give us an explicit expression for this. It will involve **denumerants**, and this will give us in turn the polynomial part in Sturmfels' theorem.

2. Early explorations [GL2-symm.tex]

The first interesting case of our problem is $k=2$. It is not difficult:

[sigma2] **2.1. Proposition.** *If $\sigma = \sigma_2$ then*

$$S^m(\sigma) = \sigma_{2m} + \sigma_{2m-4} \cdot \det^2 + \cdots + \begin{cases} \sigma_0 \cdot \det^n & \text{if } m = 2n \\ \sigma_2 \cdot \det^n & \text{if } m = 2n+1. \end{cases}$$

In other words, the weight polynomial of $S^m(\sigma_2)$ is

$$1 + q + 2q^2 + 2q^3 + 3q^4 + \cdots + 2q^{2m-2} + q^{2m-1} + q^{2m} \, .$$

Proof. The vectors $e_0 = u^2$, $e_1 = uv$, $e_2 = v^2$ span the space $S^2(\mathbb{C}^2)$, with

$$\sigma(\gamma) e_i = q^i e_i \, .$$

The eigenvectors of $\mathbf{S}_2^m(\gamma)$are then the $e_0^{m_0} e_1^{m_1} e_2^{m_2}$ with

$$0 \le m_i, \quad m_0 + m_1 + m_2 = m \, .$$

The corresponding eigenvalue is $\mu = m_1 + 2m_2$. Sort these partitions according to m_1. For a fixed m_1 these correspond to partitions of $m - m_1$.

On the other hand, the eigenvalues of $\sigma_{2m-2i} \cdot \det^i$ are the $k+i$ for $0 \le k \le 2m-2i$. They may be picked off easily from the list. For example, consider $S^3(\sigma)$. The sorted list with eigenvalues is:

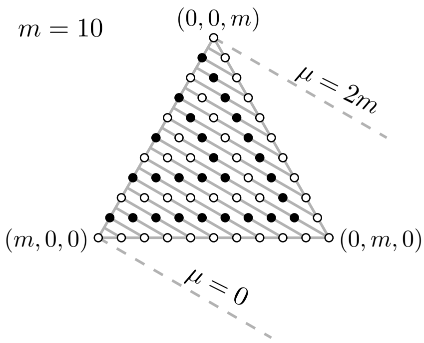
$$\begin{array}{llll} (3,0,0) : \boldsymbol{0} & (2,1,0) : \boldsymbol{1} & (1,2,0) : \boldsymbol{2} & (0,3,0) : \boldsymbol{3} \\ (2,0,1) : 2 & (1,1,1) : 3 & (0,2,1) : \boldsymbol{4} & \\ (1,0,2) : 4 & (0,1,2) : \boldsymbol{5} & & \\ (0,0,3) : \boldsymbol{6} & & & \end{array}$$

From these we can extract first the segment $(0,1,2,3,4,5,6)$ which are the eigenvalues of σ_6, and then the segment $(2,3,4)$ which are the eigenvalues of $\sigma_2 \cdot \det^2$.

This result can be interpreted geometrically in a way that should illuminate what happens in higher dimensions where a picture can be difficult if not impossible. The eigenvectors of $S^m(\sigma_2)$ are in bijection with the set of lattice points

$$\Sigma_m = \Big\{ (m_i) \in \mathbb{N}^3 \Bigm| \sum m_i = m \Big\}$$

on certain two-dimensional slices of the positive octant. The eigenvalue of (m_i) is $\mu = m_1 + 2m_2$, and points of equal eigenvalue are hence subsets of Σ_m on a line. Here is a typical picture, of the slice $x_0 + x_1 + x_2 = 10$, in which the lines are the level lines of $m_1 + 2m_2 = \mu$ for eigenvalues $\mu = 0$ to 20:



gl2-images/slice-10.eps

As the figure illustrates, and as I have already mentioned, the eigenvalues μ of γ on the slice $\sum m_i = m$ fall naturally into two ranges, $[0,m]$ and $[m,2m]$. On each, the multiplicity is approximately a linear function of μ, in the sense that the difference between it and some linear function is bounded.

The decompositions of the $\boldsymbol{S}_k^m$ with $k \ge 3$ is more interesting. I'll begin by telling how to compute them.

Molien's formula suggests a method to compute the weight multiplicities of the representations $S^m(\pi)$ for any representation π. In general, it comes down to computing the expansion of $1/P(t)$ for a polynomial $P(t)$, in which case it yields a recursive formula for the m-th coefficient of the series. Let $P(t) = \sum_{i=0}^d p_i t^i$ with $p_0 = 1$, and let $1/P(t) = S(t) = \sum_m s_m t^m$. Since $P(t)S(t) = 1$:

(2.2)

$$\begin{aligned} s_0 &= \quad 1 \\ s_1 &= -p_1 \\ s_2 &= -(p_2 + p_1 s_1) \\ &\ \ \ldots \\ s_d &= -(p_d + p_{d-1}s_1 + \cdots + p_1 s_{d-1}) \\ s_m &= -(p_d s_{m-d} + p_{d-1}s_{m-d+1} + \cdots + p_1 s_{m-1}) \quad (m > d)\,. \end{aligned}$$

[srecurse]

For GL_2 there is a special version of this. The denominator with $A = \sigma_k(\gamma)$ is

$$\det(I - \sigma_k(\gamma)t) = (1-t)(1-qt)\ldots(1-q^k t) = \det(I - \sigma_{k-1}(\gamma)t) \cdot (1 - q^k t)\,,$$

so we recover a product formula. If $\omega_k^m = \operatorname{trace} \mathbf{S}_k^m(\gamma)$, then:

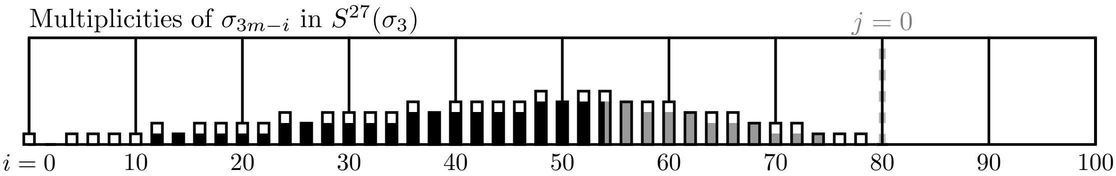
$$\Big(\sum\nolimits_m \omega_k^m t^m\Big)\big(1 - q^k t\big) = \Big(\sum\nolimits_m \omega_{k-1}^m t^m\Big)\,,$$

We have initial conditions $\omega_0^m = 1$, $\omega_k^0 = 1$, and directly from the equation above $\omega_{k-1}^m = \omega_k^m - q^k\omega_k^{m-1}$, leading to a recursion:

[tau-recursion] **(2.3)**
$$\omega_k^m = \omega_{k-1}^m + q^k\omega_k^{m-1}\,.$$

Both these methods work well when one wants, as we shall want, to compute a large initial sequence of the decomposition polynomials $\overline{\delta}_k^m$.

To get a rough idea of how things go, we can look at σ_3. The results are best exhibited graphically. For small values of m, things look a bit too random to be instructive, so I'll look at a range involving larger m:



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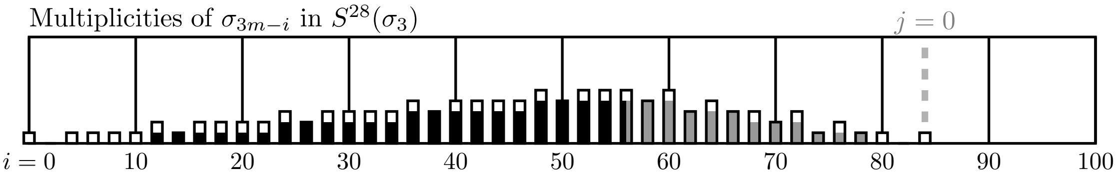

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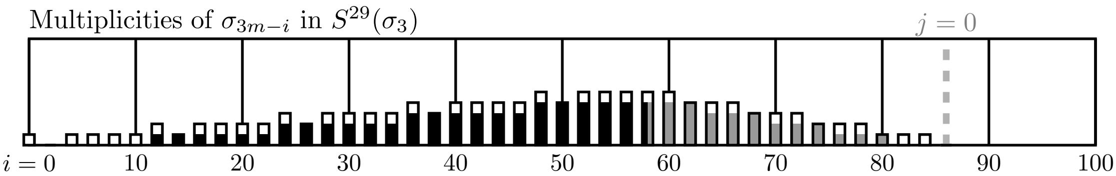

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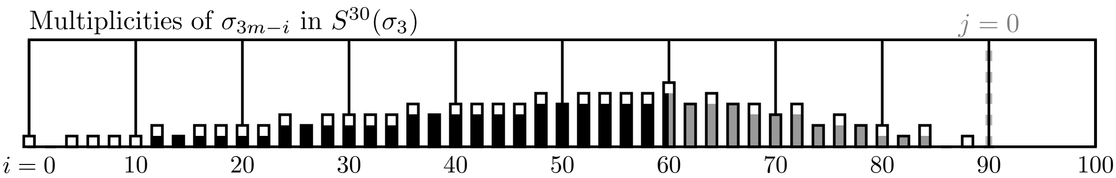
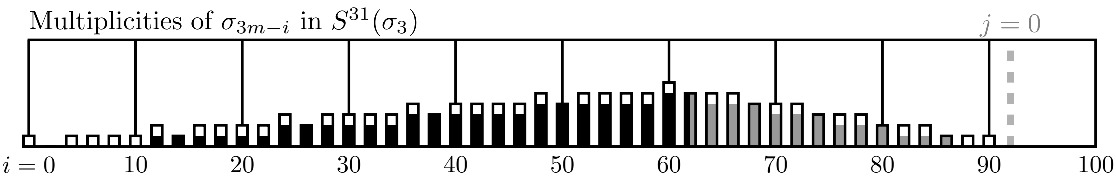


new-python/delta-30.eps



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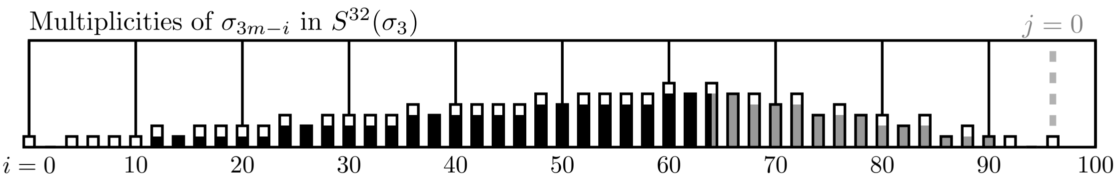


new-python/delta-32.eps

For example, we can read off from this that for $m \gg 0$

$$\delta_3^m = 1 + q^2 + q^3 + q^4 + q^5 + 2q^6 + \cdots$$

which is to say that

$$S^m(\sigma_3) = \sigma_{3m} + \sigma_{3m-4} + \sigma_{3m-6} + \cdots$$

We shall see in a moment the significance of the dotted vertical line. Very roughly, it denotes where the component is one-dimensional.

As the figures show, with the help of shading, there is a definite pattern to the multiplicity of σ_{3m-2i} in $\mathbf{S}_3^m$. The simplest feature is what occurs in the range $0 \le i \le m$ (black shading). Here, the multiplicity is approximately equal to $i/6$, and the discrepancy is periodic with period 6. Beyond $i = m$, things are slightly more complicated. If $j = \lfloor 3m/2 \rfloor - i$, the multiplicity is approximately $j/3$. Both these approximations are indicated in the figures by lighter shading. Roughly speaking, the multiplicities behave uniformly in bands of length m. But the figures also suggest something precise. Define the arrays

$$\begin{aligned} A &= [1, 0, 1, 1, 1, 1] \\ B &= [0, 1, 1] \\ C &= [1, 0, 1, 0, 1, 0] \\ D &= [0, 1, 0, 1, 0, 1] \end{aligned}$$

[sigma3-delta] **2.4. Theorem.** *Let μ_i be the multiplicity of σ_{3m-2i} in $\mathbf{S}_3^m$. If $i > m$, let $j = \lfloor 3m/2 \rfloor - i$. Then*

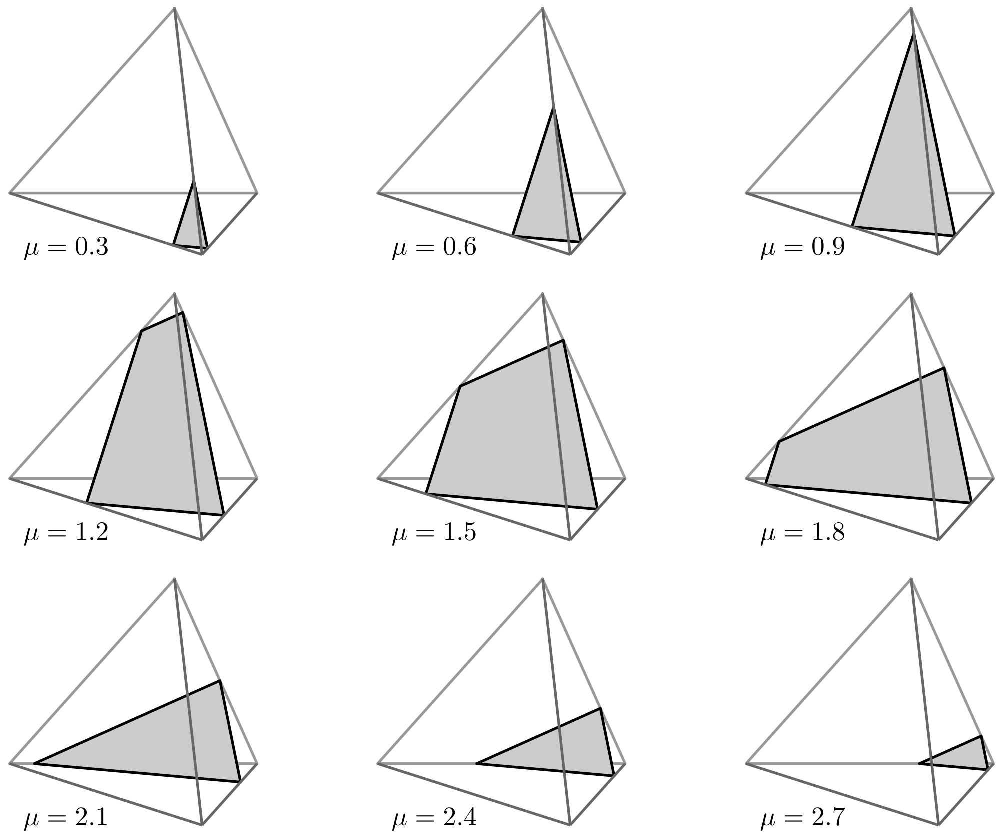
$$\mu_i = \begin{cases} \lfloor i/6 \rfloor + A_{i \bmod 6} & \text{if } i \le m \\ \lfloor j/3 \rfloor + B_{j \bmod 3} & \text{if } i > m \text{ and } m \equiv 1\ (2) \\ \lfloor j/3 \rfloor + C_{j \bmod 6} & \text{if } i > m \text{ and } m \equiv 0\ (4) \\ \lfloor j/3 \rfloor + D_{j \bmod 6} & \text{if } i > m \text{ and } m \equiv 2\ (4) \end{cases}$$

There is one curious very curious feature that is worthwhile pointing out. We have two formulas, each valid in a different range. It is a consequence of Sturmfels' theory that both formulas are valid for the boundary case $i = m$. The diagrams illustrate this by dual colouring. *More remarkably, the two formulas agree at*
《?》 $i = m - 1$. I am not aware of any theory that accounts for this, nor whether it is a general phenomenon.°

Basically, this formula is a consequence of some classical formulas due to the nineteenth century mathematicians Cauchy, Cayley, and Sylvester! I'll explain this later on, when I'll have much more to say about this matter.

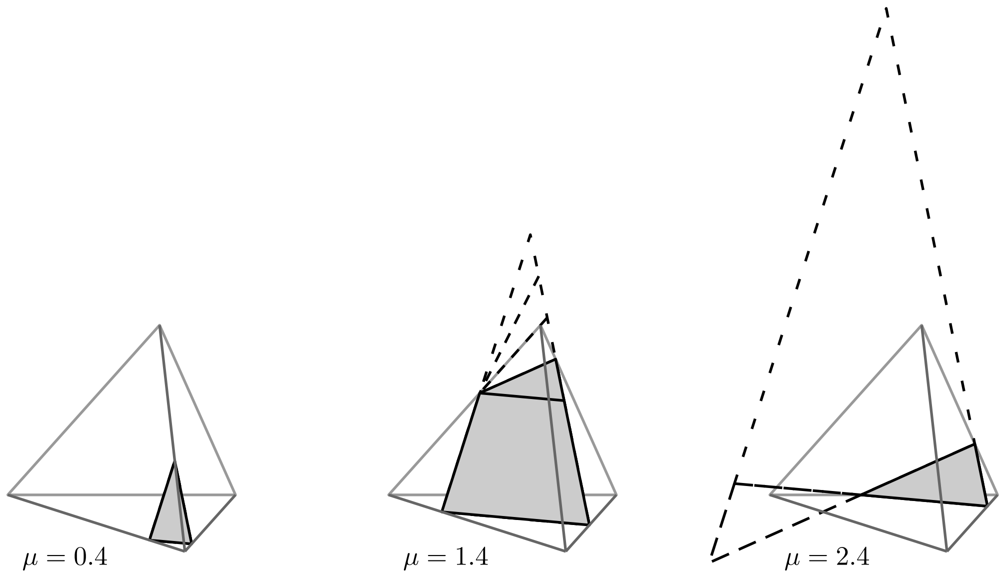
The piecewise approximate linearity might have been predicted for geometric reasons. The eigenvectors of $\mathbf{S}_3^m(\gamma)$ are the $e_0^{m_0} e_1^{m_1} e_2^{m_2} e_3^{m_3}$ with $m_i \ge 0$, $\sum m_i = m$, and the corresponding eigenweight is $\mu = m_1 + 2m_2 + 3m_3$. The range of eigenvalues is $[0, 3m]$, and as with the case $k = 2$ this range breaks up into pieces—here $[0, m]$, $[m, 2m]$, $[2m, 3m]$—on each of which things behave uniformly. The point is that the inverse images of these ranges are the regions in between slices of the tetrahedron through its vertices, and inside these regions the inverse images are shapes that are geometrically alike.

Pictures can be instructive. The region $\sum_{i=0}^{3} x_i = m$ ($x_i \ge 0$) is a regular three-dimensional tetrahedron with 4 vertices. We want to know what the intersection of this tetrahedron with the hyperplane $\sum i\, x_i = c$ looks like. It will be a two-dimensional polygon. For $0 < c \le m$ or $2m \le c < 3m$ it will be a triangle, and all of these in one range will be similar. In the middle range $m < c < 2m$ it will be a quadrilateral. These
《$\mu = \ell/m$?》 quadrilaterals will not be similar, but they are are roughly alike in appearance.°



Inside each range the area A_μ of the slice is a quadratic function of the parameter μ. For m large, the number of lattice points is approximately proportional to this area, so the coefficients of ω_3^m will also be approximately quadratic, and those of δ_3^m approximately linear.

There is in fact a simple formula for the area of each slice, and hence an approximate formula for the number of lattice points in it. This formula can best be understood by looking more closely at the slices, which I'll do now, with a little decoration. In the following figures, triangles which look like they might be congruent are in fact congruent.



To interpret this figure, you should know that regions that look as though they might be congruent are in fact congruent.

The conclusion is that the configuration at one value of μ is closely related to that for $\mu - 1$. I'll exhibit the precise relationship in a moment, but first I have to explain a certain normalization. There is no obvious canonical metric involved here, but I choose the measure of volumes so that the volume of the full tetrahedron is 1, and the area A_μ in such a way that

$$\int_0^3 A_\mu \, d\mu = 1 \, .$$

We are now led to the formula

(2.5) [slice-area]

$$\text{area of slice } \mu \ = \begin{cases} \dfrac{\mu^2}{2} & \text{if } \mu < 1 \\[2ex] \dfrac{\mu^2}{2} - 3\,\dfrac{(\mu-1)^2}{2} & \text{if } 1 \le \mu < 2 \\[2ex] \dfrac{\mu^2}{2} - 3\,\dfrac{(\mu-1)^2}{2} + 3\,\dfrac{(\mu-2)^2}{2} & \text{if } 2 \le \mu < 3. \end{cases}$$

The formula for $0 \le \mu \le 1$ is a matter of elementary geometry, and fixes the normalization. From that, the others are suggested by the figures above. The apparent connection with Pascal's triangle is significant.

As we'll see later, something similar occurs in higher dimensions.

3. The classical formula [GL2-symm.tex]

Molien's formula may be applied to the weight polynomial, setting $A = \sigma_k(\gamma)$. It tells us that

$$\frac{1}{\det(I - \sigma_k(\gamma)t)} = \frac{1}{(1-t)(1-qt)\dots(1-q^k t)} = \sum\nolimits_m \omega_k^m(q) \cdot t^m \, .$$

♡ [tau-recursion] The recursive rule (2.3) is so simple that it should not be too surprising that there is a simple formula for ω_k^m.

For any $n \ge 0$ define

$$[n]_q = 1 + q + \cdots + q^{n-1} = \frac{q^n - 1}{q - 1} \, .$$

If we set $q = 1$ then $[n]_q$ becomes n, and $[n]_q$ is known as the q-analogue of the function $f(n) = n$. As we have seen, the weight polynomial of σ_k is $[k+1]_q$.

The q-analogue of the factorial function is now naturally defined to be

$$[n]_q^! = [1]_q \cdot [2]_q \ldots [n]_q \, .$$

These are easy to compute inductively.

The q-analogue of the binomial coefficient is

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \begin{cases} \dfrac{[n]_q^!}{[m]_q^! [n-m]_q^!} & \text{if } 0 \le m \le n \\ 0 & \text{otherwise.} \end{cases}$$

This is also

$$\frac{[n]_q \ldots [n-m+1]_q}{[m]_q^!} \quad (0 \le m \le n) \, .$$

It is symmetric in m and $n - m$:

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \begin{bmatrix} n \\ n-m \end{bmatrix}_q \, .$$

Special cases are

$$\begin{bmatrix} n \\ 0 \end{bmatrix}_q = 1 \quad (n \ge 0) \quad \text{and} \quad \begin{bmatrix} n \\ 1 \end{bmatrix}_q = [n]_q \quad (n \ge 1) \, .$$

These all fit into a q-analogue of Pascal's triangle:

$$\begin{array}{lcccccc} n & m=0 & m=1 & m=2 & \ldots & & \\ 0: & 1 & & & & & \\ 1: & 1 & 1 & & & & \\ 2: & 1 & 1+q & 1 & & & \\ 3: & 1 & 1+q+q^2 & 1+q+q^2 & 1 & & \\ 4: & 1 & 1+q+q^2+q^3 & 1+q+2q^2+q^3+q^4 & \ldots & 1 & \\ 5: & 1 & 1+q+q^2+q^3+q^4 & 1+q+2q^2+2q^3+2q^4+q^5+q^6 & \ldots & \ldots & 1 \\ & \ldots & & & & & \end{array}$$

This illustrates the following, which is easily verified:

[qchoose] **3.1. Proposition.** *For* $n \ge 1$

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}_q + q^m \begin{bmatrix} n-1 \\ m \end{bmatrix}_q \, .$$

Which is to say that, as in Pascal's triangle, the expression at (n, m) is a simple linear combination of those at $(n-1, m)$ and $(n-1, m-1)$. This can be combined with the evaluation of the first row:

$$\begin{bmatrix} 0 \\ m \end{bmatrix}_q = \begin{cases} 1 & \text{if } m = 0 \\ 0 & \text{otherwise,} \end{cases}$$

to recover by induction:

[cor1] **3.2. Corollary.** *The function* $\begin{bmatrix} n \\ m \end{bmatrix}_q$ *is a polynomial in* q.

This not at all immediately apparent from the definition, just as it is not immediately apparent that $\binom{n}{m}$ is an integer.

What's the point? From now on, for $0 \le m \le n$ let

$$\lambda_n^m = \text{ the weight polynomial of } \bigwedge^m(\sigma_{n-1}) \, ,$$

and assuming the convention that $\lambda_n^0 = 1$ for all $n \ge 0$. The first few are:

n	$m = 0$	$m = 1$	$m = 2$	$\ldots$	
$0:$	1				
$1:$	1	q			
$2:$	1	$1+q$	q		
$3:$	1	$1+q+q^2$	$q+q^2+q^3$		
$4:$	1	$1+q+q^2+q^3$	$q+q^2+2q^3+q^4+q^5$	$\ldots$	1

A comparison with the table of values of $\begin{bmatrix} n \\ m \end{bmatrix}_q$ suggests:

[exterior] **3.3. Proposition.** *For* $0 \le m \le n$*,*

$$\lambda_n^m = q^{m(m-1)/2} \begin{bmatrix} n \\ m \end{bmatrix}_q \, .$$

This is consistent with the fact that since the dimension of σ_{n-1} is n, the dimension of $\bigwedge^m(\sigma_{n-1})$ is $\binom{n}{m}$.

Proof. This is clear for $n = 0$ and 1 by direct calculation, and it is trivally true for $m = 0$ and $m = n$. So suppose $n \ge 2$, $1 \le m \le n-1$.

Let the e_i for $0 \le i \le n-1$ be an eigenbasis of σ_{n-1} with respect to γ, and suppose the eigenvalue of e_k to be q^k.

For an ordered subset $I = \{i_j\}$ of size $|I| = p$ with $0 \le i_1 < \ldots < i_p \le n-1$ let $e_I = e_{i_1} \wedge \ldots \wedge e_{i_p}$. The eigenvalue of e_I is $q^{i_1 + \cdots + i_{p-1}}$. The e_I with $|I| = m$ form a basis of $\bigwedge^m(\sigma_{n-1})$.

The natural thing to do is partition these into the $e_I \wedge e_{n-1}$ with $I \subseteq [0, n-2]$ of size $m-1$ and the e_I with $I \subseteq [0, n-2]$, $|I| = m$. This causes some problems, however. Instead, partition these I into those with $i_1 = 0$ and the rest. This gives us, after some elementary shifts,

$$\lambda_n^m = q^{m-1} \lambda_{n-1}^{m-1} + q^m \lambda_{n-1}^m \, .$$

Temporarily, set

$$\ell_n^m = q^{m(m-1)/2} \begin{bmatrix} n \\ m \end{bmatrix}_q \, .$$

♡ [qchoose] If we multiply the earlier recursion formula Proposition 3.1 by $q^{m(m-1)/2}$ we get

$$\ell_n^m = q^{m-1} \ell_{n-1}^{m-1} + q^m \ell_{n-1}^m \, .$$

[symmpk] **3.4. Theorem.** *The weight polynomial* ω_k^m *of* $\mathbf{S}_k^m$ *is equal to*

$$\begin{bmatrix} m+k \\ k \end{bmatrix}_q \, .$$

This is consistent with the fact that the dimension of $\boldsymbol{S}^m(\mathbb{C}^{k+1})$ is $\binom{m+k}{k}$. It is perhaps well known, although it seems to have been rediscovered often, and is sometimes formulated equivalently in terms of Young diagrams. For example, it is Lemma 4.1.22 of [Goodman-Wallach:2009], proved by an application of Molien's formula. It ought perhaps to be thought of as a generalization of Weyl's character formula, although no generalization for other reductive groups seems to be known or even conjectured.

Proof. It suffices to prove that the polynomial λ^n_m is equal to the weight polynomial of $S^m(\sigma_{n-m})$ multiplied by $q^{m(m-1)/2}$.

There is a simple bijection of eigenvectors for γ in the two spaces $\bigwedge^m(\sigma_{n-1})$ and $S^m(\sigma_{n-m})$. The exterior product $e_{i_1} \wedge \ldots \wedge e_{i_m}$ with $0 \le i_1 < i_2 < \ldots < i_m \le n-1$ maps to the symmetric product $e_{i_1} e_{i_2-1} \ldots e_{i_m-(m-1)}$.

[decompsigmak] **3.5. Corollary.** *The decomposition polynomial $\overline{\delta}^m_k$ is the truncation of*

$$\delta^m_k = \frac{(1-q^{m+1})(1-q^{m+2})\ldots(1-q^{m+k})}{(1-q^2)\ldots(1-q^k)}$$

at terms of degree beyond $\lfloor km/2 \rfloor$.

♡ [gl2-decomp] *Proof.* From the Theorem and Lemma 1.11.

It is instructive to see what happens for $S^m(\sigma_2)$. Here

$$\delta^m_2 = \frac{(1-q^{m+1})(1-q^{m+2})}{(1-q^2)} .$$

The denominator will divide one of the factors in the numerator, but which one it divides depends on the parity of m. Taking this into account:

$$\delta^m_2 = \begin{cases} (1+q^2+\cdots+q^{2n})(1-q^{m+1}) & \text{if } m = 2n \\ (1+q^2+\cdots+q^{2n})(1-q^{m+2}) & \text{if } m = 2n+1 \end{cases}$$

which agrees exactly with what we already know.

It should be apparent even from this simple case that the expansions of ω^m_k and δ^m_k will generally depend on congruence conditions, since the divisibility of terms in the numerator of $\begin{bmatrix} m+k \\ k \end{bmatrix}_q$ by terms in the denominator will depend on them.

♡ [sigma3-delta] Let's look now at $S^m(\sigma_3)$, and take up the proof of Theorem 2.4. I first recall the statement. *Let μ_i be the multiplicity of σ_{3m-2i} in $\boldsymbol{S}^m_3$. If $i > m$, let $j = \lfloor 3m/2 \rfloor - i$. Then*

$$\mu_i = \begin{cases} \lfloor i/6 \rfloor + A_{i \bmod 6} & \text{if } i \le m \\ \lfloor j/3 \rfloor + B_{j \bmod 3} & \text{if } i > m \text{ and } m \equiv 1\ (2) \\ \lfloor j/3 \rfloor + C_{j \bmod 6} & \text{if } i > m \text{ and } m \equiv 0\ (4) \\ \lfloor j/3 \rfloor + D_{j \bmod 6} & \text{if } i > m \text{ and } m \equiv 2\ (4) \end{cases}$$

in which

$$\begin{aligned} A &= [1,0,1,1,1,1] \\ B &= [0,1,1] \\ C &= [1,0,1,1,2,1] \\ D &= [0,1,0,2,1,2] . \end{aligned}$$

《why do denominators disappear?》

♡ **[symmpk]** We know from Theorem 3.4 that°

$$\begin{aligned}
\delta_3^m &= \frac{(1-q^{m+1})(1-q^{m+2})(1-q^{m+3})}{(1-q^2)(1-q^3)} \\
&= \frac{1-q^{m+1}(1+q+q^2)+q^{2m+3}(1+q+q^2)-q^{3m+6}}{(1-q^2)(1-q^3)} \\
&= \frac{1}{(1-q^2)(1-q^3)} - q^{m+1}\cdot\frac{1+q+q^2}{(1-q^2)(1-q^3)} + q^{2m+3}\cdot\frac{1+q+q^2}{(1-q^2)(1-q^3)} - q^{3m+6}\cdot\frac{1}{(1-q^2)(1-q^3)} \\
&= \frac{1}{(1-q^2)(1-q^3)} - q^{m+1}\cdot\frac{1+q+q^2}{(1-q^2)(1-q^3)} + O(q^{2m+3})
\end{aligned}$$

Luckily we are only interested in the range between 0 and $\lfloor 3m/2 \rfloor$, so we can ignore the last term. But even more luckily, we can write the first two terms as

$$\frac{1}{(1-q^2)(1-q^3)} - q^{m+1}\cdot\frac{(1-q^3)/(1-q)}{(1-q^2)(1-q^3)} = \frac{1}{(1-q^2)(1-q^3)} - q^{m+1}\cdot\frac{1}{(1-q)(1-q^2)}\,.$$

We must now evaluate the series for the rational functions

$$\frac{1}{(1-q^2)(1-q^3)},\quad \frac{1}{(1-q)(1-q^2)}\,.$$

♡ **[srecurse]** But this is simple. The recursion formulas (2.2) give us

$$\frac{1}{(1-q^2)(1-q^3)} = \sum a_i q^i \quad \big(a_i = \lfloor i/6 \rfloor + A_{i \bmod 6}\big)\,,$$

and

$$\frac{1}{(1-q)(1-q^2)} = \sum b_i q^i \quad \big(b_i = \lfloor i/2 \rfloor + 1\big)\,.$$

♡ **[sigma3-delta]** I leave it as an exercise to verify that the formula we now have in hand agrees with that of Theorem 2.4.

This remarkable trick is actually generally valid, and leads to a very useful expansion of the formula for the ♡ **[symmpk]** trace and decomposition polynomials. We know from Theorem 3.4 that

$$\omega_k^m = \frac{(1-q^{m+1})(1-q^{m+2})\dots(1-q^{m+k})}{(1-q)(1-q^2)\dots(1-q^k)}\,.$$

♡ **[exterior]** Expanding the product in the numerator, we see by Proposition 3.3 that it becomes

$$\begin{aligned}
1 - q^{m+1}(1+q+\cdots+q^{k-1}) &+ q^{2m+2}\Big(\sum\nolimits_{0\le i<j\le k-1} q^{i+j}\Big) - \cdots \\
&= \sum\nolimits_{i\le k} (-1)^i q^{mi+i}\lambda_k^i \\
&= \sum\nolimits_{i\le k} (-1)^i q^{mi+i(i+1)/2}\cdot\frac{(1-q^k)(1-q^{k-1})\dots(1-q^{k-i+1})}{(1-q)(1-q^2)\dots(1-q^i)}\,.
\end{aligned}$$

This leads to a succinct formula for ω_k^m:

[tauelegant] **3.6. Proposition.** *We have*

$$\omega_k^m = \frac{1}{(1-q)\dots(1-q^k)}\cdot\left(\sum\nolimits_{i\le k} (-1)^i q^{mi+i(i+1)/2}\cdot\begin{bmatrix}k\\ i\end{bmatrix}_q\right)\,.$$

This is surprisingly elegant, but there are other useful ways to write it. For one thing, a great deal of cancellation in the coefficient takes place—the basic fact is that the coefficient of $q^{mi+i(i+1)/2}$ simplifies to

$$\frac{1}{(1-q)(1-q^2)\dots(1-q^i)\cdot(1-q)(1-q^2)\dots(1-q^{k-i})}\,.$$

There is evident symmetry with respect to the interchange of i with $k-i$.

The most straightforward result is this:

[taukm] **3.7. Corollary.** *We have*

$$\omega_k^m = \sum\nolimits_{i\le k} (-1)^i q^{mi+i(i+1)/2}\cdot C_{k,i}$$

with

$$C_{k,i} = \begin{cases} \dfrac{1}{(1-q)\dots(1-q^k)} & \text{if } i=0 \\ \dfrac{1}{(1-q)^2(1-q^2)^2\dots(1-q^i)^2(1-q^{i+1})\dots(1-q^{k-i})} & 0<i<k/2 \\ \dfrac{1}{(1-q)^2(1-q^2)^2\dots(1-q^i)^2} & i=k/2 \\ \dfrac{1}{(1-q)^2(1-q^2)^2\dots(1-q^{k-i})^2(1-q^{k-i+1})\dots(1-q^i)} & k/2<i<k \\ \dfrac{1}{(1-q)\dots(1-q^k)} & i=k. \end{cases}$$

I should point out right now that there is something peculiar about this formula. The q-term of highest degree is $q^{km+k(k+1)/2}$, whereas we know that ω_k^m is a polynomial of degree km. We'll see later several other manifestations of a similar problem.

In any case, what we really want to know is, *what are the q-expansions of these rational functions?* This problem, as I suggested in the opening paragraphs, is classical, with valuable solutions known already to Cayley and Sylvester.

4. Denumerants [GL2-symm.tex]

Suppose $\mathbf{a} = (a_0, a_1, \dots, a_{k-1})$ to be a sequence of positive integers. For each non-negative n, define

$$D_{\mathbf{a}}(n) = \left\{ (n_i) \in \mathbb{N}^k \,\middle|\, \sum n_i a_i = n \right\}.$$

The function $D_{\mathbf{a}}$ is called (I think first by James Joseph Sylvester) the **denumerant** function associated to $\mathbf{a}$. It is related to the problems we have seen, because of the following, which is easy to prove (and is also a special case of Molien's formula).

[denumerants-basic] **4.1. Lemma.** *For any* $\mathbf{a}$

$$\frac{1}{(1-q^{a_0})\dots(1-q^{a_{k-1}})} = \sum D_{\mathbf{a}}(n) q^n\,.$$

♡ [taukm] Corollary 3.7 can now be formulated in terms of denumerants. Let $\omega_k^m(n)$ be the coefficient of q^n in the expansion of ω_k^m. For $0 \le i \le k$ define $m_i = mi + i(i+1)/2$, and let

(4.2) [alphas]
$$\alpha_i = \begin{cases} (1,\dots,k) & \text{if } i=0 \\ (1,1,\dots,i,i,i+1,\dots,k-i) & 0<i<k/2 \\ (1,1,\dots,i,i) & i=k/2 \\ (1,1,\dots,k-i,k-i,k-i+1,\dots,i) & k/2<i<k \\ (1,\dots,k) & i=k \end{cases}$$

♡ **[taukm]** Here $D_{\mathbf{a}}(n)$ is taken to be 0 if $n < 0$. I recall that $m_i = mi + i(i+1)/2$. Then Corollary 3.7 asserts that

(4.3) **[taukmn]**
$$\omega_k^m(n) = \sum_{j=0}^{i} (-1)^j D_{\alpha_j}(n - m_j) \quad \text{if} \quad m_i \le n < m_{i+1} \,.$$

This tells us that computing $\omega_k^m(n)$ reduces to computing denumerants. *What can we say about the function $D_{\mathbf{a}}(n)$ on $\mathbb{N}$?*

One simple observation is that in investigating a denumerant $D_{\mathbf{a}}$, one may as well assume that the a_i have no non-trivial common divisor, because if they have the common divisor d one may consider instead $D_{\mathbf{a}/d}(n/d)$. This might motivate the next discussion.

QUASI-POLYNOMIALS IN 1D. A function f on $\mathbb{Z}$ or $\mathbb{N}$ is a quasi-polynomial if there exists $N > 0$ such that its restriction to each congruence class $k + (N)$ is a polynomial function. That is to say, for each k there is a polynomial function $f_k(n)$ such that
$$f(k+n) = f_k(n)$$
if n lies in the ideal (N).

Let $\zeta_N = e^{2\pi i/N}$.

[qpiff] **4.4. Proposition.** *Suppose f to be a function on $\mathbb{N}$. The following are equivalent:*

(a) the function f is quasi-polynomial;

(b) there exists a polynomial $P(n)$ and a polynomial $Q(q)$ of degree $< N$ such that
$$\sum_{n=0} f(n) q^n = Q(q) \sum_{n=0} P(n) q^{nN} \,;$$

(c) there exists $N > 0$ and for each k in $[0, N-1]$ a unique polynomial function φ_k such that
$$f(n) = \sum_{k=0}^{N-1} \zeta_N^{kn} \varphi_k(n) \,.$$

In effect, the expression in (c) assigns a polynomial φ_ω to every N-th root of unity $\omega = \zeta_N^k$.

♡ **[quasi-poly-ell]** *Proof.* This a special case of Lemma 1.4, but I'll prove it anyway. The equivalence of (a) and (b) is immediate. The requirement on the degree of Q is necessary.

To prove that (a) implies (c), let χ_k be the characteristic function of $k + (N)$. If $\zeta = \zeta_N$ then
$$\chi_k(n) = \frac{1}{N} \Big(\sum_{a=0}^{N-1} \zeta^{a(n-k)} \Big) \,.$$

Then on the one hand
$$f(n) = \sum_{k=0}^{N-1} \chi_k(n) f(n)$$

but on the other $\chi_k f = \chi_k f_k$, so that
$$\begin{aligned} f(n) &= \frac{1}{N} \sum_{k=0}^{N-1} \sum_{\ell=0}^{N-1} \zeta^{\ell(n-k)} f_k(n) \\ &= \frac{1}{N} \sum_{\ell=0}^{N-1} \zeta^{n\ell} \sum_{k=0}^{N-1} \zeta^{-k\ell} f_k(n) \\ &= \sum_{\ell=0}^{N-1} \zeta^{n\ell} \varphi_\ell(n) \,. \end{aligned}$$

in which

[varphiqp] **(4.5)** .
$$\varphi_\ell(n) = \frac{1}{N}\sum_{k=0}^{N-1} \zeta^{-k\ell} f_k(n)$$

Note that this may be solved easily to recover the polynomials f_k.

To see that (c) implies (a), it needs to be shown that if f has an expression as in (c) its restriction to $k + (N)$ agrees with a polynomial. But

$$f(a + Nn) = \sum_{k=0}^{N-1} \zeta^{ka}\varphi_k(a + Nn)$$

and for each fixed a this is a linear combination of the shifted polynomials $\varphi_k(a+Nn)$, which are polynomial functions of n.

DENUMERANTS. The basic results about denumerants originated in the nineteenth century, although they seem to have been rediscovered often since then.

[denumqp] **4.6. Theorem.** (Cayley) *Every denumerant function $D_{\mathbf{a}}(n)$ is quasi-polynomial. More precisely, let $\mathcal{D}$ be the set of all d dividing some a_i. For each d in $\mathcal{D}$ let n_d be the number of a_i divisible by d. Then for each primitive d-th root of unity ζ there exists a polynomial φ_ζ of degree $n_d - 1$ such that*

$$D_{\mathbf{a}}(n) = \sum_k \zeta^{kn}\varphi_\zeta(n)\,.$$

Proof. One can find a partial fraction decomposition

$$D_{\mathbf{a}} = \sum_{d\in\mathcal{D}}\sum_\zeta\sum_1^{n_d} \frac{\gamma_{\zeta,i}}{(1-\zeta q)^i}\,,$$

in which the inner sum is over all primitive roots of $z^d = 1$, and the $\gamma_{\zeta,i}$ are constants. It therefore suffices to prove that the coefficients in the Taylor series of

$$\frac{1}{(1-\zeta q)^k}\,,$$

in which $\zeta^d = 1$, are determined by a quasi-polynomial.

If we start with

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

and differentiate successively, we see that

$$\begin{aligned}
\frac{1}{(1-x)^2} &= 1 + 2x + 3x^2 + 4x^3 + \cdots \\
\frac{2}{(1-x)^3} &= 2 + (3.2)x + (4.3)x^2 + (5.4)x^3 + \cdots \\
\frac{3.2}{(1-x)^4} &= (3.2) + (4.3.2)x + (5.4.3)x^2 + \cdots \\
&\cdots
\end{aligned}$$

and in general:

$$\begin{aligned} \frac{1}{(1-x)^n} &= \frac{1}{(n-1)!} \cdot \sum_{i=0} (i+1)\dots(i+n-1)\cdot x^i \\ &= \sum_{i=0} \binom{n-1+i}{n-1} x^i \\ &= \sum_{i=0} \frac{i^{[n-1]}}{(n-1)!} \cdot x^i \,. \end{aligned} \tag{4.7}$$

[pfm]

Here I write

$$x^{[\ell]} = (x+1)\dots(x+\ell)\,.$$

It is often convenient to write polynomials in what I call **Newton form**

$$\sum\nolimits_{\ell=0}^{n} c_\ell \, \frac{x^{[\ell]}}{\ell!}\,.$$

One point is that any integral-valued polynomial can be expressed in this form with integral coefficients. This is because

$$\frac{x^{[\ell]}}{\ell!} - \frac{(x-1)^{[\ell]}}{\ell!} = \frac{x^{[\ell-1]}}{(\ell-1)!}\,.$$

Such polynomials can be evaluated efficiently by Horner's method, obtaining in succession constants C_i for $i = n$ to 0:

$$\begin{aligned} C_n &= c_n/n \\ C_{n-1} &= (C_n(x+n) + c_{n-1})/(n-1) \\ C_{n-2} &= (C_{n-1}(x+n-1) + c_{n-2})/(n-2) \\ &\dots \\ C_0 &= C_1(x+1) + c_0\,. \end{aligned}$$

The drawback is that this involves rational arithmetic.

Conclude by setting $x = \zeta q$.

There is a large literature concerned with computing denumerants. The paper [Bell:1943] explains how to calculate the restrictions to each congruence class without going through partial fractions. A more recent example is [Baldoni et al.:2014]. It is not clear to me what the value of the sophisticated endeavours is. In practice, only small values of k arise, and partial fraction decompositions seem to work well.

SYLVESTER'S FORMULA. There is one component of the expansion of a denumerant that is simply a polynomial function, the part arising from the factor $(1-q)^{n_1}$ in the denominator. The procedure above will find a formual for this polynomial, but will require a great deal of work to find it, and requiring cyclotomic arithmetic. In fact there exists a remarkable formula for it, found originally by the nineteenth century mathematician James Joseph Sylvester.

Recall that the Bernoulli numbers are defined by the Taylor series expansion

$$\frac{x}{e^x - 1} = \sum_{i=0} B_i \cdot \frac{x^i}{i!}\,.$$

[sylvesters] **4.8. Proposition.** (Sylvester) *The polynomial part φ_1 is the polynomial*

$$\mathfrak{S}_{\mathbf{a}}(n) = \frac{1}{a_0 \dots a_{k-1}} \cdot \sum_{i=0}^{k-1} \frac{(-1)^i n^{k-1-i}}{(k-1-i)!} \left(\sum_{i_0+\dots+i_{k-1}=i} \frac{a_0^{i_0} \dots a_{k-1}^{i_{k-1}}}{i_0! \dots i_{k-1}!} \cdot B_{i_1} \dots B_{i_{k-1}} \right).$$

Note that the exponents of n are a decreasing sequence. An efficient derivation of this can be found in [Beck et al.:2001]. [Bachmann:1910] also has a pleasant and leisurely treatment of this and other things relating to denumerants. I am not aware that anyone has even conjectured a similar formula for the components corresponding to non-trivial roots of unity, not even $\zeta = -1$.

There is a geometric interpretation of the denumerant $D_{\mathbf{a}}(n)$ as the number of lattice points in the simplex in $\mathbb{N}^k$ spanned by the points $(\ldots, n/a_i, \ldots)$. A natural guess for a first estimate is hence the volume of that simplex. This is in fact a consequence of Sylvester's formula. for any $\boldsymbol{a}$ let

$$|\mathbf{a}| = a_0 \ldots a_{k-1} .$$

[dominant-term-syl] **4.9. Corollary.** *We have*

$$D_{\mathbf{a}}(n) = \frac{n^{k-1}}{(k-1)!\,|\mathbf{a}|} + O(n^{k-2}) .$$

5. The main theorem [GL2-symm.tex]

♡ [taukmn] The expression (4.3) is promising, but it isn't quite what we expect from Sturmfels' theorem. It requires only modest modification, however.

[main-theorem] **5.1. Theorem.** *For* $mi \le n \le m(i+1)$

$$\omega_k^m(n) = \sum\nolimits_{j=0}^{i} (-1)^j D_{\alpha_j}\big(n - (mj + j(j+1)/2)\big) .$$

♡♡[taukmn][proofs] *Proof.* From (4.3) and Lemma 1.5(d), in light of Sturmfels' theorem. ▯

♡ [dominant-term-syl] As far as I can see, the most important consequence of this is an asymptotic estimate as $m \to \infty$. By Corollary 4.9, the dominant term of $D_{\alpha_j}(n)$ is

$$\frac{n^{k-1}}{(k-1)!|\alpha_j|}$$

and it is easy to see that

$$\frac{1}{|\alpha_i|} = \binom{k}{i} \cdot \frac{1}{|\alpha_0|} .$$

Now define

[normvol] **(5.2)**
$$\Phi_k(x) = \frac{1}{(k-1)!} \cdot \sum\nolimits_{i=0}^{j} (-1)^i \binom{k}{i} (x-i)^{k-1} \quad (j \le x < j+1) .$$

♡ [slice-area] Thus Φ_k has support in $[0,k]$, Φ_1 is the characteristic function of $[0,1]$, and Φ_3 is what is exhibited in (2.5).

[main-cor] **5.3. Corollary.** *We have the asymptotic estimates*

$$\frac{\omega_k^m(n)}{m^{k-1}/(k-1)!} \sim \sum \Phi_k(n/km)$$
$$\frac{\delta_k^m(n)}{m^{k-1}/(k-1)!} \sim \frac{1}{km} \sum \Phi_k'(n/km)$$

Proof. The first claim follows from the fact that a polynomial and one of its shifts share the same highest ♡ [gl2-decomp] degree terms. The claim about δ_k^m follows from Lemma 1.11. ▯

This has some intuitive significance. The function $\Phi_k(x)$ ought to be, and is, normalized volume of the slice $\mu = x$ for σ_k. We have already seen this for $k = 1, 2, 3$. It is striking that there is another more familiar interpretation of Φ_k.

[normvolconv] **5.4. Lemma.** *The function Φ_k is the same as the k-fold convolution of Φ_1.*

That is to say, $\Phi_{k+1} = \Phi_1 * \Phi_k$. The Φ_k are thsu the probability distributions for sums of uniformly distributed random variables.

Proof. Define for the moment

$$[x] = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } x \geq 0. \end{cases}$$

♡ [normvol] Thus $[x]^0$ is the Heaviside step function. In this notation, the function in (5.2) is

$$\sum_{i=0} (-1)^i \binom{k}{i} \cdot \frac{[x-i]^{k-1}}{(k-1)!} \,.$$

If $k = 1$, the Lemma is asserts that $\Phi_1 = [x]^0 - [x-1]^0$. which is clearly true everywhere in $\mathbb{R}$.

Now apply induction and the convenient equation

$$[x]^0 * \frac{[x-c]^\ell}{\ell!} = \frac{[x-c]^{\ell+1}}{(\ell+1)!} \,.$$

♡ [normvol] This implies at least that the integral of (5.2) is 1.

Convolutions seem to be ubiquitous in this theory, even for groups other than GL_2. But that is a different story.

6. *Appendix.* Computing denumerants `[GL2-symm.tex]`

In this section I'll explain in detail an algorithm to find explicit formulas for the terms in the q-series of

[denumerant] **(6.1)**
$$\mathfrak{D}_{\mathbf{a}}(q) = \frac{1}{(1-q^{a_0})\ldots(1-q^{a_{k-1}})}$$

without doing cyclotomic arithmetic. The output will be a number of series

$$\sum_{j=0} P_{d,s}(j) q^{dj+s}$$

one for each divisor d of some a_i and s in $[0, d)$. The polynomial $P_{d,s}$ will be expressed in Newton form. It is the same as the sum of terms corresponding to the primitive d-roots of unity in the previous argument.

Step 1. The first step is to factor each $(1 - q^a)$ into a product of cyclotomic polynomials. There is one for each divisor d of a. The cyclotomic polynomial P_d is the product $\prod(\zeta - q)$ over all primitive roots ζ of order d, and is of degree $\phi(d)$, the number of units in $\mathbb{Z}/d$. It is irreducible in $\mathbb{Q}[q]$. To compute it, for each d set

$$\Pi_d = \prod\nolimits_{\substack{e|d \\ e \neq d}} P_e(q)$$

and apply the recursive formula

$$P_d(q) = \begin{cases} 1 - q & \text{if } d = 1 \\ \dfrac{1-q^d}{\Pi_d} & \text{otherwise.} \end{cases}$$

I am not aware of a markedly better way to calculate Π_d other than by traversing all divisors of d. In order to do this, we need to produce for each divisor of a a list of its divisors, ordered compatibly with divisibility. This is easy, given a prime factorization, because if $n = \prod p_i^{m_i}$ we can proceed through divisors of n in lexicographic order. Thus the divisors of $12 = 2^2 \cdot 3$ would be listed as

$$\begin{aligned} 1 &: (0,0) \\ 2 = 2^1 &: (1,0) \\ 4 = 2^2 &: (2,0) \\ 3 = 3^1 &: (0,1) \\ 6 = 2\cdot 3 &: (1,1) \\ 12 = 2^2\cdot 3 &: (2,1)\,. \end{aligned}$$

Thus, we can compute all the P_d and Π_d for $d|a$, given a prime factorization of a, which gives us a list of divisors of a along with a prime factorization of each of them.

My program uses a 'divisor iterator' for this.

♡ **[denumerant]** **Step 2.** Let $\mathcal{D}$ be the set of all those d dividing some a_i. It is probably best to use one dictionary of factorizations for all the d in $\mathcal{D}$, storing for each the polynomials P_d and Π_d. The denominator of (6.1) may now be factored as

$$\prod_{d\in\mathcal{D}} P_d^{n_d} \quad \text{with} \quad n_d = \big|\{k \mid d \mid a_k\}\big|\,.$$

Thus n_d is the number of i such that P_d divides $1 - a_i$. In particular, $n_1 = k$ is the largest exponent. Store the n_d in a dictionary.

Step 3. Now set

$$Q_d = \prod_{\substack{e\in\mathcal{D}\\ e\neq d}} P_e^{n_e} = \frac{\prod(1-q^{a_i})}{P_d^{n_d}}\,.$$

for each d in $\mathcal{D}$.

These Q_d have no common non-scalar divisor, so one can find polynomials A_d such that

$$\sum A_d Q_d = 1\,.$$

♡ **[denumerant]** Dividing this by the denominator of (6.1), we deduce the partial fraction decomposition

$$\frac{1}{(1-q^{a_0})\cdots(1-q^{a_{k-1}})} = \sum_d \frac{A_d}{P_d^{n_d}}\,.$$

The A_d are not unique, and the degree of A_d might well be larger than the degree $n_d\,\phi(d)$ of the denominator, but since the rational function vanishes as $q \to \infty$ polynomial parts will cancel, and we may assume that in fact the degree of A_d is less than $n_d\,\phi(d)$. This can be done if necessary by replacing each A_d by its remainder upon division by $P_d^{n_d}$.

Step 4. One may then multiply both top and bottom of the term $A_d/P_d^{n_d}$ by $\Pi_d^{n_d}$ to get it in the form

$$\frac{B_d(q)}{(1-q^d)^{n_d}}$$

with the degree of B_d less than that of the denominator.

Step 5. We are now reduced to several rational functions, one for each d in $\mathcal{D}$, of the form

$$\frac{B(q)}{(1-q^d)^n}$$

in which the degree of B is less than nd. One may now successively divide by $(1-q^d)$ to express

$$B = \sum_{i=0}^{n-1} B_i(1-q^d)^i \, ,$$

♡ **[denumerant]** with the degree of each B_i less than d, and express (6.1) as a sum of terms

[B-sum] **(6.2)**
$$\sum_{i=0}^{n-1} \frac{B_i(q)}{(1-q^d)^{n-i}} \, .$$

♡ **[B-sum]** **Step 6.** Continue working with these terms one by one. We want to find an expression for (6.2) as a series in q. This will be a sum over congruence classes modulo d, and on each congruence class a polynomial function. Write

$$B_i = \sum_{s=0}^{d-1} B_{i,s} q^s \, .$$

It suffices now to work with a single congruence class, so I may assume

$$B_i = B_{i,s} q^s \, .$$

Thus we are looking at

$$\sum_{i=0}^{n-1} \frac{B_{i,s} q^s}{(1-q^d)^{n-i}} \, .$$

♡ **[pfm]** Apply (4.7):

$$\frac{1}{(1-q^d)^\ell} = \sum_{j=0} \frac{j^{[\ell-1]}}{(\ell-1)!} \cdot q^{dj}$$

In the end, we get an expression in Newton form:

$$\sum_{j=0} \left(\sum_{i=1}^{n} B_{n-i,s} \cdot \frac{j^{[i-1]}}{(i-1)!} \right) q^{dj+s} \, .$$

The polynomial can also be written

$$P_{d,s}(j) = \sum_{i=0}^{n-1} B_{i,s} \cdot \frac{j^{[n-1-i]}}{(n-1-i)!} \, .$$

The final series for $D_{\mathbf{a}}$ is the sum of these, one for each divisor d and each congruence class s modulo d. The coefficient of q^m is

$$\sum_d P_{d,s_d}((m-s_d)/d)$$

if $m \equiv s_d$ modulo d.

Example. Let $\mathbf{a} = (1,2)$. Then

$$\begin{aligned}
\frac{1}{(1-q)(1-q^2)} &= \frac{(3-q)/4}{(1-q)^2} + \frac{1/4}{1+q} \\
&= \frac{1/2}{(1-q)^2} + \frac{1/4}{1-q} + \frac{1/4}{1+q} \\
&= \frac{1}{4} \cdot \frac{2}{(1-q)^2} + \frac{1}{1-q} + \frac{1}{1+q} \\
&= \frac{1}{4} \cdot \big((2+4q+6q^2+8q^3+\cdots) + (1+q+q^2+q^3+\cdots) + (1-q+q^2-q^3+\cdots) \big) \, .
\end{aligned}$$

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