The Gamma function

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I attempt here a somewhat unorthodox introduction to the Gamma function. My principal references here are [Schwartz:1965]. and [Tate:1950/1967].

Contents

1. Characters as distributions

The Schwartz space $\mathcal{S}(\mathbb{R})$ is the space of all smooth functions f on $\mathbb{R}$ such that

$$f^{(n)}(x) \ll \big(1 + |x|\big)^{-N}$$

for all n, $N \ge 0$ or, equivalently, for which

$$\|f\|_{N,n} = \sup_{\mathbb{R}} \big(1 + |x|\big)^N \big|f^{(n)}(x)\big| < \infty$$

for all non-negative integers N, n. It is a Fréchet space with these semi-norms. It contains as closed subspaces the spaces $\mathcal{S}(0,\infty)$ (resp. $\mathcal{S}(-\infty,0)$) of functions that vanish identically for $x \le 0$ (resp. $x \ge 0$), and as quotient the space $\mathcal{S}[0,\infty)$ made up of restrictions to $[0,\infty)$.

The following elementary result will be useful many times:

1.1. Lemma. *If f is a smooth function defined in a neighbourhood U of 0 in $\mathbb{R}$, then for any m it may be expressed as*

$$f(x) = \sum_{k<m} f^{(k)}(0)\,\frac{x^k}{k!} + x^m f_m(x)$$

where f_m is a smooth function defined on U.

Proof. The fundamental theorem of calculus tells us that

$$f(x) - f(0) = \int_0^x f'(s)\,ds\,.$$

An easy estimate tells us that the integral is $O(x)$, but a simple trick will do better. If we set $s = tx$ this equation becomes

$$f(x) = f(0) + x \int_0^1 f'(tx)\,dt\,,$$

and the integral

$$f_1(x) = \int_0^1 f'(tx)\,dt$$

is a smooth function of x. Induction gives us

$$f(x) = \sum_{k<m} c_k x^k + x^m f_m(x)$$

with $f_m(x)$ smooth. An easy calculation tells us that $c_k = f^{(k)}(0)/k!$.

THE SCHWARTZ SPACE OF THE POSITIVE REALS.

1.2. Proposition. *The space $\mathcal{S}(0,\infty)$ is that of all f in $C^\infty(0,\infty)$ such that*

$$x^N f^{(n)}(x)$$

is bounded on $(0,\infty)$ for all $n \geq 0$, $N \in \mathbb{Z}$.

Proof. Suppose f to lie in $\mathcal{S}(0,\infty)$. Since f is in $\mathcal{S}(\mathbb{R})$, $x^N f^{(n)}(x)$ is bounded for $N \geq 0$. But Lemma 1.1 implies that it remains true for $N \leq 0$. So the condition on f is necessary.

As for sufficiency, it must be shown that if this equation holds for all n, $N \geq 0$ then f extends to a function smooth on all of $\mathbb{R}$ vanishing on $(-\infty, 0]$. This is immediate from the definition of smoothness.

Let D be the multiplicative derivative xd/dx.

1.3. Corollary. *The space $\mathcal{S}(0,\infty)$ is the same as that of all f in $C^\infty(0,\infty)$ such that*

$$x^N [D^n f](x)$$

is bounded on $(0,\infty)$ for all $n \geq 0$, $N \in \mathbb{Z}$.

1.4. Corollary. *For any s in $\mathbb{C}$ multiplication by x^s is an isomorphism of $\mathcal{S}(0,\infty)$ with itself.*

Proof. This follows from Leibniz's formula for $(x^s f)^{(n)}$.

For every s in $\mathbb{C}$ the integral

$$\langle \Phi_s, f \rangle = \int_0^\infty x^s f(x)\,\frac{dx}{x}$$

defines therefore a continuous linear functional on $\mathcal{S}(0,\infty)$—in effect a distribution.

The multiplicative group $\mathbb{R}^\times_{>0}$ of positive real numbers acts on both of the spaces $\mathcal{S}(\mathbb{R})$ and $\mathcal{S}(0,\infty)$, as well as on their continuous linear duals, by the formulas:

$$\mu_a f(x) = f(a^{-1}x), \quad \langle \mu_a \Phi, f \rangle = \langle \Phi, \mu_{a^{-1}} f \rangle \ .$$

The scale factor a^{-1} rather than a has been chosen for compatibility with linear representations of non-abelian groups. The Lie algebra of $\mathbb{R}^\times_{>0}$ is spanned by the differential operator $D = xd/dx$, and the representations are smooth in the sense that

$$\lim_{h \to 0} \frac{\mu_{1+h} f - f}{h} = -Df \ .$$

in $\mathcal{S}(\mathbb{R})$. The $-$ sign here comes about because of the choice of a^{-1} rather than a. It will continue to annoy.

Differential operators act on distributions. If Φ is a smooth function on $(0,\infty)$ then integration by parts implies that

$$\langle \Phi', f \rangle = -\langle \Phi, f' \rangle$$

so we extend the definition of derivative to distributions accordingly. Hence

$$\langle L\Phi, f\rangle = \langle \Phi, L^* f\rangle$$

for any differential operator L, where L^* is its formal adjoint.

1.5. Proposition. *The distribution Φ_s on $\mathcal{S}(0,\infty)$ is an eigendistribution for μ_a with eigencharacter a^{-s}. Furthermore $D\Phi_s = s\Phi$.*

Proof. We have

$$\begin{aligned}\langle \mu_a \Phi_s, f\rangle &= \langle \Phi_s, \mu_{a^{-1}} f\rangle \\ &= \int_0^\infty x^s f(ax)\,\frac{dx}{x} \\ &= \int_0^\infty (y/a)^s f(y)\,\frac{dy}{y} \\ &= a^{-s}\int_0^\infty y^s f(y)\,\frac{dy}{y} \\ &= a^{-s}\langle \Phi_s, f\rangle\end{aligned}$$

so that $\mu_a \Phi_s = a^{-s}\Phi_s$ as a distribution (as well as a function).

As for the second claim:

$$\begin{aligned}\langle D\Phi_s, f\rangle &= -\langle \Phi_s, Df\rangle \\ &= -\int_0^\infty x^s f'(x)\,dx \\ &= s\int_0^\infty x^{s-1} f(x)\,dx \\ &= s\langle \Phi_s, f\rangle\end{aligned}$$

This concludes the proof of the Lemma. ∎

There is a converse to this claim, and there is also a uniqueness theorem for eigendistributions. If f lies in $\mathcal{S}(0,\infty)$, its **Mellin transform** is

$$\widehat{f}(s) = \langle \Phi_s, f\rangle\,.$$

It is uniformly bounded on any horizontally bounded strip $|\mathrm{RE}(s) \le C$. It is also holomorphic in all of $\mathbb{C}$, and

$$\widehat{Df} = s\widehat{f}\,.$$

It therefore belongs to the space $PW(0,\infty)$, the space of all function $F(s)$ holomorphic on all of $\mathbb{C}$ such that $(1+|\mathrm{IM}(s)|)^N|F(s)|$ is bounded on any horizontal strip $|\mathrm{RE}(s) \le C$, for all $N \ge 0$.

1.6. Proposition. *The map $f \mapsto \widehat{f}$ is an isomorphism of $\mathcal{S}(0,\infty)$ with $PW(0,\infty)$.*

Proof. One way is because $D\Phi_s = s\Phi_s$. The other way involves shifting contours. ∎

For any fixed s_0 the image in $\mathcal{M}(0,\infty)$ of multiplication by $s-s_0$ is the subspace of F such that $F(s_0)=0$, which is of codimension one. Hencethe quotient $\mathcal{S}(0,\infty)/(D-s)\mathcal{S}(0,\infty)$ is isomorpphic to $\mathbb{C}$, and

1.7. Corollary. *The space of distributions on $(0,\infty)$ such that $D\Phi = s\Phi$ is spanned by Φ_s.*

1.8. Corollary. *The space of distributions on $(0,\infty)$ such that $\mu_a\Phi = a^{-s}Phi$ for all a in $\mathbb{R}^\times_{>0}$ is spanned by Φ_s.*

THE SCHWARTZ SPACE OF THE NON-NEGATIVE REALS. Now define $\mathcal{S}[0,\infty)$ to be the space of restrictions to the closed half-line $[0,\infty)$ of functions in $\mathcal{S}(\mathbb{R})$. It may be identified with the quotient $\mathcal{S}(\mathbb{R})/\mathcal{S}(-\infty,0)$. The space $\mathcal{S}(0,\infty)$ is embedded in it, and again the multiplicative group acts smoothly on it.

Does there exist an eigendistribution on $\mathcal{S}[0,\infty)$ extending Φ_s? An affirmative answer follows from:

1.9. Proposition. *For* $\mathrm{RE}(s) > 0$

$$\int_0^\infty x^s f(s)\,\frac{dx}{x} == \frac{(-1)^{n+1}}{s(s+1)\dots(s+n)} \int_0^\infty x^{s+n} f^{(n+1)}(x)\,dx\,.$$

Proof. Integration by parts give us

$$\begin{aligned}
\langle \Phi_s, f\rangle &= \int_0^\infty x^s f(x)\,\frac{dx}{x} \\
&= \int_0^\infty x^{s-1} f(x)\,dx \\
&= \left[\frac{f(x)x^s}{s}\right]_0^\infty - \frac{1}{s}\int_0^\infty x^s f'(x)\,dx \\
&= -\frac{1}{s}\int_0^\infty x^{s+1} f'(x)\,\frac{dx}{x} \\
&= -\frac{1}{s}\langle \Phi_{s+1}, f'\rangle
\end{aligned}$$

and continuing:

$$\begin{aligned}
&= \frac{1}{s(s+1)}\langle \Phi_{s+2}, f''\rangle \\
&\dots \\
&= \frac{(-1)^{n+1}}{s(s+1)\dots(s+n)}\langle \Phi_{s+(n+1)}, f^{(n+1)}\rangle\,.
\end{aligned}$$

As a consequence, Φ_s may be defined on $\mathcal{S}[0,\infty)$ for all s not in $-\mathbb{N}$. Thus for every s not in $-\mathbb{N}$ we have an eigendistribution with eigencharacter x^{-s}. *Is it unique? What happens for $s = -n$?* Set $s = -n + h$ in the Lemma. We get

(1.10)
$$\langle \Phi_s, f\rangle = \frac{-1}{(n-h)(n-1-h)\dots(1-h)h}\int_0^\infty x^h f^{(n+1)}(x)\,dx\,.$$

Thus $(s+n)\langle \Phi_s, f\rangle$ as $s \to -n$ has limit

$$-\frac{1}{n!}\int_0^\infty f^{(n+1)}(x)\,dx = \frac{f^{(n)}(0)}{n!}\,.$$

The distribution δ_0 is defined to take f to $f(0)$. Its derivative $\delta_0^{(n)}$ takes f to $(-1)^n f^{(n)}(0)$. The residue of Φ_s at $s = -n$ is therefore $(-1)^n\delta_0^{(n)}/n!$.

1.11. Lemmma. *The distribution $\delta_0^{(n)}$ is an eigendistribution for the character a^n.*

Proof. Since $f^{(n)}(ax) = a^n f^{(n)}(ax)$.

In other words, the character Φ_s fails to be defined precisely when another eigencharacter arises. One way to understand the situation is by considering the short exact sequence

$$0 \to \mathcal{S}(0,\infty) \to \mathcal{S}[0,\infty) \to \mathbb{C}[[x]] \to 0$$

where the last map is that taking f to its Taylor series at 0, surjective by a classic theorem of Émile Borel. If $T = D - sI$ this gives rise to a long exact sequence

$$\begin{aligned}
0 \to \mathcal{S}(0,\infty)(T) \to \mathcal{S}[0,\infty)(T) \to \mathbb{C}[[x]]\,(T) \\
\to \mathcal{S}(0,\infty)/T\cdot\mathcal{S}(0,\infty) \to \mathcal{S}[0,\infty)/T\cdot\mathcal{S}(0,\infty) \to \mathbb{C}[[x]]/T\cdot\mathbb{C}[[x]] \to 0\,.
\end{aligned}$$

Here $V(T)$ is the subspace of v in V such that $Tv = 0$. The first two terms are always 0. When s does not belong to $-\mathbb{N}$, the third and sixth terms vanish, but when s does belong to $-\mathbb{N}$ they are both of dimension one.

1.12. Proposition. *The distribution Φ_s on $\mathcal{S}[0,\infty)$ is meromorphic on all of $\mathbb{C}$ with residue $(-1)^n\delta_0^{(n)}/n!$ at n. For each s not in $-\mathbb{N}$ it is the unique eigendistribution on $\mathcal{S}[0,\infty)$ for the character a^s. For s in $-\mathbb{N}$ the distribution $\delta_0^{(n)}$ spans the space of eigendistributions for a^n.*

Any function in $\mathcal{S}[0,\infty)$ corresponds to the function (sometimes called its **Mellin transform**)

$$\widehat{f}(s) = \langle \Phi_s, f\rangle\,.$$

1.13. Proposition. *The function $\widehat{f}$ is meromorphic on $\mathbb{C}$ with simple poles on $-\mathbb{N}$. In any bounded vertical strip $|\mathrm{RE}(s)| \le C$ away from the real axis it is uniformly rapidly decreasing as a function of $\mathrm{IM}(s)$.*

This is because $D\Phi_s = s\Phi_s$. It is not hard to show that, conversely, any function satisfying these conditions is $\widehat{f}$ for some f in $\mathcal{S}[0,\infty)$.

I want now to look at (1.10) again. It can be rewritten and expanded in powers of h:

$$\begin{aligned}\langle \Phi_s, f\rangle &= -\frac{1}{h}\cdot\frac{1}{n!}\cdot\frac{1}{(1-h/n)(1-h/n-1)\ldots(1-h)}\cdot\int_0^\infty e^{h\log x} f^{(n+1)}(x)\,dx \\ &= -\frac{1}{h}\cdot\frac{1}{n!}\cdot\big(1+h\,\Lambda_n + O(h^2)\big)\cdot\left(\int_0^\infty f^{(n+1)}(x)\,dx + h\int_0^\infty (\log x) f^{(n+1)}(x)\,dx + O(h^2)\right)\end{aligned}$$

with

$$\Lambda_n = 1 + 1/2 + 1/3 + \cdots + 1/n\,.$$

We have already seen that the leading term is $f^{(n)}(0)/n!$, and now we see that the second term in the expansion is

$$-\frac{1}{n!}\left(\Lambda_n f^{(n)}(0) + \int_0^\infty (\log x) f^{(n+1)}(x)\,dx\right)\,.$$

The integral can be expressed also as the limit as $\varepsilon \to 0$ of

$$\begin{aligned}\int_\varepsilon^\infty f^{(n+1)}(x)\log x\,dx &= \Big[f^{(n)}(x)\log x\Big]_\varepsilon^\infty - \int_\varepsilon^\infty \frac{f^{(n)}(x)}{x}\,dx \\ &= -f^{(n)}(\varepsilon)\log\varepsilon - \int_\varepsilon^\infty \frac{f^{(n)}(x)}{x}\,dx \\ &= -f^{(n)}(0)\log\varepsilon - \int_\varepsilon^\infty \frac{f^{(n)}(x)}{x}\,dx\,,\end{aligned}$$

since $f(\varepsilon) - f(0) = O(\varepsilon)$ and $\lim_{\varepsilon\to 0}\varepsilon\log\varepsilon = 0$. The second term in the Laurent expansion of $\langle \Phi_s, f\rangle$ at $s = -n$ is therefore also

(1.14) $$\frac{1}{n!}\cdot\lim_{\varepsilon\to 0}\left(f^{(n)}(0)\log\varepsilon - \Lambda_n f^{(n)}(0) + \int_\varepsilon^\infty \frac{f^{(n)}(x)}{x}\,dx\right)\,.$$

THE SCHWARTZ SPACE OF THE REAL LINE. The full multiplicative group $\mathbb{R}^\times$ acts on its own Schwartz space $\mathcal{S}(\mathbb{R}^\times)$, the subspace of functions in $\mathcal{S}(\mathbb{R})$ whose Taylor series at 0 vanish. We now have distributions

$$\langle \Phi_s^{[m]}, f\rangle = \int_{-\infty}^\infty f(x)|x|^s \mathrm{sgn}^m(x)\,\frac{dx}{|x|}$$

for $\mathrm{RE}(s) > 0$ and $m = 0, 1$, which are again eigen-distributions:

$$\mu_a \Phi_s^{[m]} = \mathrm{sgn}^m(a)|a|^{-s}\Phi_s^{[m]}\,.$$

We can express

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)|x|^s \mathrm{sgn}^m(x)\,\frac{dx}{|x|} &= (-1)^m \int_{-\infty}^{0} x^s f(x)\,\frac{dx}{|x|} + \int_0^\infty x^s f(x)\,\frac{dx}{x} \\ &= (-1)^m \int_0^\infty x^s f(-x)\,\frac{dx}{x} + \int_0^\infty x^s f(x)\,\frac{dx}{x} \\ &= \langle \Phi_s, f\rangle + (-1)^m \langle \Phi_s, f^-\rangle \\ &= \frac{(-1)^n}{s(s+1)\dots(s+n-1)}\left\langle \Phi_{s+n}, f^{(n)} + (-1)^m (f^-)^{(n)}\right\rangle\end{aligned}$$

which means that $\Phi_s^{[m]}$ extends equivariantly and meromorphically to $\mathcal{S}(\mathbb{R})$ over all of $\mathbb{C}$ with residue

$$\big((-1)^m + (-1)^n\big)\,\frac{\delta_0^{(n)}}{n!}$$

at $-n$. In particular, there is no pole if the parity of m is different from the parity of n. In this case, because of (1.14) we get as value at $-n$

$$\langle \mathrm{Pf}(1/x^{n+1}), f\rangle = \frac{1}{n!}\int_0^\infty \left[\frac{f^{(n)}(x) - f^{(n)}(-x)}{x}\right] dx$$

which always makes sense because the integrand is still a smooth function. For reasons we'll see in a moment this is called the **finite part** of $1/x^{n+1}$. This defines an extension to $\mathcal{S}(\mathbb{R})$ of the integral

$$\int_{\mathbb{R}} |x|^{-n-1}\mathrm{sgn}^{n-1}(x) f(x)\,dx = \int_{\mathbb{R}} x^{-(n+1)} f(x)\,dx\,.$$

on $\mathbb{R}^\times$.

2. Parties finies

In order to understand the nature of certain eigenfunctions of D on $\mathbb{R}$, I now recall the notion of 'parties finies', introduced in [Hadamard:1923] in order to understand classical techniques for solving the wave equation in high dimensions.

The first important observation is that the Dirac distributions are eigendistributions. For $n \geq 0$

$$D\delta_0^{(n)} = -n\,\delta_0^{(n)}\,.$$

There is, however, another distribution Φ such that $D\Phi = -n\Phi$.

2.1. Proposition. *We have*

$$\begin{aligned}\mu_a \mathrm{Pf}(1/x^{n+1}) &= a^n \mathrm{sgn}(a)\mathrm{Pf}(1/x^{n+1}) \\ D\,\mathrm{Pf}(1/x^{n+1}) &= -n\,\mathrm{Pf}(1/x^{n+1}) \\ (d/dx)\mathrm{Pf}(1/x^n) &= -n\,\mathrm{Pf}(1/x^{n+1})\,.\end{aligned}$$

Proof. Left as exercise.

The two distributions $\delta_0^{(n)}$ and $\mathrm{Pf}(1/x^{n+1})$ span the space of eigendistributions Φ on $\mathbb{R}$ such that $D\Phi = -m\Phi$, or (equivalently) $\mu_a \Phi = a^m \Phi$, but they are distinguished by what μ_{-1} does to them:

$$\mu_{-1}\delta_0^{(n)} = (-1)^n \delta_0^{(n)}, \quad \mu_{-1}\mathrm{Pf}(1/x^{n+1}) = -(-1)^n \mathrm{Pf}(1/x^{n+1})\,.$$

This has to be, of course, since a cohomological argument like the one we saw earlier shows that there at most one $\mathbb{R}^\times$-equivariant extension to all of $\mathcal{S}(\mathbb{R})$ of the distribution which on $\mathcal{S}(\mathbb{R}^\times)$ is given by the formula

$$\int_{-\infty}^{\infty} \frac{f(x)}{x^{n+1}}\,dx\,.$$

I'll say more here about the construction of *parties finies* distributions. Suppose f in $\mathcal{S}(\mathbb{R})$, and let

$$f(x) = f_0 + f_1 x + f_2 x^2 + \cdots$$

be its Taylor series at 0, so $f_m = f^{(m)}(0)/m!$. Then

$$\varphi_n(x) = \frac{f - (f_0 + xf_1 + \cdots f_n x^n)}{x^{n+1}}$$

is still smooth throughout $\mathbb{R}$, although no longer in in $\mathcal{S}(\mathbb{R})$. Then

$$\begin{aligned}\int_\varepsilon^\infty \frac{f(x)}{x^{n+1}}\,dx &= \int_\varepsilon^1 \frac{f(x)}{x^{n+1}}\,dx + \int_1^\infty \frac{f(x)}{x^{n+1}}\,dx \\ &= \int_\varepsilon^1 \frac{f_0 + f_1 x + \cdots + f_n x^n}{x^{n+1}}\,dx + \int_\varepsilon^1 \varphi_n(x)\,dx + \int_1^\infty \frac{f(x)}{x^{n+1}}\,dx\,.\end{aligned}$$

The last integral is independent of ε. As $\varepsilon \to 0$, the second integral has a finite limit. The first integral is

$$\begin{aligned}&\Big[-\frac{f_0}{nx^n} - \frac{f_1}{(n-1)x^{n-1}} - \cdots - f_n \log x\Big]_\varepsilon^1 \\ &\qquad = -\frac{f_0}{n} - \frac{f_1}{(n-1)} - \cdots - f_{n-1} + \frac{f_0}{n\varepsilon^n} + + \frac{f_1}{(n-1)\varepsilon^{n-1}} + \cdots + f_n \log\varepsilon\end{aligned}$$

Therefore the limit

$$\lim_{\varepsilon\to 0} \int_\varepsilon^\infty \frac{f(x)}{x^{n+1}}\,dx - \Big(\frac{f_0}{n\varepsilon^n} + \frac{f_1}{(n-1)\varepsilon^{n-1}} + \cdots + f_n \log\varepsilon\Big)$$

exists, and agrees with $\mathrm{Pf}(1/x^{n+1})$.

The distribution $\mathrm{Pf}(1/x^{n+1})$ on $[0,\infty)$ does not behave equivariantly with respect to scalar multiplication, because of the $\log\varepsilon$ term. But on $\mathbb{R}$ the finite part is

$$\lim_{\varepsilon\to 0} \Big(\int_{-\infty}^{\varepsilon} \frac{f(x)}{x^{n+1}}\,dx + \int_\varepsilon^\infty \frac{f(x)}{x^{n+1}}\,dx\Big) - \Big(\sum_{\substack{k=1 \\ k \text{ odd}}}^{n} \frac{2f_{n-k}}{k\varepsilon^k}\Big)\,,$$

and it does behave well, because on $(-\infty, 0]$ $\log\varepsilon$ is replaced by $\log|\varepsilon|$.

3. The Gamma function

One function in $\mathcal{S}[0,\infty)$ is the restriction of $f(x) = e^{-x}$ to $[0,\infty)$. The Gamma function is defined to be the integral

$$\Gamma(s) = \int_0^\infty x^s e^{-x}\,\frac{dx}{x} = \langle \Phi_s, e^{-x} \rangle\,.$$

for $\mathrm{RE}(s) > 0$. The argument extending Φ_s in the last section is classical in this case. Since here $f'(x) = -f(x)$, we have the functional equation

$$\Gamma(s+1) = s\,\Gamma(s)$$

and since $\Gamma(1) = 1$, we see by induction that if s is a positive integer n

$$\Gamma(n) = (n-1)!$$

The extension formula can be rewritten as

$$\Gamma(s) = \frac{\Gamma(s+1)}{s}$$

so that we can extend the definition of $\Gamma(s)$ to the region $\mathrm{RE}(s) > -1$, except for $s = 0$. And so on. More explicitly we have

$$\Gamma(s) = \frac{\Gamma(s+n+1)}{(s+n)(s+n-1)\ldots(s+1)s}$$

which allows $\Gamma(s)$ to be defined for $\mathrm{RE}(s) > -n-1$, except at the negative integers, where it will have simple poles (of order one).

Proposition. *For $n \ge 0$ the residue of $\Gamma(s)$ at $-n$ is $(-1)^n/n!$*

Another formula for $\Gamma(s)$ can be obtained by a change of variables $t = \pi x^2$:

$$\Gamma(s) = 2\pi^s \int_0^\infty e^{-\pi x^2} x^{2s}\,\frac{dx}{x}$$

which can also be written as

$$\Gamma\left(\frac{s}{2}\right) = \pi^{s/2} \int_{-\infty}^\infty |x|^s e^{-\pi x^2}\,\frac{dx}{|x|}$$

or

$$\pi^{-s/2}\,\Gamma\left(\frac{s}{2}\right) = \int_{-\infty}^\infty |x|^s e^{-\pi x^2}\,\frac{dx}{|x|}\,.$$

This function of s is often expressed as $\zeta_{\mathbb{R}}(s)$ because of its role in functional equations of ζ functions.

4. The volumes and areas of spheres

If we set $s = 1$ in the formula for $\zeta_{\mathbb{R}}$ at the end of the last section, we get

$$\pi^{-1/2}\Gamma\left(\frac{1}{2}\right) = \int_{-\infty}^{\infty} e^{-\pi x^2}\,dx\,.$$

The integral on the right cannot be evaluated as an improper integral, but there is a well known trick one can use to evaluate the infinite integral. We move into two dimensions. We can shift to polar coordinates and get

$$\begin{aligned}\left(\int_{\mathbb{R}} e^{-\pi x^2}\,dx\right)^2 &= \int_{\mathbb{R}} e^{-\pi x^2}\,dx \cdot \int_{\mathbb{R}} e^{-\pi y^2}\,dy \\ &= \int_{\mathbb{R}^2} e^{-\pi(x^2+y^2)}\,dx\,dy \\ &= \int_0^{\infty} dr \int_0^{2\pi} e^{-\pi r^2} r\,d\theta \\ &= \int_0^{\infty} 2\pi r e^{-\pi r^2}\,dr \\ &= \int_0^{\infty} 2\sqrt{\pi} r e^{-\pi r^2}\,dr \\ &= \int_0^{\infty} e^{-\pi r^2}(2\pi r)\,dr \\ &= \int_0^{\infty} e^{-s}\,ds \\ &= 1\,,\end{aligned}$$

so $\pi^{-1/2}\Gamma(1/2) = 1$, and $\Gamma(1/2) = \sqrt{\pi}$.

We can use this formula and the same trick to find a formula for the volumes of spheres in n dimensions. Let S_{n-1} be the volume of the unit sphere in $\mathbb{R}^n$. Then

$$\begin{aligned}\left(\int_{\mathbb{R}} e^{-\pi x^2}\,dx\right)^n &= 1 \\ &= \int_{\mathbb{R}^n} e^{-\pi r^2}\,dx_1\,\ldots\,dx_n \\ &= \int_0^{\infty} S_{n-1}\,r^{n-1} e^{-\pi r^2}\,dr \\ &= \int_0^{\infty} S_{n-1}\,r^n e^{-\pi r^2}\,\frac{dr}{r} \\ &= S_{n-1}\frac{1}{2}\pi^{-n/2}\Gamma(n/2)\,. \\ S_{n-1} &= \frac{2\pi^{n/2}}{\Gamma(n/2)}\,.\end{aligned}$$

For example, the area of the two-sphere in $\mathbb{R}^3$ is

$$S_2 = \frac{2\pi^{3/2}}{\Gamma(3/2)} = \frac{2\pi^{3/2}}{\pi/2} = 4\pi\,.$$

The volume of the n-ball of radius R in $\mathbb{R}^n$ is

$$V_n(R) = \int_0^R S_{n-1} r^{n-1}\,dr = \frac{S_{n-1}R^n}{n}\,.$$

5. Tate's functional equation

Now I introduce the Fourier transform and its interaction with the multiplicative group. For f in $\mathcal{S}(\mathbb{R})$ its Fourier transform is

$$\widehat{f}(\lambda) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i \lambda x}\, dx$$

and this defines an isomorphism of $\mathcal{S}(\mathbb{R})$ with itself. The inverse is

$$f(x) = \int_{-\infty}^{\infty} \widehat{f}(x) e^{2\pi i \lambda x}\, dx$$

Another way to express this is that $\widehat{\widehat{f}} = \mu_{-1} f$.

How do the Fourier transform and the multiplication operators interact?

5.1. Proposition. *For $a \neq 0$*

$$\widehat{\mu_a f} = |a|\, \mu_{a^{-1}} \widehat{f}\,.$$

Proof. Because

$$\begin{aligned} \widehat{\mu_a f}(\lambda) &= \int_{-\infty}^{\infty} [\mu_a f](x) e^{-2\pi i \lambda x}\, dx \\ &= \int_{-\infty}^{\infty} f(a^{-1}x) e^{-2\pi i \lambda x}\, dx \\ &= |a| \int_{-\infty}^{\infty} f(y) e^{-2\pi i \lambda a y}\, dy \\ &= |a| \mu_{a^{-1}} \widehat{f}(\lambda)\,. \ \blacksquare \end{aligned}$$

The Fourier transform $\widehat{\Phi}$ of a distribution Φ is defined by

$$\langle \widehat{\Phi}, f \rangle = \langle \Phi, \widehat{f} \rangle\,.$$

This, as an easy calculation will show, agrees with the definition the Fourier transform on $\mathcal{S}(\mathbb{R})$.

Suppose χ to be a multiplicative character. The distribution $\Phi = \Phi_\chi$ is defiend by

$$\langle \Phi_\chi, \rangle = \int_{\mathbb{R}} \chi(x) f(x)\, \frac{dx}{x}$$

defined by convergence for certain χ and extended meromorphically. What is the Fourier transform of Φ? Since $\mu_a \Phi_\chi = \chi^{-1}(a) \Phi_\chi$ we have

$$\begin{aligned} \langle \mu_a \widehat{\Phi}, f \rangle &= \langle \widehat{\Phi}, \mu_{a^{-1}} f \rangle \\ &= \langle \Phi, \widehat{\mu_{a^{-1}} f} \rangle \\ &= \langle \Phi, |a|^{-1} \mu_a \widehat{f} \rangle \\ &= |a|^{-1} \langle \mu_{a^{-1}} \Phi, \widehat{f} \rangle \\ &= |a|^{-1} \chi(a) \langle \Phi, \widehat{f} \rangle \\ &= |a|^{-1} \chi(a) \langle \widehat{\Phi}, f \rangle \end{aligned}$$

so because of uniqueness $\widehat{\Phi}$ must be a scalar multiple $\gamma_\chi \Phi_{\widetilde{\chi}}$ where $\widetilde{\chi}(a) = |a| \chi^{-1}(a)$. To calculate the scalar γ_χ explicitly, we calculate first the Fourier transform of some particular functions.

5.2. Lemma. *The Fourier transform of $e^{-\pi x^2}$ is itself.*

Proof. Let $f(x) = e^{-\pi x^2}$. Then

$$\begin{aligned}
\widehat{f}(\lambda) &= \int_{-\infty}^{\infty} e^{-2\pi i\lambda x - \pi x^2}\,dx \\
&= e^{-\pi\lambda^2}\int_{-\infty}^{\infty} e^{\pi\lambda^2 - 2\pi i\lambda x - \pi x^2}\,dx \\
&= e^{-\pi\lambda^2}\int_{-\infty}^{\infty} e^{-\pi(x-i\lambda)^2}\,dx \\
&= e^{-\pi\lambda^2}\int_{-\infty}^{\infty} e^{-\pi x^2}\,dx \\
&= e^{-\pi\lambda^2}\,. \quad \square
\end{aligned}$$

Let now $\chi(x) = |x|^s$, $\Phi = \Phi_\chi$. Then

$$\begin{aligned}
\langle \widehat{\Phi}, e^{-\pi x^2}\rangle &= \langle \Phi, e^{-\pi x^2}\rangle \\
&= \int_{-\infty}^{\infty} |x|^s e^{-\pi x^2}\,\frac{dx}{|x|} \\
&= \pi^{-s/2}\int_{-\infty}^{\infty} |x|^s e^{-x^2}\,\frac{dx}{|x|} \\
&= \pi^{-s/2}\Gamma(s/2) \\
&= \zeta_{\mathbb{R}}(s)\,.
\end{aligned}$$

Since $\widetilde{\chi} = |x|^{1-s}$:

5.3. Proposition. *We have*

$$\widehat{\Phi}_{s,0} = \gamma_s \Phi_{1-s,0}$$

where

$$\gamma_s = \frac{\zeta_{\mathbb{R}}(s)}{\zeta_{\mathbb{R}}(1-s)}\,.$$

This formula isn't quite right for values of s where $\Phi_{s,0}$ or $\Phi_{1-s,0}$ have poles. The simplest way to formulate things is to observe that $\Phi_{s,0}/\zeta(s)$ is entire, and that this formula says that the Fourier transform of $\Phi_{s,0}/\zeta_{\mathbb{R}}(s)$ is $\Phi_{1-s,0}/\zeta_{\mathbb{R}}(1-s)$.

We can reason similarly for $|x|^s\mathrm{sgn}(x)$ with $xe^{\pi x^2}$.

5.4. Proposition. *The Fourier transform of $xe^{-\pi x^2}$ is $-i\lambda e^{-\pi\lambda^2}$.*

Proof. Differentiate the equation

$$\int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i\lambda x}\,dx = e^{-\pi\lambda^2}$$

with respect to λ. $\square$

Therefore

$$\begin{aligned}
\langle \widehat{\Phi_{s,1}}, xe^{-\pi x^2}\rangle &= \langle \Phi_{s,1}, -ixe^{-\pi x^2}\rangle \\
&= -i\int_{\mathbb{R}} |x|^{s-1}\mathrm{sgn}(x)xe^{-\pi x^2}\,dx \\
&= -i\int_{\mathbb{R}} |x|^s e^{-\pi x^2}\,dx \\
&= -i\zeta_{\mathbb{R}}(1+s) \\
\langle \Phi_{1-s,1}, xe^{-\pi x^2}\rangle &= \zeta_{\mathbb{R}}\big(1+(1-s)\big)
\end{aligned}$$

and hence:

5.5. Proposition. *We have*

$$\widehat{\Phi}_{s,1} = \lambda_s \Phi_{1-s,1}$$

where

$$\lambda_s = -i\,\frac{L_{\mathbb{R}}(s)}{L_{\mathbb{R}}(1-s)}, \quad L_{\mathbb{R}}(s) = \pi^{-(s+1)/2}\,\Gamma\left(\frac{s+1}{2}\right).$$

I conclude with a useful calculation, then I examine some special cases.

5.6. Proposition. *Suppose Φ to be a tempered distribution on $\mathbb{R}$. Then*

(a) the Fourier transform of Φ' is $2\pi i\lambda\widehat{\Phi}$;
(b) the Fourier transform of $x\Phi$ is $\widehat{\Phi}'/(-2\pi i)$.

Proof. First assume Φ to be in $\mathbb{S}(\mathbb{R})$. The first assertion follows from integration by parts, the second by differentiating

$$\int_{-\infty}^{\infty} \Phi(x)e^{-2\pi i\lambda x}\,dx = \widehat{\Phi}(\lambda)$$

with respect to λ. Proving the assertion for distributions follows from this simpler case.

The distributions defined by integrals

$$\int_{\mathbb{R}} x^n f(x)\,dx, \quad \int_{\mathbb{R}} x^n \mathrm{sgn}(x) f(x)\,dx$$

are of particular importance.

5.7. Proposition. *For $n \ge 0$*

(a) the Fourier transform of x^n is $\delta_0^{(n)}/(-2\pi i)^n$;
(b) the transform of $x^n\mathrm{sgn}(x)$ is

$$\frac{2\,n!}{(2\pi i)^{n+1}}\,\mathrm{Pf}(1/x^{n+1}).$$

As for the first, calculation shows that the transform of 1 is δ_0. But then by the previous lemma the transform of x^n is $\delta_0^{(n)}/(-2\pi i)^n$.

For the second, we can write $x^n\mathrm{sgn}(x)$ as $|x|^{n+1}\mathrm{sgn}^{n+1}(x)/|x|$, so its transform will be an eigendistribution for $|x|^{-n}\mathrm{sgn}^{n+1} = x^n\mathrm{sgn}(x)$, which means that it is a multiple of $\mathrm{Pf}(1/x^{n+1})$. To compute the constant, let's look at $n = 0$, where we want the Fourier transform of $\mathrm{sgn}(x)$ itself. Here

$$\langle \widehat{\mathrm{sgn}}, xe^{-\pi x^2}\rangle = \frac{-i}{\pi} = \frac{2}{2\pi i}$$
$$\langle \mathrm{Pf}(1/x), xe^{-\pi x^2}\rangle = 1$$

so that the transform of sgn is $(2/2\pi i)\mathrm{Pf}(1/x)$. Then

$$\widehat{x^n\mathrm{sgn}(x)} = \frac{1}{(-2\pi i)^n}\,\frac{2}{2\pi i}\big(\mathrm{Pf}(1/x)\big)^{(n)} = \frac{2n!}{(2\pi i)^{n+1}}\,\mathrm{Pf}(1/x^{n+1})$$

since $\mathrm{Pf}(1/x^n)' = -n\,\mathrm{Pf}(1/x^{n+1})$.

6. The Beta function

The Gamma function appears in a wide variety of integration formulas. One of the most useful is:

6.1. Proposition. *We have*

$$\int_0^\infty \frac{t^\alpha}{(1+t^2)^\beta}\,dt = \frac{1}{2}\,\frac{\Gamma\left(\frac{\alpha+1}{2}\right)\Gamma\left(\beta-\frac{\alpha+1}{2}\right)}{\Gamma(\beta)}$$

Proof. Start with

$$\Gamma(s) = 2\int_0^\infty e^{-x^2} x^{2s-1}\,dx\ .$$

Moving to two dimensions and switching to polar coordinates:

$$\begin{aligned}
\Gamma(u)\Gamma(v) &= 4\int\int_{s\ge 0,t\ge 0} e^{-s^2-t^2} s^{2u-1} t^{2v-1}\,ds\,dt \\
&= 4\int\int_{r\ge 0, 0\le\theta\le\pi/2} e^{-r^2} r^{2(u+v)-1}\cos^{2u-1}\theta\sin^{2v-1}\theta\ dr\,d\theta \\
&= 4\int_{r\ge 0} e^{-r^2} r^{2(u+v)-1}\,dr\ \int_0^{\pi/2}\cos^{2u-1}(\theta)\sin^{2v-1}(\theta)\,d\theta \\
&= \Gamma(u+v)B(u,v) \\
B(u,v) &= \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}\ ,
\end{aligned}$$

where

$$B(u,v) = 2\int_0^{\pi/2}\cos^{2u-1}(\theta)\ \sin^{2v-1}(\theta)\,d\theta\ .$$

If we change variables to $t = \tan(\theta)$ we get

$$\begin{aligned}
\theta &= \arctan(t) \\
d\theta &= dt/(1+t^2) \\
\cos(\theta) &= 1/\sqrt{1+t^2} \\
\sin(\theta) &= t/\sqrt{1+t^2}
\end{aligned}$$

leading to

$$\int_0^\infty \frac{t^\alpha}{(1+t^2)^\beta}\,dr = \frac{1}{2}\,\frac{\Gamma\left(\frac{\alpha+1}{2}\right)\Gamma\left(\beta-\frac{\alpha+1}{2}\right)}{\Gamma(\beta)}\ ,$$

and in particular

$$\Gamma^2(1/2) = \int_{-\infty}^\infty \frac{dr}{1+r^2} = \pi,\quad \Gamma(1/2) = \sqrt{\pi}\ .$$

7. The limit product formula

The exponential function e^{-t} can be approximated by finite products.

Lemma. *For any real t we have*

$$e^{-t} = \lim_{n\to\infty} \left(1 - \frac{t}{n}\right)^n .$$

This can be seen most easily by taking logarithms since for $0 \le t < n$

$$\begin{aligned}
\log\left(1-\frac{t}{n}\right)^n &= n\log\left(1-\frac{t}{n}\right) \\
&= n\left(-\left(\frac{t}{n}\right) - \frac{1}{2}\left(\frac{t}{n}\right)^2 - \frac{1}{3}\left(\frac{t}{n}\right)^3 - \dots\right) \\
&= -t - \frac{t^2}{2n} - \frac{t^3}{3n^2} - \dots \\
&= -t - T
\end{aligned}$$

where

$$T = \frac{t^2}{2n} + \frac{t^3}{3n^2} + \dots .$$

which converges to 0 as $n \to \infty$.

Another way of putting this is to define

$$\varphi_n(t) = \begin{cases} (1-t/n)^n & 0 \le t \le n \\ 0 & t > n \end{cases}$$

and then define for each n an approximation $\Gamma_n(s)$ to $\Gamma(s)$:

$$\begin{aligned}
\Gamma_n(s) &= \int_0^\infty t^{s-1}\varphi_n(t)\,dt \\
&= \int_0^n t^{s-1}\left(1-\frac{t}{n}\right)^n\,dt
\end{aligned}$$

On the one hand, this can be explicitly calculated through repeated integration by parts:

$$\int_0^n t^{s-1}\left(1-\frac{t}{n}\right)^n dt = \frac{1}{s}\,\frac{n-1}{n(s+1)}\,\frac{n-2}{n(s+2)}\dots\frac{1}{n(s+n-1)}\int_0^n t^{s+n-1}\,dt = \frac{n!\,n^s}{s(s+1)\dots(s+n)}$$

On the other, since for all fixed t the limit of $\varphi_n(t)$ as $n\to\infty$ is equal to e^{-t}, and both $\varphi_n(t)$ and e^{-t} are small at ∞, this is at least plausible:

Proposition. *For any s with $\mathrm{RE}(s) > 1$ the limit of $\Gamma_n(s)$ as $n\to\infty$ is equal to $\Gamma(s)$. In other words, for any s in $\mathbb{C}$*

$$\Gamma(s) = \lim_{n\to\infty} \frac{n!\,n^s}{s(s+1)\dots(s+n)}.$$

The **Euler constant** γ is defined to be the limit

$$\gamma = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \dots \frac{1}{n-1}\right) - \log n.$$

The limit product formula implies immediately a limit formula for $1/\Gamma(s)$:

$$\frac{1}{\Gamma(s)} = \lim_{n\to\infty} \left[s\left(1+\frac{s}{1}\right)\left(1+\frac{s}{2}\right)\ldots\left(1+\frac{s}{n-1}\right) n^{-s}\right]$$

but

$$n^{-s} = e^{-s\log n} = e^{-s\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n-1}\right)+s\gamma_n}$$

where $\gamma_n \to \gamma$. Therefore:

Proposition. *The inverse Gamma function has the product expansion*

$$\frac{1}{\Gamma(s)} = se^{\gamma s}\prod_1^\infty \left(1+\frac{s}{n}\right) e^{-s/n}$$

where γ is Euler's constant.

The limit product formula also implies Legendre's duplication formula:

$$\Gamma\left(\frac{1}{2}\right)\Gamma(s) = 2^{s-1}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)$$

Explicitly

$$\begin{aligned}\Gamma\left(\frac{s}{2}\right) &= \lim_{n\to\infty} \frac{2^{n+1} n!\, n^{s/2}}{s(s+2)\ldots(s+2n)}\\ \Gamma\left(\frac{s+1}{2}\right) &= \lim_{n\to\infty}\frac{2^{n+1} n!\, n^{s+1/2}}{(s+1)\ldots(s+2n+1)}\end{aligned}$$

so

$$\begin{aligned}&2^s\,\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)\Big/\Gamma(s)\\ &\qquad= \lim_{n\to\infty} 2^s \frac{2^{n+1}n!\,n^{s/2}}{s(s+2)\ldots(s+2n)}\frac{2^{n+1}n!\,n^{(s+1)/2}}{(s+1)(s+3)\ldots(s+2n+1)}\frac{s(s+1)(s+2)\ldots(s+2n)}{(2n)!\,(2n)^s}\\ &\qquad= \lim_{n\to\infty}\frac{(n!)^2}{(2n)!}\frac{2^{2n+2}\,n^{1/2}}{(s+2n+1)}\\ &\qquad= \lim_{n\to\infty}\frac{(n!)^2 2^{2n+1}}{(2n)!\,\sqrt{n}}\end{aligned}$$

but this last does not depend on s, and is finite since the limit on the left hand side exists, so we may set $s = 1/2$ to see that it is equal to $2\sqrt{\pi}$.

8. The reflection formula

The formula for the Beta function gives us

$$\begin{aligned}\Gamma(s)\Gamma(1-s) &= \int_0^1 u^{s-1}(1-u)^{-s}\,du \\ &= \int_0^\infty \frac{v^{s-1}}{1+v}\,dv \quad (u = v/1+v, v = (u/1-u), du/(1-u) = (1+v)dv)\end{aligned}$$

We can calculate this last integral by means of a contour integral in $\mathbb{C}$. Let C be the path determined by these four segments: (1) along the positive real axis, or just above it, from ϵ to R; (2) around the circle of radius R, counter-clockwise, to the point just below R; (3) along and just below the real axis to ϵ; (4) around the circle of radius ϵ, clockwise, to just above ϵ. We want to calculate the limit of the integral

$$\int_C \frac{z^{s-1}}{1+z}\,dz$$

as $\epsilon \to 0$ and $R \to \infty$.

On the one hand the integrals over the different components converge to

$$\int_0^\infty \frac{z^{s-1}}{1+z}\,dz + 0 - e^{2\pi is}\int_0^\infty \frac{z^{s-1}}{1+z}\,dz + 0 = (1-e^{2\pi is})\int_0^\infty \frac{z^{s-1}}{1+z}\,dz$$

But on the other there is exactly one pole inside the curves C, so the integral is also equal to $-2\pi i e^{\pi is}$. Therefore

$$\Gamma(s)\Gamma(1-s) = \int_0^\infty \frac{z^{s-1}}{1+z}\,dz = \frac{-2\pi i e^{\pi is}}{1-e^{2\pi is}} = \frac{\pi}{\sin \pi s}$$

Incidentally, combined with the product formula for $\Gamma(s)$ this gives the product formula for $\sin \pi s$

$$\sin \pi s = \pi s \prod_1^\infty \left(1 - \frac{s^2}{n^2}\right)$$

9. The Euler-Maclaurin formula

Define a sequence of polynomials

$$\begin{aligned}B_0(x) &= 1 \\ B_1(x) &= x - 1/2 \\ B_2(x) &= x^2 - x + 1/6 \\ &\ldots\end{aligned}$$

recursively determined by

$$B'_{n+1}(x) = nB_n(x), \qquad \int_0^1 B_n(x)\,dx = 0\,.$$

These are the **Bernoulli polynomials**. They determine in turn functions ψ_n by extension to all of $\mathbb{R}$ of period 1.

The following is a simple version of the much more interesting Euler-Maclaurin sum formula:

Proposition. *Suppose f to be a function on the interval $[k, \ell]$ which has continuous second derivatives. Then*

$$f(k) + f(k+1) + \ldots f(\ell - 1) = \int_k^\ell f(x)\,dx - \frac{1}{2}\big(f(\ell) - f(k)\big) + \frac{1}{12}\big(f'(\ell) - f'(k)\big) + R_2$$

where

$$R_2 = -\frac{1}{2}\int_k^\ell f''(x)\psi_2(x)\,dx\,.$$

The proof is very simple, a repetition of integration by parts. Suppose m to be an integer with f defined and continuously differentiable on $[m, m+1]$. Then since $\psi_0 = 1$ and $\psi_1' = \psi_0$

$$\begin{aligned}\int_m^{m+1} f(x)\,dx &= \int_m^{m+1} f(x)\psi_0(x)\,dx \\ &= [f(x)\psi_1(x)]_m^{m+1} - \int_m^{m+1} f'(x)\psi_1(x)\,dx \\ &= \frac{1}{2}\big(f(m) + f(m+1)\big) - \int_m^{m+1} f'(x)\psi_1(x)\,dx\end{aligned}$$

since $\psi_1'(x) = 1$, and of course we look at the limit of ψ_1 from above at m, the limit from below at $m+1$. Then we sum this equation over all the unit sub-intervals of $[k, \ell]$, using the periodicity of ψ_1.

$$\int_k^\ell f(x)\,dx = (1/2)f(k) + f(k+1) + \ldots + f(\ell - 1) + (1/2)f(\ell) - \int_k^\ell f'(x)\psi_1(x)\,dx$$

We can rewrite this and apply integration by parts successively:

$$\begin{aligned}&f(k) + f(k+1) + \ldots + f(\ell - 1) \\ &\quad = \int_k^\ell f(x)\,dx - \frac{1}{2}\big(f(\ell) - f(k)\big) + \int_k^\ell f'(x)\psi_1(x)\,dx \\ &\quad = \int_k^\ell f(x)\,dx - \frac{1}{2}\big(f(\ell) - f(k)\big) + \frac{1}{2}\big(\psi_2(\ell)f'(\ell) - \psi_2(k)f'(k)\big) - \frac{1}{2}\int_k^\ell f''(x)\psi_2(x)\,dx \\ &\quad = \int_k^\ell f(x)\,dx - \frac{1}{2}\big(f(\ell) - f(k)\big) + \frac{1}{12}\big(f'(\ell) - f'(k)\big) - \frac{1}{2}\int_k^\ell f''(x)\psi_2(x)\,dx\end{aligned}$$

The calculations can be continued to obtain an infinite asymptotic expansion involving the polynomials and their constant terms, the **Bernoulli numbers**.

10. Stirling's formula

We know that the Gamma function can be evaluated as a limit product

$$\begin{aligned}\Gamma(s) &= \lim_{n\to\infty} \frac{(n-1)!\,(n-1)^s}{s(s+1)\dots(s+n-1)} \\ &= \lim_{n\to\infty} \frac{(n-1)!\,n^s}{s(s+1)\dots(s+n-1)}\left(\frac{n-1}{n}\right)^s \\ &= \lim_{n\to\infty} \frac{(n-1)!\,n^s}{s(s+1)\dots(s+n-1)}\end{aligned}$$

We have proven this for s in the domain of convergence of the integral defining $\Gamma(s)$, but in fact the limit exists and defines an analytic function for all s except $s=-n$ with n a non-negative integer, so that by the principle of analytic continuation it must be valid wherever $\Gamma(s)$ is defined. As a consequence

$$\log\Gamma(s) = \lim_{n\to\infty} S_{n-1}(1) - S_n(s) + s\log n$$

where

$$S_n(s) = \log s + \log(s+1) + \ldots + \log(s+n-1)$$

We can evaluate $S_n(s)$ by the Euler-Maclaurin formula

$$\begin{aligned}&f(0)+f(1)+\ldots+f(n-1) \\ &\qquad = \int_0^n f(x)\,dx - \frac{1}{2}\big(f(n)-f(0)\big) + \frac{\beta_2}{2}\big(f'(n)-f'(0)\big) - \frac{1}{2}\int_0^n f^{(2)}(x)\psi_2(x)\,dx\end{aligned}$$

with

$$f(x) = \log(s+x), \quad f'(x) = \frac{1}{s+x}, \quad f^{(2)}(x) = -\frac{1}{(s+x)^2}$$

so

$$\begin{aligned}&\log s + \log(s+1) + \ldots + \log(s+n-1) \\ &\quad = \int_0^n \log(s+x)\,dx - \frac{1}{2}\left[\log(s+n) - \log s\right] + \frac{1}{12}\left[\frac{1}{s+n} - \frac{1}{s}\right] + \frac{1}{2}\int_0^n \frac{\psi_2(x)}{(s+x)^2}\,dx \\ &\quad = \left[x\log x - x\right]_s^{s+n} - \frac{1}{2}\left[\log(s+n) - \log s\right] + \frac{1}{12}\left[\frac{1}{s+n} - \frac{1}{s}\right] + \frac{1}{2}\int_0^n \frac{\psi_2(x)}{(s+x)^2}\,dx \\ &\quad = (s+n-1/2)\log(s+n) - (s-1/2)\log s - n + \frac{1}{12}\left[\frac{1}{s+n} - \frac{1}{s}\right] + \frac{1}{2}\int_0^n \frac{\psi_2(x)}{(s+x)^2}\,dx\end{aligned}$$

and setting $s=1$, $n-1$ for n:

$$\begin{aligned}&\log 1 + \log 2 + \ldots + \log n \\ &\qquad = (n-1/2)\log n - (n-1) + \frac{1}{12}\left[\frac{1}{n} - 1\right] + \frac{1}{2}\int_1^n \frac{\psi_2(x)}{x^2}\,dx \\ &\qquad = (n-1/2)\log n - n + \frac{11}{12} + \frac{1}{12n} + \frac{1}{2}\int_1^n \frac{\psi_2(x)}{x^2}\,dx \\ &\qquad = (n-1/2)\log n - n + C + \frac{1}{12n} - \frac{1}{2}\int_n^\infty \frac{\psi_2(x)}{x^2}\,dx\end{aligned}$$

where we define the constant

$$C = \frac{11}{12} + \frac{1}{2}\int_1^\infty \frac{\psi_2(x)}{x^2}\,dx.$$

Taking limits, therefore

$$\log \Gamma(s) = \left(s - \frac{1}{2}\right) \log s - s + C + \frac{1}{12s} - \frac{1}{2}\int_0^\infty \frac{\psi_2(x)}{(s+x)^2}\,dx.$$

This is valid for all s not on the negative real axis, and gives immediately the generalization of Stirling's formula

$$\Gamma(s) \sim \frac{e^C}{\sqrt{s}}\left(\frac{s}{e}\right)^s$$

as s goes to infinity in any region

$$-\pi + \delta \le \arg(s) \le \pi - \delta$$

since the remainder will have a uniform estimate in this region. The constant C can be evaluated by letting $t \to \pm\infty$ in the reflection formula. On the one hand

$$\begin{aligned}\Gamma(it)\Gamma(-it) &= -\frac{\pi}{it \sin \pi it} \\ &= -\frac{2\pi i}{it[e^{-\pi t} - e^{\pi t}]} \\ &\sim 2\pi t^{-1} e^{-\pi t}\end{aligned}$$

while on the other

$$\begin{aligned}\Gamma(it)\Gamma(-it) &\sim \frac{e^C}{\sqrt{it}}\left(\frac{it}{e}\right)^{it} \frac{e^C}{\sqrt{-it}}\left(\frac{-it}{e}\right)^{-it} \\ &= \frac{e^{2C}}{t}(i)^{it}(-i)^{-it} \\ &= \frac{e^{2C}}{t} e^{(it)(\pi i)/2} e^{(-it)(-\pi i)/2} \\ &= \frac{e^{2C}}{t} e^{-\pi t}\end{aligned}$$

I recall that

$$x^y = e^{y \log x}$$

where $\log$ is given its principal value. This gives

$$C = \log \sqrt{2\pi}$$

and finally the explicit version

Proposition. *(Stirling's asymptotic formula) As s goes to ∞ in the region*

$$-\pi + \delta \le \arg(s) \le \pi - \delta$$

we have the asymptotic estimate

$$\Gamma(s) \sim \sqrt{\frac{2\pi}{s}}\left(\frac{s}{e}\right)^s$$

11. References

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Chapter 1 of Book III introduces 'finite parts' of integrals. This notion is necessary in order to interpret the fundamental solutions to the wave equation in high dimensions.

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