Quadratic forms over local fields

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This essay is one of a series that I intend to be an introduction to the Weil representations of $\mathrm{SL}_2(k)$, for k a local field. It amounts to an introduction to Weil's theory of Fresnel integrals. For real fields these are classical objects in the physics of diffraction, and for $\mathfrak{p}$-adic fields they are natural generalizations of Gauss sums for forms over finite fields. Following Minkowski, I shall use them to classify quadratic forms over $\mathfrak{p}$-adic fields.

Earlier essays in this series are *Introduction to quadratic forms* and *Quadratic forms over finite fields*. In this version, I shall assume every field not to have characteristic two.

Contents

Part I. Fourier transforms and quadratic forms

1. Fourier transforms

Let F be a local field.

SCHWARTZ SPACES. Suppose V to be a vector space of dimension n over F, which I may as well take to be F^n.

• Suppose F to be $\mathbb{R}$. The Schwartz space $\mathbb{S}(V)$ is that of all smooth functions f on V for which all semi-norms

$$\|f\|_{m,n} = \sup_v \Big| [\partial^m f/\partial x^m](v) \Big| \cdot \|v\|^n$$

are bounded. These assign to $\mathcal{S}(V)$ the structure of a Fréchet space. A **tempered distribution** on V is a continuous linear function on $\mathcal{S}(V)$. To define $\mathcal{S}(\mathbb{C})$, consider $\mathbb{C}^n$ as $\mathbb{R}^{2n}$.

• Now suppose F to be $\mathfrak{p}$-adic. If $L \subseteq M$ are two lattices in V, the vector space $\mathbb{C}(M/L)$ embeds into $\mathbb{C}(V)$ as the subspace of functions with support in M and invariant under translation by elements of L. The Schwartz space of V is the union of all of these—the space of all locally constant complex-valued functions on V of compact support. This possesses the trivial topology according to which U is open if the intersection of U with each of these subspaces is open. All linear functions are continuous.

THE STANDARD ADDITIVE CHARACTERS. If ψ is a non-trivial character of the additive group F, then every other character is some $\psi_a(x) = \psi(ax)$. To parametrize the group of all characters it suffices to fix one. There are standard ways to do this.

• If $F = \mathbb{R}$, the standard additive character ψ_F takes x to $e^{2\pi i x}$. Its kernel is $\mathbb{Z}$.

• If $F = \mathbb{C}$, it takes z to $e^{2\pi i(z+\overline{z})}$.

• Suppose $F = \mathbb{Q}_p$. The group $\mathbb{Q}_p/\mathbb{Z}_p$ is isomorphic to the p-torsion in $\mathbb{Q}/\mathbb{Z}$. Hence there exists for every x in F an integer $k \ge 0$ and m in $\mathbb{Z}$ such that

$$x - m/p^k$$

lies in $\mathbb{Z}_p$. The number m is unique modulo p^k. Define

$$\psi_F(x) = e^{2\pi i m/p^k} .$$

The kernel of this character is $\mathbb{Z}_p$.

• If F is a finite extension $F/\mathbb{Q}_p$, define

$$\psi_F(x) = \psi_{\mathbb{Q}_p}\big(\mathrm{TR}_{F/\mathbb{Q}_p}(x)\big) .$$

The inverse different of the extension $F/\mathbb{Q}_p$ is the fractional ideal

$$\vartheta^{-1}_{F/\mathbb{Q}_p} = \{x \in F \mid \mathrm{TR}_{F/\mathbb{Q}_p}(x\mathfrak{o}_F) \subset \mathbb{Z}_p\} .$$

It is also $\{x \in F \mid \psi(x\mathfrak{o}_F) = 1\}$, the $\mathfrak{o}_F$-stable kernel of ψ.

• Suppose $F = \mathbb{F}_q(\!(t)\!)$. Since $\mathbb{F}_p$ is isomorphic to $\mathbb{Z}/p$, the map

$$m \longmapsto e^{2\pi i m/p}$$

defines a character of $\mathbb{F}_p$. Define ψ_F on $\mathbb{F}_q$ to be the composite of this with the trace from $\mathbb{F}_q$ to $\mathbb{F}_p$, then finally on a Laurent series $\sum c_k t^k$ set

$$\psi_F(x) = \psi_F(c_{-1})$$

(the residue).

FOURIER TRANSFORMS. A bilinear form B on V is **non-degenerate** if $B(x, V) = 0$ implies that $x = 0$. Suppose B to be a non-degenerate bilinear form. Eventually I'll deal only with symmetric forms, so in order to make notation slightly more convenient I'll assume that from now on. Given a basis λ for V and the corresponding coordinate system there exists a matrix M_B with entries in F such that if $u = \lambda x$, $v = \lambda y$ then

$$B(u, v) = {}^t x M_B y .$$

Since B is assumed to be non-degenerate, $\det(M_B) \ne 0$.

Suppose ψ to be a non-trivial unitary character of the additive group of F. The Fourier transform determined by B, ψ, and a choice of Haar measure du on V is defined formally by the specification

$$\widehat{f}(v) = \int_V f(u)\psi(-B(u, v))\, du .$$

This is well defined when f is in $\mathcal{S}(V)$, and is then itself in $\mathcal{S}(V)$—conditions of smoothness on f translate to conditions of rapid decrease of $\widehat{f}$, and conditions of rapid decrease on f translate to conditions of smoothness of $\widehat{f}$. As is well known, it is in fact an isomorphism of $\mathcal{S}(V)$ with itself. More precisely, $\widehat{\widehat{f}}(v) = c \cdot f(-v)$ for some constant $c \ne 0$. *What is $c = c(\psi, dx)$?*

I'll find it explicitly in various cases, when ψ is the standard additive character of F, because it is easy enough to modify answers to fit other cases. But as a preliminary I ask, how does c depend on the choices of ψ and dx?

1.1. Lemma. *For ψ an additive character of F, dx a Haar measure on F, a in $F^\times$*

$$c(\psi_a, dx) = |a|^{-n} c(\psi, dx)$$
$$c(\psi, a\,dx) = a^2 c(\psi, dx)\,.$$

Proof. Only the first might not be transparent. Let $\mu_a f(v) = f(a^{-1}v)$, and let Φf be the Fourier transform of f with respect to ψ_a. Then a simple computation shows that

$$[\Phi f](v) = \mu_{1/a}\widehat{f} = |a|^{-n}\widehat{\mu_a f}(v)\,,$$

which implies that $\widehat{f} = \mu_a(Ff)$. But then

$$[\Phi\Phi f] = \mu_{1/a}\widehat{\Phi f} = |a|^{-n}\widehat{\mu_a(\Phi f)} = |a|^{-n}\widehat{\widehat{f}}\,.$$

The Haar measure dx is called **self-dual** with respect to ψ if $c(\psi, dx) = 1$.

1.2. Corollary. *If dx is self-dual with respect to ψ, then $|a|^{n/2}dx$ is self-dual with respect to ψ_a.*

• First suppose that $F = \mathbb{R}$, $V = \mathbb{R}^n$.

Suppose L to be a lattice in V, which is to say a free module over $\mathbb{Z}$ of rank n. Fix a $\mathbb{Z}$-basis of L, and let M_B be the corresponding matrix. The quotient V/L is compact. Let $L^\perp$ be its dual lattice with respect to ψ and B, or in other words set

$$L^\perp = \mathrm{Ann}_{\psi,B}(L) = \{x \in V \mid \psi(B(L,x)) = 1\}\,.$$

This is the same as the lattice of all x in V such that $B(L,x) \subseteq \mathbb{Z}$. It has as basis the columns of M_B^{-1}. If B takes integral values on L, so that M_B is an integral matrix, then $L \subseteq L^\perp$ and $|L^\perp/L| = |\det M_B|$.

For example, if $V = \mathbb{R}$, $B(x,y) = xy$, and $L = \mathbb{Z}$, then $L^\perp = L$.

The measure on V determines also a measure on V/L. Any λ in $L^\perp$ determines a character of V/L:

$$v \longmapsto \psi(B(v,\lambda))\,.$$

For any smooth function F on V/L and λ in $L^\perp$ let

$$\widehat{F}(\lambda) = \int_{V/L} F(x)\psi(-B(x,\lambda))\,dx\,,$$

its Fourier coefficient at λ. The theory of Fourier series tells us that

$$F(x) = \frac{1}{\mathrm{meas}(V/L)} \cdot \sum_{\lambda\in L^\perp} \widehat{F}(\lambda)\psi(B(x,\lambda))\,,$$

and in particular that

(1.3)
$$F(0) = \frac{1}{\mathrm{meas}(V/L)} \cdot \sum_{\lambda\in L^\perp} \widehat{F}(\lambda)\,.$$

For f in $\mathcal{S}(V)$ apply this to

$$F(x) = f_L(x) = \sum_{\ell\in L} f(x+\ell)\,.$$

Then (1.3) can be reinterpreted as the Poisson sum formula

(1.4).
$$\sum_{\ell\in L} f(\ell) = \frac{1}{\mathrm{meas}(V/L)}\cdot\sum_{\lambda\in L^\perp}\widehat{f}(\lambda)\,.$$

If we apply this to $\widehat{f}$ on the left hand side, we get

$$\sum_{\ell}\widehat{\widehat{f}}(\ell) = \frac{1}{\mathrm{meas}(V/L)\,\mathrm{meas}(V/L^\perp)}\cdot\sum_{\ell} f(\ell)\,,$$

so that

$$c = \frac{1}{\mathrm{meas}(V/L)\,\mathrm{meas}(V/L^\perp)}\,.$$

If $L\subseteq L^\perp$ then $|L^\perp/L| = |\det(M_B)|$, so

$$\mathrm{meas}(V/L^\perp) = |\det(M_B)|^{-1}\mathrm{meas}(V/L)\,.$$

1.5. Proposition. *In these circumstances self-duality is equivalent to the condition that*

$$\mathrm{meas}(V/L) = |\det(M_B)|^{1/2}\,.$$

Example. Let B be the bilinear form $2x_1x_2 + 2y_1y_2$, whose matrix is

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

As we shall see, it is the bilinear form associated to the quadratic form $x^2 + y^2$. Say $L = \mathbb{Z}^2$, then $L^\perp = (1/2)\,L$. The measure on V is such that $\mathrm{meas}(L) = 2$, which is to say $2\,dx\,dy$.

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• Now suppose F to be a $\mathfrak{p}$-adic local field, ψ again the standard additive character of F, and B a non-degenerate (symmetric) bilinear form on the n-dimensional vector space V. If L is a lattice in V, let

$$L^\perp = \{v\in V \mid \psi(B(v,L)) = 1\}$$

Because B is non-degenerate, they are both lattices. They can be calculated explicitly given the matrix M_B determined by a basis of L.

It is relevant here because of:

1.6. Lemma. *Suppose f in $\mathcal{S}(V)$. Then*

(a) the function f has support in L if and only if $\widehat{f}$ is constant under translation by $L^\perp$;
(b) the function f is constant under translation by L if and only if $\widehat{f}$ has support on $L^\perp$.

Proof. Let f be the characteristic function of $v + L$. Then

$$\widehat{f}(y) = \int_L f(x)\psi(-B(x-v,y))\,dx = \psi(B(v,y))\int_L f(x)\psi(-B(x,y))\,.$$

If y is not in $L^\perp$ then $x\mapsto\psi(-B(x,y))$ is a non-trivial character on L and the integral vanishes. ▯

This leads to an explicit formula for the Fourier transform:

1.7. Corollary. *If f is constant under translation by L with support in $M \supseteq L$ then*

$$\widehat{f}(y) = \begin{cases} \mathrm{meas}(L)\big(\sum_{M/L} f(x)\psi(-B(x,y))\big) & \textit{if } y \textit{ is in } L^\perp \\ 0 & \textit{otherwise.} \end{cases}$$

It is constant under translation by elements of $M^\perp$.

Therefore the Fourier transform of f is $\mathrm{meas}(L)$ times the characteristic function of $L^\perp$. Its Fourier transform in turn is $\mathrm{meas}(L)\mathrm{meas}(L^\perp)$ times the characteristic function of L. The self-dual measure is that for which

$$\mathrm{meas}(L) \cdot \mathrm{meas}(L^\perp) = 1 .$$

QUADRATIC FORMS. A quadratic form Q on V is a homogeneous function of order two. It determines an associated bilinear form

$$\nabla(u,v) = Q(u+v) - Q(u) - Q(v) .$$

For example, if the dimension is 1 and $Q(x) = x^2$ then $\nabla(x,y) = 2xy$.

The form is said to be non-degenerate if ∇ is non-degenerate. This means that $\nabla(u,V) = 0$ if and only if $u = 0$.

• *From now on, I assume Q to be non-degenerate.*

The group GO_Q of **similarities** is that of all linear transformations g of V such that

$$Q(g(v)) = \mu(g)Q(v)$$

for all v in V. Here $g \mapsto \mu(g)$ is a homomorphism from GO to $F^\times$. If we have chosen a coordinate system, M_Q is the matrix of Q, and X is the matrix of g in GO then

$${}^tX\, M_Q X = \mu(g) M_Q$$

which implies

(1.8)
$$\det{}^2(g) = \mu^n(g) .$$

In particular, if n is odd, $\mu(g)$ is always a square in $F^\times$.

For g in $\mathrm{GO}(Q)$

(1.9)
$$B(g(u), g(v)) = \mu(g)\underline{(}u, v) .$$

For a function f on V and g in $\mathrm{GL}_n(F)$ let $[\lambda_g f](v) = f(g^{-1}(v))$.

1.10. Proposition. *If f lies in $\mathcal{S}(V)$ and g in GO_Q, then*

$$\widehat{\lambda_g f} = |\mu(g)|^{n/2} \cdot \lambda_{g/\mu(g)} \widehat{f} .$$

Scalar multiplication is in $\mathrm{GO}(Q)$, and this is consistent with Corollary 1.2.

Proof. Because

$$\begin{aligned} \widehat{\lambda_g f}(y) &= \int_V f(g^{-1}(v))\psi(-\nabla(v,y))\, dv \\ &= |\det(g)| \cdot \int_V f(u)\psi(-\nabla(g(u),y))\, du \\ &= |\det(g)| \cdot \int_V f(u)\psi(-\nabla(g(u), g(g^{-1}(y))))\, du \\ &= |\det(g)| \cdot \int_V f(u)\psi(-\nabla(u, \mu(g) g^{-1}(y))\, du \\ &= |\det(g)| \cdot \widehat{f}(\mu(g)\, g^{-1}(y)) . \end{aligned}$$

Apply (1.8) .

2. Fresnel integrals

Suppose F to be an arbitrary local field, (V,Q) a nondegenerate space over F with bilinear form ∇, and ψ a non-trivial additive character of F.

2.1. Lemma. *For every a in F, multiplication by $\psi(aQ(x))f(x)$ is an isomorphism of $\mathcal{S}(V)$ with itself.*

Proof of the Lemma. In the $\mathfrak{p}$-adic case there is no difficulty. In the real case, it follows from the fact that the derivatives of $\psi(aQ(x))$ are of the form $P(x)\psi(aQ(x))$ with P a polynomial.

As a consequence, for every a in F the distribution defined by the equation

$$\langle \psi(aQ(x)), f\rangle = \int_V f(x)\psi(aQ(x))\,dx$$

is tempered.

The following is a special case of Theorem 2 of §14 in [Weil:1964], and the proof is the same as his.

2.2. Theorem. *For $a \neq 0$, the Fourier transform of the distribution $\psi(aQ(x))$ is equal to a constant multiple of the distribution $\psi(-Q(x)/a)$.*

This is motivated by a simple calculation. Let $\Phi(x) = \psi(aQ(x))$. Since

$$aQ(x-y/a) = aQ(x) - \nabla(ax, y/a) + aQ(y/a) = aQ(x) - \nabla(x,y) + Q(y)/a$$

we have, at least formally,

$$\begin{aligned}\widehat{\Phi}(y) &= \int_V \psi(aQ(x))\psi(-\nabla(x,y))\,dx \\ &= \psi(-Q(y)/a)\int_V \psi(Q(y)/a)\psi(aQ(x))\psi(-\nabla(x,y))\,dx \\ &= \psi(-Q(y)/a)\int_V \psi(aQ(x-y/a))\,dx \\ &= \psi(-Q(y)/a)\int_V \psi(aQ(x))\,dx\,.\end{aligned} \tag{2.3}$$

The problem is to make sense of the integral factor, which is called a **Fresnel integral**. Later on, we shall see how to do this explicitly in different cases.

Proof. We want to prove that

$$\int_V \psi(aQ(x))\widehat{f}(x)\,dx = \gamma\int_V \psi(-Q(x)/a)f(x)\,dx$$

for some constant γ. According to Lemma 2.1, we may set $f(x) = \psi(Q(x)/a)\varphi(x)$ with φ in $\mathcal{S}(V)$, and the equation to verify becomes

$$\int_V \psi(aQ(x))\cdot\big(\psi(Q(x)/a)\varphi(x)\big)\widehat{\ }(x)\,dx = \gamma\int_V \varphi(x)\,dx \tag{2.4}$$

for all φ in $\mathcal{S}(V)$. To see this, according to Proposition 3.1 it has only to be shown that the left hand side defines a translation-invariant distribution.

The Fourier transform of $\varphi(x)\psi(Q(x)/a)$ is

$$\begin{aligned}\int_V \varphi(x)\psi(Q(x)/a)\psi(-\nabla(x,y)\,dx &= \psi(-aQ(y))\int_V \varphi(x)\psi(Q(x-ay)/a)\,dx \\ &= \psi(-aQ(y))\int_V \varphi(x+ay)\psi(Q(x)/a)\,dx\end{aligned}$$

so that the left hand side of (2.4) is

$$\int_V \left(\int_V \varphi(x+ay)\psi(Q(x)/a)\,dx \right)\,dy$$

which is manifestly a translation-invariant function of φ.

Define $\gamma_{\psi,Q}(a)$ to be that scalar, under the assumption that V is given the self-dual measure with respect to ψ. We shall see later explicit formulas for the scalar appearing in Theorem 2.2. All we can say immediately is that $|\gamma_{\psi,Q}(1)| = 1$, since $\psi(Q(x))$ is the conjugate of $\psi(-Q(x))$. Finding a formula for $\mathbb{R}$ or $\mathbb{C}$, or for $\mathfrak{p}$-adic fields, is very different.

Formally, as we have seen, a Fresnel integral is defined to be the function

$$\gamma_{\psi,Q}(a) = \int_V \psi(aQ(x))\,dx\,,$$

Later, this formula will be justified by showing that the integral is conditionally convergent.

2.5. Theorem. *If ψ is a non-degenerate character of F and measures are chosen to be self-dual:*

(a) *if $Q = Q_1 \oplus Q_2$ then $\gamma_{\psi,Q} = \gamma_{\psi,Q_1}\cdot\gamma_{\psi,Q_2}$;*
(b) *if $c = \mu(g)$ for some g in GO_Q then $\gamma_{\psi,Q}(cx) = |c|^{-n/2}\gamma_{\psi,Q}(x)$;*
(c) *for all Q, $|\gamma_{\psi,Q}(a)| = |a|^{-n/2}$;*
(d) *$\gamma_{\psi_a,Q}(x) = \gamma_{\psi,aQ}(x) = |a|^{n/2}\gamma_{\psi,Q}(ax)$;*
(e) *for all ψ and Q, $\gamma^8_{\psi,Q} = 1$.*

Proof. In sequence:

(a) The first claim is immediate from Theorem 2.2.

(b) Proposition 1.10 implies that

$$\widehat{\lambda_{g^{-1}}\Phi} = |\mu(g)|^{-n/2}\lambda_{\mu(g)g^{-1}}\widehat{\Phi}$$

for any tempered distribution Φ. Applied to $\psi(aQ(x))$ with $c = \mu(g)$, this gives us

$$\begin{aligned}\psi(ac\,Q(x))\widehat{\ } &= \psi(aQ(g(x)))\widehat{\ } \\ &= |\mu(g)|^{-n/2}\gamma_{\psi,Q}(a)\psi(-Q(cg^{-1}x)/a) \\ &= |c|^{-n/2}\gamma_{\psi,Q}(a)\psi(-Q(x)/ac)\,.\end{aligned}$$

Compare this with the equation

$$\psi(ac\,Q(x))\widehat{\ } = \gamma_{\psi,Q}(ac)\psi(-Q(x)/ac)\,.$$

(c) Since $\psi(-aQ(x))$ is the conjugate of $\psi(aQ(x))$, this follows from the equation

$$\gamma_{\psi,Q}(a)\cdot\gamma_{\psi,Q}(-a) = |a|^{-n}\,.$$

This is a consequence of the corollary to be proved in a moment, since $Q\oplus(-Q)$ is equivalent to sum of hyperbolic planes.

(d) These differ only because of a choice of measures. Apply (c).

(e) This will be proved later on, separately for real and $\mathfrak{p}$-adic fields.

2.6. Corollary. *The quotient*

$$\mathrm{sgn}Q(x) = \frac{\gamma_{\psi,Q}(x)}{|x|^{-n/2}}$$

is a constant of absolute value 1 on each coset of $(F^\times)^2$.

This quotient is the **signature** of the form.

The hyperbolic plane is the two dimensional quadratic form $H(x,y) = xy$, and a hyperbolic form of dimension $n = 2m$ if the direct sum of m hyperbolic planes. The matrix

$$\begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix}$$

is in the group GO_H. Hence

$$\gamma_{\psi,Q}(a) = |a|^{-n/2}\gamma_{\psi,H}^{n/2}(1)\,.$$

Since the characteristic of F is not equal to two, the quadratic form xy is equivalent to $x^2 - y^2$. But $\gamma_{\psi,-x^2}$ is the conjugate of γ_{ψ,x^2}, which implies that $\gamma_{\psi,H}(1) = 1$. Hence:

2.7. Corollary. *For Q a hyperbolic form of dimension n, $\gamma_{\psi,Q}(a) = |a|^{-n/2}$.*

As another immediate consequence of the theorem:

2.8. Corollary. *In every dimension, the function $\gamma_{\psi,Q}(a)|a|^{n/2}$ on $F^\times$ is a sum of characters of order two.*

Remark. We shall also see later that for $\mathfrak{p}$-adic fields the function $\gamma_{\psi,Q}(a)$ determines the quadratic form completely. That is certainly not true for $\mathbb{R}$ since, as we shall also see later, the functions γ for $\sum_1^n x_k^2$ and $-\sum_1^n x_k^2$ are the same for all $n \equiv 0 \bmod 8$. This is a relatively simple calculation, but there is also a subtle reason—it is related to the existence of derivatives of the Dirac δ distribution at 0 on $\mathcal{S}(\mathbb{R})$. Is it also related to the periodicity of Clifford algebras responsible for Bott periodicity of homotopy groups?

3. *Appendix.* Characterization of integration

A **tempered distribution** on V is a continuous linear function on $\mathcal{S}(V)$. The Fourier transform of a tempered distribution is defined by duality:

$$\langle \widehat{\Phi}, f\rangle = \langle \Phi, \widehat{f}\rangle\,.$$

This is compatible with the embedding of $\mathcal{S}(V)$ into the space of tempered distributions defined by integration. For example, suppose I to be the distribution defined by a bilinear form B and integration with respect to the self-dual measure on V:

$$\langle I, f\rangle = \int_V f(v)\,dv\,.$$

Its Fourier transform is the Dirac delta δ_0, since

$$\int_V \widehat{f}(v)\,dv = f(0)\,.$$

For any f in $\mathcal{S}(V)$ and u in V, let

$$\lambda_v f(x) = f(x-v)\,.$$

This also extends compatibly to a translation operator on tempered distribution:

$$\langle \lambda_v \Phi, f\rangle = \langle \Phi, \lambda_{-v} f\rangle$$

Integration is invariant under translation. It is essentially unique:

3.1. Proposition. *Any continuous linear function on $\mathcal{S}(V)$ invariant under translation is a multiple of integration.*

Proof. The Proposition will follow from the claim that the kernel of integration is the same as the closure of linear combinations of functions of the form $\lambda_v f - f$ with f in $\mathcal{S}(V)$.

The proof will divide into two cases. First, suppose that F is $\mathfrak{p}$-adic. The Proposition will follow from the strong assertion that the kernel of integration is equal to the space of such linear combinations.

3.2. Lemma. *If f lies in $\mathcal{S}(V)$, then*

$$\int_V f(v)\,dv = 0$$

if and only if

$$f = \sum\nolimits_u \lambda_u f_u$$

in which u traverses a finite set in V, with each f_u in $\mathcal{S}(V)$.

Proof. Because the Haar measure is translation-invariant

$$\int_V \big([\lambda_u f](v) - f(v)\big)\,dv = 0\,.$$

This proves one half of the Proposition.

So now suppose that f lies in $\mathcal{S}(V)$, with $\lambda_k f = f$ for all k in some compact open subset K of V. Suppose also that

$$\int_V f(v)\,dv = 0\,.$$

This means that

$$\sum\nolimits_i f(v_i) = 0$$

if the support of f is $\sqcup(v_i + K)$. If $\mathfrak{char}_i$ is the characteristic function of $v_i + K$, then $\mathfrak{char}_i = \lambda_{v_i}\mathfrak{char}_K$ and

$$\begin{aligned} f &= \sum\nolimits_i f(v_i)\mathfrak{char}_i \\ f &= \sum\nolimits_i f(v_i)\mathfrak{char}_i - \sum\nolimits_i f(v_i)\mathfrak{char}_K \\ f &= \sum\nolimits_i f(v_i)(\lambda_{v_i}\mathfrak{char}_K - \mathfrak{char}_K)\,. \end{aligned}$$

Now assume $F = \mathbb{R}$, $V = \mathbb{R}^n$. The strong assertion is not valid in this case, and is replaced by:

3.3. Lemma. *If f lies in $\mathcal{S}(V)$, then*

$$\int_V f(v)\,dv = 0$$

if and only if f is a linear combination of partial derivatives $\partial f_i/\partial x_i$.

Proof. Since

$$f'(x) = \lim\nolimits_{h\to 0} \frac{f(x+h) - f(x)}{h}\,,$$

one half follows from invariance and continuity.

As is well known, the Fourier transform takes $\partial f/\partial x_i$ to $2\pi i y_i \widehat{f}(y)$. The Lemma will then follow from this:

3.4. Lemma. *If f lies in $\mathcal{S}(V)$ then $f(0) = 0$ if and only if f is a sum of the form*

$$\sum\nolimits_i y_i f_i$$

with each f_i in $\mathcal{S}(V)$.

Proof of the Lemma. One way is easy. For the other, I shall start by proving that *if f lies in $\mathcal{S}(V)$ and $f(x_1, \ldots, x_{n-1}, 0) = 0$ then f/x_n lies in $\mathcal{S}(V)$.*

The proof of this claim will be induction. Say $n = 1$ and $f(0) = 0$. For each x in $\mathbb{R}$ let $\varphi_x(t) = f(tx)$. Then

$$\begin{aligned} f(x) &= \varphi_x(1) - \varphi_x(0) \\ &= \int_0^1 \varphi_x'(t)\, dt \\ &= x \int_0^1 f'(tx)\, dt \\ &= x f_1(x)\,. \end{aligned}$$

Away from 0, say in the region $|x| > 1$, $f_1(x) = f(x)/x$. This quotient and all its derivatives are rapidly decreasing since f lies in $\mathcal{S}(V)$, so $f_1(x)$ also lies in $\mathcal{S}(V)$.

Now suppose $n > 1$. By induction, the restriction of f to the hyperplane $x_n = 0$ may be expressed as

$$\sum_1^{n-1} x_i f_{i,0}$$

with each $f_{i,0}$ in $\mathcal{S}(\mathbb{R}^{n-1})$. Each of these may be extended to a function I'll all also call f_i in $\mathcal{S}(V)$. Then $F = f - \sum_0^{n-1} x_i f_i$ vanishes on $x_n = 0$. But then the argument for the case $n = 1$ shows that $F(x) = x_n f_n(x)$ for some F in $\mathcal{S}(V)$, giving finally

$$f = \sum_1^n x_i f_i\,.$$

Part II. Real Fresnel integrals

4. The classic Fresnel integral

Suppose $V = \mathbb{R}$ and $Q(x) = x^2$, hence $\nabla(x, y) = 2xy$. Also fix $\psi(x) = e^{\pi i x}$. The Fourier transform is now

$$\widehat{f}(y) = \int_{\mathbb{R}} f(x) e^{-2\pi i x y}\, dx\,.$$

The corresponding self-dual measure is dx. We know from Theorem 2.2 that the Fourier transform of the tempered distribution $e^{\pi i x^2}$ is equal to $\gamma e^{-\pi i x^2}$ for some constant γ. What is γ? Formally, according to the derivation of (2.3) , it is equal to

$$\int_{\mathbb{R}} e^{\pi i x^2}\, dx\,.$$

But this integral is only conditionally convergent, and this makes rigourous manipulations difficult.

The way around difficulties is through analytic continuation. Integration against $e^{-\pi\lambda x^2}$ is a tempered distribution for any λ in $\mathbb{C}$ with $\mathrm{RE}(\lambda) \ge 0$. For $\mathrm{RE}(\lambda) > 0$ this function decreases exponentially, and in that case finding its Fourier transform is relatively straightforward.

There are a couple of steps in this process.

Step 1. One of the simplest functions in $\mathcal{S}(\mathbb{R})$ is $f(x) = e^{-\pi x^2}$.

4.1. Lemma. *The function $e^{-\pi x^2}$ is its own Fourier transform.*

This amounts to another verification that the measure dx is self-dual with respect to this Fourier transform.

Proof. We have

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-\pi x^2} e^{2\pi i x y}\,dx &= \int_{-\infty}^{\infty} e^{-\pi x^2+2\pi i x y+y^2} e^{-\pi y^2}\,dx \\ &= \int_{-\infty}^{\infty} e^{-\pi(x+iy)^2} e^{-\pi y^2}\,dx \\ &= e^{-\pi y^2}\int_{-\infty}^{\infty} e^{-\pi(x+iy)^2}\,dx \\ &= e^{-\pi y^2}\int_{-\infty}^{\infty} e^{-\pi x^2}\,dx\,.\end{aligned}$$

The substitution can be justified by a simple change of contour in $\mathbb{C}$. As is well known, the Gaussian integral in this can be computed explicitly by transferring to polar coordinates in $\mathbb{R}^2$:

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\pi(x^2+y^2)}\,dx\,dy &= \left(\int_{-\infty}^{\infty} e^{-\pi x^2}\,dx\right)^2 \\ &= \int_0^{2\pi} d\theta \int_0^{\infty} r e^{-\pi r^2}\,dr \\ &= 2\pi\int_0^{\infty} r e^{-\pi r^2}\,dr \\ &= \int_0^{\infty} e^{-s}\,ds \\ &= 1\,.\end{aligned}$$

Step 2. The next move is to apply contour integration to show that for any λ with $\mathrm{RE}(\lambda) \geq 0$ the limit

$$\lim_{R\to\infty}\int_0^R e^{-\pi\lambda x^2}\,dx$$

exists and is equal to

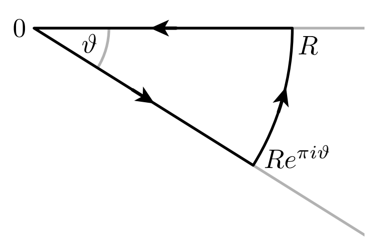
$$\frac{1}{\sqrt{\lambda}}\int_0^{\infty} e^{-\pi x^2}\,dx\,.$$

The square root here is the principal one. To be explicit, with $\lambda = \cos 2\vartheta + i\sin 2\vartheta$ I set

$$\sqrt{\lambda} = \omega = \cos\vartheta + i\sin\vartheta$$

with $0 \leq |\vartheta| \leq \pi/4$.

Consider the complex integral of $e^{-\pi z^2}$ over the closed path shown in the following diagram.



It has three segments: (i) the ray from 0 to $Re^{i\vartheta}$; (ii) the arc of radius R from $Re^{i\vartheta}$ to R; the segment from R back to 0. The integral over the real component is

$$-\int_0^R e^{-\pi x^2}\,dx\,,$$

and as $R \to \infty$ it has limit $1/2$. If we set $z = \omega x$, and consequently $dz = \omega\, dx$, the integral over the complex ray becomes

$$\omega \cdot \int_0^R e^{-\pi\lambda x^2}\, dx$$

We deduce that

$$\int_0^\infty e^{-\pi\lambda x^2}\, dx = \frac{1}{\omega} \cdot \frac{1}{2}$$

if we can prove

4.2. Lemma. *As R goes to ∞, the integral over the arc tends to 0.*

Proof. It suffices to show that the integral over the entire arc from the ray with angle $\pi/4$ to 0 tends to 0. This arc is parametrized as $z = Re^{i(\pi/4-\alpha)}$ for α ranging over $[0, \pi/4]$. Since $dz = -iz\, d\alpha$, the integral is thus

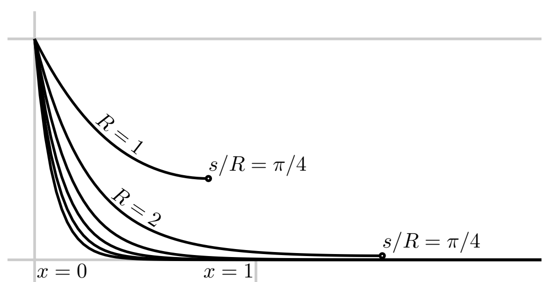
$$-i \int_0^{\pi/4} ze^{-\pi R^2(\cos(\pi/2-2\alpha)+i\sin(\pi/2-2\alpha))}\, d\alpha = \int_0^{\pi/4} Re^{i(\pi/4-\alpha)} \cdot e^{-\pi R^2(\cos(2\alpha)+i\sin(2\alpha))}\, d\alpha\,.$$

We therefore want to estimate

$$\int_0^{\pi/4} Re^{-\pi R^2\cos(2\alpha)}\, d\alpha = \int_0^{\pi R/4} e^{-R^2\cos(2s/R)}\, ds \quad (s = R\alpha)\,.$$

Here s is arc length along the path.

Although the value of the integrand is 1 at the start of this path segment, it decreases very rapidly to 0, and the more rapidly the larger R is. Here are the graphs of that magnitude for various R, over the range



Rigourous estimates of the integral can be made by using the estimate

$$\cos x \geq 1 - 2x/\pi \quad (0 \leq x \leq \pi/2)\,.$$

We have seen that the Fourier transform of the Schwartz function $e^{-\pi x^2}$ is itself. A simple change of variables allows us to see that the Fourier transform of $e^{-\pi\lambda x^2}$ for $\lambda > 0$ is equal to $(1/\sqrt{\lambda})e^{-\pi y^2/\lambda}$. But the equation

$$\frac{1}{\sqrt{\lambda}} \cdot \langle e^{-\pi x^2/\lambda}, f\rangle = \langle e^{-\pi\lambda x^2}, \widehat{f}\rangle$$

is analytic in the region $\mathrm{RE}(\lambda) < 0$ and continuous for $\mathrm{RE}(\lambda) \leq 0$, so it remains true for all λ with $\mathrm{RE}(\lambda) \geq 0$:

4.3. Proposition. *The Fourier transform of $e^{-\pi\lambda x^2}$ is $(1/\sqrt{\lambda})e^{-\pi x^2/\lambda}$ whenever $\mathrm{RE}(\lambda) \geq 0$.*

All of the assertions of Theorem 2.5 when $F = \mathbb{R}$. are now straightforward, given the calculations above.

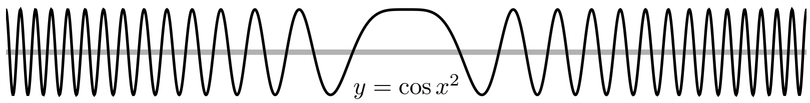
4.4. Corollary. *If $\psi(x) = e^{\pi i x}$ and $Q(x) = x^2$, then for $a > 0$*

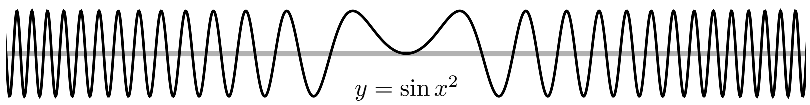
$$\gamma_{\psi,Q}(\pm a) = \frac{1 \pm i}{2} a^{-1/2}\,.$$

I'll now say more about the classic Fresnel integral

$$\begin{aligned}\int_{\mathbb{R}} e^{-\pi i x^2}\,dx &= 2\int_0^\infty e^{\pi i x^2}\,dx\ . \\ &= 2\int_0^\infty (\cos \pi x^2 + i \sin \pi x^2)\,dx\ .\end{aligned}$$

This integral converges conditionally. Here are the graphs of the real and imaginary parts of the integrand:





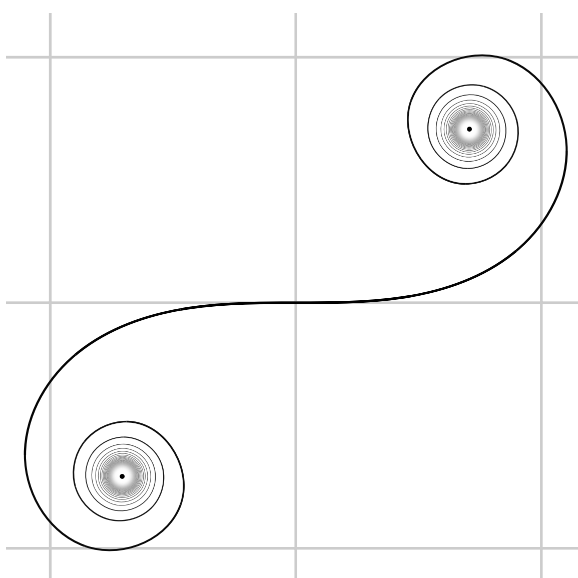
If we set $s = x^2$ in these integrals we get

$$\begin{aligned}\int_0^\infty \cos \pi x^2\,dx &= \int_0^\infty \frac{\cos \pi s}{2\sqrt{s}}\,ds \\ \int_0^\infty \sin \pi x^2\,dx &= \int_0^\infty \frac{\sin \pi s}{2\sqrt{s}}\,ds\ .\end{aligned}$$

The integrals therefore converge, by the alternating sign test. When the finite integral

(4.5) $$\int_0^R e^{\pi i x^2}\,dx$$

is plotted as a path in the complex plane as R varies from $-\infty$ to ∞, we get the familiar Cornu spiral:



The illustration agrees with the evaluation $e^{2\pi i/8} = (1+i)/\sqrt{2}$.

5. Anisotropic real forms

Suppose (V, Q) to be anisotropic over $\mathbb{R}$—i.e. if if $v \neq 0$ then $Q(v) \neq 0$. Its spheres are compact, and there is a geometric interpretation of Fresnel integrals.

Define the function on $F^\times$:

$$\nu_Q(x) = \lim_{U \to x} \frac{\text{meas}(Q^{-1}(U))}{\text{meas}(U)} .$$

We shall see an explicit example in a moment. For f in $\mathcal{S}(F)$ we have the basic equation

$$\int_F f(x)\nu_Q(x)\, dx = \int_V f(Q(x))\, dx ,$$

so that $\nu_Q(x)$ is well defined as a tempered distribution on $\mathbb{R}$. It is of interest here because, at least formally, its Fourier transform evaluated at $a \neq 0$ is

$$\widehat{\nu}_Q(a) = \int_{\mathbb{R}} \psi(-ay)\nu_q(y)\, dy = \int_V \psi(-aQ(y))\, dy = \gamma_{\psi,Q}(a) .$$

This formal calculation is in fact legitimate, but it doesn't tell the whole story. The full Fourier transform also involves derivatives of the distribution δ_0.

Example. Suppose Q to be the positive definite form $\sum x_k^2$ on $\mathbb{R}^n$, $\psi(x) = e^{2\pi i x}$, dx the Lebesgue measure on V, which is self-dual. On $\mathbb{R}^\times$

$$\nu_Q(y) = \lim_{h \to 0} \text{meas}\left(\frac{Q^{-1}(y-h, y+h)}{2h}\right) .$$

If Γ_{n-1} is the volume of a unit sphere in $\mathbb{R}^n$, the volume of the inverse image is

$$\int_{\sqrt{y-h}}^{\sqrt{y+h}} \Gamma_{n-1} r^{n-1}\, dr = \left[\Gamma_{n-1} \frac{r^n}{n}\right]_{\sqrt{y-h}}^{\sqrt{y+h}} = \frac{\Gamma_{n-1}}{n}\big((y+h)^{n/2} - (y-h)^{n/2}\big)$$

which implies that

$$\nu_Q(y) = \begin{cases} \Gamma_{n-1} y^{n/2-1}/2 & \text{if } y > 0 \\ 0 & \text{if } y \leq 0. \end{cases}$$

Even if $n = 1$ and the exponent of y is $-1/2$, this is integrable around 0, thus defining the distribution ν_Q as an integral. If $\chi_\pm$ are the characters of $\mathbb{R}^\times$ defined by

$$\begin{aligned} \chi_+(y) &= |y|^{n/2-1} \\ \chi_-(y) &= \text{sgn}(y)|y|^{n/2-1} \end{aligned}$$

then we can also express

$$\nu_Q = \frac{\Gamma_{n-1}}{4}(\chi_+ + \chi_-) .$$

Its Fourier transform is known to involve some derivative of δ_0. As I mentioned earlier, this is related to the fact that the Fresnel integral does not distinguish $\sum x_i^2$ from $-\sum x_i^2$ in dimension eight.

Remark. There is a paradox implicit in these assertions. If ψ is the standard character $e^{2\pi i x}$ on $\mathbb{R}$, the scalar γ_{ψ,x^2} is equal to $\omega = e^{\pi i/4}$. The scalar for the sum Q of 4 squares is therefore -1. On the other hand, that for $H \oplus H$ is 1. The Fourier transform of $\gamma_{\psi,Q}(x)$ is ν_Q, which is of course a non-negative function. That of $\gamma_{\psi,H\oplus H}$ is therefore negative. Is there any way to relate it to measures on its 'spheres'? Perhaps by some regularization procedure?

∘ ——— ∘

Part III. p-adic Fresnel integrals

6. p-adic Fresnel integrals

Now let F be a $\mathfrak{p}$-adic field, $\mathfrak{o}$ its ring of integers. Let (V, Q) be a strictly non-degenerate quadratic space over F with V of dimension n, ∇ be its associated bilinear form.

If F has characteristic two, the requirement that ∇ be non-degenerate requires that V have even dimension. In order to avoid special treatment in that case

- *I shall assume from now on that the characteristic of F is not two.*

With this assumption, the squares are open in $F^\times$ and every non-degenerate quadratic form is a sum of one-dimensional forms.

Assign V the Haar measure self-dual with respect to ψ and ∇_Q. Formally, the associated $\mathfrak{p}$-adic Fresnel integral defined earlier is the function

$$\gamma_{\psi,Q}(a) = \int_V \psi(aQ(x))\,dx\,.$$

We shall now see how to interpret this as a confitionally convergent integral.

A lattice in V is a free $\mathfrak{o}$-submodule of V of rank n. For any $\mathfrak{o}$-lattice L let

$$L^\perp = \mathrm{Ann}_\psi(L) = \{x \in V \mid \psi(\nabla(x,y)) = 1 \text{ for all } y \in L\}\ .$$

This is also a module over $\mathfrak{o}$. Because ∇ is assumed to be non-degenerate, it is a lattice in V.

6.1. Proposition. *If $\psi(aQ(L)) \equiv 1$, then for any lattice M containing $a^{-1}L^\perp$*

$$\int_M \psi(aQ(x))\,dx = \int_{a^{-1}L^\perp} \psi(aQ(x))\,dx = \mathrm{meas}(L) \sum_{a^{-1}L^\perp/L} \psi(aQ(x))\,.$$

This guarantees at least that the limit

$$\gamma_{\psi,Q} = \lim_M \int_M \psi(aQ(x))\,dx$$

exists and even expresses it as a finite sum. It also implies that $\gamma_{\psi,Q}(1) = 1$ if $L^\perp = L$.

Proof. Suppose $L^\perp \subseteq M$. If $\psi(aQ(L)) \equiv 1$ then $L \subseteq L^\perp \subseteq M$. Let $m_1, \ldots, m_r$ be coset representatives for L in M. Then

$$\begin{aligned}
\int_M \psi(Q(x))\,dx &= \sum_i \int_L \psi(aQ(m_i + x)\,dx \\
&= \sum_i \int_L \psi(aQ(m_i)\psi(a\nabla(m_i,x))\psi(aQ(x))\,dx \\
&= \sum_i \psi(aQ(m_i)) \int_L \psi(\nabla(am_i,x))\,dx \\
&= \int_{a^{-1}L^\perp} \psi(aQ(x))\,dx\,,
\end{aligned}$$

since by definition of $L^\perp$

$$\int_L \psi(\nabla(am,x))\,dx = 0$$

unless am is in $L^{\perp}$.

6.2. Corollary. *For every $\varepsilon > 0$ there exits a lattice M in V such that*

$$\gamma_{\psi,Q}(a) = \int_M \psi(aQ(x)\,dx$$

for all $|a| \ge \varepsilon$.

Up to a positive scalar measuring the ratio of two self-dual Haar measures, the function $\gamma_{\psi,Q}(a)$ is the same as $\gamma_{\psi_a,Q}(1)$. I'll call the function $\gamma_{\psi,Q}(a)$ the **characteristic** of Q. It depends on ψ, but this is usually fixed in any one discussion.

I recall from Theorem 2.5:

$$\begin{aligned} \gamma_{Q\oplus R} &= \gamma_Q \gamma_R \\ |\gamma_Q(a)| &= |a|^{-n/2} \\ \gamma_Q(\mu x) &= |\mu|^{-n/2}\gamma_Q(x) \text{ if } \mu = \mu_g. \end{aligned}$$

Example. In this and the next example, let $V = \mathbb{Q}_p$, $Q(x) = x^2$, $L = \mathbb{Z}_p$.

At first, suppose p odd. Then $L^{\perp} = L$, so $\gamma(1) = 1$. But if $a = p$ then

$$\gamma(p) = \sum_{m \bmod p} e^{2\pi m^2/p}\,.$$

It is easy to see that $\gamma^2 = \varepsilon p$ with

$$\varepsilon = \begin{cases} 1 & \text{if } p \equiv 1 \bmod 4 \\ -1 & \text{if } p \equiv 3 \bmod 4. \end{cases}$$

A well known result of Gauss tells us that

$$\gamma = \begin{cases} \sqrt{p} & \text{if } p \equiv 1 \bmod 4 \\ \sqrt{-p} & \text{otherwise.} \end{cases}$$

Example. Now suppose $p = 2$. Then $L^{\perp} = (1/2)\mathbb{Z}_2$. If $a = 1$ then

$$\gamma(a) = \frac{1}{\sqrt{2}}\big(1 + e^{2\pi i/4}\big) = \frac{1+i}{\sqrt{2}}\,.$$

If $a = 2$ then $2^{-1}L^{\perp}/L \cong (1/4)\mathbb{Z}/\mathbb{Z}$ and

$$\gamma(a) = \frac{1}{\sqrt{2}}\sum_0^3 e^{2\cdot 2\pi i m^2/16} = \frac{1}{\sqrt{2}}\big(1 + \zeta + \zeta^4 + \zeta^9\big) = \frac{2\zeta}{\sqrt{2}} = \sqrt{2}\zeta$$

with $\zeta = e^{2\pi i/8}$. These examples are typical in that, as I shall verify later, $\gamma_Q(1)$ is always an eighth root of unity.

7. Anisotropic p-adic forms

One might well wonder whether γ_Q has some intuitive interpretation. The answer is that it does, at least in certain cases.

I recall that an anisotropic quadratic space (V, Q) is one for which $Q(v) = 0$ only if $v = 0$. As is well known, every quadratic space is a direct sum $U \oplus nH$, in which U is anisotropic. The decomposition is unique up to equivalence. The classification of quadratic spaces thus reduces to the classification of anisotropic spaces. This can be done by applying a tool Minkowski introduced to count points on spheres of quadratic forms over finite fields.

7.1. Lemma. *If (V, Q) is anisotropic over a $\mathfrak{p}$-adic field then each ball*

$$B(C) = \{v \,|\, |Q(v)| \le C\}$$

is compact.

Proof. Fix an arbitrary lattice L. Then $\Omega L - \mathfrak{p}L$ is compact, and by assumption $(Q(v)$ is bounded away from 0 on it—there exists $A > 0$ such that $|Q(v)| > A$ for v in Ω.

Suppose now that $|Q(v)| \le C$. Then for some k the multiple $\varpi^k V$ lies in Ω, and hence

$$q^{-2k}|Q(v)| \ge A, \quad q^{2k} \le Q(v)/A \le C/A\,.$$

Hence k is bounded by $\ell = (1/2)\log_q(C/A)$. Therefor $|Q(v)| \le C$ implies that v is in $\varpi^{-\ell}L$.

This will motivate the following discussion.

DIRECT AND INVERSE IMAGES. In this sub-section, suppose V to be F^n, with F a $\mathfrak{p}$-adic field, and Q a non-degenerate quadratic form. Let $d\boldsymbol{v}$, $d\boldsymbol{u}$ be the differential forms on V, F determining the Haar measures $dv = |d\boldsymbol{v}|$, $du = |d\boldsymbol{u}|$. The map Q is submersive except at 0, which means that at any point $v \ne 0$ in V the quotient differential form $d\boldsymbol{v}/d\boldsymbol{u}$ on the 'sphere' on which v lies is well defined.

7.2. Proposition. *If f is in $\mathcal{S}(V)$, then for every $u \ne 0$ in F the limit*

$$[Q_* f](u) = \lim_{U \to \{u\}} \frac{\int_{Q^{-1}(U)} f(x)\,dx}{\text{meas}\,U}$$

exists. The function defined in this way on $F^\times$ is locally constant. For c in $(F^\times)^2$, $x \ne 0$

$$\nu_Q(cx) = |c|^{n/2-1}\nu_Q(x)\,.$$

The U in the limit are open neighbourhoods of u. Explicit formulas for the limit can be found by applying Hensel's Lemma.

Proof. For $x \in 4\mathfrak{p}_F$ the binomial series for $\sqrt{1+x}$ converges. For ε small the inverse image of $u(1+\varepsilon)^{\pm 1}$ is therefore the direct product $Q^{-1}(u) \times \left(\sqrt{1+\varepsilon}\right)^{\pm 1}$.

The primary property of ν_Q is this an immediate consequence:

7.3. Corollary. *For f in $\mathcal{S}(F^\times)$*

$$\int_V f(Q(x))\,dx = \int_F f(x)\nu_Q(x)\,dx\,.$$

In particular, ν_Q is locally integrable on all of F, and hence defines a distribution. What is its Fourier transform? To answer this, I have to recall something about Fourier transforms and multiplicative characters. The integral

$$f \longmapsto \int_{F^\times} f(x)\chi(x)\,dx/|x|$$

is well defined for any character χ of $F^\times$, with f in $\mathcal{S}(F^\times)$, and defines a χ-equivariant distribution φ_χ on $\mathcal{S}(F^\times)$. In any of several ways it may be shown to extend uniquely to a χ-equivariant distribution on all of $\mathcal{S}(F)$, except for χ equal to the trivial character. Its Fourier transform is a scalar multiple of $\varphi_{\nu\chi^{-1}}$ in which

$$\nu\colon\ x \longmapsto |x|\,.$$

In the exceptional case, the invariant distributions are all multiples of the Dirac δ_0 with

$$\langle \delta_0, f\rangle = f(0)\,.$$

7.4. Theorem. *Assume (V,Q) to be anisotropic. The restriction of the Fourier transform of ν_Q to $F^\times$ is equal to $\gamma_{\psi,Q}(-y)$.*

Proof. This means that

$$\int_F \widehat{f}(-y)\nu_Q(y)\,da = \int_F f(x)\gamma_{\psi,Q}(x)\,dx\,.$$

if f lies in $\mathcal{S}(F)$ with $f(0)=0$. The lattice support L of f is bounded away from 0. By Corollary 6.2 we can therefore choose M large enough in V so that the right hand side is

$$\int_L f(x)\left(\int_M \psi(xQ(v))\,dv\right)\,dx\,.$$

We may also choose M large enough so that for y in the support of $\widehat{f}$ the inverse image $Q^{-1}(y)$ (known to be bounded) is contained in M. But then by Corollary 7.3 this expression is

$$\int_L f(x)\left(\int_F \psi(xy)\nu_Q(y)\,dy\right)\,da = \int_F \nu_Q(y)\widehat{f}(-y)\,dy\,.$$

What about the full Fourier transform? In dimension n the function ν_Q is a sum of characters $\rho|x|^{n/2-1}$. In all dimensions except two, these have as their Fourier transforms sums of other characters, so $\gamma_{\psi,Q}$ has a canonical distribution to $\mathcal{S}(F)$, and these are the Fourier trnsforms of ν_Q. I'll say what happens in dimension two in a moment. That discussion will complete the proof of:

7.5. Corollary. *Two anisotropic forms are equivalent if and only if they have the same characteristic function.*

Proof. By induction on dimension. The theorem implies that if two anisotropic forms have the same characteristic function then they have the same images in F. One can split off a certain one-dimensional form from each, then apply induction and Theorem 2.5.

Remark. This is the $\mathfrak{p}$-adic analogue of an observation of Minkowski's in his prize memoir [Minkowski:1911], in which he used an analogous trick to compute the sizes of spheres in quadratic spaces over finite fields. Siegel applied the same trick to prove his extension of Minkowski's formula. We can use it to compute $\gamma_{\psi,Q}$ explicitly for a large class of quadratic spaces, and to classify quadratic forms over F.

DIMENSION TWO. Any quadratic form of dimension two can be written as a sum

$$ax^2 + by^2 = a(x^2 + (b/a)y^2$$

which is $aN_{E/F}$ when $E = F(\sqrt{-b/a}$. If $-b/a$ is a square in F this is equivalent to the hyperbolic plane H. In this case, we have seen that $\gamma_{\psi,H}(a) = |a|^{-1}$.

Otherwise, E is a quadratic field extension. By local class field theory, the image of $N_{E/F}$ in $F^\times$ has index two. Let $\mathrm{sgn} = \mathrm{sgn}_{E/F}$ be the non-trivial multiplicative character which is trivial on the image.

7.6. Theorem. *If $(V,Q) = (E, N_{E/F})$ then $\gamma_{\psi,Q}(ax) = \mathrm{sgn}_{E/F}(a)|a|^{-1}\cdot\gamma_{\psi,Q}(x)$.*

Proof. Suppose x in $F^\times$, and let e be the ramification degree of E/F. Thus $\mathfrak{p}_F = \mathfrak{p}_E^e$. It follows from Proposition 7.2 that

$$\nu_Q(x) = \begin{cases} \dfrac{\text{meas}(\mathfrak{o}_E^\times)}{(1/e)\text{meas}(\mathfrak{o}_F^\times)} & \text{if } x \text{ is in } NE^\times \\ 0 & \text{otherwise.} \end{cases}$$

As a consequence, ν_Q is equal to a linear combination of 1 and $\text{sgn}_{E/F}$. These are both distributions, and the Fourier transform of ν_Q will be a linear combination of their Fourier transforms, That of 1 is the Dirac delta δ_0, and that of $\text{sgn}_{E/F}$ is the principal value defined by the function $\text{sgn}_{E/F}|x|^{-1}$. Because of Theorem 7.4, the function $\gamma_{\psi,Q}$ is a scalar multiple of $\text{sgn}_{E/F}|x|^{-1}$. ▯

7.7. Corollary. *If $Q = aN_{E/F}$ then*

$$\gamma_{\psi,Q}^2 = \text{sgn}_{E/F}(-1)\,.$$

Proof. Because

$$\gamma_{\psi,-Q} = \text{sgn}(-1)\,\gamma_{\psi,Q}$$

but also $\gamma_{\psi,-Q} = \overline{\gamma}_{\psi,Q}$ and $|\gamma_{\psi,Q}| = 1$. ▯

7.8. Corollary. *For any Q, $\gamma_{\psi,Q}(1)$ is an eighth root of unity.*

Proof. It suffices to show this for cx^2.

Let $\gamma = \gamma_{\psi,Q}(1)$. The form $Q = cx^2 \oplus cx^2$ is a non-degenerate form of dimension two, and $\gamma_{\psi,Q}(1) = \gamma^2$. It is either the hyperbolic plane or $aN_{E/F}$ for some quadratic field extension. In the first case $\gamma_{\psi,Q}(1) = 1$. In the second, because of the previous corollary, it is a fourth root of unity. ▯

8. Quaternion algebras and quadratic forms

The classification of $\mathfrak{p}$-adic quadratic forms is closely related to the structure of quaternion algebras. I'l begin with a discussion of quaternion algebras over general fields.

QUATERNION ALGEBRAS. For the moment, suppose F to be an arbitrary field of characteristic other than two, E/F to be a separable quadratic extension algebra. This means that $E = F[x]/P(x)$ in which $P(x)$ is a quadratic polynomial whose roots are distinct. If the polynomial $P(x)$ factors in F then $E = F \oplus F$, and otherwise E is a field.

Choose α in $F^\times$, and let $B = B_{E,\alpha}$ be the algebra over E with basis 1, σ and relations

$$x\sigma = \sigma\overline{x}, \quad \sigma^2 = \alpha\,.$$

Replacing α by $c\overline{c}\alpha$ produces an isomorphic algebra, so that the algebra B depends essentially only on the image of α in $F^\times/(F^\times)^2$.

The field E acts on the right on this, so the identification with E^2 is the map

$$(x, y) \longmapsto x + \sigma y\,.$$

Acting by multiplication on the left, B commutes with E. This gives us an embedding of B into $M_2(E)$. Explicitly, $x + \sigma y$ takes

$$\begin{aligned} 1 &\longmapsto x + \sigma y \\ \sigma &\longmapsto x\sigma + \sigma y\sigma \\ &\quad = x\sigma + \sigma^2\overline{y} \\ &\quad = \sigma\overline{x} + \alpha\overline{y}\,. \end{aligned}$$

In other words, if I choose σ, 1 as basis $\lambda = x + \sigma y$ corresponds to the matrix

$$\iota(\lambda) = \begin{bmatrix} \overline{x} & y \\ \alpha\overline{y} & x \end{bmatrix}.$$

8.1. Lemma. *The image of ι is made up of all matrices X in $M_2(E)$ such that*

$$\begin{bmatrix} 0 & 1 \\ \alpha & 0 \end{bmatrix} X \begin{bmatrix} 0 & 1 \\ \alpha & 0 \end{bmatrix}^{-1} = \overline{X}\,.$$

In other language, this tells us that B is defined as an inner form of $M_2(F)$.

The trace $x + \overline{x}$ of $\iota(x + \sigma y)$ defines the trace map TR from B to F. Its determinant is

$$\mathrm{NM}_{B/F}\colon\ x + \sigma y \longmapsto x\overline{x} - \alpha y\overline{y}\,.$$

This lies in F, and defines the norm map from B to F. Considering E as a vector space over F, this gives us a non-degenerate quadratic form of dimension 4. If λ is any element of B, it is a root of the quadratic equation

$$x^2 - \mathrm{TR}(\lambda)\,x + \mathrm{NM}(\lambda) = 0\,.$$

The bilinear form ∇ associated to NM is

(8.2) $$(x + y)(\overline{x} + \overline{y}) = x\overline{x} - y\overline{y} = \mathrm{TR}(y\overline{x})\,.$$

The **conjugate** of $x + \sigma y$ is $\overline{x} - \overline{y}\sigma$. This definition is motivated by the requirement that it be conjugation on E and $F(\sigma)$.

8.3. Proposition. *This conjugation is an involutory anti-automorphism.*

Proof. In $M_2(E)$ it takes

$$\begin{bmatrix} \overline{x} & y \\ \alpha\overline{y} & x \end{bmatrix} \longmapsto \begin{bmatrix} x & -y \\ -\alpha\overline{y} & \overline{x} \end{bmatrix},$$

which is the composition of matrix transposition, conjugation by

$$\begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix}$$

and conjugation of matrix entries.

The norm map can be expressed as

$$\mathrm{NM}(x + \sigma y) = (x + \sigma y)(\overline{x} - \overline{y}\sigma) = (x + \sigma y)(\overline{x + \sigma y})\,.$$

8.4. Proposition. *If α lies in $NE^\times$ then $B_{E,\alpha}$ is isomorphic to $M_2(F)$, and otherwise it is a division algebra.*

In particular, if $E = F \oplus F$ then B is isomorphic to $M_2(F)$. If E is a field, then $E \otimes E$ is isomorphic to $E \oplus E$, so that $B \otimes K$ is isomorphic to $M_2(K)$ for any extension field K/F into which E embeds.

Proof. To prove the first claim, it suffices to do it for any element of $\mathrm{NM}(E^\times)$.

Suppose λ to be a generator of E/F, satisfying the quadratic equation

$$\lambda^2 - a\lambda + b = 0\,.$$

Take λ, 1 as a basis of E/F. Since

$$\begin{aligned} \lambda\cdot 1 &= \lambda \\ \lambda\cdot\lambda &= a\lambda - b \end{aligned}$$

we get an embedding of E into $M_2(F)$:

$$\varphi\colon \lambda \longmapsto \begin{bmatrix} a & 1 \\ -b & 0 \end{bmatrix} .$$

But then

$$\overline{\lambda} = a - \lambda \longmapsto \begin{bmatrix} 0 & -1 \\ b & a \end{bmatrix}$$

If

$$\sigma = \begin{bmatrix} 0 & 1 \\ b & 0 \end{bmatrix}$$

then $\sigma^2 = b \cdot I$ and

$$\sigma\varphi(\lambda)\sigma^{-1} = \varphi(\overline{\lambda}) .$$

But b is the norm of λ, so the algebra generated by E and σ is the same as $B_{E,1}$, which is hence isomorphic to $M_2(F)$.

If α does not lie in $NE^\times$, then $\mathrm{NM}(z) \neq 0$ if and only if $z = 0$, and $z \neq 0$ has inverse $\overline{z}/\mathrm{NM}(z)$.

If $E = F(\sqrt{\beta})$, the norm form on B becomes

$$x_1^2 - \beta x_2^2 - \alpha x_3^2 + \alpha\beta x_4^2 .$$

Its discriminant is $(\alpha\beta)^2 \sim 1$.

$\mathfrak{p}$-ADIC QUATERNION ALGEBRAS. Now suppose F to be a $\mathfrak{p}$-adic field.

If B is a division algebra, define an **integer** in B to be a z such that $\mathrm{NM}(z)$ is in $\mathfrak{o}_F$.

8.5. Proposition. *Suppose E to be the unramified quadratic extension of F, and $\alpha = \varpi_F$, ι the associated embedding of B into $M_2(E)$. Then $z = x + \sigma y$ is an integer in B if and only if x, y are in $\mathfrak{o}_E$.*

In particular, the integers in B form a ring.

8.6. Proposition. *If B is a division algebra, then*

$$\gamma_{\psi,\mathrm{NM}}(a) = -|a|^2$$

for all a in $F^\times$, and NM is surjective.

What is $\gamma_{\psi,\mathrm{NM}}$?

Proof. Well,

$$\mathrm{NM} = N_{E/F} \oplus -\alpha N_{E/F}$$

so that by Corollary 7.7

$$\begin{aligned} \gamma_{\psi,\mathrm{NM}}(a) &= \gamma_{\psi,N_{E/F}}(a) \cdot \gamma_{\psi,-\alpha N_{E/F}}(a) \\ &= \gamma^2_{\psi,N_{E/F}}(a)\mathrm{sgn}(-\alpha) \\ &= |a|^{-2}\mathrm{sgn}^2(-1)\mathrm{sgn}(\alpha) \\ &= -|a|^2 . \end{aligned}$$

This is independent of E. Furthermore, it implies that $\nu_Q(a) = |a|$ for all a, which proves the second claim. ▯

8.7. Proposition. *Up to equivalence, there are exactly two quadratic forms of dimension four with discriminant 1, $H \oplus H$ and a unique anisotropic form.*

Proof. This is because every form of discriminant 1 is the norm form of some quaternion algebra, which is either $M_2(F)$ or anisotropic. In the second case, all are equivalent, since they all have the same function $\gamma_{\psi,Q}$. ▯

8.8. Corollary. *There is up to isomorphism exactly one quaternion division algebra over F.*

Proof. Fix E and α such that $B_{E,\alpha}$ is a division algebra.

Let B be any quaternion division algebra. By Proposition 8.6, the norm of B is surjective. Therefore B is equal to the orthogonal sum of x^2 and its restriction to the elements of B with trace 0. So is the analogous restriction form for $B_{E,\alpha}$. These two restrictions are equivalent, since their characteristics are the same. So B also contains an embedded copy of E, which implies that B and $B_{E,\alpha}$ are isomorphic. ∎

9. Classification of $\mathfrak{p}$-adic quadratic forms

In this section I'll complete the classification of $\mathfrak{p}$-adic quadratic forms. As I have already mentioned, this reduces to the classification of anisotropic forms, which I now list.

Dimension one. These are the forms $Q(x) = cx^2$, where c ranges over representatives of $F^\times/(F^\times)^2$. The function ν_Q is the characteristic function of one coset of this group in $F^\times$, and is therefore equal to some explicit linear combination of quadratic characters of $F^\times$. This leads to an expression for $\gamma_{\psi,Q}$ as a similar linear combination whose coefficients are the root numbers occurring in Tate's local functional equation. The only simple fact is that $\gamma_{\psi,Q}(1)$ is an eighth root of unity.

Dimension two. The anisotropic forms of dimension two are the $Q = cN_{E/F}$, where E ranges over the quadratic field extensions of F and c over representatives of $F^\times/N(E^\times)$. The function $\gamma_{\psi,Q}$ satisfies the equations

$$\gamma^2_{\psi,Q}(1) = \mathrm{sgn}_{E/F}(-1)$$
$$\gamma_{\psi,Q}(a) = \mathrm{sgn}_{E/F}(a)\gamma_{\psi,Q}(1)\,.$$

The function $\nu_{cN_{E/F}}$ is a positive multiple of the characteristic function of one coset of $N(E^\times)$ in $F^\times$.

Dimension four (a). There is exactly one anisotropic form of dimension four, the norm of the unique quaternion algebra B over F. The function $\gamma_{\psi,Q}$ is equal to $-|x|^{-2}$, and $\nu_Q = |x|$. In particular, as I have already noted, the image of ν_Q is all of F.

this contrastS with the form $H \oplus H$, for which $\gamma_{\psi,Q} = |x|^{-2}$.

In both cases, the functions $\gamma_{\psi,Q}/|x|^2$ are invariant under multiplication, and sgn_Q is the trivial character.

Dimension three. Suppose (V, Q) to be an anisotropic quadratic space of dimension three, say of discriminant d. The orthogonal sum $Q + dx^2$ has discriminant 1. It cannot be $H \oplus H$, so it must be equivalent to the norm formof the quaternion algebra. Therefore $\gamma_Q = -\gamma^{-1}_{dx^2}|x|^{-2}$, and two such forms with the same discriminant are equivalent.

On the other hand, suppose d given. I claim that we can choose a, b, c such that

$$ax^2 - aby^2 - cz^2 = aN_{E/F} \oplus (-cz^2)$$

with $E = F(\sqrt{b})$ is anisotropic and has discriminant d. For this we require that (1) $d = a^2bc$ and (2) b is not a square in $F^\times$, and (3) c is not in $aN(E^\times)$. For this, choose b to be element of $F^\times$ that is not a square. Then choose a such that $db^{-1}(F^\times)^2$ is contained in $aNE^\times$. Finally, set $c = db^{-1}a^{-2}$.

All in all:

9.1. Proposition. *The map assigning to Q its discriminant induces a bijection of equivalence classes of anisotropic quadratic forms of dimension three with $F^\times/(F^\times)^2$.*

Dimension four (b). Now suppose that Q is of dimension four but does not have discriminant 1. In this case, it must be isotropic.

Why? Suppose

$$Q = cx^2 \oplus R\,,$$

with R of dimension three, does not have discriminant 1. If R is isotropic we are through. So suppose R to be anisotropic. The condition on the discriminant and the discussion of the three-dimensional case then tells us we can find an isotropic vector for Q.

Hence any quadratic form Q of dimension four and non-trivial discriminant is of the form

$$Q = \alpha N_{E/F} \oplus H$$

where E is distinguished by the discriminant of Q, and α is taken modulo $NE^\times$. Again, forms are distinguished by their characteristics.

In this case, the signature character is $\mathrm{sgn}_{E/F}$.

Dimensions more than four.

9.2. Proposition. *All quadratic forms of dimension more than four are isotropic.*

Proof. It suffices to show this when the dimension is five. Suppose $Q = R \oplus cx^2$. If R does not have discriminant 1, then it is isotropic, and so is Q. Otherwise, it is either H^2 or the norm of a quaternion division algebra. In either case, the image of R is all of $F^\times$, so there exists x such that $R(x) = -c$, and Q is again isotropic.

One conclusion that we can see in retrospect is:

9.3. Proposition. *Two quadratic forms over F are equivalent if and only if their characteristics are the same.*

GO(Q). Suppose V to have even dimension, and let χ_Q be its signature character.

9.4. Proposition. *In even dimension, the image of the homomorphism μ from GO_Q to $F^\times$ is a coset of the subgroup in which $\chi_Q = 1$.*

Proof. If $\chi(a) = 1$, then aQ and Q have the same characteristic, so that (V, Q) and (V, aQ) are isomorphic. But this means that there exists g in GO_Q with $\mu(g) = a$.

Part IV. References

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