Essays in analysis

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The Hardy-Littlewood maximal inequality (discrete version)

In this essay, I'll present the proof in [Bollobas:2006] (solution to Problem 85) of a well known result of [Hardy-Littlewood:1930], which amounts to the discrete case of a more famous theorem. In fact, this discrete version was for them a preliminary to the later continuous one. The illustrations are my main contribution, but I have also made some effort to make the obvious a little more obvious.

My motive for taking up this subject is the approach in [Brislawn:1988] and [Brislawn:1990] to trace formulas.

Contents

1. Finite arrays

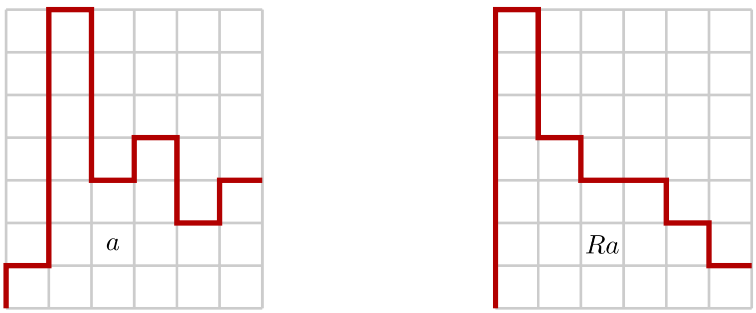
Suppose a to be an array (a_i) for $0 \le i < n$.

I'll associate to it a function defined on all of $[0, n]$, which I'll also designate as a:

$$a(x) = a_i \ \text{ if } \ i \le x < i + 1 \,.$$

For convenience, I'll assume all the a_i to be non-negative. The graph of the array will then be some kind of bar graph.

For example, the figure on the left below displays the graph of the extended a when the original array is $(1, 7, 3, 4, 2, 3)$:



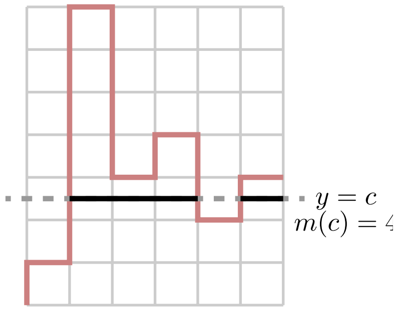
Define Ra to be the array a rearranged so as to be in (weakly) decreasing order. Thus $Ra = (7, 4, 3, 3, 2, 1)$ if $a = (1, 7, 3, 4, 2, 3)$. The figure on the right above displays its graph—the bars of the graph of a are just shifted around horizontally.

To every array is associated the set

$$\mathfrak{M}_a(c) = \{ i \,|\, a_i \ge c \}$$

and the associated function $m_a(c) = |\mathfrak{M}_a(c)|$. This is also the one-dimensional measure of the intersection of the line $y = c$ and the region

$$\{(x, y) \mid 0 \leq y \leq a(x)\}\,.$$

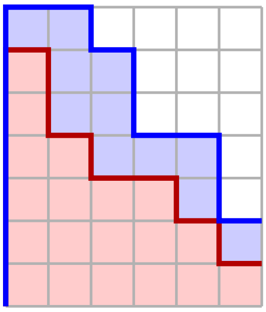


The functions m_a and m_{Ra} are the same.

1.1. Proposition. *Suppose a and b to be two non-negative weakly decreasing arrays. The following are equivalent:*

(a) $a_i \leq b_i$ *for all* i*;*
(b) $m_a(y) \leq m_b(y)$ *for all* y*;*
(c) the bar graph of a is contained in that of b.

The following figure illustrates what's going on.



I'll write $a \leq b$ if these conditions hold.

1.2. Corollary. *If $a \leq b$ then $Ra \leq Rb$.*

Proof. Because $m_a(y) \leq m_b(y)$ for all y. ▯

The following is a trivial observation:

1.3. Lemma. *If $b = Ra$, then for every set I of size n*

$$b_0 + \cdots + b_{n-1} \geq \sum\nolimits_{i \in I} a_i\,.$$

2. The maximal function

Continue to let a be an array with indices in $[0, n)$. I now associate to it a new array Ma. Define it by the specification

$$Ma_i = \max_{0 \le j \le i} \frac{a_j + \cdots + a_i}{(i-j)+1} \,.$$

This can be calculated by hand, but also very easily in a spreadsheet. For convenience of notation in the table below, let

$$S^i_j = \left(\sum_{k=i-j+1}^{i} a_k\right) \Big/ j \,.$$

Then the tableau looks like this:

i	a_i	S^i_1	S^i_2	S^i_3	S^i_4	S^i_5	S^i_6	max
0	1	1						1
1	7	7	4					7
2	3	3	5	11/3				5
3	4	4	7/2	14/3	15/4			14/3
4	2	2	3	3	4	17/5		4
5	3	3	5/2	3	3	19/5	20/6	19/5

In dealing with averages, it will be convenient to have in front of us a trivial observation:

2.1. Lemma. *Suppose the finite set A to be the disjoint union of subsets A_i, and for each i let $s_i = |A_i|/|A|$. Then the average of a function f over A is equal to the weighted sum of the averages over the A_i:*

$$\frac{\sum_{a \in A} f(a)}{|A|} = \sum_i s_i \cdot \frac{\sum_{a \in A_i} f(a)}{|A_i|} \,.$$

One consequence is that

- *if the average over each A_i is in the interval $[a, b]$, so is the average over A.*

There are a couple of simple facts about the array Ma. First of all, since a_i itself is among the averages, $Ma_i \ge a_i$. Second, if a is a weakly decreasing array then Ma_i is simply the average

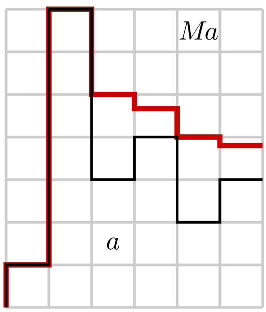
$$\frac{a_0 + a_1 + \cdots + a_i}{i+1}$$

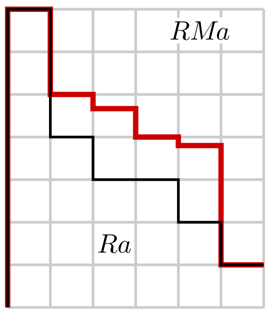
of all preceding entries. This should be intuitively clear, but follows also from Lemma 2.1.

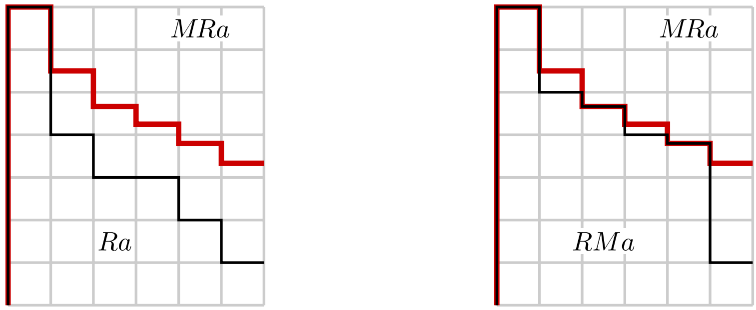
The main result of this essay:

2.2. Theorem. (Hardy-Littlewood) *For all a, $RMa \le MRa$.*

For example, with a as above, here are some relevant graphs:







Proof. In several steps.

Step 1. It is to be shown that $RMa_i \le MRa_i$. According to Proposition 1.1, for this it suffices to show that for every y less than the maximum value of a_i

(2.3)
$$\big|\mathfrak{M}(y)\big| = \big|\{i \,|\, RMa_i \ge y\}\big| = \big|\{i \,|\, Ma_i \ge y\}\big| \le \big|\{i \,|\, MRa_i \ge y\}\big| \,.$$

So suppose that the right hand side is equal to k. We want to deduce that $|\mathfrak{M}(y)| \le k$.

If $k = n$, this is immediate. So from now on suppose $k < n$. To reduce notational complexity, let $b = Ra$. Since b is decreasing,

$$MRa_i = \frac{b_0 + \cdots + b_i}{i+1} \,.$$

for all i. The assumption about k therefore directly translates to the condition

$$\frac{b_0 + \cdots + b_{k-1}}{k} \ge y > \frac{b_0 + \cdots + b_k}{k+1} \,.$$

The average value of any $k+1$ values of a_i is therefore less than y. Equivalently,

- *if I is any subset of $[0, n)$ on which the average value of a_i is $\le y$, then $|I| \le k$.*

The proof will show that this holds for $\mathfrak{M}(y)$.

Step 2. If $Ma_i \ge y$ there will be some largest index $\mu = \mu(i) \le i$ such that

$$Ma_i \ge \frac{a_\mu + \cdots + a_i}{(i-\mu)+1} \ge y \,.$$

That is to say, the interval $I_i = [\mu, i]$ is the shortest possible satisfying this condition. In these circumstances

$$\frac{a_j + \cdots + a_i}{(i-j)+1} \begin{cases} < y & \text{if } \mu < j \le i \\ = y & \text{if } \mu = j. \end{cases}$$

2.4. Lemma. *Suppose i, j both in $[0, n)$. Then either I_i and I_j are disjoint, or one is contained in the other.*

Proof. We may suppose $j < i$. If I_i and I_j overlap, then $\mu(i) \le j < i$. If I_j is not contained in I_i, then $\mu(j) < \mu(i)$. The average of Ra on the interval $[\mu(i), h]$ is less than y, by choice of $\mu(j)$. But so is the average over the interval $[h+1, i]$, by choice of $\mu(i)$. By Lemma 2.1, this implies that the average over $[\mu(i), i]$ is less than y, a contradiction. ∎

Step 3. Let $\mathcal{I}$ be the union of the all the intervals $[\mu(i), i]$, which, according to the Lemma, is the same as the union of all the maximal intervals. Since the average of a_i over each is $\ge y$, so is the average over all of $\mathcal{I}$. According to an earlier observation, this implies that $|\mathcal{I}| \le k$.

Step 4. However, the set $\mathfrak{M}(y)$ is contained in $\mathcal{I}$, and therefore $|\mathfrak{M}(y)| \le k$. ∎

3. References

1. Béla Bollobas, **The art of mathematics: coffee time in Memphis**, Cambridge University Press, 2006.

2. Christopher Brislawn, 'Kernels of trace class operators', *Proceedings of the American Mathematical Society* **104** (1988), 1181–1190.

3. ——, 'Trace class integral kernels', pp. 61–62 in Part 2 of **Operator theory: operator algebras and applications**, *Proceedings of Symposia in Pure Mathematics* **51**, American Mathematical Society, 1990.

4. G. H. Hardy and John E. Littlewood, 'A maximal theorem with function-theoretic applications', *Acta Mathematica* **54**, 1930.