Essays in analysis

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Hilbert spaces

This essay is a fairly complete and almost self-contained account of that part of the theory of operators on Hilbert spaces that's needed in the theory of automorphic forms. My principal reference is the admirable manual [Reed-Simon:1972], particularly Chapters VI and VIII. I have also used [Treves:1967]. Much of this essay is copied from one or the other of these, but neither should be blamed for my sometimes eccentric approach to well known material.

Some of the theory applies without much modification to Banach spaces, and this too will useful in the applications to automorphic forms. At little apparent extra cost, I give a short account of this material as well. Occasionally, I refer to [Reed-Solomon:1972] for proofs there that are essentially self-contained.

All vector spaces will be assumed to be over $\mathbb{C}$ unless mentioned otherwise.

Contents

1. Banach spaces

A **norm** on a vector space V is a function $\|v\|$ satisfying the conditions

(a) $\|cv\| = |c|\,\|v\|$;
(b) $\|u+v\| \le \|u\| + \|v\|$;
(c) $\|v\| = 0$ if and only if $v = 0$.

A Banach space is a vector space assigned a norm with respect to which it is complete. Completeness means that Cauchy sequences converge.

Any book on functional analysis (and certainly [Reed-Simon:1972], in Chapter III) will offer many examples, but for the applications I have in mind there is one that is particularly relevant. If X is a locally compact topological space, define $C(X)$ to be the vector space of all continuous bounded complex-valued functions on X. For f in $C(X)$ define

$$\|f\| = \sup_x \big|f(x)\big| \,.$$

I leave the following as an exercise (start with IV.8 of [Reed-Simon:1972]):

1.1. Proposition. *The space $C(X)$ with this norm is a Banach space.*

Banach spaces are locally convex topological vector spaces (TVS), and all results about the more general class apply to them. The following two results are examples:

1.2. Theorem. (Hahn-Banach)*If U is a closed subspace of the Banach space V, then any continuous linear function on U may be extended to one on V.*

1.3. Theorem. *Every finite-dimensional Banach space is isomorphic to the standard one of the same dimension, and every finite-dimensional vector subspace of a Banach space is closed.*

The proofs of these are no simpler for Banach spaces than for general TVS. I refer elsewhere (say, [Reed-Simon:1972]) for these things. Some results, while valid for a larger class of TVS, are significantly easier to prove for Banach spaces, and I include their proofs below.

If V is a Banach space, by convention its dual V^* is defined to be its **conjugate-linear dual**, the space of all continuous conjugate-linear functions $\mathbb{V} \to \mathbb{C}$, satisfying $f(cv) = \overline{c} f(v)$. Thus for v in V, v_* in V^*

$$\langle cv, v_* \rangle = \overline{c} \langle v, v_* \rangle, \quad \langle v, cv_* \rangle = c \langle v, v_* \rangle \,.$$

In order to simplify notation throughout this essay, and following standard notation for Hilbert spaces, I shall write

$$u \bullet v_* = \text{the conjugate of } \langle u, v_* \rangle \,.$$

Thus

$$cu \bullet v_* = c(u \bullet v_*), \quad u \bullet cv_* = \overline{c}(u \bullet v_*) \,.$$

As we'll see in the next section, this agrees with how duality for Hilbert spaces is dealt with.

By definition of continuity, there exists for every f in V^* a constant $C > 0$ such that

$$\big| f(v) \big| \le C \|v\|$$

for all v in V. Define

$$\|f\| = \sup_{\|v\|=1} \big| f(v) \big| \,.$$

1.4. Proposition. *The space V^* with norm $\|f\|$ is a Banach space.*

Proof. See III.2 of [Reed-Simon:1972]. ▯

Since V^* is a Banach space, so is its dual V^{**}. There exists a canonical map from V to V^{**}.

1.5. Proposition. *The canonical map from V to V^{**} is an isometric embedding.*

Proof. See III.4 of [Reed-Simon:1972]. ▯

1.6. Corollary. *The canonical image of V in V^{**} is closed.*

If U is a subspace of V, define

$$U^{\perp} = \{ v_* \in V^* \mid u \bullet v_* = 0 \text{ for all } u \in U \} \,.$$

It is the **annihilator** of U in V^*. Thus $U^{\perp\perp}$ is a subspace of V^{**}.

1.7. Corollary. *If U is a closed subspace of V then*

$$U^{\perp\perp} \cap V = U \,.$$

Proof. By Hahn-Banach. ▯

2. Hilbert spaces

Suppose V to be a complex vector space given a positive definite Hermitian inner product $u \bullet v$. Associated to this is the function $\|u\| = (u \bullet u)^{1/2}$

2.1. Proposition. (Cauchy-Schwarz) *For any u and v*

$$|u \bullet v| \le \|u\|\,\|v\|\ .$$

Proof. For any ζ with $|\zeta| = 1$ the quadratic function

$$\|t\zeta u + v\|^2 = t^2\|u\|^2 + 2t\,\mathrm{RE}(\zeta u \bullet v) + \|v\|^2$$

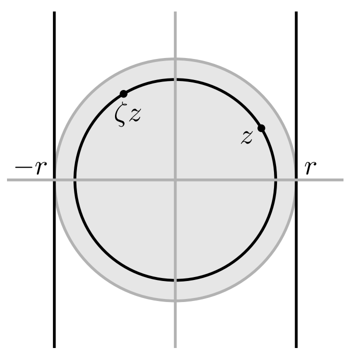
take non-negative values for all t, which implies that its roots are either complex conjugate or equal, hence that its discriminant

$$\big(\mathrm{RE}(\zeta u \bullet v)\big)^2 - \|u\|^2\|v\|^2$$

is non-positive and therefore

$$\big|\mathrm{RE}(\zeta u \bullet v)\big| \le \|u\|\,\|v\|$$

for all ζ.



But if z is any complex number and $\big|\mathrm{RE}(\zeta z)\big| \le r$ for all $|\zeta| = 1$ then the circle around the origin of radius $|z|$ lies inside the vertical strip $\big|\mathrm{RE}(w)\big| \le r$, and in particular $|z| \le r$. ∎

2.2. Corollary. *For any u and v*

$$\|u + v\| \le \|u\| + \|v\|\ .$$

Equivalently,

$$\|u - v\| \ge \|u\| - \|v\|\,.$$

Proof. It suffices to show that

$$\|u + v\|^2 = \|u\|^2 + 2\,\mathrm{RE}(u \bullet v) + \|v\|^2 \le \big(\|u\| + \|v\|\big)^2 = \|u\|^2 + 2\|u\|\,\|v\| + \|v\|^2\ ,$$

which follows immediately from the Proposition. ∎

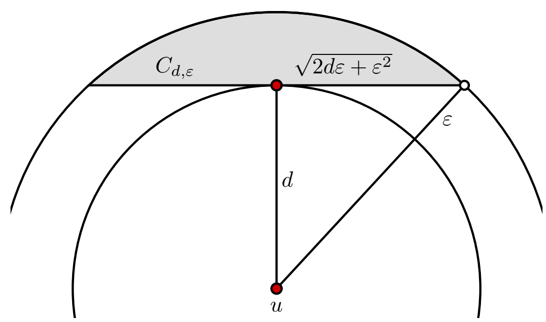
A **Hilbert space** will be defined in this essay to be a complex vector space assigned a positive-definite inner product with respect to which it is complete, and which possesses a countable dense subset. (This is what is often called in the literature a **separable** Hilbert space.)

2.3. Corollary. *A Hilbert space is a Banach space.*

2.4. Proposition. *If C is any closed convex set in a Hilbert space H and u a point of H, then there exists a unique point $\overline{u}$ in C closer to u than any other point in C.*

Proof. We may assume that u does not lie in C itself. Let d be the greatest lower bound of all the distances from u to points of C. This means that there are no points of C inside the ball of radius d centred at u, but there are some inside that of radius $d+\varepsilon$ for every $\varepsilon > 0$. We want to find a point in C at distance exactly d, and we shall do this by exhibiting it as the limit of a Cauchy sequence.

Let $C_{d,\varepsilon}$ be the intersection of C with the interior of the ball of radius $d+\varepsilon$ and the exterior of that of radius d. We want to know that the sets $C_{d,\varepsilon}$ form a Cauchy sequence, or equivalently that ε becomes small, so does the diameter of $C_{d,\varepsilon}$. The following figure suggests how this happens. Any two points in a convex subset of the ring in between C_d and $C_{d+\varepsilon}$ have to lie inside a 'cap' region like the shaded one.



What the picture suggests, since the arc is close to a parabola, is that the diameter of the cap is of the order of $\sqrt{\varepsilon}$. More precisely:

2.5. Lemma. *If v_1 and v_2 are two points inside a sphere of radius $d+\varepsilon$ centred at the origin and the line segment between v_1 and v_2 lies completely outside the sphere of radius d then $\|u-v\| \le 2\sqrt{2d\varepsilon+\varepsilon^2}$.*

Proof of the Lemma. If $\mu = (v_1+v_2)/2$ is the midpoint between v_1 and v_2, then

$$\begin{aligned}\|\mu\|^2 + \frac{\|v_1-v_2\|^2}{4} &= \frac{\|v_1\|^2+\|v_2\|^2}{2} \\ \|v_1-v_2\|^2 &= 2\big(\|v_1\|^2+\|v_2\|^2\big) - 4\|\mu\|^2 \\ &\le 4(d+\varepsilon)^2 - 4d^2 \\ \|v_1-v_2\| &\le 2\sqrt{(d+\varepsilon)^2-d^2} \\ &= 2\sqrt{2d\varepsilon+\varepsilon^2}\,. \ \square\end{aligned}$$

Conclusion of the proof of Proposition 2.4. According to Lemma 2.5, for each positive integer n the set C_n of all points in C of distance at most $d+1/n$ isn't empty and has diameter at most $2\sqrt{2d/n+1/n^2}$. This maximum diameter is a monotonic decreasing function of n with limit 0. Furthermore $C_{n+1} \subseteq C_n$, so the sets C_n form a Cauchy sequence, hence converge to a unique point of H. It lies in C since C is closed. $\square$

If U is a linear subspace of a Hilbert space, let $U^\perp$ be the subspace of vectors perpendicular to it.

2.6. Corollary. *If U is a closed affine subspace of a Hilbert space and u a point in the space, then there exists on U a unique point $\overline{u}$ nearest to u. The vector $u-\overline{u}$ is perpendicular to U.*

Proof. Let $\overline{u}$ be the nearest point, $\Delta u = u-\overline{u}$, w a vector parallel to the given affine subspace. For $v = \overline{u}+\zeta t w$ in U with $|\zeta| = 1$, $t > 0$ real

$$\|u-v\|^2 = \|\Delta u\|^2 - 2t\,\mathrm{RE}(\zeta w \bullet \Delta u) + t^2\|w\|^2 = \|\Delta u\|^2 - t\big(2\,\mathrm{RE}(\zeta w \bullet \Delta u) - t\|w\|^2\big)\,.$$

If $w \bullet \Delta u \neq 0$ we can choose ζ so that $\zeta w \bullet \Delta u = \zeta(w \bullet \Delta u) > 0$. If we then choose t small we can make $2\,\mathrm{RE}(\zeta w \bullet \Delta u) - t\|w\|^2$ positive, a contradiction. ∎

2.7. Corollary. *If U is a closed linear subspace of a Hilbert space H then every v in H is the sum of unique vectors in U and $U^\perp$.*

Finite dimensional Hilbert spaces offer no surprises.

2.8. Proposition. *Any finite-dimensional subspace of a Hilbert space H is closed.*

Proof. Suppose E of dimension n in H, v a point of H not in E. It must be shown that there is a positive minimum distance from v to points of E.

Since the formula for a Hermitian dot-product is algebraic, the norm on E induces the standard topology on E. Say $\|v\| = R$. Then the minimum distance from v to E is less than R. If $\|e\| > 2R$ then

$$\|e - v\| \geq 2R - R = R\,.$$

So the minimum m of the distance $\|e - v\|$ (for e in E) is achieved on the compact ball $\|e\| \leq 2R$. Since v does not lie in E, $m > 0$. ∎

2.9. Corollary. *A locally compact Hilbert space has finite dimension.*

Proof. Let U be a compact neighbourhood of 0. Fix $0 < c < 1$. The set U will be covered by a finite number of translates $e_i + cU$. Let E be the subspace spanned by the e_i. I claim that $H = E$. Since E is closed, $H = E \oplus E^\perp$. Let $\overline{U}$ be the projection of U onto $E^\perp$. Since U is covered by the $e_i + cU$, for every n we can find a finite set F of f in E with U covered by translates $f + c^n U$. Therefore $\overline{U} \subseteq c^n \overline{U}$ for all n, and $\overline{U}$ is at once homogeneous and compact. It can only be $\{0\}$, and hence $E^\perp = \{0\}$. ∎

2.10. Proposition. *A Hilbert space possesses a countable orthogonal basis.*

Proof. I have assumed H to possess a countable dense subset E. Apply Gram-Schmidt orthogonalization to it. ∎

If v is a vector in a Hilbert space then the dot product $f_v\colon u \mapsto u \bullet v$ is a continuous (or **bounded**) linear function with values in $\mathbb{C}$, since

$$\bigl|f_v(u)\bigr| \leq C\,\|u\|$$

with $C = \|v\|$. The converse to this is the most characteristic property of Hilbert spaces:

2.11. Proposition. (Riesz Lemma) *If H is a Hilbert space and $f\colon H \to \mathbb{C}$ a bounded linear function, there exists a unique v in H such that $f(u) = u \bullet v$ for all v in H.*

Proof. We may assume f does not vanish identically. The linear space where $f(u) = 0$ is closed and by Corollary 2.7 possesses an orthogonal complement, which is necessarily one-dimensional. The vector v can be taken to be a suitably normalized vector in that complement. ∎

In other words, and in somewhat informal terms:

2.12. Corollary. *A Hilbert space H may be identified with its own conjugate linear dual H^*.*

It is this that makes Hilbert spaces so much simpler than general topological vector spaces, for which duality causes major complications.

3. Bounded operators on a Banach space

If $T\colon V_1 \to V_2$ is a linear transformation from one Banach space to another then it is continuous if and only if for some $r > 0$ the image under T of the ball

$$B_r = B_r(0) = \{v \mid \|v\| \le r\}$$

of radius r in V_1 is contained in the unit ball of V_2. Clearly equivalent is the condition that it be **bounded**—i.e. that

$$\|T(u)\| \le C\|u\|$$

for some constant C. The **norm** $\|T\|$ is defined to be the supremum of the ratios $\|T(u)\|/\|u\|$ as u varies over all non-zero vectors in the domain space of T, or equivalently the supremum of $\|T(u)\|$ for $\|u\| = 1$. Thus by definition

$$\|T(u)\| \le \|T\|\,\|u\|$$

for all u.

The bounded linear operators from one Banach space to another are not the most interesting maps associated to the two spaces. In particular, it is another class of operators that arises in spectral analysis of differential operators in one and more dimensions. But in the end just about all questions reduce to one about bounded operators of some kind or another.

Going from finite-dimensional vector spaces to infinite-dimensional ones often induces a somewhat queasy feeling, as if all were not what it seems, and certainly not what it should be. The significance of the next result, although at first sight quite technical, is that things aren't all that bad.

3.1. Proposition. (Open Mapping Theorem) *Every surjective, bounded, linear operator $T\colon V_1 \to V_2$ is an open mapping.*

Proof. It has to be shown that the image under T of a ball B_r in V_1 contains a ball in the image space. In order to prove this, we have to use the Baire Category Theorem, which asserts that if a countable union of closed subsets of a separable metric space contains an open set, then one of the closed sets already has to contain an open set. If this doesn't seem quite obvious, that's perhaps because its proof depends on the Axiom of Choice, which often offers surprises in its many manifestations. At any rate, I'll just sweep this little difficulty under the rug and not prove it here.

Its application here should be clear. We want to show that the image $T(B_1)$ contains an open neighbourhood of 0. It will suffice to show this for any B_n. The union of the closures of the images $T(B_n)$ of the balls B_n is all of V_2, hence according to Baire one of the closures $\overline{T(B_n)}$ must contain an open subset of V_2, hence some $v + B_\varepsilon$. But then it will also contain $(v + B_\varepsilon) - (v + B_\varepsilon) = B_{2\varepsilon}$.

The final step of the proof is to show that $T(B_{4n})$ contains $\overline{T(B_{2n})}$ and thus B_ε in turn. Equivalently, after rescaling, we just need to show that $T(B_2)$ contains $\overline{T(B_1)}$, given that $\overline{T(B_1)}$ contains B_ε. Suppose y to lie in $\overline{T(B_1)}$. Choose x_1 in B_1 such that

$$y_1 = y - T(x_1) \in B_{\varepsilon/2} \subseteq \overline{T(B_{1/2})}\,.$$

Then choose x_2 in $B_{1/2}$ so that

$$y_1 - T(x_2) \in B_{\varepsilon/4}\ \ldots$$

In the limit

$$y = T(x_1 + x_2 + \cdots\ )\,. \qquad \square$$

This has a remarkable and reassuring consequence:

3.2. Corollary. *A bounded bijective operator is a topological isomorphism.*

In other words, its inverse is also bounded.

If $T\colon V_1 \to V_2$ is a bounded operator, its **graph** is the set of all pairs $\big(x, T(x)\big)$ in their direct sum, with x ranging over all vectors in V_1. It is a closed linear subspace of $V_1 \oplus V_2$, one which projects isomorphically onto the first factor. Conversely, as an immediate consequence of the previous result:

3.3. Corollary. (Closed Graph Theorem) *If Γ is a closed linear subspace of the direct sum $V_1 \oplus V_2$ that projects bijectively onto the first summand then it is the graph of a bounded operator.*

The condition of course means that (a) the image of the projection is all of V_1 and (b) the inverse image of 0 is just $(0, 0)$. Both of these conditions will be relaxed in the next section, where more interesting operators are considered.

4. Unbounded operators

A bounded operator from the Banach space V_1 to another one V_2 is determined by its graph, a linear subspace in $V_1 \oplus V_2$ that projects isomorphically onto V_1. But it is useful to consider other linear subspaces of the sum as defining analogues of the bounded operators. Phenomena appear, and are important, that have no analogue in finite dimensions. A typical and relevant example is the operator $D = d/dx$. It is not in any straightforward way defined on all of the Hilbert space $\mathrm{L}^2(\mathbb{R})$, but it is defined on a dense subspace, that of all smooth functions of compact support. This definition can be extended to all functions f such that f', when considered as a distribution on $\mathbb{R}$, lies in $\mathrm{L}^2(\mathbb{R})$. Integration by parts tells us that the equation

$$f' \bullet g = -f \bullet g'$$

holds for all f, g in $C_c^1(\mathbb{R})$. The theory to come will imply that it also holds whenever f, f', g, g' lie in $\mathrm{L}^2(\mathbb{R})$. This not an obvious fact.

We start with a slightly more general situation. A **correspondance** Γ between two Banach spaces V_1, V_2 is *any* linear subspace of the direct sum $V_1 \oplus V_2$. There are interesting cases in which it need not even be closed. Its **domain** $\mathrm{Dom}(\Gamma)$ is its projection onto the first factor, and I define its **ambiguity** $\mathrm{Amb}(\Gamma)$ to be the kernel of this projection, which may (and will) be identified with a linear subspace of V_2. If its ambiguity is trivial, I call Γ **well defined** or **unambiguous**. If the graph is closed, as we have seen in the previous section, an unambiguous correspondance whose domain is all of V_1 is the graph of a bounded operator.

An unambiguous correspondance Γ defines a linear transformation T_Γ from its domain to V_2—for any x in this domain its image $T_\Gamma(x)$ with respect to the operator is the unique y in V_2 such that (x, y) lies in Γ. An unambiguous correspondance is usually denoted by the pair (D, T), where D is the domain and T the linear operator, but sometimes one or the other is not explicitly mentioned. Sometimes also, D is referred to as $\mathrm{Dom}(T)$. A correspondance is rarely of interest unless both of these conditions hold:

(1) it is unambiguous;
(2) its domain is dense in V_1.

Without the second condition there is not much tie to the Banach space V_1, after all. Correspondances satisfying these two conditions I call **operators**.

As suggested by opening remarks, a typical example of an operator would be the pair $C_c^\infty(\mathbb{R})$ and the graph of $f \mapsto df/dx$, with $V_1 = V_2 = \mathrm{L}^2(\mathbb{R})$.

CLOSED OPERATORS. A correspondance Γ is called **closed** when it is a closed subspace of the direct sum. In this case $\mathrm{Amb}(\Gamma)$ is a closed subspace of H_2. Closed operators are to be thought of as the true generalizations of bounded operators, since one cannot expect too much from operators with no topological properties imposed. If Γ is any correspondance then one may define its **graph closure** $\overline{\Gamma}$ to be the correspondance which is the closure of its graph. However, even if Γ is an operator this new correspondance may not be an operator—that is to say, it may not be unambiguously defined. An operator is called **closable** if its closure is unambiguous—i.e. again an operator. Operators that aren't closable aren't very interesting, but it is not always easy to tell whether one is or isn't. We shall see later a large class of interesting operators that are closable. This will include the operator $\big(d/dx, C_c^\infty(\mathbb{R})\big)$ mentioned above.

4.1. Proposition. *If $\overline{\Gamma}$ is unambiguous, its domain is the completion of the domain D of Γ with respect to the metric $\|x\| + \|T_\Gamma(x)\|$.*

The Banach space norm $\|x\| + \|T_\Gamma(x)\|$, or any norm equivalent to it, is called for obvious reasons a **graph norm**. In the case of an operator on Hilbert spaces, an equivalent norm is $\big(\|x\|^2 + \|T_\Gamma(x)\|^2\big)^{1/2}$

Proof. Because according to Corollary 3.2 its domain is isomorphic under projection to its graph. ∎

If you believe the claim that d/dx defines a closable operator on L^2, this at least gives you a non-trivial example of an operator, since it should be pretty clear that the completion of $C_c^\infty(\mathbb{R})$ with respect to the norms $\|f\|^2_{\mathrm{L}^2}$ and $\|f\|^2_{\mathrm{L}^2} + \|f'\|^2_{\mathrm{L}^2}$ are very different—the first is $\mathrm{L}^2(\mathbb{R})$, while as we shall see later the second consists of all f in $\mathrm{L}^2(\mathbb{R})$ such that f', considered as a distribution, is also in $\mathrm{L}^2(\mathbb{R})$.

> *If T is a closable operator, then v lies in the domain of its closure with $\overline{T}(v) = w$ if and only if there exists a sequence v_i in the domain of T with $\lim v_i = v$, $\lim T(v_i) = w$.*

ADJOINTS. Suppose T to be a bounded operator $V_1 \to V_2$. Then for every v_* in V_2^* we may define an element $T^*(v_*)$ in V_1^* by the condition

$$u_1 \bullet T^*(v_*) = T(u_1) \bullet v_* \,.$$

This map

$$T^*\colon\ V_2^* \to V_1^* \,,$$

is called the **adjoint** of T. If the V_i are Hilbert spaces then according to the Riesz Lemma (Proposition 2.11) this gives a map

$$T^*\colon\ V_2 \to V_1 \,.$$

4.2. Proposition. *If $T\colon V_1 \to V_2$ is a bounded operator, its adjoint $T^*\colon V_2^* \to V_1^*$ is also bounded.*

Proof. Because

$$u \bullet T^*(v_*) = T(u) \bullet v_* \le \|T\|\,\|u\|\,\|v_*\| \,. \qquad ∎$$

The pair (v_*, w_*) lies in the graph $\Gamma(T^*)$, which is a linear subspace of $V_2^* \oplus V_1^*$, if and only if $T(x) \bullet v_* - x \bullet w_* = 0$ for all x in V_1, or yet again if and only if (v_*, w_*) in $V_2^* \oplus V_1^*$ lies in the annihilator of all $(-y, x)$ for (x, y) in the graph of T.

Define the signed swap

$$\sigma\colon\ (x, y) \longmapsto (-y, x) \,.$$

Consistent with the observation just made, define the **adjoint of an arbitrary correspondance** $\Gamma \subseteq V_1 \oplus V_2$ to be the correspondance

$$\Gamma^* = (\Gamma^\sigma)^\perp \subseteq V_2^* \oplus V_1^* \,.$$

That is to say, it is the correspondance between the duals whose graph is the subspace annihilating all pairs $(-y, x) \in V_2 \oplus V_1$ with $(x, y) \in \Gamma$. These properties are immediate from the definition:

(a) this definition agrees with that for bounded operators;
(b) the adjoint of any correspondance is closed;
(c) a correspondance and its closure have the same adjoint.

I give quickly one simple example, in whch $D = C_c^\infty(\mathbb{R})$ and $T = d/dt$. The domain of its adjoint is the space of all F in $\mathrm{L}^2(\mathbb{R})$ for which there exists F_* in $\mathrm{L}^2(\mathbb{R})$ with $T(f) \bullet F = f \bullet F_*$ for all f in C_c^∞. Now any function in $\mathrm{L}^2(\mathbb{R})$ may be considered as a distribution, and its derivative F' is defined as a distribution by the equation

$$\langle f, F' \rangle = -\langle f', F \rangle .$$

So the domain of T^* is exactly the space of F in $\mathrm{L}^2(\mathbb{R})$ for which F', defined as a distribution, lies also in $\mathrm{L}^2(\mathbb{R})$.

Suppose now that Γ is a correspondance in $V_2^* \oplus V_1^*$. Its adjoint Γ^* lies in $V_1^{**} \oplus V_2^{**}$, and since the image of $V_1 \oplus V_2$ is closed in this, the intersection of Γ^* with $V_1 \oplus V_2$ is closed, defining a correspondance I'll call $\Gamma^\diamond$. If V is a Hilbert space, $\Gamma^\diamond = \Gamma^*$.

4.3. Proposition. *If Γ is a correspondance in $V_1 \oplus V_2$ then $\Gamma^{*\diamond} = \overline{\Gamma}$.*

Proof. Because $U^{\perp\perp} \cap V = \overline{U}$ if U is a vector subspace of a Banach space V, by Proposition 8.4. ∎

But this leaves completely open the question: *When is T^* an operator?* This breaks up into two questions: *When is it unambiguous? Densely defined?* The first question is simple to answer:

4.4. Proposition. *For any correspondance Γ*

$$\mathrm{Dom}(\Gamma)^\perp = \mathrm{Amb}(\Gamma^*) .$$

Proof. If x_* is in V_1^* then it lies in $\mathrm{Amb}(\Gamma^*)$ if and only if

$$(-y, x) \bullet (0, x_*) = 0$$

for every (x, y) in Γ, or if and only if $x \bullet x_* = 0$ for every x in $\mathrm{Dom}(T)$. ∎

4.5. Corollary. *If Γ is densely defined, its adjoint Γ^* is unambiguous.*

Equally simple is this:

4.6. Proposition. *If Γ^* has a dense domain, Γ is closable.*

To decide if Γ^* is densely defined is not always easy. In practice, exactly one criterion is useful. Suppose (D, T) an operator. Since T is densely defined, T^* is unambiguous. A formal adjoint $(D_*, T^\#)$ is an operator $T^\#$ with domain $D_* \subseteq V_2^*$ such that

$$T(u) \bullet v_* = u \bullet T^\#(v_*)$$

for all u in $\mathrm{Dom}(T)$, v_* in $\mathrm{Dom}(T^\#)$. The domain of T^* contains the domain of any formal adjoint. Therefore:

4.7. Corollary. *An operator T with a densely defined formal adjoint is closable.*

RANGE. Analysts define the **range** $\mathrm{Ran}(T)$ of an operator (D, T) to be what everybody else would call its image, the linear subspace $T(D)$.

The following is immediate:

4.8. Proposition. *For any operator T*

$$\mathrm{Ran}(T)^\perp = \mathrm{Ker}(T^*) .$$

As a special case:

4.9. Corollary. *Suppose T to be a bounded operator T from one Banach space to another. The range of T is dense if and only if T^* is injective. A map T from one Hilbert space to another is injective if and only if the range of T^* is dense.*

There is a slightly more conceptual way to define the adjoint, one that is very close to the method actually used to determine it in practice. The domain $\overline{D}$ of the closure is the completion of the domain of T with respect to the graph norm $\|v\| + \|T(v)\|$, a Banach space in its own right. Under the assumption that the closure is unambiguous, we have the injection

$$\iota : \overline{D} \hookrightarrow V_1$$

and under the assumption that the domain is dense we have a dual injection

$$\widehat{\iota} : V_1^* \hookrightarrow D^* .$$

The map $T : \overline{D} \to V_2$ determines a dual map

$$\widehat{T} : V_2^* \longrightarrow D^*$$

4.10. Proposition. *In these circumstances, the domain* $\mathrm{Dom}(T^*)$ *is the inverse image in* V_2^* *of* $V_1^* \subseteq D^*$*, and on this domain* $T^* = \widehat{T}$.

This is nothing more than a restatement of the definition. It can be summarized in a diagram:

$$\begin{array}{ccc} D & \stackrel{\iota}{\hookrightarrow} & V_1 \\ \downarrow T & & \\ V_2 & & \end{array} \qquad \begin{array}{ccc} D^* & \stackrel{\widehat{\iota}}{\hookleftarrow} & V_1^* \\ \uparrow \widehat{T} & & \uparrow T^* \\ V_2^* & \stackrel{\iota^*}{\hookleftarrow} & \mathrm{Dom}(T^*) \end{array}$$

This fits in with the earlier discussion of d/dt acting on distributions. There $V_1 = V_2 = \mathrm{L}^2(\mathbb{R})$, $D = C_c^\infty(\mathbb{R})$, and D^* is the space of distributions. The domain of D^* is the subs[pace of f in $\mathrm{L}^2(\mathbb{R})$ such that the distribution df/dx lies in $\mathrm{L}^2(\mathbb{R}|)$.

5. Self-adjoint operators

There is one situation where closeability is straightforward. An operator T on the Hilbert space H with values in H is called **symmetric** if

$$T(x) \bullet y = x \bullet T(y)$$

for all x, y in $\mathrm{Dom}(T)$. It follows from Corollary 4.7 that any symmetric operator T is closable. A simple argument about limits proves that its closure is still symmetric, and that T^* extends T. A **self-adjoint** operator is one which in fact coincides with its adjoint. Self-adjoint operators and operators related to them might be reasonably considered to be the real justification for the whole machinery of unbounded operators.

5.1. Lemma. *If T is a symmetric operator, it is closable, and the domain of any symmetric extension T_0 of T lies between* $\mathrm{Dom}(T)$ *and* $\mathrm{Dom}(T^*)$.

Proof. From Corollary 4.7, as I have already remarked, and the definition of the adjoint.

If the closure of a symmetric operator T is the same as its adjoint, the adjoint will be symmetric and in fact self-adjoint. In this case T is said to be **essentially self-adjoint**. One example of this, as we shall see, is the operator id/dt with domain $C_c^\infty(\mathbb{R})$ in $\mathrm{L}^2(\mathbb{R})$. It may also happen, however, that T has *no* self-adjoint extensions at all. An example of this is id/dt with domain $C_c^\infty(0,\infty)$ in $\mathrm{L}^2(0,\infty,dx)$.

There is in principle a way to characterize all self-adjoint extensions of a given symmetric operator. Suppose T to be symmetric and closed. The domain of T^* is a Hilbert space with squared norm $\|v\|^2 + \|T^*v\|^2$. The subspace $\mathrm{Dom}(T)$ is closed in it. The quotient $\mathrm{Dom}(T^*)/\mathrm{Dom}(T)$ possesses a natural Hilbert space structure, and on it one may define the non-degenerate skew-Hermitian form

$$T^*(x) \bullet y - x \bullet T^*(y) \, .$$

The possible self-adjoint extensions of T correspond bijectively, in the obvious way, to the closed subspaces of this quotient which are, with respect to this form, their own complements. If $\mathrm{Dom}(T^*)/\mathrm{Dom}(T)$ is finite dimensional, these are just the classical Lagrangian subspaces. For such subspaces to exist, this form must be hyperbolic in a strong sense. It is not hard to see that this is always true if H and T arise by complexification from a real Hilbert space.

Example. It should help if I explain right now one example, even though without proof.

Let H be $\mathrm{L}^2[0,1]$, the space of all square-integrable functions on $[0,1]$. Let $T = d^2/dx^2$ with domain $D = C_0^\infty[0,1]$, the linear space of all smooth functions on $[0,1]$ vanishing of infinite order at 0 and 1. The adjoint is the space of all f in H such that f' and f'', considered as distributions, are square-integrable functions. Such a function is continuous, and so is f', and in effect evaluation at 0 and 1 are continuous on $\mathrm{Dom}(T^*)$. (This will be proved later.) On the other hand, the domain of the closure of T is the subspace where $f(0) = f(1) = 0$, and the quotient of $\mathrm{Dom}(T^*)/\mathrm{Dom}(T)$ may be identified with $\mathbb{C}^4$ via evaluation of f and f' at 0 and 1.

We have the integral formula

$$
\begin{aligned}
\int_0^1 f(x)g''(x)\,dx &= [f(x)g'(x)]_0^1 - \int_0^1 f'(x)g'(x)\,dx \\
\int_0^1 f''(x)g(x)\,dx &= [f'(x)g(x)]_0^1 - \int_0^1 f'(x)g'(x)\,dx \\
\int_0^1 f(x)g''(x)\,dx - \int_0^1 f''(x)g(x)\,dx &= [f(x)g'(x) - f'(x)g(x)]_0^1 \,.
\end{aligned}
$$

What this means is that self-adjoint extensions correspond to the Lagrangian subspaces of $\mathbb{C}^4$ with respect to the skew-Hermitian form

$$[f(x)\overline{g}'(x) - f'(x)\overline{g}(x)]_0^1 \,.$$

This is because each Lagrangian subspace give rise to certain homogeneous conditions on boundary values. for example, one is determined by boundary conditions $f(0) = f(1) = 0$, another by periodicity.

I have not proved these claims, but I hope this will convey to you the association between self-adjoint extensions and boundary-values. To be essentially self-adjoint often means that a geometric region has no boundaries, at least not in the analytic sense.

Example. Let's look now at one case which is in some sense a model for all. Let dx be the usual measure on $\mathbb{R}$, and let M_φ be multiplication by the continuous function $\varphi(x)$ acting on the domain of all functions f in $\mathrm{L}^2(\mathbb{R})$ such that $\varphi(x)f(x)$ is also in $\mathrm{L}^2(\mathbb{R})$.

5.2. Proposition.. *The operator M_φ is self-adjoint.*

Proof. The operator M_φ is certainly symmetric, hence closable, so M_φ^* is an operator. Suppose g to be in the domain of M_φ^*. This means that there exists h in $\mathrm{L}^2(\mathbb{R})$ such that

$$\varphi f \bullet g = f \bullet h$$

for all f with $f, \varphi f$ in $\mathrm{L}^2(\mathbb{R})$. It must be shown that $h = \varphi g$, or equivalently that φg lies in L^2.

Let χ_N be the characteristic function of $[-N, N]$. For every f in $\mathrm{L}^2(\smallsetminus R)$ the function $\chi_N f$ lies in $\mathrm{Dom}(T)$. If g lies in $\mathrm{Dom}(T^*)$ then by definition there exists h in $\mathrm{L}^2(\mathbb{R})$ such that

$$(\varphi \chi_N f) \bullet g = (\chi_N f) \bullet h$$

for all f in $\mathrm{L}^2(\mathbb{R})$. This implies that if only φg lie sin $\mathrm{L}^2(\mathbb{R})$ then $h = M_\varphi^*(g) = \varphi g$.

If f is any locally L^2 function then f will be globally L^2 if and only if the limit of the $\|\chi_N f\|$ is bounded, and in that case $\|f\|$ will be the limit of the $\|\chi_N f\|$ as $N \to \infty$. But:

$$\begin{aligned}
\left\|\chi_N M_\varphi^*(g)\right\| &= \sup_{\|f\|=1} f \bullet \chi_N M_\varphi^*(g) \\
&= \sup_{\|f\|=1} \chi_N f \bullet M_\varphi^*(g) \\
&= \sup_{\|f\|=1} M_\varphi(\chi_N f) \bullet g \\
&= \sup_{\|f\|=1} \varphi \chi_N f \bullet g \, d\mu \\
&= \sup_{\|f\|=1} f \bullet \chi_N \, \varphi \, g \\
&= \|\chi_N \, \varphi \, g\|
\end{aligned}$$

so that

$$\left\|M_\varphi^* g\right\| = \lim_{N\to\infty} \|\chi_N \, \varphi \, g\|$$

and $\|\varphi g\| < \infty$. ▯

6. The spectrum

Suppose in this section that $H_1 = H_2 = H$. An **eigenvector** of an operator T from H to itself is a vector $v \neq 0$ in the domain of T such that $T(v)$ is a scalar multiple of v. An **eigenvalue** is one of the scalars that occurs. If H has finite dimension then the following conditions are equivalent:

(a) λ is an eigenvalue of T;
(b) $T - \lambda I$ is not invertible.

This is because in finite dimensions the dimension of the kernel is also the dimension of the cokernel. In infinite dimensions things are more complicated. We no longer have an equivalence of these conditions, as already the example of multiplication by x on $\mathrm{L}^2(\mathbb{R})$ shows. The set of eigenvalues of an operator T is replaced by its spectrum, which is defined negatively—that is to say, it is defined only in terms of properties of its complement.

Example. Consider $T = id/dx$ as an operator on $\mathrm{L}^2(\mathbb{R})$. Its domain is the space of all f in $\mathrm{L}^2(\mathbb{R})$ such that f' is also in $\mathrm{L}^2(\mathbb{R})$, and it is self-adjoint. The Fourier transform, formally defined as

$$\widehat{f}(s) = \int_{-\infty}^{\infty} f(x) e^{-sx} \, dx \quad (\mathrm{RE}(s) = 0)$$

identifies $\mathrm{L}^2(\mathbb{R})$ with $\mathrm{L}^2(i\mathbb{R})$, but takes T into multiplication by is. There are no square-integrable eigenfunctions of T, but the function e^{sx} for $\mathrm{RE}(s) = 0$ is at least bounded, and taken into a real multiple of itself by T. It is hard to make this observation very general, but what does generalize nicely is this observation: for c with $\mathrm{IM}(c) \neq 0$ the operator $T - cI$ is an isomorphism of $\mathrm{Dom}(T)$ with $\mathrm{L}^2(\mathbb{R})$, as you can easily check by applying the Fourier transform.

The **resolvent set** $\rho(T)$ of a closed operator T is defined to be the set of all $c \in \mathbb{C}$ such that $T - cI$ is a bijection between $\mathrm{Dom}(T)$ and H. When this occurs, then the inverse of $T - cI$ is a bounded operator from H onto $\mathrm{Dom}(T)$, and in particular into itself. The **spectrum** $\sigma(T)$ of T is the complement of its resolvent set. For finite-dimensional linear transformations the spectrum is equal to the set of its eigenvalues, but in general the set of eigenvalues is a proper subset.

6.1. Proposition. *The resolvent set of any closed operator is open in* $\mathbb{C}$.

Proof. Let T be the operator and assume $c \in \rho(T)$. Let $S = T - cI$, so that by assumption $R = S^{-1}$ is a bounded linear transformation. Formally for $x = c + h$ we have

$$\begin{aligned}(T - xI)^{-1} &= (S - hI)^{-1} = S^{-1}(I - hS^{-1})^{-1} \\ &= R(I - cR)^{-1} = R(I + hR + h^2R^2 + \cdots)\end{aligned}$$

and in fact the series converges for h small to an operator that one can easily verify to be $(T - xI)^{-1}$. ▯

6.2. Lemma. *Suppoes T to be a symmetric operator. Then:*

- *(a) for any $c \in \mathbb{C} - \mathbb{R}$, $\mathrm{Ker}(T - cI) = 0$;*
- *(b) if T is closed then $\mathrm{Ran}(T - cI)$ is closed in H for every $c \in \mathbb{C} - \mathbb{R}$;*
  Itemc conversely if $\mathrm{Ran}(T - cI)$ is closed in H for some $c \in \mathbb{C} - \mathbb{R}$ then T is closed.

Proof. If $c = s + it$, then for every x in $\mathrm{Dom}(T)$

$$\left\|(T - cI)(x)\right\|^2 = \left\|(T - sI)(x)\right\|^2 + t^2\left\|x\right\|^2 \geq t^2\left\|x\right\|^2 .$$

Assertion (a) follows immediately. For (b), this same formula implies that if

$$z = \lim_n (T - cI)(y_n)$$

then the y_n converge also to some element y of H with $(T - cI)(y) = z$. The proof of the converse is similar. ▯

6.3. Lemma. *Suppose T to be a symmetric closed operator. Then the following are equivalent:*

- *(a) the operator T is self-adjoint;*
- *(b) the operator T^* is symmetric;*
- *(c) $\mathrm{Ker}(T^* - cI) = 0$ for all $c \in \mathbb{C} - \mathbb{R}$;*
- *(d) $\mathrm{Ker}(T^* - cI) = 0$ for some pair of conjugate $c \in \mathbb{C} - \mathbb{R}$;*
- *(e) there exists a conjugate pair (possibly a single element—i.e. a real number) in $\rho(T)$;*
- *(f) the spectrum of T is contained in $\mathbb{R}$.*

Some of the implications are trivial. What is important is that (d) implies (a).

Proof. It is easy to see that (a) implies (d). To see that (d) implies (e) and (f): according to Proposition 4.8 the image of $T - \lambda I$ is dense and according to Lemma 6.2 it is closed. To see that (e) implies (a), let c be some element of $\rho(T)$ which is not real and such that its conjugate is also in $\rho(T)$. This is guaranteed by assumption (e), together with Proposition 6.1 if this number is real. Let x be in $\mathrm{Dom}(T^*)$. Then there exists $y \in \mathrm{Dom}(T)$ with

$$(T - cI)(y) = (T^* - cI)(x) .$$

Since T is symmetric, $\mathrm{Dom}(T) \subseteq \mathrm{Dom}(T^*)$, so $(x - y)$ lies in the kernel of $(T^* - cI)$. But by Proposition 4.8 and the assumption that the conjugate of c is also in $\rho(T)$, this means that $x = y$. Thus $\mathrm{Dom}(T^*) \subseteq \mathrm{Dom}(T)$. ▯

6.4. Proposition. *If $\mathrm{Ker}(T^* - cI) = 0$ for some pair of conjugate $c \in \mathbb{C} - \mathbb{R}$ then T is essentially self-adjoint.*

This is a fairly practical criterion.

Proof. Because of Lemma 6.3, condition (d) of the previous Lemma with the stronger assumption that the conjugate pair is not real implies self-adjointness even without the preliminary condition that T be closed. ▯

A symmetric operator is called **positive** if $T(v) \bullet v \geq 0$ for all $v \in \mathrm{Dom}(T)$.

6.5. Lemma. *Suppose T to be a positive symmetric operator.*

(a) if T is closed then $\mathrm{Ran}(T + cI)$ *is closed for all* $c > 0$ *and conversely*
(b) if $\mathrm{Ran}(T + cI)$ *is closed for some* $c > 0$ *then* T *is closed.*

6.6. Proposition. *The spectrum of a positive self-adjoint operator is contained in* $[0, \infty)$.

Proof. For any $\lambda > 0$ the operator $T + \lambda I$ is closed according to the previous proposition, and the space perpendicular to the image, which is $\mathrm{Ker}(T + \lambda I)$ because of self-adjointness, is clearly null by positivity. $\square$

I leave this as an exercise:

6.7. Proposition. *If T is a positive operator and* $\mathrm{Ker}(T^* + cI) = \{0\}$ *for some $c > 0$ then T is essentially self-adjoint.*

Let's look again at the example we looked at before.

6.8. Proposition. *Suppose $\varphi(x)$ to be a continuous real-valued function on $\mathbb{R}$. If T is multiplication by φ on* $\mathrm{L}^2(\mathbb{R})$ *with domain equal to that of all f such that φf lies again in* L^2*, its spectrum is the closure of the image of φ in $\mathbb{R}$.*

Proof. Lemma 6.3 implies that the spectrum is contained in $\mathbb{R}$. Suppose c is real. Since the spectrum is always closed, we must show that (a) if $c = \varphi(x)$ then the range of $T - cI$ is not all of $\mathrm{L}^2(\mathbb{R})$ and (b) if c is in the complement of the closure of the image of φ then $T - cI$ is bijective.

Suppose $c = \varphi(x)$. I claim that the characteristic function χ of any interval $[c - \delta, c + \delta]$ is not in the image of $T - cI$. If it were, then let $h = (T - cI)^{-1}\chi$. If χ_ε is the characteristic function of the union of the intervals $[c - \delta, c - \varepsilon]$ and $[c + \varepsilon, c + \delta]$ then χ_ε converges to χ as $\varepsilon \to 0$, so $h_\varepsilon = (T - cI)^{-1}\chi_\varepsilon$ will converge to h. But as $\varepsilon \to 0$ the L^2-norm of h_ε passes off to infinity.

As for (b), if c is not in the closure of the image of φ, let $d > 0$ be the minimum value of $\varphi(x) - c$ as x ranges over $\mathbb{R}$. Then $|\varphi - c|^{-1}$ is bounded by d^{-1}, so multiplication by it is a bounded operator with norm $\le d^{-1}$. $\square$

7. The Friedrichs extension

It is not true that every symmetric operator possesses self-adjoint extensions, but a positive symmetric operator (D, T) on a Hilbert space H always has at least one, called its **Friedrichs extension**.

Define a Hilbert space which for the moment I'll call H_F to be the completion of $\mathrm{Dom}(T)$ with respect to the norm $\|x\|_F^2 = x \bullet x + T(x) \bullet x$.

7.1. Lemma. *The inclusion of* $\mathrm{Dom}(T)$ *in H induces an inclusion of H_F as well.*

Proof. Since $x \bullet x \le \|x\|_F$, the inclusion of $\mathrm{Dom}(T)$ extends to a continuous map $\iota\colon\ H_F \to H$ satisfying the condition

$$\|\iota(x)\| \le \|x\|_F$$

for all x in H_F. It is to be shown that ι is injective. Now for every x, y in $\mathrm{Dom}(T)$

$$\langle x, y\rangle_F = x \bullet \big(y + T(y)\big), \tag{1}$$

and by continuity this translates to

$$\langle x, y\rangle_F = \iota(x) \bullet \big(y + T(y)\big) \tag{2}$$

for every $x \in H_F$, $y \in \mathrm{Dom}(T)$. If $\iota(x) = 0$ then this implies that $\langle x, y\rangle_F = 0$ for every $y \in \mathrm{Dom}(T)$, which in turn implies that $x = 0$ since $\mathrm{Dom}(T)$ is dense in H_F. Q.E.D.

I will now identify H_F with its (dense) image in H. We have the two equations

$$\|x\| \le \|x\|_F$$

$$\langle x, y\rangle_F = x \bullet \big(y + T(y)\big)$$

for all x in H_F, y in $\mathrm{Dom}(T)$.

Let τ be the canonical map from H_F to its conjugate dual H_F^*—i.e. to itself—taking x to the conjugate-linear functional $y \mapsto \langle y, x\rangle_F$. This is an isomorphism. We then have a commutative diagram

$$\begin{array}{ccc} \mathrm{Dom}(T) & \hookrightarrow & H_F \\ \downarrow{\scriptstyle I+T} & & \downarrow{\scriptstyle \tau_F} \\ H & \hookrightarrow & H_F^* \end{array}$$

I now define the operator T_F by specifying its domain to be the elements of H_F whose image under τ_F lies in the image of H, and for x in this domain define $T_F(x)$ to be $y - x$ if y in H is such that the image of y in $\widehat{H}_F$ is $\tau_F(x)$.

As the follwoing diagram indicates, the domain of T_F may also be specified as the intersection of H_F with the domain of T^*:

$$\begin{array}{ccccc} \mathrm{Dom}(T) & \hookrightarrow & H_F & \hookrightarrow & H \\ \downarrow{\scriptstyle I+T} & & \downarrow{\scriptstyle \tau_F} & & \downarrow \\ H & \hookrightarrow & H_F^* & \hookrightarrow & \mathrm{Dom}(T)^* \end{array}$$

7.2. Proposition. *The operator T_F defined in this way is a positive self-adjoint extension of T.*

Proof. We have

$$\langle x, y\rangle_F = \langle x, y + T(y)\rangle$$

for every $x \in H_F$, $y \in \mathrm{Dom}(T)$, and by transposition and continuity this implies that

$$\langle T_F x, y\rangle = \langle x, y\rangle_F - \langle x, y\rangle$$

for every $x \in \mathrm{Dom}(T_F)$, $y \in H_F$. From this it is clear that T_F is positive and symmetric.

It is immediate from the definition that the range of $T_F + I$ is all of H, which implies by Lemma 6.3 that it is self-adjoint. ▯

Example. Let $Df = df/dx$ for f in $C_c^\infty(0, \infty)$, and let $\Delta f = -D^2 f = -d^2 f/dx^2$. Integration by parts shows that Δ is a positive operator, and more precisely that

$$f \bullet \Delta f = Df \bullet Df\ .$$

If now $H = \mathrm{L}^2(0, \infty)$, we get:

7.3. Proposition. *The Friedrichs extension of $(\Delta, C_c^\infty(0, \infty))$ has as domain those f in $\mathrm{L}^2(0, \infty)$ such that df/dx lies in $\mathrm{L}^2(\mathbb{R})$ and $d^2 f/dx^2$ lies in $\mathrm{L}^2(0, \infty)$.*

Such functions are in fact continuous on $\mathbb{R}$, so in effect the Friedrichs extension corresponds to the boundary value condition $f = 0$.

8. Derivatives on $\mathbb{R}$

I'll discuss here several classical examples involving analysis on the real line. Let D be differentiation id/dx. Integration by parts gives

$$\int_{-\infty}^{\infty} f(x)g'(x)\,dx = -\int_{-\infty}^{\infty} f'(x)g(x)\,dx$$

for f and g in $C_c^\infty(\mathbb{R})$. This implies that $f \bullet Dg = Df \bullet g$, so that the operator D, and in fact every D^k, with domain $C_c^\infty(\mathbb{R})$, is symmetric. Most of this section will be concerned with the proof of the following result, which is along essentially the same lines as the earlier analysis of $\big(d/dx, C^\infty(\mathbb{S})\big)$.

8.1. Proposition. *For every $k \geq 0$ the operator $\big(D^k, C_0^\infty(\mathbb{R})\big)$ is essentially self-adjoint.*

The proof will show how powerful the criterion of Lemma 6.3 is. This is a special case of a far more general result about the Laplacian on complete Riemannian manifolds, to be found in [Gaffney:1951], but I cannot begin to give any idea of how his proof goes.

Proof. The domain of the adjoint of $T = \big(D^k, C_c^\infty(\mathbb{R})\big)$ is the space of all f in $\mathrm{L}^2(\mathbb{R})$ such that for some h in $\mathrm{L}^2(\mathbb{R})$

$$\langle h, \varphi\rangle = \langle f, D^k\varphi\rangle$$

for all φ in $C_c^\infty(\mathbb{R})$—in other words, all f such that the distribution $D^k f$ lies in $\mathrm{L}^2(\mathbb{R})$.

It must be shown that all eigenvalues of T^* are real, which suffices by Lemma 6.3.

Suppose $T^* f = \lambda f$, f not identically 0. For φ in $C_c^\infty(\mathbb{R})$ let

$$f_\varphi = R_\varphi f = \int_{\mathbb{R}} \varphi(x)\lambda_x f\,dx$$

where

$$\lambda_x f(y) = f(y-x)\,.$$

Then f_φ will have these properties: (a) it is smooth on $\mathbb{R}$ and (b) all of its derivatives lie in $\mathrm{L}^2(\mathbb{R})$—it lies in $\mathrm{L}^{2,\infty}(\mathbb{R})$. By choosing φ close to δ_0, we may assume $f_\varphi \neq 0$. But since T commutes with translations, f_φ is also an eigenvector of D^k. However, the eigenfunctions of D^k are exponentials, none of which are square-integrable. $\square$

In the rest of this section I'll look more closely at the domain of D^k. Here is the basic fact:

8.2. Proposition. *If f and $D^k f$ both lie in $\mathrm{L}^2(\mathbb{R})$ then so do all $D^\ell f$ with $0 < \ell < k$.*

Proof. I recall the **Fourier transform**. For f in the Schwartz space of $\mathbb{R}$ its Fourier transform is

$$\widehat{f}(s) = \int_{\mathbb{T}} f(x)e^{-sx}\,dx\,.$$

for s in $i\mathbb{R}$. It induces an isomorphism of $\mathcal{S}(\mathbb{R})$ with $\mathcal{S}(i\mathbb{R})$, since the Fourier transform of f' is

$$\int_{\mathbb{R}} f'(x)e^{-sx}\,dx = s\int_{\mathbb{R}} f(x)e^{-sx}\,dx\,.$$

The Fourier transform on $\mathcal{S}(\mathbb{R})$ extends to a dual map of tempered distributions and also an isomorphism of $\mathrm{L}^2(\mathbb{R})$ with $\mathrm{L}^2(i\mathbb{R})$. Thus $D^\ell F$ will belong to $\mathrm{L}^2(\mathbb{R})$ if and only if $|s|^\ell \widehat{F}(s)$ does. If $\widehat{F}$ and $|s|^k\widehat{F}$ are square-integrable, then of course every $|s|^k$ with $1 < \ell < k$ is as well (partition $i\mathbb{R}$ into regions where $|s| \leq 1$ and $|s| > 1$). $\square$

The space of all f in $\mathrm{L}^2(\mathbb{R})$ such that every distribution $D^\ell f$ for $\ell \le k$ lies in $\mathrm{L}^2(\mathbb{R})$ is called the k-th **Sobolev space** $\mathbf{H}^k(\mathbb{R})$. With the norm

$$\|f\|_{\mathbf{H}^k} := \Big(\sum_{\ell \le k} \|D^\ell f\|^2\Big)^{1/2}$$

it becomes a Hilbert space. The domain in the previous Proposition turns out to be the Sobolev space.

8.3. Lemma. *Every f in $\mathbf{H}^k(\mathbb{R})$ has continuous derivatives of order up to $(k-1)$. Furthermore, for all $\ell \le k-1$, $f^{(\ell)}(x) \to 0$ as $|x| \to \infty$.*

Proof. By recursion, it suffices to show this for $k = 1$. Let

$$\Phi(s) := \lim_{T\to\infty} \int_{-T}^{T} e^{-sx} f(x)\,dx \qquad (s \in i\mathbb{R})$$

be the Fourier transform of f. It lies in $\mathrm{L}^2(\mathbb{R})$ as does also the Fourier transform $s\Phi(s)$ of Df. Thus $(1+s)\Phi(s)$ lies in $\mathrm{L}^2(i\mathbb{R})$ and $\Phi(s)$ lies in fact in $\mathrm{L}^1(i\mathbb{R})$. But f is the inverse Fourier transform of Φ. It is easy to see that the inverse Fourier transform of a function in $\mathrm{L}^1(\mathbb{R})$ is continuous. The Riemann-Lebesgue Lemma, which I'll recall in a moment, finishes off the proof. ∎

Now for the Riemann-Lebesgue Lemma. Define $C_\infty(\mathbb{R})$ to be the space of continuous functions on $\mathbb{R}$ vanishing at ∞. It is a Banach space with norm

$$\|f\|_\infty = \sup_{\mathbb{R}} |f(x)|\,.$$

It is a closed subspace in the space of all bounded continuous functions on $\mathbb{R}$.

8.4. Proposition. (Riemann-Lebesgue Lemma) *The Fourier transform of f in $\mathrm{L}^1(\mathbb{R})$ lies in $C_\infty(i\mathbb{R})$.*

Proof. For f in $\mathcal{S}(\mathbb{R})$ the transform $\widehat{f}$ lies in C_∞. Also

$$\big\|\widehat{f}\,\big\|_\infty \le \|f\|_1\,.$$

By the extension theorem for quasi-complete spaces, it extends to a continuous map from L^1 to C_∞. ∎

This Fourier transform of a function in L^1 is the same as that of the same function considered as a distribution.

The space $C_c^\infty(\mathbb{R})$ is dense in each $\mathbf{H}^k(\mathbb{R})$. The inclusion induces by duality an injection from the dual $\mathbf{H}^{-k}(\mathbb{R})$ of $\mathbf{H}^k(\mathbb{R})$ into the space of distributions on $\mathbb{R}$. The Lemma amounts to the assertion that the δ functions $\delta_x^{(\ell)}$ for any $\ell \le (k-1)$ lie in the image. It is at this point an exercise that these functionals are continuous in the topology of $\mathbf{H}^k(\mathbb{R})$. Conversely:

8.5. Proposition. *The linear combinations of the $\delta_x^{(\ell)}$ with $\ell \le (k-1)$ are the only distributions in $\mathbf{H}^{-k}$ with support at x.*

Proof. Choose an integer $r \ge 0$ and let

$$F = \sum_{\ell=0}^{r} a_\ell \delta_x^{(\ell)}$$

with $a_r \ne 0$. Let f in $C_c^\infty(\mathbb{R})$ be such that $f^{(\ell)}(x) = 0$ for $\ell < r$, but $f^{(r)}(x) \ne 0$. For any $n > 0$ let

$$f_n(y) = n^{-r} f\big(x + n(y-x)\big)\,.$$

Then for $r > k$, $f_n \to 0$ in $\mathbf{H}^k(\mathbb{R})$ but $\langle F, f_n\rangle$ does not tend to 0. ∎

As a last topic in this section, I'll consider the differential operators D^k with domain $C_c^\infty(0,\infty)$ in $\mathrm{L}^2(0,\infty)$.

8.6. Proposition. *The domain of the adjoint of the operator $(D^k, C_c^\infty(0,\infty))$ in $\mathrm{L}^2(0,\infty)$ may be characterized in two ways:*

(a) *those* f *in* $\mathrm{L}^2(0,\infty)$ *obtained as the restriction of a function* g *in* $\mathbf{H}^k(\mathbb{R})$ *to* $(0,\infty)$*;*
(b) *those* f *in* $\mathrm{L}^2(0,\infty)$ *which when extended to all of* $\mathbb{R}$ *as null in* $(-\infty,0)$ *satisfy the condition that for some* h *in* $\mathrm{L}^2(0,\infty)$

$$D^k f = h + \text{ a linear combination of } \delta_0^{(\ell)} (\ell \le k-1) \,.$$

These two descriptions are connected by the relation that the restriction of $D^k g$ to $(0,\infty)$ is h.

Proof. A function f in $\mathrm{L}^2(0,\infty)$ lies in the domain of the adjoint of $(D^k, C_c^\infty(0,\infty))$ if and only if for some h in $\mathrm{L}^2(0,\infty)$ we have

$$\langle h, \varphi \rangle = \langle f, D^k \varphi \rangle$$

for all φ in $C_c^\infty(0,\infty)$—i.e. if and only if the restriction of the distribution $D^k f$ to $(0,\infty)$ lies in $\mathrm{L}^2(0,\infty)$. The distribution $(-1)^k D^k f - h$ then has support at 0. But it also lies in $\mathbf{H}^{-k}(\mathbb{R})$ (as can be seen by exhibiting it as a functional) so that (b) follows from Proposition 8.5. To prove (a), it must be shown that given an f as in (b) there exists an extension of it to all of $\mathbb{R}$ which lies in $\mathbf{H}^k(\mathbb{R})$. If φ lies in $C_c^\infty(\mathbb{R})$ and χ for the moment is the characteristic function of $(-\infty,0)$, then as a distribution

$$D(\varphi\chi) = \sum_0^k (-1)^{k-\ell} \binom{k}{\ell} D^\ell \varphi \delta_0^{(k-\ell-1)} .$$

Choose φ so that this is the same as $D^k f$, and then $f - \varphi\chi$ is the extension necessary. ∎

9. Consequences of the spectral theorem

The main theorem of this section will be:

9.1. Theorem. *If* T *is a self-adjoint operator and* c *lies in the resolvent set of* T*, then*

$$\|T - cI\|^{-1} \le \frac{1}{\mathrm{dist}\big(c, \sigma(T)\big)} \,.$$

This an extremely useful result. Its proof depends on one of the avatars of the spectral theorem. I'll not prove the spectral theorem itself, but just explain how it works. I begin with:

9.2. Proposition. *Suppose* $(M, d\mu)$ *to be a finite measure space,* $\varphi(m)$ *a measurable real-valued function on* M*. Let* Φ_φ *be multiplication by* φ*, with domain the space of* f *in* $\mathrm{L}^2(M, d\mu)$ *such that* φf *is again in* $\mathrm{L}^2(M, d\mu)$*. The operator* Φ_φ *is self-adjoint, and its spectrum is the essential range of* φ*.*

The essential range of φ is the set of c for which the measure of $\varphi^{-1}(c-\varepsilon, c+\varepsilon)$ is positive for all ε.

Proof. The proof of the first part is almost word for word the same as that of Proposition. 5.2, except that now χ_N is the characteristic function of the m with $|f(m)| \le N$. That of the second is almost exactly the same as that of Proposition 6.8. ∎

The spectral theorem itself asserts that the operators defined in this result are universal models.

9.3. Theorem. *If* T *is any self-adjoint operator on a Hilbert space* H*, then it is isomorphic to some* Φ_φ*.*

How does Theorem 9.1 follow from it?

Proof of Theorem 9.1. Because

$$\int_M \left| \frac{f(m)}{\varphi(m) - c} \right|^2 d\mu \le \sup_{\mathrm{supp}(d\mu)} \frac{1}{\big|\varphi(x) - c\big|^2} \int_M \big|f(m)\big|^2 \, d\mu \,.$$ ∎

As for the proof of the spectral theorem, it is rather long, and well explained in [Reed-Simon:1972]. The first step is to prove it for bounded self-adjoint operators (221–227 in Chapter VII of [Reed-Simon]); then for bounded normal ones—i.e. ones that commute with their adjoints— (problems 3,4, 5 of Chapter VII); and finally unbounded ones (259–261 of [Reed-Simon]).

10. References

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