Integration

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The main goal of this note is to explain how (and what) to integrate on Lie groups and homogeneous spaces. It will begin with a discussion of integration on arbitrary manifolds. Some care will be taken to explain in particular how this works for non-orientable manifolds, which in practice appears the most confusing.

For a slightly different account, look at [Tao:2018].

Contents

1. Integration of measures on Euclidean space

If $f(x_1, x_2, \dots, x_n)$ is a continuous function of support in the relatively compact open set $X \subset \mathbb{R}^n$, its Riemann integral

$$\int_X f(x)\,dx_1 \dots dx_n$$

is defined to be the limit of sums over a mesh of smaller and smaller n-dimensional cubes sitting in X. The starting point of integration in $\mathbb{R}^n$ is thus the assignment of a volume to cubes, one that's invariant under translation. Explicit calculation of integrals is usually by iterated one-dimensional integration, where one can apply the fundamental theorem of calculus. I'll say more about that in the next section.

If we make a change of variables $x = \varphi(y)$ where φ is an invertible smooth function, the integral becomes

$$\int_{\varphi^{-1}(X)} f\big(\varphi(y)\big)\,|\det(\partial x/\partial y)|\,dy_1 \dots dy_n\,.$$

For example

$$\int_{\mathbb{R}} f(-x)\,dx = \int_{\mathbb{R}} f(x)\,dx\,.$$

The significant thing is that this formula involves the *absolute value* of the determinant of the Jacobian matrix $\partial x/\partial y$, not the determinant itself. This is ultimately because the volume of the image of the unit cube with respect to a linear map A is $|\det(A)|$. With this in mind, and in view of the next section, it seems to me that preferable notation might be

$$\int_X f(x)\,|dx_1 \dots dx_n|\,.$$

This idea is reinforced by considerations of what happens when integrating real- or complex-valued functions on p-adic spaces. In this case, a p-adic differential form ω and the associated measure $|\omega|$ are very different objects (as explained in the lecture notes [Weil:1962/1982]).

2. Integration of differential forms on Euclidean space

If ω is a smooth n-form on $X \subseteq \mathbb{R}^n$, it can be expressed as $f(x)\,dx_1 \wedge \ldots \wedge dx_n$ and then its integral is the integral of measures defined in the previous section:

$$\int_X \omega = \int_X f(x)\,dx_1 \ldots dx_n \,.$$

The point is that we first have to arrange the formula for ω so as to match the *standard orientation* of $\mathbb{R}^n$.

The integration we teach in calculus courses is an integration of measures, but integration of forms enters into consideration implicitly. The point is that *the two integrals*

$$\int_{[a,b]} f(x)\,dx$$

and

$$\int_a^b f(x)\,dx$$

are not quite the same. The first is an integration of measures, while the second is one of forms. Implicit in the second integral is the assignment of orientation of the interval $[a,b]$, in the direction from a to b. This is why the formula

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

is a special case of de Rham's theorem. Put another way, if one swaps a and b in these formulas, the first integral does not change, but the second changes sign.

This is all taken into account easily and automatically in Leibniz' marvelous notation. For example, if we change variables $y = -x$ then we have $dx = -dy$, and write

$$\int_{-\infty}^{\infty} f(x)\,dx = -\int_{\infty}^{-\infty} f(-y)\,dy = \int_{-\infty}^{\infty} f(-y)\,dy$$

The change of sign that goes with the change of limits is necessary to manage orientation of forms correctly, and the final formula

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} f(-y)\,dy$$

is that for integration of measures. The integral is, after all, the area underneath the graph of $f(x)$, a quantity independent of orientation. But there is something subtle going on—in writing the integral

$$\int_{\infty}^{-\infty} f(x)\,dx$$

we are now assuming that $\mathbb{R}$ *is given the orientation opposite to the standard one*. As I say, this is all somewhat implicit. The basic fact is that $x \mapsto -x$ changes the sign of dx, but preserves the measure $|dx|$. and this is what ultimately justifies the procedure involved in a change of variables.

Suppose we are given a change of variables $y = h(x)$, where $x = (x_i)$ is the standard coordinate system. If $\varphi(y)\,dy_1 \wedge \ldots \wedge dy_n$ is an n-form expressed in terms of the coordinates y (instead of x), then

$$\varphi(y)\,dy_1 \wedge \ldots \wedge dy_n = \varphi\big(h(x)\big)\,\det(\partial y/\partial x)\,dx_1 \wedge \ldots \wedge dx_n$$

leading to the formula for change of variables

$$\int_{\mathbb{R}^n} \varphi(y)\,dy_1 \wedge \ldots \wedge dy_n = \int_{\mathbb{R}^n} \varphi\big(h(x)\big)\;\det(\partial y/\partial x)\,dx_1 \wedge \ldots \wedge dx_n \,.$$

It is the determinant itself, not its absolute value, that now figures here.

3. Oriented manifolds

A smooth manifold M is said to be **orientable** if it can be covered by coordinate patches that are consistent in orientation in the sense that the Jacobian matrix on any overlap has positive sign. It is **oriented** if one consistent family of coordinates has been chosen. On each connected component there are at most two choices of orientation.

Given an oriented coordinate covering $\{X_i\}$, we can find a partition of unity φ_i subordinate to it—that is to say the support of φ_i lies in X_i, $0 \le \varphi_i \le 1$, and $\sum \varphi_i = 1$. If ω is an n-form on M then each $\omega_i = \varphi_i \omega$ may be identified with an n-form on $\mathbb{R}^n$ and

$$\int_M \omega = \sum_i \int_{X_i} \omega_i \,.$$

This is essentially the definition of the left hand side. The right hand side does not depend on the choice of coordinate patches or partitions of unity.

If M is orientable, we can integrate forms over M, but only after making a choice of orientation. Reversing the orientation reverses the sign of an integral. *There is no way to integrate forms on M independently of a choice of orientation.*

4. Arbitrary manifolds

But what if M is not orientable? Things become more interesting. Suppose M to be an arbitrary manifold of dimension n. At each point x of M, the exterior power $\bigwedge^n T_x$ of the tangent space T_x has dimension one. This defines a real line bundle over M. Choosing an orientation of M is equivalent to choosing at each point of M an orientation of the line $\bigwedge^n T_x$, and in a continuous way.

Let me make this more precise. The complement of the origin in $\bigwedge^n T_x$ has two connected components, the orbits of the multiplicative group of positive real numbers. Let $\widetilde{M}$ be the set of these two components, or equivalently the quotient of the bundle $\bigwedge^n T - \{0\}$ by the positive reals. It is the fibre bundle of local orientations on M, with each fibre of exactly two points. If y on $\widetilde{M}$ lies over x on M, then projection identifies T_y canonically with T_x. To y also, by definition, is assigned an orientation of T_x, hence of T_y. Hence:

4.1. Lemma. *The space $\widetilde{M}$ of local orientations of M possesses a canonical orientation.*

An orientation of M amounts to a continuous section of the fibring of $\widetilde{M}$ over M. If M is connected, this happens if and only if $\widetilde{M}$ has two components, in which case each of them projects isomorphically to M.

There is a canonical way to integrate forms on $\widetilde{M}$, since it has a canonical orientation. The two-fold covering $\widetilde{M} \to M$ has a canonical involution, which swaps orientations at each point of M. If we pull back a form from M to $\widetilde{M}$, it will be taken into itself by the involution, and conversely every involution-invariant n-form is obtained from one on M. Unfortunately, if we integrate it we'll get 0, since the integrals in the neighbourhood of opposite points cancel.

In contrast to this, a **twisted** n-form or **density** $\widetilde{\omega}$ on M is defined to be one which is taken into its negative by the involution. We define its integral over M to be

$$\int_M \widetilde{\omega} = \frac{1}{2} \int_{\widetilde{M}} \widetilde{\omega} \,.$$

How does this relate to what I have already said about oriented manifolds? An orientation of M gives rise to a section of $\widetilde{M}$ over M, and an n-form lifts to a form on the image of the section. Taking its negative on the other component gives a twisted n-form whose integral, because of the factor $1/2$, is the same as that of the original form over M. So this definition is consistent with the earlier one. In other words, on an oriented manifold, n-forms may be identified with twisted n-forms.

In terms of coordinate patches, the bundle of twisted forms is that with transformation $|\det(\partial y/\partial x)|$ as in our original observation, whereas that for n-forms is the determinant $\det(\partial y/\partial x)$ itself.

Differential forms of degree n are sections of a certain real line bundle over M. Twisted n-forms are also sections of a real line bundle over M. Suppose $\widetilde{\omega}$ to be a twisted n-form at x. If $\eta \neq 0$ is an n-vector in $\bigwedge^n T_x$, then we can lift η to a unique point $\tilde{x}$ of $\widetilde{M}$ assigning it a positive orientation. The number $\langle \widetilde{\omega}, \eta \rangle_{\tilde{x}}$ is the same for η and $-\eta$. In this way, the fibre of the bundle of twisted n-forms at x may be identified with the one-dimensional vector space

$$\left|\bigwedge^n\right| T_x^* = \left\{ f \colon \bigwedge^n T_x \longrightarrow \mathbb{R} \,\middle|\, f(cv) = |c| f(v) \right\}.$$

To each non-zero map f of this kind is associated the pair η, $-\eta$ with $|f(\eta)| = 1$, and the set of non-zero maps of this kind may therefore be identified with the quotient of $\bigwedge^n T_x - \{0\}/\{\pm 1\}$, a contractible set. The bundle is therefore trivial. In particular, on any manifold there always exists a non-vanishing twisted n-form.

Suppose we are given a connected, oriented manifold M and an involution σ that acts freely on M. If σ preserves the orientation, then we may assign a quotient orientation to the quotient M/σ which will thus again be oriented. But if σ reverses orientation, the covering $M \to M/\sigma$ will be the canonical orientation fibring of the non-orientable manifold M/σ.

Two examples are relevant to representation theory. The unit circle $|z| = 1$ is orientable with two possible orientations, clock-wise or anti-clockwise. The projective line $\mathbb{P}^1(\mathbb{R})$ is the quotient of the unit circle by the involution $z \mapsto -z$, and is again orientable since it is also a circle.

The unit two-sphere $\mathbb{S}^2$ is also orientable, with orientations determined by either the left- or right-hand rule in three dimensions. The projective plane $M = \mathbb{P}^2(\mathbb{R})$ is the quotient of $\mathbb{S}^2$ by the antipodal involution. But this involution reverses orientation, so M is non-orientable, and $\mathbb{S}^2$ may be identified with $\widetilde{M}$.

In general, $z \mapsto -z$ preserves orientation on odd-dimensional spheres, but reverses it on even-dimensional ones, since the determinant of scalar multiplication by -1 in n dimensions is $(-1)^n$.

5. Densities on homogeneous manifolds

Suppose now G to be a Lie group and H a closed Lie subgroup. The group acts on the right on the homogeneous space $X = H\backslash G$. For g in G let $[g]$ be the coset Hg considered as a point of X.

A remark about conventions of left and right—topologists usually work with homogeneous spaces on which the group acts on the left, whereas analysts usually follow the convention I use. Neither choice is insane—the point of both conventions is to simplify notation. Analysts work with function spaces, and the right regular action

$$[R_g f](x) = f(xg)$$

is a left action of G on functions on $H\backslash G$:

$$R_{gh}(f) = R_g(R_h f),$$

whereas for topologists the action on the space itself occurs most often.

Suppose V to be a vector bundle over X on which G acts compatibly. That is to say, for each g in G there is a transformation of V compatible with projection onto X. In particular, there is for each x in X an invertible transformation $\sigma_x(g) \colon V_x \to V_{xg^{-1}}$. Thus $\sigma_x(g_1 g_2) = \sigma_{xg_2^{-1}}(g_1)\sigma_x(g_2)$. Since H fixes $[1]$ in X there is in particular a representation σ of H on the fibre U at $[1]$. If (v_x) is a section of V, then to each g in G one may associate the function F from G to U taking g to $F(g) = \sigma(g)v_{[g]}$. For h in H we then have

$$F(hg) = \sigma(h)F(g)\,.$$

It is easy to verify that in fact:

5.1. Proposition. *The map from (v_x) to F is an isomorphism of the space of continuous sections of V with that of continuous functions $F\colon G \to U$ such that $F(hg) = \sigma(h)F(g)$ for all h in H, g in G.*

Conversely, if (σ, U) is a finite-dimensional representation of H, then there is associated to it a fibre bundle V over $H\backslash G$ whose fibre at any point is non-canonically equal to U. Geometrically V is the quotient of $U \times G$ by the group H taking (u, g) to $(\sigma(h)u, hg)$. The bundle acquires a product structure over any open subset Y of $H\backslash G$ over which one can find a splitting of the projection from G to $H\backslash G$.

An example of relevance to us is that where $G = \mathrm{SO}_2$ and $H = \{\pm 1\}$. If σ is the one-dimensional representation of H taking -1 to -1, the associated line bundle is an incarnation of the Möbius strip as a fibre bundle over the circle, which may be identified with $\mathbb{P}^1(\mathbb{R})$.

One representation of the subgroup $H \subseteq G$ is the adjoint representation on the tangent space at $[1]$ of $H\backslash G$, which may be identified with $\mathfrak{h}\backslash\mathfrak{g}$. The bundle associated to this is the tangent bundle. In other words, a vector field on $H\backslash G$ may be identified with a function

$$X\colon\ G \longrightarrow \mathfrak{h}\backslash\mathfrak{g}$$

such that

$$X(hg) = \mathrm{Ad}(h)X(g)$$

for all h in H, g in G.

If $n = \dim \mathfrak{h}\backslash\mathfrak{g}$, twisted n-forms may therefore be identified with functions F from G to $\left|\bigwedge^n\right|(\mathfrak{h}\backslash\mathfrak{g})^\vee$ such that

$$F(hg) = \delta_{H\backslash G}^{-1}(h)F(g)$$

where

$$\delta_{H\backslash G}(h) = |\mathrm{det}\mathrm{Ad}_{\mathfrak{h}\backslash\mathfrak{g}}(h) = \frac{\left|\det \mathrm{Ad}_{\mathfrak{g}}(h)\right|}{\left|\det \mathrm{Ad}_{\mathfrak{h}}(h)\right|}\,,$$

and if G is unimodular, so $\mathrm{det}\mathrm{Ad}_{\mathfrak{g}} = 1$, then

$$\delta_{H\backslash G}^{-1} = \delta_H = \left|\mathrm{det}\mathrm{Ad}_{\mathfrak{h}}(h)\right|\,.$$

As an example, suppose $G = \mathrm{GL}_n(\mathbb{R})$, and P is the subgroup of matrices

$$\begin{bmatrix} a & v \\ 0 & A \end{bmatrix}\,,$$

where a is a non-zero scalar, v a row vector of dimension $n-1$, A an invertible $(n-1) \times (n-1)$ matrix. The quotient $P\backslash G$ may be identified with $\mathbb{P}^1(\mathbb{R})$. What is the adjoint action of P on $\mathfrak{p}$? The Lie group N acts trivially, and the diagonal part takes v to avA^{-1}. The determinant of the adjoint action on $\mathfrak{p}\backslash\mathfrak{g}$ is therefore by the character

$$\begin{bmatrix} a & v \\ 0 & A \end{bmatrix} \longmapsto a^{n-1}\mathrm{det}^{-1}(A)\,.$$

This is the same as the modulus character if and only if n is even—i.e. the cases where $\mathbb{P}^{n-1}(\mathbb{R})$ is orientable, as one might expect.

For an example that is even more explicit, take $G = \mathrm{SL}_2(\mathbb{R})$ and $H = P$. Here the adjoint action of A on $\mathfrak{p}\backslash\mathfrak{g}$ is as multiplication

$$\begin{bmatrix} a & x \\ 0 & 1/a \end{bmatrix} \longmapsto a^{-2}$$

since as an A-representation $\mathfrak{p}\backslash\mathfrak{g}$ is $\overline{\mathfrak{n}}$. The twisted n-forms therefore correspond to the inverse character

$$\delta_P\colon \begin{bmatrix} a & x \\ 0 & 1/a \end{bmatrix} \longmapsto a^2 \,.$$

Since $a^2 > 0$ these do not differ from ordinary n-forms, as already pointed out.

Since the dimension of $\bigwedge^n(\mathfrak{h}\backslash\mathfrak{g})$ is one, the space of twisted forms may be identified (non-canonically) with H-covariant scalar-valued functions on G. Thus a continuous real twisted n-form on $P\backslash G$ may be identified with an element of

$$|\Omega|(P\backslash G) = \big\{ f \in C(G,\mathbb{R}) \,\big|\, f(pg) = \delta_P(p) f(g) \text{ for all } p \in P, g \in G \big\} \,.$$

That $\bigwedge^n(\mathfrak{h}\backslash\mathfrak{g})$ is only non-canonically to be identified with $\mathbb{C}$ means that explicitly integrating functions in $|\Omega|(H\backslash G)$ can be done only after making some choices—usually choosing measures on H and G. If f is a smooth function of compact support on G, then

$$\overline{f}(g) = \int_H f(hg)\, d_\ell h = \int_H \delta_H^{-1}(h) f(hg)\, d_r h$$

may be identified with a twisted form on $H\backslash G$: $\overline{f}(hg) = \delta_H(h)\overline{f}(g)$. The composite taking f to $\overline{f}$ and then to its integral is a G-invariant linear functional, hence a multiple of integration over G. With suitable normalizations

$$\int_G f(g)\, dg = \int_{H\backslash G} \overline{f}(x)\, dx \,.$$

6. Parabolic quotients—an example

Now I'll specialize to the case where G is a reductive group and $H = P$ is a parabolic subgroup of G. There are a number of things that make this case special:

(1) If K is any maximal compact subgroup of G, then $G = PK$;
(2) there exists a parabolic subgroup $\overline{P} = M\overline{N}$ **opposite** to P—i.e. with the property that $P \cap \overline{P} = M$, a reductive Levi component for either;
(3) there exists a single open orbit of $\overline{P}$ acting on $P\backslash G$, hence an embedding of $\overline{N}$ as an open subset of $P\backslash G$.

I'll not prove these, but go on in a moment to sketch what happens for the simplest case, $\mathrm{SL}_2(\mathbb{R})$. But first I'll draw some consequences from these things.

Since $G = PK$, the quotient $P\backslash G$ may be identified with $K \cap P\backslash K$, and Ω^∞ may be identified as a K-space with functions on $K \cap P\backslash K$. Integration of twisted forms is K-invariant as well as G-invariant. Linear functionals that are K-invariant are unique up to scalars, and are given by integration over K. If we assign K a total measure 1 integration of twisted forms on $P\backslash G$ may therefore be identified with integration over K:

$$\begin{aligned} \int_G f(g)\, dg &= \int_{P\backslash G} \overline{f}(x)\, dx \\ &= \int_K dk \int_P f(pk)\, d_\ell p \\ &= \int_K dk \int_A \delta_P(a)^{-1}\, da \int_N f(nak)\, dn \end{aligned}$$

The tangent space at the identityon $P\backslash G$ is the quotient $\mathfrak{p}\backslash\mathfrak{g}$, which may be identified with $\overline{N}$. The space of densities is induced from its dual. But G is reductive, which means that there is a canonical inner product on G identifying the dual of $\mathfrak{n}$ with $\overline{\mathfrak{n}}$.

Suppose now that $G = \mathrm{SL}_2(\mathbb{R})$, $P = AN$ its subgroup of upper triangular matrices, $K = \mathrm{SO}(2)$ a maximal compact subgroup. In this case I can prove everything in an elementary way.

6.1. Lemma. *Every element g in G can be written as $g = pk$ with p in P, k in K.*

Proof. Since the group G acts transitively by linear fractional transformations on the upper half plane $\mathcal{H}$ with K fixing i, this is another way to say that P acts transitively on $\mathcal{H}$. Since

$$\begin{bmatrix} a & b \\ 0 & 1/a \end{bmatrix}$$

takes i to $a^2 i + ab$, we can solve $a^2 i + ab = x + iy$ to get $a = \sqrt{y}$, $b = x/\sqrt{y}$.

This leads to the explicit formula

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \dfrac{1}{\sqrt{c^2+d^2}} & \dfrac{ac+bd}{\sqrt{c^2+d^2}} \\ 0 & \sqrt{c^2+d^2} \end{bmatrix} \cdot \begin{bmatrix} \dfrac{d}{\sqrt{c^2+d^2}} & \dfrac{-c}{\sqrt{c^2+d^2}} \\ \dfrac{c}{\sqrt{c^2+d^2}} & \dfrac{d}{\sqrt{c^2+d^2}} \end{bmatrix}$$

We'll see this used in just a moment.

6.2. Lemma. *The group G is the union of the two disjoint subsets $P \cup PwP$ where*

$$w = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Proof. The group G acts transitively on $\mathbb{R}^2 - \{0\}$ with g fixing $(1,0)$ if and only if it lies in N. The complement of $[1,0]$ turns out to be the orbit of $(0,1) = w(1,0)$ under P. Explicitly, if

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then

$$\begin{aligned} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} &= \begin{bmatrix} a \\ c \end{bmatrix} \\ &= \begin{bmatrix} A & B \\ 0 & 1/A \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} B \\ 1/A \end{bmatrix} \end{aligned}$$

if and only if $A = 1/c$, $B = a$. This tells us that $g = pwn_*$ for some n_* if and only if $c \neq 0$, in which case we can write $n_* = p^{-1}w^{-1}g$.

Explicitly

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1/c & a \\ 0 & c \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & d/c \\ 0 & 1 \end{bmatrix}.$$

In particular, the subset PwN is open in G, as its shift $PwPw = P\overline{N}$. The two open sets PwP and $P\overline{N}$ cover G.

Integration over K is a very explicit way to integrate over $P\backslash G$, but there is another and slightly more subtle formula for integration.

6.3. Lemma. *For f in $C(P\backslash G)$ the integral*

$$\int_{\overline{n}} f(\overline{n})\, d\overline{n}\,.$$

converges absolutely.

Proof. We can write $f = f_1 + f_w$ where f_1 has compact support on $P\overline{N}$ and f_w has support on PwN. For f_1 the integral is certainly well defined. As for f_w, since

$$\overline{N}_x = \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1/x & 1 \end{bmatrix} \begin{bmatrix} 1/x & 0 \\ 0 & x \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1/x \\ 0 & 1 \end{bmatrix}$$

we have

$$\int_{\overline{N}} f_w(\overline{n})\, d\overline{n} = \int_{\mathbb{R}} |x|^{-2} f_w(wn_{1/x})\, dx$$

which is convergent since the integrand has support only for x near infinity.

This gives, up to a constant, another valid formula for integration over $P\backslash G$. To find the constant, let f be the function in $\Omega^\infty(P\backslash G)$ which is 1 when restricted to K. Then the integral of f over K is 1, while because

$$\begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix} = \begin{bmatrix} 1 & \ldots \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{x^2+1} & 0 \\ 0 & \sqrt{x^2+1} \end{bmatrix} \begin{bmatrix} 1/\sqrt{x^2+1} & -x/\sqrt{x^2+1} \\ x/\sqrt{x^2+1} & 1/\sqrt{x^2+1} \end{bmatrix}$$

the other integral is

$$\int_{\mathbb{R}} \frac{dx}{x^2+1} = \pi\,.$$

7. Integration and the Cartan factorization

Let G be an arbitrary reductive algebraic group over $\mathbb{R}$, K the maximal compact subgroup fixed by the Cartan involution θ. Let A be the unique maximal split real torus stabilized by θ. Let Σ be the set of roots of G with respect to A, Δ a basis of Σ, W the Weyl group. The subset A^{++} of a in A with $|\alpha(a)| \ge 0$ for all α in Δ is a fundamental domain for W. Every w in W has a representative in K. For $\Theta \subseteq \Delta$ let A_Θ be the intersection of all α in Θ, M_Θ the centralizer of A_Θ, and $K_\Theta = K \cap M_\Theta$.

The connected component $|A|$ embeds into $K\backslash G$. The K-orbit of the image is all of G. The isotropy subgroup in K of a in $|A_\Theta|$ is K_Θ. The orbit for regular a is isomorphic to $K_\emptyset\backslash K$. The product map from $A \times K_\emptyset\backslash K$ is generically a $|W|$-fold covering of G.

This factorization $K\backslash G = A^{++} \times (K_\emptyset\backslash K)$ amounts to a radial decomposition of the Riemannian space $K\backslash G$, similar to the radial factorization $O(n)/O(n-1) \times \mathbb{R}^{\text{pos}}$ of $\mathbb{R}^n$. At every point of $\mathbb{R}^n$ except the origin, an $n-1$-sphere lies transverse to the radius. Something similar happens here at regular points of $|A|$—for a regular in $|A|$ we can find an explicit subspace of T_a tangent to the K-orbit through a and transverse to $|A|$ in $K\backslash G$.

7.1. Lemma. *Let λ be a root, $\nu \in \mathfrak{n}_\lambda$, and $a \in A$ such that $\lambda^2(a) \ne 1$. The element $\kappa = \nu + \nu^\theta$ lies in $\mathfrak{k}$ and*

$$\nu = \frac{1}{\lambda(a)^{-1} - \lambda(a)} \big(\kappa^a - \lambda(a)\kappa\big).$$

Here $\kappa^a = \mathrm{Ad}(a^{-1})\kappa$.

Proof. We calculate

$$\begin{aligned}
\kappa &= \nu + \nu^\theta \\
\kappa^a &= \mathrm{Ad}(a^{-1})\kappa \\
&= \lambda(a)^{-1}\nu + \lambda(a)\nu^\theta \\
\lambda(a)\kappa &= \lambda(a)\nu + \lambda(a)\nu^\theta \\
\big(\lambda(a)^{-1} - \lambda(a)\big)\nu &= \kappa^a - \lambda(a)\kappa \\
\nu &= \frac{1}{\lambda(a)^{-1} - \lambda(a)}\,\big(\kappa^a - \lambda(a)\kappa\big)\,.
\end{aligned}$$

Let $\mathfrak{q} \subseteq \mathfrak{k}$ be the image of

$$I + \theta\colon\ \mathfrak{n} \to \mathfrak{k}\,.$$

It is a linear complement in $\mathfrak{k}$ to $\mathfrak{k}_M$. We have

$$\begin{aligned}
&k_1 \exp(tX) a k_2 = k_1 a k_2 \cdot k_2^{-1} a^{-1} \exp(tX) a k_2 \quad (X \in \mathfrak{k}) \\
&k_1 a \exp(tX) k_2 = k_1 a k_2 \cdot k_2^{-1} \exp(tX) k_2 \quad (X \in \mathfrak{a}) \\
&k_1 a k_2 \exp(tX) = k_1 a k_2 \exp(tX) \quad (X \in \mathfrak{k})\,.
\end{aligned}$$

The differential of the product map from $\mathfrak{k}_\lambda$ is therefore the composite of $\mathrm{Ad}(k_2^{-1})$ with the transformation

$$\kappa_1 \longmapsto x^{-2}\nu_+ - x^2\nu_-\,.$$

The total modulus is then

$$D(a) = \prod_{\lambda>0} \big|\lambda(a) - \lambda^{-1}(a)\big|^{\dim \mathfrak{g}_\lambda}$$

Therefore

$$\int_G f(g)\,dg = \frac{1}{|W|}\int_K\int_A\int_K f(k_1 a k_2)\,D(a)\,dk_1\,da\,dk_2\,.$$

In the special case $G = \mathrm{SL}_2(\mathbb{R})$ this amounts to the claim that the non-Euclidean circumference at radius r is $2\pi\sinh(r)$.

8. Conjugacy classes

Suppose T to be a maximal torus in the reductive group G. An element t of T is called **regular** if $\det \mathrm{Ad}_{\mathfrak{g}/\mathfrak{t}}(t)$ does not have 1 as an eigenvalue. For regular t, there is a canonical splitting of the short exact sequence

$$0 \longrightarrow \mathfrak{t} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{t} \longrightarrow 0\,,$$

since $\mathfrak{t}$ is the space on which $\mathrm{Ad}(t)$ is trivial.

Let G_T^{reg} be the open subset of G comprised of elements in G conjugate to a regular element of T. The map from $G \times T$ to G taking (g,t) to gtg^{-1} has image G_T, and induces a map from $G/T \times T$ to G. It is a local isomorphism on G_T^{reg}, and if W_T is the normalizer of T in G the order of the covering is $|W_T|$. Let $\Delta(t) = |\det(\mathrm{Ad}(t^{-1}) - I)|$. It vanishes on the complement of T^{reg}.

8.1. Proposition. (Weyl integration formula) *For a continuous function f with support in G_T*

$$\int_{G_T} f(g)\,dg = \frac{1}{|W_T|}\int_T \Delta(t)\left(\int_{G/T} f(gtg^{-1})\,dg\right)dt\,.$$

Proof. For X in $\mathfrak{g}/\mathfrak{t}$ we have the first order calculation

$$\begin{aligned} g(I+\varepsilon X)t(I+\varepsilon X)^{-1}g^{-1} &= gtg^{-1}\cdot g\cdot (t^{-1}(I+\varepsilon X)t(I+\varepsilon X)^{-1})\cdot g^{1} \\ &= gtg^{-1}\cdot g\cdot \big(I+\varepsilon(tXt^{-1}-X)\big)\cdot g^{1}\,. \end{aligned}$$

Similarly, for Y in $\mathfrak{t}$ we have

$$gt(T+\varepsilon Y)g^{-1} = gtg^{-1}\cdot g(I+\varepsilon Y)g^{-1}\,.$$

The Jacobian map of $\mathrm{Ad}(t)$ from $\mathfrak{g}/\mathfrak{t}\oplus\mathfrak{t}$ to $\mathfrak{g}=\mathfrak{g}/\mathfrak{t}\oplus\mathfrak{t}$ therefore has matrix

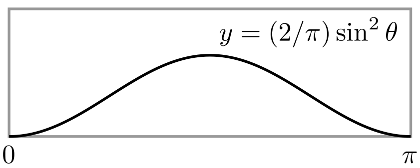
$$\begin{bmatrix} \mathrm{Ad}(t^{-1})-I & 0 \\ 0 & I \end{bmatrix}$$

which has determinant $\det(\mathrm{Ad}(t^{-1})-I)$. $\square$

This is a variation on the original integration formula of Weyl for $SU(n)$. For $n=2$ every element is conjugate to a diagonal matrix t with diagonal entries $e^{\pm i\theta}$. It is unique if we specify $0\le\theta\le\pi$. The centralizer of t is the whole group T of diagonal matrices unless $t=\pm I$, and the map $(g,t)\mapsto gtg^{-1}$ is a two-fold covering of G by $G/T\times T$, except on the set $\{\pm I\}$. The integration formula says that

$$\int_G = \frac{1}{2}\int_T \sin^2\theta\left(\int_{G/T} f(xtx^{-1})\,dx\right)dt$$

where measures are chosen compatibly with the quotient structure. The $1/2$ is there because in $SU(2)$ the order of eigenvalues doesn't matter. One thing the formula means is that if you choose a 2×2 unitary matrix with determinant 1 randomly you are more likely to get one with eigenvalues around i than around ± 1. In terms of density:



For SL_2 it is also simple to calculate $\Delta_T(t)$. There are two tori, the split torus A and the compact one T. For x in $\mathbb{R}^\times$ let

$$a_x = \begin{bmatrix} x & 0 \\ 0 & 1/x \end{bmatrix}$$

and for $0\le\theta<2\pi$ let

$$t_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Then for A

$$|D(a_x)| = |x^2-1|\,|x^{-2}-1| = |x-x^{-1}|^2$$

while for K

$$|D(t_\theta)| = 4\sin^2\theta\,.$$

The first formula can be made to look like the second if we set $x=e^\theta$ in which case $|x-x^{-1}|^2 = 4\sinh^2\theta$. One difference between the two tori is that $|W_A|=2$ while $|W_T|=1$.

9. References

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