

Representations of $SL_2(\mathbb{R})$

Bill Casselman
 University of British Columbia
 cass@math.ubc.ca

In this essay I hope to explain what is needed about representations of $SL_2(\mathbb{R})$ in the elementary parts of the theory of automorphic forms. On the whole, I have taken the most straightforward approach, even though the techniques used are definitely not valid for other groups.

This essay is about representations. In the current version it says almost nothing about what is usually called invariant analysis, which is to say harmonic analysis of distributions invariant under conjugation in $SL_2(\mathbb{R})$. It says very little about orbital integrals, and equally little about characters of representations. Nor does it prove anything deep about the relationship between representations of G and those of its Lie algebra, or about versions of Fourier transforms. These topics involve too much analysis to be dealt with in this largely algebraic essay. I shall deal with those matters elsewhere, if not in a subsequent version of this essay.

From now on, unless specified otherwise:

$G = SL_2(\mathbb{R})$
 $K =$ the maximal compact subgroup SO_2 of G
 $A =$ the subgroup of diagonal matrices
 $N =$ subgroup of unipotent upper triangular matrices
 $P =$ subgroup of upper triangular matrices
 $\quad = AN$
 $\mathfrak{g}_{\mathbb{R}} =$ Lie algebra $\mathfrak{sl}_2(\mathbb{R})$
 $\mathfrak{g}_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}_{\mathbb{R}}$.

Most of the time, it won't matter whether I am referring to the real or complex Lie algebra, and I'll skip the subscript.

Much of this essay is unfinished. There might well be some annoying errors, for which I apologize in advance. Constructive complaints will be welcome. There are also places where much has yet to be filled in, and these will be marked by one of two familiar signs:



The first should be self-explanatory. The second means that at the moment the most convenient reference is elsewhere.

Contents

Introduction

I. Bargmann's classification

1. Representations of the group and of its Lie algebra	5
2. The Lie algebra	9
3. Differential operators and the Lie algebra	12
4. Admissibility	14
5. Classification of irreducible representations	17
6. The algebraic construction	19
7. Duals	20
8. Unitarity	21

II. The principal series

9. Vector bundles	22
10. The principal series	24
11. Frobenius reciprocity and its consequences	27
12. Intertwining operators and the smooth principal series	32
13. The complementary series	33
14. Normalization	33
15. <i>Appendix.</i> Characters as distributions	34
16. <i>Appendix.</i> The Gamma function	35
17. <i>Appendix.</i> Characters and the Fourier transform	38
18. <i>Appendix.</i> Invariant integrals on quotients	40

III. The discrete series

19. Projective spaces and representations	42
20. Restriction to K	43
21. In classical terms	44
22. Relation with functions on the group	46
23. Holomorphic automorphic forms	49
24. Square-integrability	50

IV. Matrix coefficients and differential equations

25. Matrix coefficients	52
26. Differential equations	53
27. The non-Euclidean Laplacian	56
28. Realizations of admissible modules	59

V. Langlands' classification

29. The Weil group	59
30. Langlands' classification for tori	62
31. Langlands' classification for SL_2	64

VI. Verma modules and representations of G

32. Verma modules	65
33. Verma modules and intertwining operators	66
34. The Bruhat filtration	67
35. The Jacquet module	68
36. The Paley-Wiener theorem for spherical functions	68
37. The canonical pairing	68
38. Whittaker models	69
39. The module	74
40. Whittaker models for $SL(2)$	75

VII. References

Introduction

Suppose G for the moment to be an arbitrary semi-simple group defined over \mathbb{Q} . It will have good reduction at all but a finite number of primes p , which means that G defines a smooth group scheme over \mathbb{Z}_p . Suppose that for each prime p the group K_p is a compact open subgroup of $G(\mathbb{Q}_p)$, and that for all but a finite number of p it is $G(\mathbb{Z}_p)$. Let Γ be the subgroup of γ in $G(\mathbb{Q})$ such that γ lies in K_p for all p . Then Γ is discrete in $G(\mathbb{R})$ and $\Gamma \backslash G(\mathbb{R})$ has finite volume. The group Γ is called a congruence subgroup of $G(\mathbb{Q})$.

The group $G(\mathbb{R})$ acts on the right on $\Gamma \backslash G(\mathbb{R})$, and this gives rise to representations on several different spaces of functions and distributions, among them $L^2(\Gamma \backslash G(\mathbb{R}))$. Among the most interesting questions in this business is this: *What irreducible representations of $G(\mathbb{R})$ occur in the decomposition of $L^2(\Gamma \backslash G(\mathbb{R}))$?* There is a simple conjecture concerning necessary conditions for this, analogous to Ramanujan's conjecture for the coefficients of certain modular forms. This might also be considered in some sense a problem in number theory, since it is known not to be true for some discrete subgroups that are not congruence subgroups, and a resolution of the conjecture will follow from conjectures of Langlands regarding certain L -functions associated to automorphic forms. A commutative ring of certain algebraic correspondences called Hecke operators also acts on this quotient and also interesting is this: *What are the eigenvalues of Hecke operators on these?* Associated to the group G is its adèle group $G(\mathbb{A})$. It contains the discrete subgroup $G(\mathbb{Q})$ and the quotient $G(\mathbb{Q}) \backslash G(\mathbb{A})$ also has finite volume. The questions above are subsumed in the more comprehensive one: *What irreducible representations of the adèle group $G(\mathbb{A})$ occur in the right regular representation of $G(\mathbb{A})$ on $L^2(G(\mathbb{Q}) \backslash G(\mathbb{A}))$?* These questions are related, since if K_f is a compact open subgroup of the finite adèle group $G(\mathbb{A}_f)$ then $G(\mathbb{Q}) \backslash G(\mathbb{A}) / K_f$ is a union of quotients of the form $\Gamma \backslash G(\mathbb{R})$.

The problems raised by these questions are extremely difficult, even perhaps beyond answering in any complete fashion. They have motivated much investigation into harmonic analysis of reductive groups defined over local fields. Representations of semi-simple (and reductive) groups over \mathbb{R} have been studied since about 1945, and much has been learned about them, but much remains to be done. The literature is vast, and it is difficult to know where to begin in order to get any idea of what it is all about.

For p -adic groups, the problems arising that are the most difficult—and the most intriguing—are those of number theory and algebraic geometry. But for real groups there are in addition problems of analysis that do not occur for p -adic groups. These have caused much annoyance and some serious difficulties.

For one thing, there are a number of somewhat technical issues that complicate things. Perhaps the first is that one does not usually deal with representations of a real reductive group, but rather with certain representations of its Lie algebra. This has been true ever since the origins of the subject, although not really understood until a few years later. This is justified by some relatively deep theorems in analysis, and after an initial period this shift in attention is easily absorbed. The representations of the Lie algebra, which are called **Harish-Chandra modules**, are representations simultaneously of \mathfrak{g} and a maximal compact subgroup K of G . These are assumed to be compatible in that the two representations thus associated to the Lie algebra \mathfrak{k} are the same. In making this shift, one has to make a choice of maximal compact subgroup K , but the dependence on K is weak, since all choices are conjugate in G . The important condition on these representations is that the restriction to K be a direct sum of irreducible representations, each with finite multiplicity. In view of the original questions posed above, it is good to know that the classification of irreducible unitary representations of G is equivalent to that of irreducible unitarizable, Harish-Chandra modules. One does not lose much by looking at representations of the Lie algebra.

The classification of unitarizable Harish-Chandra modules has not yet been carried out for all G , and the classification that does exist is somewhat messy. The basic idea for the classification goes back to the origins of the subject—first classify all irreducible Harish-Chandra modules, and then decide which are unitarizable. The classification of all irreducible Harish-Chandra modules asked for here has been known for a long time. One useful fact is that every irreducible Harish-Chandra module may be embedded in one induced from a finite-dimensional representations of a minimal parabolic subgroup. This gives a great deal of information, and in particular often allows one to detect unitarizability.

This technique gives a great deal of information, but some important matters require another approach. The notable exceptions are those representations of G that occur discretely in $L^2(G)$, which make up the **discrete series**. These are generally dealt with on their own, and then one can look also at representations induced from discrete series representations of parabolic subgroups. Along with some technical adjustments, these are the components of Langlands' classification of all irreducible Harish-Chandra modules. One of the adjustments is that one must say something about embeddings of arbitrary Harish-Chandra modules into $C^\infty(G)$, and say something about the asymptotic behaviour of certain functions in the image. This requires examining solutions of differential equations.

This essay is mostly just about the group $G = SL_2(\mathbb{R})$. Although this is a relatively simple group, it is instructive to see that nearly all interesting phenomena mentioned already appear. I shall touch, eventually, on all themes mentioned above, but in the present version a few are left out.

Representations of compact groups have been examined for a long time, but the study of unitary representations of non-compact semi-simple groups begin with the classic [Bargmann:1947]. Bargmann's initial classification of irreducible representations of $SL_2(\mathbb{R})$ already made the step from G to (\mathfrak{g}, K) , if not rigorously. Even now it is worthwhile to include an exposition of Bargmann's paper, and I shall do so, because it is almost the only case where results can be obtained without introducing sophisticated tools. But another reason for looking at $SL_2(\mathbb{R})$ closely is that the sophisticated tools one does need eventually are for $SL_2(\mathbb{R})$ relatively simple, and it is valuable to see them in that simple form. That is my primary goal in this essay.

In Part I I shall essentially follow Bargmann's calculations to classify all irreducible Harish-Chandra modules over $(\mathfrak{sl}_2, SO(2))$. The unitarizable ones among these will be found. The techniques used here will just be relatively simple calculations in the Lie algebra \mathfrak{sl}_2 .

Bargmann's techniques become impossibly difficult for groups other than $SL_2(\mathbb{R})$. One needs in general a way to classify representations that does not require such explicit computation. There are several ways to do this. One is by means of induction from parabolic subgroups of G , and this is what I'll discuss in Part II (but just for $SL_2(\mathbb{R})$). These representations are now called **principal series**. (This is a term used by Bargmann, but for only a subset of them.)

It happens, even for arbitrary reductive groups, that every irreducible Harish-Chandra module can be embedded in one induced from a finite-dimensional representation of a minimal parabolic subgroup. This is important, but does not answer many interesting questions. Some groups, including $SL_2(\mathbb{R})$ possess certain irreducible unitary representations said to lie in the discrete series. They occur discretely in the representation of G on $L^2(G)$, and they require new methods to understand them. This is done for $SL_2(\mathbb{R})$ in Part III.

For $SL_2(\mathbb{R})$ the discrete series representations can be realized in a particularly simple way in terms of holomorphic functions. This is not true for all semi-simple groups, and it is necessary sooner or later to understand how arbitrary representations can be embedded in the space of smooth functions on the group. This is in terms of matrix coefficients, explained later in Part III. One corollary of this investigation will be that any admissible representation of $(\mathfrak{sl}_2, SO(2))$ can be extended to a smooth representation of G , thus extending a result about irreducible representations due to Harish-Chandra.

In order to understand the role of representations in the theory of automorphic forms, it is necessary to understand Langlands' classification of representations, and this is explained in Part IV.

Eventually I'll include in this essay an account of how D -modules on the complex projective line can be used to explain many phenomena that appear otherwise mysterious. Other topics I've not yet included are the relationship between the asymptotic behaviour of matrix coefficients and embeddings into principal series, and the relationship between representations of $SL_2(\mathbb{R})$ and Whittaker functions.

The best standard reference for this material seems to be [Knapp:1986] (particularly Chapter II, but also scattered references throughout), although many have found enlightenment elsewhere. A very different approach is Chapter I of [Vogan:1981]. Many books give an account of Bargmann's results, but for most of the rest I believe the account here has many new features.

Part I. Bargmann's classification

1. Representations of the group and of its Lie algebra

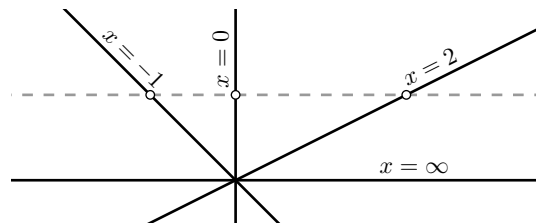
What strikes many on first sight of the theory of representations of $G = SL_2(\mathbb{R})$ is that it is very rarely concerned with representations of G ! Instead, one works almost exclusively with certain representations of its Lie algebra. And yet . . . the theory is ultimately about representations of G . In this first section I'll try to explain this paradox, and summarize the consequences of the substitution of Lie algebra for Lie group.

I want first to say something first about why one is really interested in representations of G , or at least something closely related to them. The main applications of representation theory of groups like G are to the theory of automorphic forms. These are functions of a certain kind on arithmetic quotients $\Gamma \backslash G$ of finite volume, for example when $\Gamma = SL_2(\mathbb{Z})$. The group G acts on this quotient on the right, and the corresponding representation of G on $L^2(\Gamma \backslash G)$ is unitary. *Which unitary representations of G occur as discrete summands of this representation?* As I have already mentioned, there is a conjectural if partial answer to this question, at least for congruence groups, which is an analogue for the real prime of the Ramanujan conjecture about the Fourier coefficients of holomorphic forms. It was Deligne's proof of the Weil conjectures that gave at the same time a proof of the Ramanujan conjecture, and from this relationship the real analogue acquires immediately a certain cachet.

As for why one winds up looking at representations of \mathfrak{g} , this should not be unexpected. After all, even the classification of finite-dimensional representations of G comes down to the classification of representations of \mathfrak{g} , which are far easier to work with. For example, one might contrast the action of the unipotent upper triangular matrices of G on the symmetric power $S^n(\mathbb{C}^2)$ with that of its Lie algebra.

The representations one winds up looking at are infinite-dimensional. What might be surprising is that, unlike what happens for finite-dimensional representations, these representations of \mathfrak{g} are not usually at the same time representations of G . It is this that I want to explain.

I'll begin with an example that should at least motivate the somewhat technical aspects of the shift from G to \mathfrak{g} . The projective space $\mathbb{P} = \mathbb{P}^1(\mathbb{R})$ is by definition the space of all lines in \mathbb{R}^2 . The group G acts on \mathbb{R}^2 by linear transformations, and it takes lines to lines, so it acts on \mathbb{P} as well. There is a standard way to assign coordinates on \mathbb{P} by thinking of this space as \mathbb{R} together with a point at ∞ . Formally, every line but one passes through a unique point $(x, 1)$ in \mathbb{R}^2 , and this is assigned coordinate x . The exceptional line is the x -axis, and is assigned coordinate ∞ .



In terms of this coordinate system $SL_2(\mathbb{R})$ acts by fractional linear transformations:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} : x \mapsto (ax + b)/(cx + d),$$

as long as we interpret $x/0$ as ∞ . This is because

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} z \\ 1 \end{bmatrix} = \begin{bmatrix} az + b \\ cz + d \end{bmatrix} = (cz + d) \begin{bmatrix} \frac{az + b}{cz + d} \\ 1 \end{bmatrix}.$$

The isotropy subgroup fixing ∞ is P , and \mathbb{P} may be identified with the quotient G/P . Since $K \cap P = \{\pm I\}$ and K acts transitively on \mathbb{P} , as a K -space \mathbb{P} may be identified with $K/\{\pm 1\}$.

The action of G on \mathbb{P} gives rise to a representation of G on functions on \mathbb{P} : $L_g f(x) = f(g^{-1}x)$. (The inverse is necessary here in order to have $L_{g_1 g_2} = L_{g_1} L_{g_2}$, as you can check.) But *there are in fact several spaces of functions available*—for example real analytic functions, smooth functions, continuous functions, functions that are locally square-integrable, or the continuous duals of any of these infinite-dimensional topological vector spaces. *Which of these do we really want to look at?* We can make life a little simpler by restricting ourselves to smooth representations (ones that are stable with respect to derivation by elements of the Lie algebra \mathfrak{g}), in which case the spaces of continuous and locally square-integrable functions, for example, are ruled out. But even with this restriction there are several spaces at hand.

Here is the main point: *All of these representations of G should be considered more or less the same*, at least for most purposes. The representation of K on \mathbb{P} is as multiplication on $K/\{\pm 1\}$. Among the functions on this space are those which transform by the characters ε^{2n} of K where

$$\varepsilon: \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \mapsto c + is.$$

The way to make the essential equivalence of all these spaces precise is to *replace all of them by the subspace of functions which when restricted to K are a finite sum of eigenfunctions*. This subspace is the same for all of these different function spaces, and in some sense should be considered the ‘essential’ representation. However, it has what seems to be at first one disability—it is *not* a representation of G , since g takes eigenvectors for K to eigenvectors for the conjugate gKg^{-1} , which is not generally the same. To make up for this, it is stable with respect to the Lie algebra $\mathfrak{g}_{\mathbb{R}} = \mathfrak{sl}_2(\mathbb{R})$, which acts by differentiation.

The representation of $\mathrm{SL}_2(\mathbb{R})$ on spaces of functions on \mathbb{P} is a model for other representations and even other reductive groups G . By definition, a continuous representation (π, V) of a Lie group G on a topological vector space V is a homomorphism π from G to $\mathrm{Aut}(V)$ such that the associated map $G \times V \rightarrow V$ is continuous. The TVS V is always assumed in this essay to be locally convex, Hausdorff, and quasi-complete. This last, somewhat technical, condition guarantees that if F is a continuous function of compact support on G with values in V then the integral

$$\int_G F(g) dg$$

is well defined. One basic fact about this integral is that it contained in the convex hull of the image of f , scaled by the measure of its support. If f is in $C_c^\infty(G)$, one can hence define the operator-valued integral

$$\pi(f): v \mapsto \int_G f(g)\pi(g)v dg$$

(i.e. taking $F(g) = f(g)\pi(g)v$).

If (π, V) is a continuous representation of a maximal compact subgroup K , let $V_{(K)}$ be the subspace of vectors that are contained in a K -stable subspace of finite dimension. These are called the **K-finite vectors** of the representation. If G is $\mathrm{SL}_2(\mathbb{R})$, it is known that any finite-dimensional space on which K acts continuously will be a direct sum of one-dimensional subspaces on each of which K acts by a character, so $V_{(K)}$ is particularly simple. Some technical problems arise because K might not be connected, but aside from that the irreducible representations of K are well understood (and more or less completely classified).

(1) *If (π, V) is any continuous representation of K , the subspace $V_{(K)}$ is dense in V .*

This is a basic fact about representations of a compact group, and a consequence of the Stone-Weierstrass Theorem.

A vector v in the space V of a continuous representation of G is called **differentiable** if the limit

$$\pi(X)v = \lim_{t \rightarrow 0} \frac{\pi(\exp(tX))v - v}{t}$$

exists in V for every X in \mathfrak{g} . The representation is called **smooth** if all vectors in V are differentiable, in which case $\pi(X)$ is an operator on V for every X in $U(\mathfrak{g})$.

(2) *Smooth vectors are dense in any continuous representation.*

If f lies in $C_c^\infty(G)$ and v in V , then $\pi(f)v$ is smooth. The space of such vectors is called the **Gårding subspace**. In some circumstances, it is the same as the subspace of smooth vectors. Applying a Dirac sequence, any vector may be approximated by such vectors.

A continuous representation of G on a complex vector space V is called **admissible** if $V_{(K)}$ is a direct sum of characters of K , each occurring with *finite multiplicity*.

(3) *If (π, V) is a smooth representation of \mathfrak{g} , the subspace $V_{(K)}$ is stable under \mathfrak{g} .*

If π is smooth, then there is a canonical map from $\mathfrak{g} \otimes V$ to V : $X \otimes v \mapsto \pi(X)v$. If U is K -stable then so is the image of $\mathfrak{g} \otimes U$, which contains all $\pi(X)u$ for u in U .

A continuous representation of G on a complex vector space V is called **admissible** if $V_{(K)}$ is a direct sum of characters of K , each occurring with *finite multiplicity*.

(4) *If (π, V) is an admissible representation of G , the vectors in $V_{(K)}$ are smooth.*

Suppose (σ, U) to be an irreducible representation of K , ξ the corresponding projection operator in $C^\infty(K)$. For example, if $K = SO_2$ and σ is a character, then ξ amounts to integration against σ^{-1} over K . The σ -component V_σ of V is the subspace of vectors fixed by $\pi(\xi)$. If (f_n) is chosen to be a Dirac sequence of smooth functions on G (i.e. non-negative, having limit the Dirac distribution δ_1), then $\pi(f_n)v \rightarrow v$ for all v in V . The operators $\pi(\xi)\pi(f)\pi(\xi) = \pi(\xi * f * \xi)$ are therefore dense in the finite-dimensional space $\text{End}(V_\sigma)$, hence make up the whole of it. But $\xi * f * \xi$ is also smooth and of compact support. Any vector v in V_σ may therefore be expressed as $\pi(f)v$ for some f in $C_c^\infty(G)$.

As for the second assertion, if v lies in V_σ then $\pi(\mathfrak{g})v$ lies in the direct sum of spaces V_τ as τ ranges over the irreducible components of the finite-dimensional K -stable space $\mathfrak{g} \otimes V_\sigma$.

A **unitary** representation of G is one on a Hilbert space whose norm is G -invariant. One of the principal goals of representation theory is to classify representations that occur as discrete summands of arithmetic quotients. This is an extremely difficult task. But these representations are unitary, and classifying unitary representations of G is a first step towards carrying it out.

(5) *Any irreducible unitary representation of G is admissible.*

This is the most difficult of these claims, requiring serious analysis. It is usually skipped over in expositions of representation theory, probably because it is difficult and also because it is not often needed in practice. One accessible reference is §4.5 (on ‘large’ compact subgroups) of [Warner:1970]. Other references are [Atiyah:1988] and my own notes [Casselman:2014] on unitary representations.

For $G = SL_2(\mathbb{R})$, every irreducible unitary representation of G restricts to a direct sum of characters of SO_2 , each occurring at most once. We shall see a proof of this later on.

In fact, admissible representations are ubiquitous. At any rate, we obtain from an admissible representation of G a representation of \mathfrak{g} and K satisfying the following conditions:

- (a) as a representation of K it is a direct sum of smooth irreducible representations (of finite dimension), each with finite multiplicity;
- (b) the representation of \mathfrak{k} associated to that as subalgebra of \mathfrak{g} and that as Lie algebra of K are the same;

(c) for k in K , X in \mathfrak{g}

$$\pi(k)\pi(X)\pi^{-1}(k) = \pi(\text{Ad}(k)X).$$

Any representation of the pair (\mathfrak{g}, K) satisfying these is called admissible. Most of the time I shall be most interested in those which are finitely generated as modules over the enveloping algebra $U(\mathfrak{g})$. These are of finite length, in the sense that they possess a finite filtration by irreducible representations.

Let me make some more remarks on this definition. It is plainly natural that we require that \mathfrak{g} act. But why K ? There are several reasons. • What we are really interested in are representations of G , or more precisely representations of \mathfrak{g} that come from representations of G . But a representation of \mathfrak{g} can only determine the representation of the connected component of G . The group K meets all components of G , so that requiring K to act fixes this problem. For groups like $SL_2(\mathbb{R})$, which are connected, this problem does not occur, but for the group $PGL_2(\mathbb{R})$, with two components, it does. • One point of having K act is to distinguish $SL_2(\mathbb{R})$ from other groups with the same Lie algebra. For example, $PGL_2(\mathbb{R})$ has the same Lie algebra as $SL_2(\mathbb{R})$, but the standard representation of its Lie algebra on \mathbb{R}^2 does not come from one of $PGL_2(\mathbb{R})$. Requiring that K act picks out a unique group in the isogeny class, since K contains the centre of G . • Any continuous representation of K , such as the restriction to K of a continuous representation of G , decomposes in some sense into a direct sum of irreducible finite-dimensional representations. This is not true of the Lie algebra \mathfrak{k} , which is after all the same as the Lie algebra of \mathbb{R} or \mathbb{R}^\times . So this requirement picks out from several possibilities the class of representations we want.

Admissible representations of (\mathfrak{g}, K) are not generally semi-simple. For example, the space V of smooth functions on \mathbb{P} contains the constant functions, but they are not a G -stable summand of V .

If (π, V) is a smooth representation of G assigned a G -invariant Hermitian inner product, the corresponding equations of invariance for the Lie algebra are that

$$Xu \bullet v = -u \bullet Xv$$

for X in $\mathfrak{g}_{\mathbb{R}}$, or

$$Xu \bullet v = -u \bullet \bar{X}v$$

for X in the complex Lie algebra $\mathfrak{g}_{\mathbb{C}}$. An admissible $(\mathfrak{g}_{\mathbb{C}}, K)$ -module is said to be unitary if there exists a positive definite Hermitian metric satisfying this condition. In some sense, unitary representations are by far the most important. At present, the major justification of representation theory is in its applications to automorphic forms, and the most interesting ones encountered are unitary.

Here is some more evidence that the definition of an admissible representation of $(\mathfrak{g}_{\mathbb{C}}, K)$ is reasonable:

- (a) admissible representations of (\mathfrak{g}, K) that are finite-dimensional are in fact representations of G ;
- (b) if the continuous representation (π, V) of G is admissible, the map taking each (\mathfrak{g}, K) -stable subspace $U \subseteq V_{(K)}$ to its closure in V is a bijection between $(\mathfrak{g}_{\mathbb{C}}, K)$ -stable subspaces of $V_{(K)}$ and closed G -stable subspaces of V ;
- (c) every admissible representation of (\mathfrak{g}, K) is $V_{(K)}$ for some continuous representation (π, V) of G ;
- (d) every unitary admissible representation of $(\mathfrak{g}_{\mathbb{C}}, K)$ is $V_{(K)}$ for some unitary representation of G ;
- (e) there is an exact functor from admissible representations of (\mathfrak{g}, K) to smooth representations of G .

Item (a) is classical.

For (b), refer to Théorème 3.17 of [Borel:1972].

I am not sure what a good reference for (c) is. For irreducible representations, it is a consequence of a theorem of Harish-Chandra that every irreducible admissible (\mathfrak{g}, L) -module is a subquotient of a principal series representation. This also follows from the main result of [Beilinson-Bernstein:1982].

The first proof of (d) was probably by Harish-Chandra, but I am not sure exactly where.

For (e), refer to [Casselman:1989] or [Bernstein-Krötz:2010].

Most of the rest of this essay will be concerned only with admissible representations (π, V) of (\mathfrak{g}, K) , although it will be good to keep in mind that the motivation for studying them is that they arise from representations of G .

Curiously, although nowadays representations of $SL_2(\mathbb{R})$ are seen mostly in the theory of automorphic forms, the subject was introduced by the physicist Valentine Bargmann, in pretty much the form we see here. I do not know what physics problem led to his investigation.

2. The Lie algebra

The Lie algebra of G is the vector space $\mathfrak{sl}_2(\mathbb{R})$ of all matrices in $M_2(\mathbb{R})$ with trace 0. Its Lie bracket is $[X, Y] = XY - YX$. There are two useful bases of the complexified algebra $\mathfrak{g}_{\mathbb{C}}$, mirroring a duality that exists throughout representation theory.

THE SPLIT BASIS. The simplest basis is this:

$$\begin{aligned} h &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ \nu_+ &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ \nu_- &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \end{aligned}$$

with defining relations

$$\begin{aligned} [h, \nu_{\pm}] &= \pm 2\nu_{\pm} \\ [\nu_+, \nu_-] &= h. \end{aligned}$$

The group G possesses an involutory automorphism θ , taking g to ${}^t g^{-1}$. The involution induced on the Lie algebra takes X to $-{}^t X$. This fixes elements of \mathfrak{k} and acts as multiplication by -1 on the symmetric matrices. Thus $\nu_+^{\theta} = -\nu_-$. It is sometimes more convenient to have the basis stable with respect to θ and, for this reason, sometimes in the literature $-\nu_-$ is used as part of a standard basis instead of ν_- .

THE COMPACT BASIS. This is a basis that's useful when we want to do calculations involving K . The first element of the new basis is

$$\kappa = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

which spans the real Lie algebra \mathfrak{k} of K . The rest of the new basis is to be made up of eigenvectors of K . The group K is compact. Its characters are all complex, not real, so in order to decompose the adjoint action of \mathfrak{k} on \mathfrak{g} we so we must extend the real Lie algebra to the complex one. The $\mathfrak{k}_{\mathbb{R}}$ -stable space complementary to $\mathfrak{k}_{\mathbb{R}}$ in $\mathfrak{g}_{\mathbb{R}}$ is that of symmetric matrices

$$\begin{bmatrix} a & b \\ b & -a \end{bmatrix}.$$

and its complexification decomposes into the sum of two conjugate eigenspaces for κ spanned by

$$\begin{aligned} x_+ &= \frac{1}{2} \cdot \begin{bmatrix} 1 & -i \\ -i & -1 \end{bmatrix} \\ x_- &= \frac{1}{2} \cdot \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix} \end{aligned}$$

with relations

$$\begin{aligned} [\kappa, x_{\pm}] &= \pm 2i x_{\pm} \\ [x_+, x_-] &= -i \kappa. \end{aligned}$$

By definition, the space of an admissible representation of (\mathfrak{g}, K) is spanned by eigenvectors for K . The following is an elementary consequence of the formulas above:

2.1. Lemma. *Suppose V to be any representation of (\mathfrak{g}, K) , and suppose v to be an eigenvector for K with eigencharacter ε^n . Then $\pi(\kappa)v = niv$ and $\pi(\kappa)\pi(x_{\pm})v = (n \pm 2)i\pi(x_{\pm})v$.*

Proof. Since

$$\pi(\kappa)v = \left. \frac{d}{dt} \right|_{t=0} e^{int}v = niv$$

and

$$\pi(\kappa)\pi(x_{\pm})v = \pi(x_{\pm})\pi(\kappa)v + \pi([\kappa, x_{\pm}])v. \quad \color{red}{\blacksquare}$$

THE CAYLEY TRANSFORM. The group G possesses two maximal tori, the subgroup A of diagonal matrices and the compact subgroup K . They are certainly not conjugate in G , but they become conjugate in $G(\mathbb{C}) = SL_2(\mathbb{C})$. This can be seen geometrically. The group $G(\mathbb{C})$ acts by fractional linear transformations on $\mathbb{P}^1(\mathbb{C})$. The diagonal matrices fix the origin, and the complex matrices

$$\begin{bmatrix} c & -s \\ s & c \end{bmatrix}$$

fix i . The Cayley transform

$$C: z \mapsto (z - i)/(z + i)$$

takes i to 0, and therefore

$$C K(\mathbb{C}) C^{-1} = A(\mathbb{C}) \quad \left(C = \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \right).$$

In the Lie algebra

$$\text{Ad}_C \kappa = -hi, \quad \text{Ad}_C x_{\pm} = i\nu_{\mp}.$$

In classifying algebraic tori in arbitrary semi-simple groups defined over \mathbb{R} , copies of the Cayley matrix C are crucial.

THE CASIMIR. let $Z(\mathfrak{g})$ be the centre of $U(\mathfrak{g})$. Suppose (π, V) to be an irreducible admissible representation of (\mathfrak{g}, K) . Then any element X of the centre of $U(\mathfrak{g})$ will act as a scalar multiplication, say by $\zeta(X)$, on all of V . The homomorphism from $Z(\mathfrak{g})$ taking X to $\zeta(X)$ is an important characteristic of π , called (for reasons that escape me) its **infinitesimal character**.

What is the centre of $U(\mathfrak{g})$?

2.2. Proposition. *The centre of $U(\mathfrak{g})$ is the polynomial algebra generated by the Casimir operator*

$$\Omega = \frac{h^2}{4} + \frac{\nu_+\nu_-}{2} + \frac{\nu_-\nu_+}{2}.$$

It has alternate expressions

$$\begin{aligned} \Omega &= \frac{h^2}{4} - \frac{h}{2} + \nu_+\nu_- \\ &= \frac{h^2}{4} + \frac{h}{2} + \nu_-\nu_+ \\ &= \frac{h^2}{4} - \frac{h}{2} + \nu_+^2 - \nu_+\kappa \\ &= -\frac{\kappa^2}{4} + \frac{x_-x_+}{2} + \frac{x_+x_-}{2} \\ &= -\frac{\kappa^2}{4} - \frac{\kappa i}{2} + x_-x_+ \\ &= -\frac{\kappa^2}{4} + \frac{\kappa i}{2} + x_+x_- . \end{aligned}$$

The alternate expressions follow easily from the first, basic, one. For example, since

$$\nu_- \nu_+ = \nu_+ \nu_- + [\nu_-, \nu_+] = \nu_+ \nu_- - h$$

we derive

$$\Omega = \frac{h^2}{4} + \frac{\nu_+ \nu_-}{2} + \frac{\nu_- \nu_+}{2} = \frac{h^2}{4} + \nu_+ \nu_- - \frac{h}{2}.$$

I shan't prove all of this result, but just explain why one can construct the Casimir from general principles.

The **Killing form** of a Lie algebra is the inner product

$$K(X, Y) = \text{trace}(\text{ad}_X \text{ad}_Y).$$

If G is any Lie group, the Killing form of its Lie algebra is invariant with respect to the adjoint action of the group (as well as its Lie algebra). According to a well known criterion, this Lie algebra is semi-simple if and only if the Killing form is non-degenerate. For example, if $\mathfrak{g} = \mathfrak{sl}_n$ it is straightforward to verify that the Killing form is $2n \text{ trace}(XY)$. In particular for $\mathfrak{g} = \mathfrak{sl}_2$ with basis h, ν_{\pm} its matrix is

$$\begin{bmatrix} 8 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 4 & 0 \end{bmatrix}.$$

The Killing form gives an isomorphism of \mathfrak{g} with its linear dual $\widehat{\mathfrak{g}}$, and thus induces an isomorphism of $\widehat{\mathfrak{g}} \otimes \widehat{\mathfrak{g}}$ with $\widehat{\mathfrak{g}} \otimes \mathfrak{g}$, which may be identified with $\text{End}(\mathfrak{g})$, and $\mathfrak{g} \otimes \mathfrak{g}$. The Casimir element Ω is the image of the Killing form itself through the sequence of maps

$$\widehat{\mathfrak{g}} \otimes \widehat{\mathfrak{g}} \longrightarrow \mathfrak{g} \otimes \mathfrak{g} \longrightarrow U(\mathfrak{g}).$$

Explicitly, it is

$$\Omega = 2 \sum X_i X_i^{\vee}$$

with the sum over a basis X_i and the corresponding dual basis elements X_i^{\vee} . It lies in the centre of $U(\mathfrak{g})$ precisely because the Killing form is \mathfrak{g} -invariant. For \mathfrak{sl}_2 it is now a straightforward calculation to see that

$$\Omega = (1/4)h^2 + (1/2)\nu_+\nu_- + (1/2)\nu_-\nu_+,$$

which, as we have seen, may be manipulated to give other expressions.

3. Differential operators and the Lie algebra

Elements of \mathfrak{g} act as differential operators on smooth function on any space on which G acts. Suppose π is a continuous homomorphism from G to the group of smooth automorphisms of a smooth differentiable manifold X . Then

$$[\pi(X)f](x) = \lim_{\varepsilon \rightarrow 0} \frac{f(\pi(\exp(\varepsilon X))x) - f(x)}{\varepsilon} = \left. \frac{d}{dt} \right|_{t=0} f(\pi(\exp(tX))x).$$

The group G acts on itself on the right and left, giving rise to the left and right regular representations:

$$[R_g f](x) = f(xg), \quad [L_g f](x) = f(g^{-1}x).$$

The exponent in the left action is often a nuisance in computation, so it is convenient to have also the left regular action of the *opposite* group:

$$[\Lambda_g f](x) = f(gx).$$

The definition of $\pi(X)$ might seem sometimes to involve a formidable calculation, and it is often useful to use Taylor series to simplify it. The point is that it is essentially a first order computation in which terms of second order can be neglected. Roughly speaking, up to first order $\exp(\varepsilon X) = I + \varepsilon X$, so we can in practice restrict ourselves to the simplified expression

$$\frac{(I + \varepsilon X) \cdot m - m}{\varepsilon},$$

in which we may assume $\varepsilon^2 = 0$. We must just compute the coefficient of ε in the expression for $(I + \varepsilon X) \cdot x$.

Let's look at $SL_2(\mathbb{R})$ acting on the upper half plane

$$\mathcal{H} = \{x + iy \mid y > 0\}.$$

by Möbius transformations

$$\pi(g): z \mapsto \frac{az + b}{cz + d} \quad \left(g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right).$$

The group is acting holomorphically, so the natural result of these calculations will be a complex-valued function $f(z)$. How does this correspond to a vector field? The flow $z \mapsto \pi(\exp(tX))z$ will have as Taylor series $z \mapsto z + tf(z) + \dots$, so $f(z) = p + iq$ is to be interpreted as as the real vector field $p \partial/\partial x + q \partial/\partial y$.

3.1. Proposition. *We have*

$$\begin{aligned} \Lambda_{\nu_+} &= \frac{\partial}{\partial x} \\ \Lambda_h &= 2x \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial y} \end{aligned}$$

Proof. • The simpler is ν_+ . Here

$$I + \varepsilon \nu_+ = \begin{bmatrix} 1 & \varepsilon \\ 0 & 1 \end{bmatrix}$$

and this takes

$$z \mapsto \frac{z + \varepsilon}{1} = z + \varepsilon, \quad (x, y) \mapsto (x + \varepsilon, y)$$

Therefore

$$\nu_+ \rightsquigarrow \partial / \partial x.$$

• Now for h . Here

$$I + \varepsilon h = \begin{bmatrix} 1 + \varepsilon & 0 \\ 0 & 1 - \varepsilon \end{bmatrix}$$

and this takes

$$\begin{aligned} z &\mapsto \frac{(1 + \varepsilon)z}{(1 - \varepsilon)} \\ &= z(1 + \varepsilon)(1 + \varepsilon + \varepsilon^2 + \cdots) \\ &= z(1 + 2\varepsilon) = z + 2\varepsilon z \\ (x, y) &\mapsto (x + 2\varepsilon x, y + 2\varepsilon y) \end{aligned}$$

so

$$h \rightsquigarrow 2x \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial y}.$$

The Casimir operator Ω commutes with K , hence it induces a differential operator through the right regular action on G/K . But this space may be identified with \mathcal{H} since K is the isotropy subgroup fixing i . What is the expression for Ω as a combination of partial derivatives $\partial/\partial x$ and $\partial/\partial y$?

We shall see many problems of a similar nature, and there is one extremely simple but nonetheless basic tool that can be applied in all of them.

3.2. Proposition. For X in \mathfrak{g} , g in G

$$[R_X f](g) = [\Lambda_{gXg^{-1}} f](g).$$

Informally, this says that $g \cdot X = gXg^{-1} \cdot g$.

Proof. This follows from the observation that

$$g \cdot \exp(tX) = \exp(t[\text{Ad}(g)](X)) \cdot g.$$

Let

$$n_x = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}, \quad a_t = \begin{bmatrix} t & 0 \\ 0 & 1/t \end{bmatrix}.$$

3.3. Corollary. If $p = n_x a_t$ then

$$\begin{aligned} [R_{\nu_+} f](p) &= t^2 [\Lambda_{\nu_+} f](p) \\ [R_h f](p) &= [\Lambda_h f](p) - 2x [\Lambda_{\nu_+} f](p). \end{aligned}$$

Proof. The first because $p\nu_+p^{-1} = t^2\nu_+$. As for the second:

$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -x \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2x \\ 0 & -1 \end{bmatrix}$$

so $php^{-1} = h - 2x\nu_+$.

3.4. Corollary. If f is a smooth function on \mathcal{H} then

$$\Lambda_\Omega f = \Delta f = y^2 \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right).$$

The operator Δ is the Laplacian of the non-Euclidean metric on \mathcal{H} .

Proof. We know that

$$\Omega = h^2/4 - h/2 + \nu_+^2 - \nu_+\kappa.$$

Furthermore, $\Lambda_\Omega = R_\Omega$. But if $p = n_x a_t$ then $p(i) = x + it^2$, and f may be identified with a function on G annihilated by R_κ . Apply Proposition 3.1.

4. Admissibility

Suppose V to be a Hilbert space. A unitary representation of $G = SL_2(\mathbb{R})$ is a continuous homomorphism from G to $U(V)$. Continuity here means in the **strong topology**. To each v in V corresponds the semi-norm $\|T(v)\|$ on the ring of bounded operators on H , and the map from G to $U(H)$ is continuous with respect to the topology on $U(H)$ defined by these semi-norms.

I'll not offer details about basic facts of such representations, since they are covered widely in the literature and are on the whole quite intuitive. The most important thing to know is that as a representation of $K = SO_2$ the space V is an orthogonal Hilbert sum of isotypic components on which K acts by characters.

Let V_χ be the χ -component of V . It is the image of the self-adjoint projection operator

$$p_\chi: v \longmapsto \int_K \chi^{-1}(k)\pi(k)v dk.$$

It is a therefore a closed K -stable subspace of V .

4.1. Theorem. *If (π, V) is an irreducible unitary representation of $SL_2(\mathbb{R})$, then for any character χ of SO_2 the subspace V_χ has dimension at most one.*

Proof. Fix (π, V) and χ .

Step 1. The first step is to verify this for irreducible finite-dimensional representations. Although this will be done independently in a later section, as part of a more general argument, I'll include a proof here.

Let V_n be the subspace on which

$$\pi(\kappa)v = ni v.$$

It is to be shown that each V_n has dimension one.

Suppose n to be maximum such that $V_n \neq 0$. Now if

$$\pi(\kappa)v = ki v$$

then

$$\begin{aligned} \pi(\kappa)\pi(x_\pm)v &= \pi(x_\pm)\pi(\kappa)v + \pi([\kappa, x_\pm])v \\ &= \pi(x_\pm)\pi(\kappa)v + \pi(\pm 2i x_\pm)v \\ &= (k \pm 2i)v. \end{aligned}$$

Therefore if $u_k = \pi(x_-^k)v_n$ then

$$\pi(x_+)u_0 = 0, \quad \pi(\kappa)\pi(x_-^k)v_n = (n - 2k)iu_k.$$

Therefore the Theorem will be demonstrated if I can show that the space spanned by the u_k is stable under \mathfrak{g} , in which case it must be all of V .

Recall that the Casimir element of $U(\mathfrak{g})$ is

$$\begin{aligned} \Omega &= -\frac{\kappa^2}{4} - \frac{\kappa i}{2} + x_-x_+ \\ &= -\frac{\kappa^2}{4} + \frac{\kappa i}{2} + x_+x_-, \end{aligned}$$

The first tells us that

$$\Omega u_0 = (n^2/4 + n/2)u_0.$$

Since Ω commutes with \mathfrak{g} , the Casimir element therefore acts on all of V as $(n^2/4 + n/2)I$.

The second then gives us

$$x_+x_- = \Omega + \frac{\kappa^2}{4} - \frac{\kappa i}{2}.$$

This allows us to prove by induction that $\pi(x_+)u_k$ is a scalar multiple of u_{k-1} , which proves the claim.

Step 2. The next step is to define the **Hecke algebra** \mathfrak{H}_χ . It is the ring of functions F in $C_c(G)$ such that

$$F(kg) = \chi^{-1}(k)F(g) = F(gk)$$

for all g in G . Multiplication is convolution. If F lies in \mathfrak{H}_χ then $\pi(F)$ commutes with p_χ , hence takes V_χ to itself and contains V_χ^\perp in its kernel.

The $F \mapsto \pi(F)$ is a homomorphism of \mathfrak{H}_χ into $\text{End}(V_\chi)$, the ring of bounded operators on the Hilbert space V_χ . Let H be its image, and let $\mathcal{C}(H)$ be the subring of all T in $\text{End}(V_\chi)$ that commute with all of H .

4.2. Lemma. (Schur's Lemma) *Any bounded operator on V_χ that commutes with all operators in H is a scalar multiplication.*

Proof. Recall that $f^\vee(x) = f(x^{-1})$. If f lies in \mathfrak{H}_χ , then so does $\overline{f^\vee}$, since $\chi(k^{-1}) = \overline{\chi(k)}$. Furthermore, $\pi(\overline{f^\vee})$ is the adjoint of $\pi(f)$. Thus H is stable under adjoints, and so is $\mathcal{C}(H)$. If T lies in $\mathcal{C}(H)$ then so do $(T + T^*)/2$ and $(T - T^*)/2i$, and

$$T = \frac{T + T^*}{2} + i \frac{T - T^*}{2i},$$

so we may assume T to be self-adjoint. But then the projection operators associated to T by the spectral theorem also commute with H . In order to prove that T is scalar, it suffices to prove that any one of these projection operators is the identity. So I may now assume T to be a projection operator. Let V_T be the image of T , and let U be the smallest G -stable subspace containing V_T . Since V_T is stable under H , the intersection of U with V_χ is V_T , and since π is irreducible this must be all of V_χ . Thus T is the identity operator. ■

Step 3. In finite dimensions, it is easy to see from the Lemma in the previous step that the image of \mathfrak{H}_χ is the entire ring of bounded operators on V_χ . In infinite dimensions, we can deduce only a slightly weaker fact:

4.3. Lemma. *If (π, V) is an irreducible unitary representation of G , the image of \mathfrak{H}_χ is dense in the ring of bounded operators on V_χ .*

The topology referred to here is the strong operator topology.

Proof. The image H_χ of \mathfrak{H} in $\text{End}(V_\chi)$ is a ring of bounded operators stable under the adjoint map. Since π is irreducible, the Lemma in the previous step implies its commutant is just the scalar multiplications. The operators in H embed in turn into $\mathcal{C}\mathcal{C}(H)$, the operators which commute with $\mathcal{C}(H)$.

Because every function in $C_c(G)$ is a limit of a Dirac sequence, the idempotent p_χ is the limit of functions in H . Therefore this Lemma follows from the following rather more general result. If R is a set of bounded operators on a Hilbert space, let $\mathcal{C}(R)$ be the ring of operators that commute with all operators in R .

4.4. Lemma. (Von Neumann's double commutant theorem) *Suppose R to be a subring of the ring of bounded operators on a Hilbert space V which (a) is stable under the adjoint map $T \mapsto T^*$ and (b) contains the identity I in its strong closure. Then R is dense in the ring $\mathcal{C}\mathcal{C}(R)$ of all operators that commute with $\mathcal{C}(R)$.*

Proof. I follow the short account at the very beginning of Chapter 2 in [Topping:1968]. It must be shown that if T commutes with all operators α in $\mathcal{C}(R)$, then T is in the strong operator closure of R within $\text{End}(V)$.

I recall exactly what the Lemma means. The strong topology is defined by semi-norms

$$\|T\|_\chi = \sup_{v \in \chi} \|Tv\|$$

for finite subsets χ of V . An operator τ of $\text{End}(V)$ is in the strong closure of a subset X if and only if every one of its neighbourhoods in the strong topology intersects X . Hence τ is in the strong closure of X if and

only if for every finite subset S of V and $\varepsilon > 0$ there exists a point x in X with $\|x(v) - \tau(v)\| < \varepsilon$ for all v in S .

I do first the simplest case in which there is a single x in S . Suppose T lies in $\mathcal{CC}(R)$. Let \overline{Rx} be the closure of Rx in V . Since I is in the strong closure of R , it contains x . Since \overline{Rx} is invariant with respect to R and R is stable under adjoints, its orthogonal complement is also stable under R . Therefore the projection p_x onto \overline{Rx} satisfies the relations

$$p_x r p_x = r p_x$$

for every r in R , and also

$$p_x r^* p_x = r^* p_x.$$

If we take the adjoint of this we get

$$p_x r p_x = p_x r,$$

concluding that $p_x r = r p_x$, so p_x lies in $\mathcal{C}(R)$. By assumption, T commutes with p_x and hence also takes \overline{Rx} into itself. But since x lies in \overline{Rx} , Tx is in \overline{Rx} .

If $S = \{x_1, \dots, x_n\}$, consider V^n , the orthogonal direct sum of n copies of V . The algebra $\text{End}(V)$ acts on it component-wise. Let R_n be the diagonal image of R . Its commutant in this is the matrix algebra $M_n(\mathcal{C}(R))$, and the double commutant is the diagonal image of $\mathcal{CC}(R)$. Apply the case $n = 1$ to this, and get r in R such that $\|r(x_i) - T(x_i)\| < \varepsilon$ for all i . This concludes the proof of Lemma 4.4. ▣

Step 4. In this step, I show that the claim of the theorem reduces to that for finite-dimensional representations of G . It is here where the fact that G is an algebraic group plays a role.

4.5. Lemma. *If G is the group of \mathbb{R} -rational points on any affine algebraic group defined over \mathbb{R} , the map from $C_c(G)$ to $\prod \text{End}_{\mathbb{C}}(E)$, with the product over all irreducible finite-dimensional algebraic representations E of G , is injective.*

By Stone-Weierstrass.

Since every finite-dimensional representation of $SL_2(\mathbb{R})$ has K -multiplicity one, we deduce that \mathfrak{H} is commutative.

Step 5. Now the last step. The Hecke ring H is dense in $\text{End}(V_\chi)$. Since its image in $\text{End}(V_\chi)$ is both dense and commutative, the subspace V_χ has dimension at most one. This concludes the proof of Theorem 4.1. ▣

5. Classification of irreducible representations

In this section and the next I shall classify all irreducible admissible representations (π, U) of (\mathfrak{g}, K) and classify all maps in $\text{Hom}_{\mathfrak{g}, K}(U, V)$ for arbitrary (\mathfrak{g}, K) -modules V .

At first, suppose (π, V) to be an arbitrary (\mathfrak{g}, K) module. According to Lemma 2.1, the space V will contain an eigenvector v_n for K such that $\pi(\kappa)v_n = ni v_n$. The $U(\mathfrak{g})$ submodule U of V generated by v_n will also be spanned by K -finite vectors. I now pose the questions:

- Under what circumstances is U an irreducible admissible representation of (\mathfrak{g}, K) ? And if it is, how can it be characterized?

Step 1. First suppose U to be an irreducible admissible representation of (\mathfrak{g}, K) . The Casimir Ω commutes with K , hence takes each K -eigenspace into itself. Since each eigenspace has finite dimension, there exists an eigenvector $v \neq 0$ for Ω , say $\pi(\Omega)v = \gamma v$. Since π is irreducible:

- The vector v_n must be an eigenvector for Ω .

If $\pi(\Omega)v_m = \gamma v_m$ then, because Ω commutes with all of $U(\mathfrak{g})$, $\pi(\Omega) = \gamma \cdot I$ on all of U .

Step 2. There are some useful criteria from which one can infer the eigenvalue of v_m with respect to Ω . Recall that

$$(5.1) \quad \begin{aligned} x_+x_- &= \Omega + \kappa^2/4 - \kappa i/2 \\ x_-x_+ &= \Omega + \kappa^2/4 + \kappa i/2. \end{aligned}$$

If $v_m \neq 0$ and $\pi(x_-)v_m = 0$ then

$$\pi(x_+x_-)v_m = 0 = (\pi(\Omega) - m^2/4 + m/2)v_m, \quad \pi(\Omega)v_m = (m^2/4 - m/2)v_m.$$

Something similar can be said for $\pi(x_+)v_m$. Hence I have proved one half of this:

5.2. Lemma. Suppose U to be a (\mathfrak{g}, K) -module, v_m an eigenvector for κ with eigenvalue mi . If $\pi(x_{\pm})v_m = 0$ then v_m is an eigenvector for Ω with eigenvalue $m^2/4 \pm m/2$. Conversely, if $\pi(\Omega)v_m = (m^2/4 \pm m/2)v_m$ and $m \neq 0$ then $\pi(x_{\pm})v_m = 0$.

Proof. Suppose

$$\pi(\Omega)v_m = (m^2/4 + m/2)v_m.$$

Then $\pi(x_-x_+)v_m = 0$ as well. Let $v_{m+2} = \pi(x_+)v_m$, also an eigenvector for $\pi(\kappa)$. Then on the one hand $\pi(x_-)v_{m+2} = 0$, so that

$$\pi(\Omega)v_{m+2} = \left(\frac{(m+2)^2}{4} - \frac{m+2}{2} \right) v_{m+2} = 0.$$

while on the other

$$\pi(\Omega)v_{m+2} = \left(\frac{m^2}{4} - \frac{m}{2} \right) v_{m+2}.$$

These equations are consistent if and only if $m = 0$. ▮

Step 3. From now on, suppose that $\pi(\Omega)v_n = \gamma v_n$, hence that $\pi(\Omega) = \gamma \cdot I$ on U . For $k > 0$ set

$$(5.3) \quad \begin{aligned} v_{n+2k} &= \pi(x_+^k)v_n \\ v_{n-2k} &= \pi(x_-^k)v_n \end{aligned}$$

so that $\pi(\kappa)v_m = mi v_m$ for all m .

Note that v_m is defined only for m of the same parity as n .

- The space U is spanned by the v_m .

Proof. It suffices to prove that the space spanned by the v_m is stable under $U(\mathfrak{g})$. Since each v_m is an eigenvector for κ , this reduces to showing it to be stable under x_{\pm} . Let $U_{\leq n}$ be the space spanned by the v_m with $m \leq n$, and similarly for $U_{\geq n}$. Since $\pi(x_+)U_{\geq n} \subseteq U_{>n}$ and $\pi(x_-)U_{\leq n} \subseteq U_{<n}$, the claim will follow from these two others: (i) $\pi(x_+)U_{<n} \subseteq U_{\leq n}$; (ii) $\pi(x_-)U_{>n} \subseteq U_{\geq n}$.

By (5.1)

$$\begin{aligned} v_{n-2k-2} &= \pi(x_-)v_{n-2k} \\ \pi(x_+)v_{n-2k-2} &= \pi(x_+x_-)v_{n-2k} \\ &= (\gamma - m^2/4 + m/2)v_{n-2k} \\ v_{n+2k+2} &= \pi(x_+)v_{n+2k} \\ \pi(x_-)v_{n+2k+2} &= \pi(x_+x_-)v_{n+2k} \\ &= (\gamma - m^2/4 - m/2)v_{n+2k}. \end{aligned}$$

Step 4. As a consequence of Step 3, the dimension of the eigenspace for each ε^n is at most one. Hence

- Whenever $v_\ell \neq 0$ and $\ell \leq m$, say $m = \ell + 2k$, then v_m is a scalar multiple of $\pi(x_+^k)v_\ell$.

Step 5. It may very well happen that some of the v_m vanish.

- The set of m with $v_m \neq 0$ is an interval, possibly infinite, in the subset of \mathbb{Z} of some given parity.

Say $v_m \neq 0, v_{m+2} = 0$. If $m \geq n$, then $v_m = \pi(x_+^{(m-n)/2})v_n$ and $v_{m+4} = \pi(x_+)v_{m+2} = 0$ and by induction $v_{m+2k} = 0$ for all $k \geq 1$. Similarly for $m \leq n$.

I'll call such an interval a *parity interval*, and I'll call the set of m with $v_m \neq 0$ the K -spectrum of π .

Step 6. Suppose that $\pi(x_{\pm})v_m = 0$. Then U is equal to the span of the $\pi(x_{\mp}^k)v_m$.

Step 7. There are now four possibilities:

(a) The spectrum of κ is a finite parity interval.

In this case, choose $v_n \neq 0$ with $\pi(x_+)v_n = 0$. Then $\gamma = n^2/4 + n/2$ and V is spanned by the v_{n-2k} . These must eventually be 0, so for some m we have $v_m \neq 0$ with $\pi(x_-)v_m = 0$. But then also $\gamma = m^2/4 - m/2$, which implies that $m = -n$. Therefore $n \geq 0$, the space V has dimension $n + 1$, and it is spanned by $v_n, v_{n-2}, \dots, v_{-n}$. I call this representation FD_n .

(b) The spectrum of κ is some parity subinterval $(-\infty, n]$. That is to say, some $v_n \neq 0$ is annihilated by x_+ and U is the infinite-dimensional span of the $\pi(x_-^k)v_n$. Under what circumstances is U in addition irreducible?

Say $v_n \neq 0$ and $\pi(x_+)v_n = 0$. Here $\gamma = n^2/4 + n/2$. If $n \geq 0$ then v_{-n} would be annihilated by x_- . Therefore, if U is irreducible $n < 0$.

Conversely, suppose $n < 0$. The space U is spanned by the non-zero vectors v_{n-2k} with $k \geq 0$. If W is any \mathfrak{g} -stable subspace of U then there must exist in W a vector v_m with $\pi(x_+)v_m = 0$. But then because of the eigenvalue of Ω , m has to be n and $W = U$. Hence U is irreducible.

I call this representation $DS_{|n|}^-$, for reasons that will appear later.

(c) The spectrum of κ is some parity subinterval $[n, \infty)$. That is to say, some $v_m \neq 0$ is annihilated by x_- but none is annihilated by x_+ .

By reasoning almost exactly the same as in the previous case, U is irreducible if and only if $n > 0$. I call this representation DS_n^+ .

(d) The spectrum of κ is some parity subinterval $(-\infty, \infty)$. That is to say, no $v_m \neq 0$ is annihilated by either x_+ or x_- .

I claim that in these circumstances U is always irreducible.

In this case $v_n \neq 0$ for all n . We cannot have $\gamma = ((m+1)^2 - 1)/4$ for any m of the same parity of the K -weights occurring. Furthermore, $\pi(x_+)$ is both injective and surjective. We therefore choose a new basis (v_m) such that

$$\pi(x_+)v_m = v_{m+2}$$

for all m . And then

$$\begin{aligned} \pi(x_-)v_m &= \pi(x_-x_+)v_{m-2} \\ &= \gamma - (m-2)^2/4 - (m-2)/2)v_{m-2} \\ &= (\gamma - m^2/4 - m/2)v_{m-2}. \end{aligned}$$

I call this representation $PS_{\gamma, n}$. The isomorphism class depends only on the parity of n .

Step 8. I now summarize.

Suppose U to be a (\mathfrak{g}, K) -module generated over $U(\mathfrak{g})$ by the K -eigenvector v_n with eigencharacter ε^n . Suppose in addition that $\pi(\Omega) = \gamma \cdot I$.

1. Suppose that

$$\gamma = \frac{m^2 - 1}{4}$$

for some integer $m \geq 0$ of parity opposite to n . For example, suppose that $\gamma = -1/4$ (i.e. $m = 0$) and that $n = 1$, or that $\gamma = 0$ ($m = 1$) and that $n = 0$ or 2 . There are several cases to deal with.

(a) If $|n| < m$ then U is isomorphic to the unique irreducible finite dimensional representation of degree m . (This case does not occur for $m = 0$.) For example, if $m = 1$ and $n = 0$, U is the trivial representation.

(b) If $n > m$ then U is isomorphic to DS_{m+1}^+ . For example, if $m = 0$ and $n = 1$, we are looking at the representation with K -spectrum $1, 3, 5, \dots$, while if $m = 1$ and $n = 2$ we are looking at the representation with K -spectrum $2, 4, 6, \dots$.

(c) If $n < -m$ then U is isomorphic to DS_{m+1}^- .

2. Otherwise, U is isomorphic to the irreducible representation $PS_{\gamma, n}$.

These are supplemented by the remark that if $\pi(x_{\pm})v_n = 0$ with then $\gamma = n^2/4 \pm n/2$.

6. The algebraic construction

In the previous section it has been shown that any irreducible admissible representation of (\mathfrak{g}, K) falls into one of the types listed. This is a statement of uniqueness. It is not immediately clear that every one of these does in fact define a representation of $\mathfrak{sl}_2(\mathbb{R})$. It can be shown directly, without much trouble, that in every case the formulas implicit in the previous argument above define a representation of (\mathfrak{g}, K) , but in this section I'll sketch an algebraic construction.

The basic fact for this purpose is this:

6.1. Proposition. Suppose I to be an ideal in $\mathbb{C}[\Omega]$ of finite codimension, U a finite-dimensional representation of K . If $M = U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} U$ then M/IM is an admissible representation of (\mathfrak{g}, K) .

Proof. Since $Z(\mathfrak{g})/I$ has finite dimension, it suffices to assume $I = (\Omega - \gamma)$. In this case, I claim that in effect M/IM is the direct sum of spaces

$$U, \quad x_+^p \otimes U, \quad x_-^p \otimes U \quad (p > 0)$$

For this, it suffices to assume U to be one-dimensional, in which case linear independence is immediate. For the rest, it suffices to show that the space spanned by these modulo I is stable under \mathfrak{g} . This is an easy induction argument, very similar to one we have seen in the previous section. ▣

For generic γ , the representations $PS_{\gamma, n}$ are of this form.

There is a simple converse to this. Suppose (π, V) to be any finitely generated (\mathfrak{g}, K) representation, say generated by the finite-dimensional K -stable space U , and annihilated by the ideal $I \subseteq Z(\mathfrak{g})$ of finite codimension. Then V is the natural quotient of M/IM , with M as in the Proposition.

The representations DS_n^\pm require a different construction. Let \mathfrak{p} be the Lie algebra $\mathfrak{k} \oplus \mathbb{C}[x_+]$. If \mathbb{C}_m is the representation of K on which it acts as ε^m , it becomes a representation of \mathfrak{p} through projection onto \mathfrak{k} . Consider the (\mathfrak{g}, K) -module

$$V_m = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \mathbb{C}_m.$$

Since \mathfrak{g} is the direct sum of $\mathbb{C}[x_-]$, \mathfrak{k} , and $\mathbb{C}[x_+]$, it may be identified as a linear space with $\mathbb{C}[x_-]$. The image of \mathbb{C}_m is an eigenspace for κ with eigenvalue mi , and x_+ annihilates this image. When $m < 0$, this representation is therefore a model for DS_m^- . Suppose (π, V) to be any representation of (\mathfrak{g}, K) . By standard arguments about tensor products, the space of maps

$$\text{Hom}_{(\mathfrak{g}, K)}(DS_m^-, V)$$

may be identified with the space of all v in V such that (1) $\pi(\kappa)v = mi v$ and (2) $\pi(x_+)v = 0$.

It remains to see whether these representations come from representations of G , and which are unitary.

7. Duals

Suppose (π, V) to be an admissible (\mathfrak{g}, K) -module. Its **admissible dual** is the subspace of K -finite vectors in the linear dual \widehat{V} of V . The group K acts on it so as to make the pairing K -invariant:

$$\langle \pi(k)\widehat{v}, \pi(k)v \rangle = \langle \widehat{v}, v \rangle, \quad \langle \pi(k)\widehat{v}, v \rangle = \langle \widehat{v}, \pi(k^{-1})v \rangle,$$

and the Lie algebra \mathfrak{g} acts on it according to the specification that matches G -invariance if G were to act:

$$\langle \widehat{\pi}(X)\widehat{v}, v \rangle = -\langle \widehat{v}, \pi(X)v \rangle.$$

If K acts as the character ε on a space U , it acts as ε^{-1} on its dual. So if the representation of K on V is the sum of characters ε^k , then that on \widehat{V} is the sum of the ε^{-k} . Thus we can read off immediately that the dual of FD_n is FD_n (this is because the longest element in its Weyl group happens to take a weight to its negative), and the dual of DS_n^\pm is DS_n^\mp . What about $PS_{\gamma, n}$?

7.1. Lemma. *If $\pi(\Omega) = \gamma \cdot I$ then $\widehat{\pi}(\Omega) = \gamma \cdot I$.*

Proof. Just use the formula

$$\Omega = \frac{\hbar^2}{4} + \frac{\nu_+ \nu_-}{2} + \frac{\nu_- \nu_+}{2}$$

and the definition of the dual representation. ▣

So $PS_{\gamma, n}$ is isomorphic to its dual.

The **Hermitian dual** of a (complex) vector space V is the linear space of all conjugate-linear functions on V , the maps

$$f: V \rightarrow \mathbb{C}$$

such that $f(cv) = \bar{c}f(v)$. If K acts on V as the sum of ε^k , on its Hermitian dual it also acts like that—in effect, because K is compact, one can impose on V a K -invariant Hermitian norm. I'll write Hermitian pairings as $u \bullet v$. For this the action of $\mathfrak{g}_{\mathbb{C}}$ satisfies

$$\pi(X)u \bullet v = -u \bullet \pi(\bar{X})v.$$

The Hermitian dual of FD_n is itself, the Hermitian dual of DS_n^{\pm} is itself, and the Hermitian dual of $PS_{\gamma,n}$ is $PS_{\bar{\gamma},n}$. A unitary representation is by definition isomorphic to its Hermitian dual, so a necessary condition that $PS_{\gamma,n}$ be unitary is that γ be real. It is not sufficient, since the Hermitian form this guarantees might not be positive definite.

8. Unitarity

Which of the representations above are unitary? I recall that (π, V) is isomorphic to its Hermitian dual if and only if there exists on V an Hermitian form which is $\mathfrak{g}_{\mathbb{R}}$ -invariant. For $SL_2(\mathbb{R})$ this translates to the conditions

$$\begin{aligned}\pi(\kappa)u \bullet u &= -u \bullet \pi(\kappa)u \\ \pi(x_+)u \bullet u &= -u \bullet \pi(x_-)u\end{aligned}$$

for all u in V . It is unitary if this form is positive definite:

$$u \bullet u > 0 \text{ unless } u = 0.$$

To determine unitarity, it suffices to construct the Hermitian form on eigenvectors of κ and find whether it is positive definite or not. We know from the previous section that π is its own Hermitian dual for all FD_n , DS_n^{\pm} , and for $PS_{\gamma,n}$ when γ is real. I'll summarize without proofs what happens in the first three cases: (1) FD_n is unitary if and only if $n = 0$ (the trivial representation). (2–3) Given a lowest weight vector v_n of DS_n^+ , there exists a unique invariant Hermitian norm on DS_n^+ such that $v_n \bullet v_n = 1$. Hence DS_n^+ is always unitary. Similarly DS_n^- . I leave verification of these claims as an exercise. The representations $PS_{\gamma,n}$ are more interesting. Let's first define the Hermitian form when γ is real.

The necessary and sufficient condition for the construction of the Hermitian form $v \bullet v$ is that

$$\pi(x_+)v_m \bullet v_{m+2} = -v_m \bullet \pi(x_-)v_{m+2}$$

for all m . This translates to

$$v_{m+2} \bullet v_{m+2} = -(\gamma - m^2/4 - m/2)v_m \bullet v_m.$$

So we see explicitly that γ must be real. But if the form is to be unitary, we require in addition that this coefficient be positive, requiring

$$4\gamma < (m+1)^2 - 1$$

for all m . Hence, taking $m = -1$ and $m = 0$:

8.1. Proposition. *The representation $PS_{\gamma,m}$ is unitary precisely when*

- (a) m is odd and $\gamma < -1/4$;
- (b) m is even and $\gamma < 0$.

This conclusion might seem a bit arbitrary, but we shall see later a clearer reason for it.

Part II. The principal series

Do the representations of (\mathfrak{g}, K) that we have constructed come from representations of G ?

9. Vector bundles

In the introduction I discussed the representation of G on $C^\infty(\mathbb{P})$. It is just one in an analytic family. The others are on spaces of sections of real analytic line bundles. This is such a common notion that I shall explain it here, at least in so far as it concerns us. There are two versions—one real analytic, the other complex analytic. Both are important in representation theory although the role of the bundle itself, as opposed to its space of sections, is often hidden in the mechanism of induction of representations.

Suppose G to be a Lie group, H a closed subgroup. For an example with some intuitive appeal, I'll take $G = SO(3)$, $H = SO(2)$. The group G acts by rotations on the two-sphere $S = \mathbb{S}^2$ in \mathbb{R}^3 , and H can be identified with the subgroup of elements of G fixing the north pole $P = (0, 0, 1)$. Elements of H amount to rotations in the (x, y) plane around the z -axis. The group G acts transitively on S , so the map taking g to $g(P)$ identifies G/H with S .

The group G also acts on tangent vectors on the sphere—the element g takes a vector v at a point x to a tangent vector $g_*(v)$ at the point $g(x)$. For example, H just rotates tangent vectors at P . This behaves very nicely with respect to multiplication on the group:

$$g_*h_*(v) = g_*(h_*(v)).$$

The point now is that vector fields on S may be identified with certain functions on G . Let T be the tangent space at P . If v is a tangent vector at $g(P)$, then $g_*^{-1}(v)$ is a tangent vector at P . If we are given a smoothly varying vector field on S —i.e. a tangent vector v_x at every point of the sphere, varying smoothly with x —then we get a smooth function Θ from G to T by setting

$$\Theta(g) = g_*^{-1}(v_{g(P)}).$$

Recall that $H = SO(2)$ fixes P and therefore acts by rotation on T . This action is just h_* , which takes T to itself. The function $\Theta(g)$ satisfies the equation

$$\begin{aligned} \Theta(gh) &= h_*^{-1}g_*^{-1}v_{gh(P)} \\ &= h_*^{-1}g_*^{-1}v_{g(P)} \\ &= h_*^{-1}g_*^{-1}v_{g(P)} \\ &= h_*^{-1}(\Theta(g)). \end{aligned}$$

It is straightforward to verify that *in this way smoothly varying vector fields on S may be identified with smooth functions $\Theta: G \rightarrow T$ such that $\Theta(gh) = h_*^{-1}\Theta(g)$.*

The space T_S of all tangent vectors on the sphere is a **vector bundle** on it. To be precise, every tangent vector is a pair (x, v) where x is on S and v is a tangent vector at x . The map taking (x, v) to x is a projection onto S , and the inverse image is the tangent space at x . There is no canonical way to choose coordinates on S , or to identify the tangent space at one point with that at another. But if we choose a coordinate system in the neighbourhood of a point x we may identify the local tangent spaces with copies of \mathbb{R}^2 . Therefore locally in the neighbourhood of every point the tangent bundle is a product of an open subset of S with \mathbb{R}^2 . These are the defining properties of a vector bundle.

A vector bundle of dimension n over a smooth manifold M is a smooth manifold B together with a projection $\pi: B \rightarrow M$, satisfying the condition that locally on M the space B and the projection π possess the structure of a product of a neighbourhood U and \mathbb{R}^n , together with projection onto the first factor.

The **fibre** over a point x of M is the inverse image $\pi^{-1}(x)$. A **section** of the bundle is a function $s: M \rightarrow B$ that assigns to every x a point in its fibre or, equivalently, such that $\pi(s(x)) = x$. For one example, smooth functions on a manifold are sections of the trivial bundle whose fibre at every point is just \mathbb{C} . For another, a vector field on S amounts to a section of its tangent bundle. A vector bundle is said to be trivial if it is isomorphic to a product of M and some fibre. One reason vector bundles are interesting is that they can be highly non-trivial. For example, the tangent bundle over S is definitely not trivial because every vector field on S vanishes somewhere, whereas if it were trivial there would be lots of non-vanishing fields. In other words, vector bundles can be *topologically* interesting.

Now let $H \subseteq G$ be an arbitrary closed subgroup. If (σ, U) is a smooth representation of H , we can define the **associated vector bundle** on $M = G/H$ to be the space B of all pairs (g, u) in $G \times U$ modulo the equivalence relation $(gh, \sigma^{-1}(h)u) \sim (g, u)$. This maps onto G/H and the fibre at the coset H is isomorphic to U . The space of all sections of B over M is then isomorphic to the space

$$\Gamma(M, B) = \{f: G \rightarrow U \mid f(gh) = \sigma^{-1}(h)f(g) \text{ for all } g \in G, h \in H\}.$$

This is itself a vector space, together with a natural representation of G :

$$L_g s(x) = s(g^{-1}(x)).$$

This representation is that **induced** by σ from H to G . In most situations it is not really necessary to consider the vector bundle itself, but just its space of sections. In some, however, knowing about the vector bundle is useful.

If G and H are complex groups and (σ, U) is a complex-analytic representation of H , one can define a complex structure on the vector bundle. In this case the space of holomorphic sections may be identified with the space of all complex analytic maps f from G to U satisfying the equations $f(gh) = \sigma^{-1}(h)f(g)$.

If the group G acts transitively on M , B is said to be a **homogeneous** bundle if G acts on it compatibly with the action on M . If x is a point of M , the isotropy subgroup G_x acts on the fibre U_x at x . The bundle B is then isomorphic to the bundle associated to G_x and the representation on U_x .

One example of topological interest will occur in the next section. Let sgn be the character of A taking a to $\text{sgn}(a)$. This lifts to a character of P , and therefore one can construct the associated bundle on $\mathbb{P} = P \backslash G$. Now \mathbb{P} is a circle, and this vector bundle is topologically a Möbius strip with this circle running down its middle.

One may define vector bundles solely in terms of the **sheaf** of their local sections. This turns out to be a surprisingly valuable idea, but I postpone explaining more about it until it will be immediately useful.

◊ In the next section I will follow a different convention than the one I follow here. Here, the group acts on spaces on the left, so if G acts transitively with isotropy group G_x the space is isomorphic to G/G_x . In the next section, it will act on the right, so the space is now $G_x \backslash G$. Sections of induced vector bundle then becomes functions $F: G \rightarrow U$ such that $F(hg) = \sigma(h)F(g)$. Topologists generally use the convention I follow in this section, but analysts the one in the next. Neither group is irrational or particularly stubborn—the intent of each is to avoid cumbersome notation. Topologists are interested in actions of groups on spaces, analysts in actions on functions.

One important line bundle on a manifold is that of **one-densities**. If the manifold is orientable, these are the same as differential forms of degree equal to the dimension of the manifold, but otherwise not. The point of one-densities is that one may integrate them canonically. Because they are important for representation theory, I say more in an appendix about one-densities on homogeneous spaces.

10. The principal series

In this essay, a **character** of any locally compact group H is a continuous homomorphism from H to \mathbb{C}^\times . Often in the literature this is called a quasi-character, and a character is what I call a **unitary character**, one whose image lies in the circle $|z| = 1$.

In particular, we have the **modulus character** of P

$$\delta = \delta_P: p \mapsto |\det_{\text{Ad}_n}(p)|, \quad \begin{bmatrix} t & x \\ 0 & 1/t \end{bmatrix} \mapsto t^2.$$

For any character χ of P define the smooth representation induced by it to be the right regular representation of G on

$$\text{Ind}^\infty(\chi) = \text{Ind}^\infty(\chi|P, G) = \{f \in C^\infty(G) \mid f(pg) = \chi\delta^{1/2}(p)f(g) \text{ for all } p \in P, g \in G\}.$$

Define $\text{Ind}(\chi) = \text{Ind}(\chi|P, G)$ (no superscript) to be the subspace of its K -finite vectors. The first is a smooth representation of G , the second a representation of (\mathfrak{g}, K) . We'll see later the point of the normalization factor $\delta^{1/2}$.

CHARACTERS OF P . Since

$$\begin{bmatrix} t & 0 \\ 0 & 1/t \end{bmatrix} \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/t & 0 \\ 0 & t \end{bmatrix} \begin{bmatrix} 1 & -x \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & (t^2 - 1)x \\ 0 & 1 \end{bmatrix},$$

the commutator subgroup of P is N . Therefore every character of P lifts from one of P/N , which is isomorphic to \mathbb{R}^\times .

10.1. Lemma. Any character of $\mathbb{R}_{>0}^\times$ is of the form $x \mapsto x^s$ for some unique $s \in \mathbb{C}$.

Proof. Working locally and applying logarithms, this reduces to the claim that any continuous additive function f from \mathbb{R} to itself is linear. For this, say $\alpha = f(1)$. It is easy to see that $f(m/n) = (m/n)\alpha$ for all integers $m, n \neq 0$, from which the claim follows by continuity. ▮

The characters of P are therefore all of the form

$$\chi = \chi_s \text{sgn}^n: \begin{bmatrix} t & x \\ 0 & 1/t \end{bmatrix} \mapsto |t|^s \text{sgn}^n(t).$$

The dependence on n is only through its parity, even or odd.

RESTRICTION TO K . It is natural to ask, what is the restriction of one of these induced representations to various subgroups of G ? At the moment, we'll look at the restriction of a representation $\text{Ind}^\infty(\chi)$ to K , and later on we'll look at its restriction to P .

10.2. Lemma. Every g in G can be factored uniquely as $g = nak$ with n in N , a in $|A|$, k in K . Explicitly:

$$(10.3) \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & (ac + bd)/r^2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} d/r & -c/r \\ c/r & d/r \end{bmatrix} \quad (r = \sqrt{c^2 + d^2}).$$

Here by $|A|$ I mean the connected component of A .

Proof. This is undoubtedly familiar, but I'll need the explicit factorization eventually. The key to deriving it is to let G act on the right on $\mathbb{P}^1(\mathbb{R})$ considered as non-zero row vectors modulo non-zero scalars. The group P is the isotropy group of the line $\langle\langle 0, 1 \rangle\rangle$, so it must be shown that K acts transitively. But both

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad k = \begin{bmatrix} d/r & -c/r \\ c/r & d/r \end{bmatrix}$$

take $\langle\langle 0, 1 \rangle\rangle$ to $\langle\langle c, d \rangle\rangle$. ▣

Because $G = PK$ and $K \cap P = \{\pm 1\}$, restriction to K induces a K -isomorphism of $\text{Ind}^\infty(\chi)$ with the space of smooth functions f on K such that $f(-k) = \chi(-1)f(k)$. Define the character

$$\varepsilon: \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \mapsto c + is$$

of K . According to the theory of Fourier series, the subspace $\text{Ind}(\chi)$ spanned by eigenfunctions of K is a direct sum $\oplus \varepsilon^{2k}$ if $n = 0$ and $\oplus \varepsilon^{2k+1}$ if $n = 1$. The functions

$$\varepsilon_s^m(g) = \chi_s \text{sgn}^n \delta^{1/2}(p) \varepsilon^m(k) \quad (g = pk, m \equiv n)$$

form a basis of the K -finite functions in $\text{Ind}(\chi_s \text{sgn}^n)$. This notation is justified since m determines n .)

DUALITY. There is no G -invariant measure on $P \backslash G$, but instead the things one can integrate are one-densities, which are for this P and this G the same as one-forms on $P \backslash G$. The space of smooth one-densities on $P \backslash G$ may be identified with the space of functions

$$\{f: G \rightarrow \mathbb{C} \mid f(pg) = \delta(p)f(g) \text{ for all } p \in P, g \in G\}.$$

This is not a canonical identification, but it is unique up to a positive scalar multiplication.

One often wants to integrate over $P \backslash G$ explicitly. As is explained in an appendix, there are two useful ways to do this. One is to interpret integration on $P \backslash G$ as integration over K :

$$\int_{P \backslash G} f(x) dx = \int_K f(k) dk.$$

If $\chi = \delta^{-1/2}$ then $\text{Ind}^\infty(\chi)$ is the space $C^\infty(P \backslash G)$ of smooth functions on $P \backslash G \cong \mathbb{P}_\mathbb{R}$, and for $\chi = \delta^{1/2}$ it is the space of smooth one-densities on $P \backslash G$, the linear dual of the first. More generally, if f lies in $\text{Ind}^\infty(\chi)$ and φ in $\text{Ind}^\infty(\chi^{-1})$ the product will lie in $\text{Ind}^\infty(\delta_P^{1/2})$ and the pairing

$$\langle f, \varphi \rangle = \int_{P \backslash G} f(x) \varphi(x) dx$$

determines an isomorphism of one space with the smooth dual of the other. In particular, if $|\chi| = 1$ so $\chi^{-1} = \bar{\chi}$ the induced representation is unitary—i.e. possesses a G -invariant positive definite Hermitian form. As we'll see in a moment, on these the Casimir acts as a real number in the range $(-\infty, -1/4]$. This accounts for some of the results in a previous section about unitary representations, but not all.

HOW THE LIE ALGEBRA ACTS. In terms of the basis of K -eigenvectors, how does (\mathfrak{g}, K) act? Here is the basic formula in this sort of computation:

10.4. Lemma. For X in \mathfrak{g} , g in G we have

$$(10.5) \quad [R_X f](g) = [L_{-gXg^{-1}} f](g).$$

10.6. Proposition. We have

$$\begin{aligned} R_\kappa \varepsilon_s^m &= mi \varepsilon_s^m \\ R_{x_+} \varepsilon_s^m &= (s + 1 + m) \varepsilon_s^{m+2} \\ R_{x_-} \varepsilon_s^m &= (s + 1 - m) \varepsilon_s^{m+2}. \end{aligned}$$

Proof. Apply Proposition 3.2. In our case $f = \varepsilon_s^m$ is the only K -eigenfunction f in the representation for ε^m with $f(1) = 1$, and we know that $\pi(x_+)$ changes weights by $2i$, so we just have to evaluate $R_{x_+}\varepsilon_s^m$ at 1. Since

$$2x_+ = \alpha - \kappa i - 2i\nu_+$$

we have

$$[R_{x_+}\varepsilon_s^m](1) = \chi_s \operatorname{sgn}^n(\alpha) + 1 - i(mi) = (s + 1 + n).$$

The formula for x_- is similar. ▮

10.7. Proposition. *The Casimir element acts on $\operatorname{Ind}(\chi_s \operatorname{sgn}^n)$ as $(s^2 - 1)/4$.*

Proof. This is a corollary of the previous result, but may also be seen more directly. Again, we just have to evaluate $(R_\Omega f)(1)$ for f in $\operatorname{Ind}(\chi_s \operatorname{sgn}^n)$:

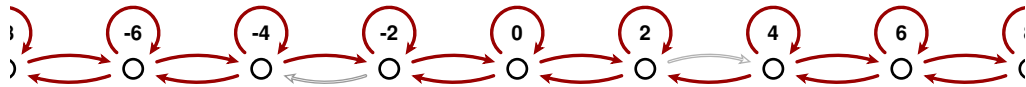
$$(R_\Omega f)(1) = (L_h^2/4 - L_h/2 + L_{\nu_+}R_{\nu_-})f(1) = (s + 1)^2/4 - (s + 1)/2 = s^2/4 - 1/4. \quad \text{▮}$$

These formulas have consequences for irreducibility. Let's look at one example, the representation $\pi = \operatorname{Ind}(\chi_s \operatorname{sgn}^n)$ with $n = 0, s = -3$. We have in this case

$$\begin{aligned} R_{x_+}\varepsilon_s^{2k} &= (-2 + 2k)\varepsilon_s^{2k+2} \\ R_{x_-}\varepsilon_s^{2k} &= (-2 - 2k)\varepsilon_s^{2k-2} \end{aligned}$$

We can make a labeled graph out of these data: one node μ_k for each even integer $2k$, an edge from μ_k to μ_{k+1} when x_+ does not annihilate ε_s^{2k} , and one from μ_k to μ_{k-1} when x_- does not annihilate ε_s^{2k} . The only edges missing are for x_+ when $n = 1$ and for x_- for $n = -1$.

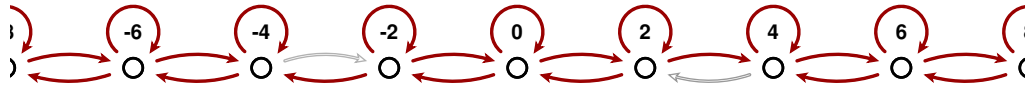
The graph we get is this:



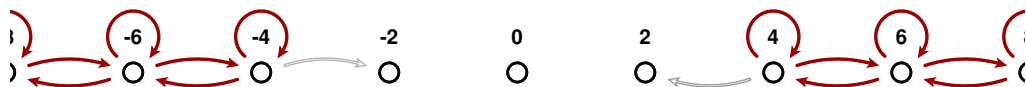
You can read off from this graph that the subgraph



represents an irreducible representation of dimension 3 embedded in π . The dual of π is $\tilde{\pi} = \operatorname{Ind}(\chi_{3,0})$, whose graph looks like this:



You can read off from this graph that the subgraph

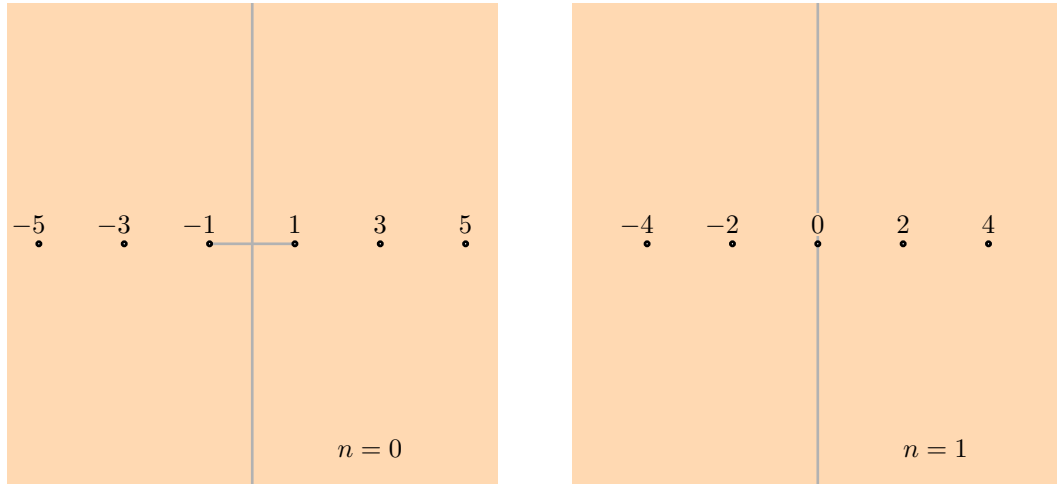


represents the sum of two irreducible representations contained in $\tilde{\pi}$, each of infinite dimension. We have similar pictures whenever s is an integer of the same parity as $n - 1$. In all other cases, the graph corresponding to $\text{Ind}(\chi_s \text{sgn}^n)$ connects all nodes in both directions. This leads to:

10.8. Proposition. *The representation $\text{Ind}(\chi_s \text{sgn}^n)$ is irreducible except when s is an integer m of the same parity as $n - 1$. In the exceptional cases:*

- (a) *if $s = -m \leq -1$, then $\text{Ind}(\chi_s \text{sgn}^{m-1})$ contains the unique irreducible representation of dimension m . The quotient is the direct sum of two infinite dimensional representation, D_{m+1}^- with weights $-(m + 1) - 2k$ for $k \geq 0$ and D_{m+1}^+ of weights $m + 1 + 2k$ with $k \geq 0$;*
- (b) *if $s = 0$ and $n = 1$ then $\text{Ind}(\chi_s \text{sgn}^n)$ itself is the direct sum of two infinite dimensional representation, D_1^- with weights $-2k - 1$ for $k \geq 0$ and D_1^+ of weights $2k + 1$ with $k \geq 0$;*
- (c) *for $s = m$ with $m \geq 1$ we have the decompositions dual to these.*

In these diagrams the points of irreducibility and the unitary parameters are shown:



Points of unitarity and reducibility of the principal series

11. Frobenius reciprocity and its consequences

For generic s the two representations $\text{Ind}(\chi_s \text{sgn}^m)$ and $\text{Ind}(\chi_{-s} \text{sgn}^m)$ are isomorphic. This section will explain this apparent accident, and in a way that perhaps makes it clear that a similar phenomenon will arise also for other reductive groups.

Suppose $H \subseteq G$ to be finite groups. If (σ, U) is a representation of H , then the representation induced by σ from H to G is the right regular representation of G on the finite-dimensional space

$$\text{Ind}(\sigma | H, G) = \{f: G \longrightarrow U \mid f(hg) = \sigma(h)f(g) \text{ for all } h \in H, g \in G\}.$$

The statement of Frobenius reciprocity in this situation is that a given irreducible representation π of G occurs as a constituent of $I = \text{Ind}(\sigma)$ as often as σ occurs in the restriction of π to H . Now representations of a finite group decompose into irreducible representations. Hence the number of occurrences of π in I is equal to the dimension of $\text{Hom}_G(V, I)$, and the number of occurrences of σ in π is likewise the dimension of $\text{Hom}_H(U, V)$. But because of semi-simplicity, this last is also the dimension of $\text{Hom}_H(V, U)$. Frobenius reciprocity is hence a consequence of an isomorphism of $\text{Hom}_G(V, I)$ with $\text{Hom}_H(V, U)$.

Such an isomorphism can be given explicitly. Let Λ_1 be the map taking f in $\text{Ind}(\sigma)$ to $f(1)$. It is an H -equivariant map from $\text{Ind}(\sigma)$ to U . If (π, V) is a representation of G and F a G -equivariant map from V to $\text{Ind}(\sigma)$, then the composition $\Lambda_1 \circ F$ is an H -equivariant map from V to U . This gives us a map

$$\text{Hom}_G(\pi, \text{Ind}(\sigma)) \longrightarrow \text{Hom}_H(\pi, \sigma).$$

Frobenius reciprocity asserts that this is an isomorphism. The proof simply specifies the inverse—to F on the right hand side we associate the map on the left taking v to $g \mapsto F(\pi(g)v)$.

Something similar holds for the principal series representations, but there are two versions, one for Ind^∞ and one for Ind . They begin in the same way. Define

$$\Lambda_1: \text{Ind}^\infty(\chi|P, G) \longrightarrow \mathbb{C}, \quad f \longmapsto f(1).$$

It is P -equivariant onto the one-dimensional representation $\chi\delta_P^{1/2}$.

11.1. Theorem. (Frobenius reciprocity for smooth representations) *If (π, V) is a smooth representation of G , composition with Λ_1 induces an isomorphism*

$$\text{Hom}_G(V, \text{Ind}^\infty(\chi)) \cong \text{Hom}_P(V, \mathbb{C}_{\chi\delta^{1/2}}).$$

Here the homomorphisms are taken to be continuous ones. The proof is essentially the same as for finite groups.

But we have also:

11.2. Theorem. (Frobenius reciprocity for admissible representations) *If (π, V) is an admissible representation of (\mathfrak{g}, K) , composition with Λ_1 induces an isomorphism*

$$\text{Hom}_{(\mathfrak{g}, K)}(V, \text{Ind}(\chi)) \cong \text{Hom}_{(\mathfrak{p}, K \cap P)}(V, \mathbb{C}_{\chi\delta^{1/2}}).$$

Proof. Here, because the group itself doesn't act, one has to be careful. The group K acts and $G = PK$, so we associate to F in $\text{Hom}_{(\mathfrak{p}, K \cap P)}(V, \mathbb{C}_{\chi\delta^{1/2}})$ the map from V to $\text{Ind}(\chi)$ taking v to the function

$$F_v(pk) = \chi\delta^{1/2}(p)F(\pi(k)v).$$

I leave it as an exercise to verify that this is inverse to composition with Λ_1 . ▣

So, how do we find maps

$$\Lambda: V \longrightarrow \mathbb{C} \text{ such that } \langle \Lambda, R_p f \rangle = \delta_P^{1/2} \chi(p) \langle \Lambda, f \rangle ?$$

The basic fact about such maps is that $\langle \Lambda, R_n f \rangle = \langle \Lambda, f \rangle$ for all n in N . Equivalently, $\langle \Lambda, R_x f \rangle = 0$ for x in the Lie algebra \mathfrak{n} . So we first look at all maps satisfying this property. The group A acts on it, since $aNa^{-1} = N$ for all a in A . Therefore the set of all maps annihilated by \mathfrak{n} is a representation of A . It will turn out in practice to be finite-dimensional, and we will want to see how A acts.

We are now interested in finding maps from $\text{Ind}(\chi)$ to some other $\text{Ind}(\rho)$, or for that matter from $\text{Ind}^\infty(\chi)$ to $\text{Ind}^\infty(\rho)$. What are the linear functions in the dual of $\text{Ind}^\infty(\chi)$ annihilated by \mathfrak{n} ? To answer this, we shall first say something about the restriction of $\text{Ind}^\infty(\chi)$ to P .

Let

$$w = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}.$$

The basic tool in understanding the restriction of $\text{Ind}^\infty(\chi)$ to P is this:

11.3. Lemma. (Bruhat decomposition) *The group G is the disjoint union of P and PwN . More explicitly, suppose*

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then if $c = 0$ the matrix g lies in P , and otherwise

$$g = \begin{bmatrix} 1/c & a \\ 0 & c \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & d/c \\ 0 & 1 \end{bmatrix}.$$

Proof. You can easily verify this. In order to discover the formula, you only had to check that

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & d/c \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & d/c \end{bmatrix}$$

takes $\langle\langle 0, 1 \rangle\rangle$ to $\langle\langle c, d \rangle\rangle$. ▣

Define $I_w(\chi)$ to be the subspace of functions f in $\text{Ind}^\infty(\chi)$ that vanish of infinite order along P , and define $I_1(\chi)$ to be the corresponding quotient. We have therefore a short exact sequence of P -stable spaces

$$0 \longrightarrow I_w(\chi) \longrightarrow \text{Ind}_w^\infty(\chi) \longrightarrow I_1(\chi) \longrightarrow 0.$$

We shall say more about this sequence later on. What does it imply about N -invariant distributions on $\text{Ind}^\infty(\chi)$? First all, the map Λ_1 is trivial on I_w , hence factors through I_1 . We shall see later a very explicit description of I_1 , and it will turn out that there may be other N -invariant elements in its dual.

It will also turn out, as we shall see later, that any f in $I_w(\chi)$ has the property that the function $f(wn)$ on N lies in the Schwartz space of N (which is isomorphic to \mathbb{R}). The integral

$$\langle \Lambda_w, f \rangle = \int_N f(wn) \, dn$$

therefore converges and defines an N -invariant linear function on $I_w(\chi)$. The natural question is, *does it extend to an N -invariant linear function on all of $\text{Ind}^\infty(\chi)$?*

11.4. Proposition. *The integral for Λ_w converges absolutely for any f in $\text{Ind}^\infty(\chi_s \text{sgn}^n)$ if $\text{RE}(s) > 0$, and defines a continuous linear functional such that $\Lambda_w(R_p f) = \chi_{-s, n}(p) \delta^{1/2}(p) \Lambda_w(f)$.*

Proof. We can apply the Iwasawa decomposition $G = PK$ explicitly to see that

$$\begin{aligned} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} &= \begin{bmatrix} 0 & -1 \\ 1 & x \end{bmatrix} \\ &= \begin{bmatrix} 1 & -x/(1+x^2) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} x/r & -1/r \\ 1/r & x/r \end{bmatrix} \quad (r = \sqrt{1+x^2}). \end{aligned}$$

Suppose f in $\text{Ind}(\chi_s \text{sgn}^n)$ with $\sup_K |f(k)| = M$. We then have

$$\left| \int_N f(wn) \, dn \right| = \left| \int_N f(wnw^{-1}w) \, dn \right| \leq M \int_{-\infty}^{\infty} (1+x^2)^{-(s+1)/2} \, dx.$$

The integrand is asymptotic to $|x|^{-s-1}$ as $|x| \rightarrow \infty$ and hence the integral converges for $\text{RE}(s) > 0$. Also:

$$\begin{aligned} \langle \Lambda_w, R_a f \rangle &= \int_N R_a f(wn) \, dn \\ &= \int_N f(wna) \, dn \\ &= \int_N f(wa \cdot a^{-1}na) \, dn \\ &= \delta_P(a) \int_N f(waw^{-1} \cdot wn) \, dn \\ &= \chi^{-1} \delta_P(a) \langle \Lambda_w, f \rangle. \end{aligned}$$
▣

The map

$$T_w: f \longrightarrow [g \mapsto \Lambda_w(R_g f)]$$

is therefore a continuous G -equivariant map from $\text{Ind}^\infty(\chi_s \text{sgn}^n)$ to $\text{Ind}^\infty(\chi_{-s} \text{sgn}^n)$ when the integral makes sense.

Can we extend Λ_w beyond the region $\text{RE}(s) > 0$? Let's see some explicit formulas. This map takes ε_s^n to some $c_{s,n} \varepsilon_{-s}^n$. What is the factor $c_{s,n}$? This is the same as $\Lambda_w(\varepsilon_s^n)$. Evaluating this amounts, essentially, to comparing the Bruhat and Iwasawa factorizations:

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & x \end{bmatrix} = \begin{bmatrix} 1/r & * \\ 0 & r \end{bmatrix} \begin{bmatrix} x/r & -1/r \\ 1/r & x/r \end{bmatrix} \quad (r = \sqrt{1+x^2}).$$

Therefore

$$\varepsilon_s^n(w_n) = \left(\frac{1}{\sqrt{x^2+1}} \right)^{s+1} \left(\frac{x+i}{\sqrt{x^2+1}} \right)^n,$$

and

$$(11.5) \quad c_{s,n} = \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{x^2+1}} \right)^{s+1} \left(\frac{x+i}{\sqrt{x^2+1}} \right)^n dx.$$

This integral is evaluated in an appendix in terms of Gamma functions. I'll look later at the case $|n| > 0$, but do $n = 0$ now, in which case we get

$$\begin{aligned} \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{x^2+1}} \right)^{s+1} dx &= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)} \\ &= \frac{\zeta_{\mathbb{R}}(s)}{\zeta_{\mathbb{R}}(s+1)} \end{aligned}$$

where I write here

$$\zeta_{\mathbb{R}}(s) = \pi^{-s/2} \Gamma(s/2)$$

as the factor of the Riemann ζ function contributed by the real place of \mathbb{Q} . It is not an accident that this factor appears in this form—a similar factor appears in the theory of induced representations of p -adic SL_2 , and globally these all contribute to a 'constant term' $\xi(s)/\xi(s+1)$ in the theory of Eisenstein series.

There is something puzzling about this formula. The poles of $c_{s,0}$ are at $s = 0, -2, -4, \dots$. The pole at $s = 0$ is related, as we shall see later, to the irreducibility of the unitary representation $\text{Ind}(\chi_{0,0})$. But there is nothing special about the principal series representations at the other points, so one must wonder, what is the significance of these poles? There is one interesting question that arises in connection with these poles. The poles are simple, and the residue of the intertwining operator at one of these is again an intertwining operator, also from $\text{Ind}(\chi_{s,n})$ to $\text{Ind}(\chi_{s,n}^{-1})$. What is that residue? It must correspond to an n -invariant linear functional on $\text{Ind}^\infty(\chi_{s,n})$. It is easy to see that in some sense to be explained later that the only n -invariant linear functional with support on PwN is Λ_w , so it must have support on P . It will be analogous to one of the derivatives of the Dirac delta. These matters will all be explained when I discuss the Bruhat filtration of principal series representations.

For $n = \pm 1$ we get

$$\begin{aligned} \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{x^2+1}} \right)^{s+1} \left(\frac{x \pm i}{\sqrt{x^2+1}} \right) dx &= \int_{-\infty}^{\infty} \frac{x \pm i}{(\sqrt{x^2+1})^{s+2}} dx \\ &= \int_{-\infty}^{\infty} \frac{\pm i}{(x^2+1)^{s/2+1}} dx \\ &= \pm i \cdot \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{s+2}{2}\right)} \\ &= \pm i \cdot \frac{\zeta_{\mathbb{R}}(s+1)}{\zeta_{\mathbb{R}}(s+2)}. \end{aligned}$$

We now know $c_{s,0}$ and $c_{s,\pm 1}$. The others can be calculated by a simple recursion, since

$$T_w \pi(x_+) \varepsilon_{s,n} = \pi(x_+) T_w \varepsilon_{s,n}, \quad (s+1+n)c_{s,n+2} = (-s+1+n)c_{s,n},$$

leading first to

$$\begin{aligned} c_{s,n+2} &= \frac{-s+1+n}{s+1+n} \cdot c_{s,n} \\ &= -\frac{s-(n+1)}{s+(n+1)} \cdot c_{s,n} \end{aligned}$$

and then upon inverting:

$$c_{s,n-2} = -\frac{s+(n-1)}{s-(n-1)} \cdot c_{s,n}.$$

Finally:

$$\begin{aligned} c_{s,2n} &= (-1)^n c_{n,0} \cdot \prod_{k=1}^n \frac{s-(2k+1)}{s+(2k+1)} \\ c_{s,-2n} &= (-1)^n c_{n,0} \cdot \prod_{k=1}^n \frac{s+(2k+1)}{s-(2k+1)}. \end{aligned}$$

Something similar holds for n odd. These intertwining operators give us explicit isomorphisms between generic principal series $\text{Ind}(\chi_s \text{sgn}^n)$ and $\text{Ind}(\chi_{-s} \text{sgn}^n)$.

There is one curious feature of these formulas—as $n \rightarrow \infty$ the ratio has as limit 1. I have no idea whether this is anything more than a curiosity.

This product formula for $c_{s,n}$ is simple enough, but other forms are also useful. The formula (11.5) can be rewritten as

$$c_{s,n} = \int_{\mathbb{R}} (1+ix)^{-(s+1-n)/2} (1-ix)^{-(s+1+n)/2} dx.$$

If we set $\alpha = (s+1-n)/2$ and $\beta = -(s+1+n)/2$ in Proposition 16.8, we see that this is

$$(11.6) \quad c_{s,n} = \frac{2^{1-s} \pi \Gamma(s)}{\Gamma\left(\frac{s+1+n}{2}\right) \Gamma\left(\frac{s+1-n}{2}\right)}.$$

One curious thing about this formula is that it is not immediately apparent that it agrees with the formula (11.5) for $c_{s,0}$. In effect, I have proved the Legendre duplication formula

$$2(2\pi)^{-s} \Gamma(s) = \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \cdot \pi^{-(1+s)/2} \Gamma\left(\frac{s+1}{2}\right) = \zeta_{\mathbb{R}}(s) \zeta_{\mathbb{R}}(1+s).$$

The left hand side here is called, with reason, $\zeta_{\mathbb{C}}(s)$. This is an analogue for \mathbb{C}/\mathbb{R} of the local Hasse-Davenport equation.

12. Intertwining operators and the smooth principal series

Proposition 11.4 tells us that the intertwining operator defined by \mathcal{I}_s converges in a right-hand half plane, for all functions in the smooth principal series. The explicit formulas in the previous section tell us that they continue meromorphically on the K -finite principal series. In this section I'll show that the meromorphic continuation is good on all of the smooth principal series.

There are two possible approaches, one by looking at K -finite functions, and the other by restricting functions in the principal series to the unipotent subgroup N° .

I'll be a bit vague about the first. The main point is that functions in the smooth principal series restrict to smooth functions on K , and may be expressed in terms of Fourier series with rapidly decreasing coefficients. Now

$$(12.1) \quad c_{s,n+2} = \frac{(n+1) - s}{(n+1) + s} \cdot c_{s,n},$$

and as a consequence $c_{s,n}$ is of moderate growth in n , more or less uniformly in s . Restriction to K is an isomorphism of $\text{Ind}^\infty(\chi_{s,n})$ with the space of smooth functions f on K such that $f(\pm k) = (-1)^n f(k)$, which is a fixed vector space independent of s . Let ι_s be this isomorphism. Thus for f on K

$$[\iota_s f](pk) = \chi_{s,n} \delta^{1/2}(p) f(k)$$

identically. The smooth vectors in $\text{Ind}(\chi_s \text{sgn}^n)$ may be expanded in Fourier series when restricted to K , which implies that the Fourier coefficients decrease rapidly with n , which implies what we want.

This proof uses the identification of $\text{Ind}^\infty(\chi_{s,n})$ as a space of functions on K , independent of s . Incidentally, the ratios in (12.1) tend to 1 as $n \rightarrow \infty$. Is this significant?

Let's look at the second approach. The Bruhat decomposition tells us that G is the union of two open sets PN° and PwN . Any function f in $\text{Ind}^\infty(\chi)$ may be expressed as a sum $f_1 + f_w$, each with corresponding compact support. The functional Λ_w evaluated on f_w is holomorphic in s , so all difficulties lie in the evaluation of

$$\langle \Lambda_w, f \rangle$$

where f has compact support on N° .

So say, for example, that

$$f(pn^\circ) = \chi_{s+1} f(n^\circ)$$

with f smooth on N° . Since

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1/x & -1 \\ 0 & x \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1/x & 1 \end{pmatrix},$$

The integral is

$$\int_{\mathbb{R}} |x|^{-s-1} f(1/x) dx = \int_{\mathbb{R}} |x|^s f(x) \frac{dx}{|x|}$$

which has indeed a meromorphic continuation.

13. The complementary series

We have seen that the imaginary line $\operatorname{RE}(s) = 0$ parametrizes unitary representations. In addition the trivial representation is unitary, and later on we'll see the holomorphic discrete series and—implicitly—the anti-holomorphic discrete series. There are, however, some more unitary representations to be found. These are the principal series $\operatorname{Ind}(\chi_{s,0})$ with s in the interval $(-1, 1)$, the so-called **complementary series** whose Casimir operator acts by a scalar in the range $(-1/4, 0)$.

The intertwining operator $T_{s,0}$ is multiplication by

$$c_{s,0} = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)}$$

which has a pole of order 1 at $s = 0$. The other constants are determined by the rule

$$c_{s,n+2} = \frac{-s+1+n}{s+1+n} \cdot c_{s,n}$$

$$c_{s,n-2} = \frac{-s+1-n}{s+1-n} \cdot c_{s,n}.$$

These both imply that the **normalized** operator $T_w/c_{s,0}$ has limit the identity operator as $s \rightarrow 0$. Furthermore, it is an isomorphism for all s in $(-1, 1)$. But in this interval the representation $\operatorname{Ind}(\chi_{-s,0})$ is the Hermitian as well as linear dual of $\operatorname{Ind}(\chi_{s,0})$, so T_w determines in effect an Hermitian inner product on $\operatorname{Ind}(\chi_{s,0})$. It does not change signature anywhere in the interval and since it is positive definite for $s = 0$ it is positive definite throughout. Therefore

13.1. Proposition. *For all s in $(-1, 1)$ the representation $\operatorname{Ind}(\chi_{s,0})$ is unitary.*

Of course we have only recovered in a different form what we deduced in an earlier section about unitary admissible modules over (\mathfrak{g}, K) .

14. Normalization

One major application of these intertwining operators is to questions of reducibility. The basic idea is to divide $T_{w,s}$ by some constant to get a normalized operator $\tau_{w,s}$ in such a way that

$$\tau_{w,-s}\tau_{w,s} = I$$

In particular, if $s = 0$ (or, equivalently, $\chi = w\chi$) then $\tau_{w,0}^2 = I$. It turns out that in fact these normalized operators are unitary for unitary characters, and span the space of endomorphisms commuting with G when $s = 0$. (This last point will be shown when we look at Verma modules.) They hence determine the irreducibility of $\operatorname{Ind}^\infty(\chi)$.

What happens is quite differently for the two cases $n = 0, n = 1$ separately.

$n = 0$. Define the normalized intertwining operator to be $\tau_w = T_w/c_{s,0}$. This is normalized so as to be 1 on the K -fixed vectors. In doing this, new poles, largely without significance, are introduced, but this does not happen in the neighbourhood of the unitary axis $\operatorname{RE}(s) = 0$. What happens is that $\tau_0 = I$, and this correlates with the fact that $\operatorname{Ind}(\chi_{0,0})$ is irreducible.

$n = 1$. Here, there is no single K -stable space distinguished in $\operatorname{Ind}(\chi)$. Instead, we divide by

$$\zeta_{\mathbb{R}}(s)/\zeta_{\mathbb{R}}(s+1).$$

This gives us

$$\tau_{0,1}\varepsilon_{s,\pm 1} = \pm i \cdot \varepsilon_{s,\pm 1}.$$

And this correlates with the decomposition of $\operatorname{Ind}(\chi)$ into two irreducible components. This is because if we divide τ by $\pm i$ we get $\tau^2 = I$, and the representation decomposes into its ± 1 -eigenspaces. There is more to this than might first appear, and I'll come back to this matter later.

15. Appendix. Characters as distributions

For any f in $C^\infty(\mathbb{R})$ define

$$\|f\|_{n,m} = \sup_{\mathbb{R}} (1 + |x|^n) |f^{(m)}(x)|.$$

We'll be interested in these Schwartz spaces:

$$\begin{aligned} \mathcal{S}(\mathbb{R}) &= \{f \in C^\infty(\mathbb{R}) \mid \|f\|_{n,m} < \infty \text{ for all } n, m \geq 0\} \\ \mathcal{S}[0, \infty) &= \text{restrictions of } f \in \mathcal{S}(\mathbb{R}) \text{ to } [0, \infty) \\ \mathcal{S}(\mathbb{R}^\times) &= \{f \in \mathcal{S}(\mathbb{R}) \mid f^{(n)}(0) = 0 \text{ for all } n \geq 0\}. \end{aligned}$$

In other words, $\mathcal{S}(\mathbb{R})$ is the space of smooth functions on \mathbb{R} all of whose derivatives vanish rapidly at infinity, and $\mathcal{S}(\mathbb{R}^\times)$ is the subspace of functions whose Taylor series vanishes 0.

Any character χ of \mathbb{R}^\times defines a distribution on the space $\mathcal{S}(\mathbb{R}^\times)$:

$$\langle \chi, f \rangle = \int_{\mathbb{R}} \chi(x) f(x) \frac{dx}{|x|},$$

and generically these extend to all of $\mathcal{S}(\mathbb{R})$. The integral

$$\int_0^\infty x^s f(x) \frac{dx}{x}$$

is defined and analytic in s for all $\text{Re}(s) > 0$, f in $\mathcal{S}[0, \infty)$. Integrating by parts gives us

$$\int_0^\infty x^s f(x) \frac{dx}{x} = -\frac{1}{s} \int_0^\infty x^{s+1} f'(x) \frac{dx}{x},$$

which means that the integral may be meromorphically continued to $\text{Re}(s) > -1$, with a possible pole at 0. The residue there is

$$-\int_0^\infty f'(x) dx = f(0).$$

This may be repeated many times to get

$$\int_0^\infty x^s f(x) \frac{dx}{x} = (-1)^n \cdot \frac{1}{s(s+1)\dots(s+n-1)} \int_0^\infty x^{s+n} f^{(n)}(x) \frac{dx}{x}.$$

for all n . The integral may therefore be extended to all of \mathbb{C} , with at most simple poles in $-\mathbb{N}$. The residue at $-n$ is $f^{(n)}(0)/n!$.

The integral

$$\int_{\mathbb{R}} |x|^s \text{sgn}^n(x) f(x) \frac{dx}{|x|} = \int_0^\infty |x|^s (f(x) + \text{sgn}^n(-1) f(-x)) \frac{dx}{x}$$

converges for $\text{Re}(s) > 0$, f in $\mathcal{S}(\mathbb{R})$. It may also be meromorphically extended to \mathbb{C} . Its poles depend on the parity of n . For $n = 0$ they are at $0, -2, -4, \dots$ while for $n = 1$ they are at $-1, -3, \dots$. The residues at poles are again scalar multiples of $f^{(n)}(0)$. At the points of $-\mathbb{N}$ where there are no poles, the values of the integral are principal values. For example, the distribution at $s = 0$ is

$$\lim_{\varepsilon \rightarrow 0} \int_{|x| \geq \varepsilon} \frac{f(x)}{x} dx = \int_0^\infty \frac{f(x) - f(-x)}{x} dx,$$

which makes sense because $(f(x) - f(-x))/x$ is smooth.

16. Appendix. The Gamma function

As we have seen, explicit formulas for intertwining operators reduce to classical integral formulas involving Gamma functions. In this section I'll review what will be needed. My principal reference for this brief account is Chapter VIII of [Schwartz:1965].

Basically, the Gamma function interpolates the factorial function $n!$ on integers to make a meromorphic function on all of \mathbb{C} . The definition of $\Gamma(s)$ for $\text{Re}(s) > 0$ is

$$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x} dx .$$

Integration by parts shows that

$$\Gamma(s + 1) = s\Gamma(s) .$$

This is an example of what we have seen in the previous appendix, since according to a well known lemma found originally in [Borel:1995] the function e^{-x} lies in $\mathcal{S}[0, \infty)$.

It is easy to calculate $\Gamma(1) = 1$, and then from this you can see that $\Gamma(n + 1) = n!$ for all positive integers n . This functional equation also allows us to extend the definition to all of \mathbb{C} with simple poles at the non-positive integers:

$$\Gamma(s) = \frac{\Gamma(s + 1)}{s} = \frac{\Gamma(s + 2)}{s(s + 1)} = \frac{\Gamma(s + 3)}{s(s + 1)(s + 2)} = \dots$$

The Gamma function is especially useful in evaluating certain Fourier and Laplace transforms. The integral

(16.1)
$$\int_0^\infty x^{s-1} f(x) dx$$

defines for generic s a tempered distribution with support on $[0, \infty)$.

16.2. Proposition. *The Laplace transform of x^{s-1} is $\Gamma(s)\lambda^{-s}$.*

Up to now I have referred to x^s only when x was real. But in this, λ is complex, with $\text{Re}(\lambda) > 0$. From now on I'll take $\log re^{i\theta}$ to be $\log r + i\theta$ whenever $\theta \in (-\pi, \pi)$, and λ^s to be defined whenever λ does not lie in $(-\infty, 0]$, in which case

$$\lambda^s = e^{s \log \lambda} .$$

Proof. The Laplace transform of x^{s-1} is

$$\int_0^\infty x^{s-1} e^{-x\lambda} dx .$$

Set $z = x\lambda$ to see that this is

$$\lambda^{-s} \int_0^{\lambda\infty} z^{s-1} e^{-z} dz .$$

A simple argument about contour integrals will show that this is the same as

$$\lambda^{-s} \int_0^\infty y^{s-1} e^{-y} dy = \lambda^{-s} \Gamma(s) .$$



Applying the inverse Laplace transform:

16.3. Corollary. *If $\sigma > 0$*

$$\frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \lambda^{-s} e^{\lambda x} d\lambda = \begin{cases} \frac{x^{s-1}}{\Gamma(s)} & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

A bit later we shall need this consequence:

16.4. Corollary. *The inverse Fourier transform of $(y - i)^{-\alpha}$ is*

$$\begin{cases} \frac{(2\pi ix)^\alpha e^{-2\pi x}}{x\Gamma(\alpha)} & x > 0 \\ 0 & \text{otherwise.} \end{cases} .$$

Proof. The inverse Fourier transform is

$$\begin{aligned} \int_{\mathbb{R}} \frac{e^{2\pi ixy} dy}{(y - i)^\alpha} &= \frac{i^\alpha e^{-2\pi x}}{i} \int_{1-i\infty}^{1+i\infty} \frac{e^{2\pi x\lambda} d\lambda}{\lambda^\alpha} \quad (-i\lambda = y - i, \lambda = 1 + iy) \\ &= (2\pi i)^\alpha e^{-2\pi x} \cdot \frac{1}{2\pi i} \int_{2\pi-i\infty}^{2\pi+i\infty} \frac{e^{\mu x} d\mu}{\mu^\alpha} \quad (\mu = 2\pi\lambda). \end{aligned}$$

But the integral is the formula for the inverse Laplace transform of $\mu^{-\alpha}$. ▮

16.5. Proposition. *We have*

$$\Gamma(s) = 2c^s \int_0^\infty y^{2s-1} e^{-cy^2} dy .$$

Proof. Set $x = cy^2$ in the definition of Γ . ▮

Recall the Beta function

$$B(u, v) = 2 \int_0^{\pi/2} \cos^{2u-1}(\theta) \sin^{2v-1}(\theta) d\theta .$$

16.6. Proposition. *We have*

$$B(u, v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)} .$$

Proof. Start with Proposition 16.5, $c = 1$. Moving to two dimensions and switching to polar coordinates:

$$\begin{aligned} \Gamma(u)\Gamma(v) &= 4 \int \int_{s, t \geq 0} e^{-s^2-t^2} s^{2u-1} t^{2v-1} ds dt \\ &= 4 \int \int_{r \geq 0, 0 \leq \theta \leq \pi/2} e^{-r^2} r^{2(u+v)-1} \cos^{2u-1} \theta \sin^{2v-1} \theta dr d\theta \\ &= \int_{r \geq 0} e^{-r^2} r^{2(u+v)-1} dr^2 \cdot 2 \int_0^{\pi/2} \cos^{2u-1}(\theta) \sin^{2v-1}(\theta) d\theta \\ &= \Gamma(u+v)B(u, v) . \end{aligned} \quad \text{▮}$$

16.7. Corollary. *We have*

$$\int_0^\infty \frac{t^\alpha}{(1+t^2)^\beta} \frac{dt}{t} = \frac{1}{2} \frac{\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\beta - \frac{\alpha}{2}\right)}{\Gamma(\beta)}$$

Proof. In the formula for the Beta function, change variables to $t = \tan(\theta)$ to get

$$\begin{aligned}\theta &= \arctan(t) \\ d\theta &= dt/(1+t^2) \\ \cos(\theta) &= 1/\sqrt{1+t^2} \\ \sin(\theta) &= t/\sqrt{1+t^2}\end{aligned}$$

leading to

$$\int_0^\infty \frac{t^{\alpha-1}}{(1+t^2)^\beta} dt = \frac{1}{2} \frac{\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\beta - \frac{\alpha}{2}\right)}{\Gamma(\beta)}.$$

In particular with $\alpha = 1$

$$\Gamma^2(1/2) = \int_{-\infty}^\infty \frac{dt}{1+t^2} = \pi, \quad \Gamma(1/2) = \sqrt{\pi}.$$

I learned a slightly different version of the following formula from [Garrett:2009], but Alok Shukla has pointed out to me that Garrett's version appears (without justification) on the last page of [Cauchy:1825]. He also came up with a proof along classical lines (see [Shukla:2016]).

Define

$$I_{\alpha,\beta} = \int_{\mathbb{R}} \frac{dy}{(y-i)^\alpha (y+i)^\beta}.$$

The integral converges for $\operatorname{Re}(\alpha + \beta) > 1$ and is analytic in that region.

16.8. Proposition. For $\operatorname{Re}(\alpha + \beta) > 1$

$$I_{\alpha,\beta} = \frac{2\pi i^{\alpha-\beta} \Gamma(\alpha + \beta - 1)}{2^{\alpha+\beta-1} \Gamma(\alpha) \Gamma(\beta)}.$$

As Garrett has pointed out, this is a consequence of the Plancherel formula for \mathbb{R} . It suffices to prove the formula when $\alpha > 1, \beta > 1$ are both real. But when α and β are real, this integral is

$$\int_{\mathbb{R}} a(y) \overline{b(y)} dx \quad \text{with} \quad a(y) = (y-i)^{-\alpha}, \quad b(y) = (y+i)^{-\beta}.$$

According to the Plancherel formula it is

$$\int_{\mathbb{R}} A(y) \overline{B(y)} dy = \int_{\mathbb{R}} a(x) \overline{b(x)} dx \quad (A(y) = \widehat{a}(y), \quad B(y) = \widehat{b}(y)).$$

By the Plancherel formula and Corollary 16.4, the integral is

$$\begin{aligned}\int_{\mathbb{R}} (y-i)^{-\alpha} (y+i)^{-\beta} dy &= \int_0^\infty (2\pi i x)^\alpha e^{-2\pi x/x} \Gamma(\alpha) (-2\pi i x)^\beta e^{-2\pi x/x} \Gamma(\beta) dx \\ &= \frac{2\pi i^{\alpha-\beta}}{\Gamma(\alpha) \Gamma(\beta)} \int_0^\infty u^{\alpha+\beta-1} e^{-2u} \\ &= \frac{2\pi i^{\alpha-\beta} \Gamma(\alpha + \beta - 1)}{2^{\alpha+\beta-1} \Gamma(\alpha) \Gamma(\beta)}.\end{aligned}$$

17. Appendix. Characters and the Fourier transform

We have seen that a generic multiplicative character is a tempered distribution, which is to say a continuous linear function from $\mathcal{S}(\mathbb{R})$ to \mathbb{C} . The Fourier transform

$$\widehat{f}(y) = \int_{\mathbb{R}} f(x) e^{-2\pi ixy} dx$$

is an isomorphism of $\mathcal{S}(\mathbb{R})$ with itself. Proving this reduces to:

17.1. Lemma. *The Fourier transform of $f'(x)$ is $2\pi iy\widehat{f}(y)$.*

Proof. Because

$$\int_{\mathbb{R}} f'(x) e^{-2\pi ixy} dx = (2\pi iy) \int_{\mathbb{R}} f(x) e^{-2\pi ixy} dx = (2\pi iy)\widehat{f}(y).$$

Any function of the form $P(x)e^{-cx^2}$ with $c > 0$ and P a polynomial lies in $\mathcal{S}(\mathbb{R})$.

17.2. Lemma. *Under the Fourier transform*

- (a) $f(x) = e^{-\pi x^2}$ is taken to $f(x)$;
- (b) $f(x) = xe^{-\pi x^2}$ is taken to $-if(x)$.

Proof. If $f(x) = e^{i\pi x^2}$ then

$$\begin{aligned} \widehat{f}(y) &= \int_{\mathbb{R}} e^{-\pi x^2} e^{-2\pi ixy} dx \\ &= e^{-\pi y^2} \int_{\mathbb{R}} e^{-\pi(x+iy)^2} dx \\ &= e^{-\pi y^2} \int_{\mathbb{R}+iy} e^{-\pi z^2} dz \\ &= e^{-\pi y^2} \int_{\mathbb{R}} e^{-\pi z^2} dz \\ &= e^{-\pi y^2}. \end{aligned}$$

Apply Lemma 17.1 to see that the transform of $-2\pi x e^{-\pi x^2}$ is $2\pi iy e^{-\pi y^2}$ and that of $x e^{-\pi x^2}$ is $-iy e^{-\pi y^2}$.

If f and F are both in $\mathcal{S}(\mathbb{R})$ then

$$\int_{\mathbb{R}} f(x) \widehat{F}(x) dx = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x) F(y) e^{-2\pi ixy} dx dy = \int_{\mathbb{R}} \widehat{f}(y) F(y) dy.$$

It is hence natural, even inevitable, to extend the Fourier transform to tempered distributions by duality:

$$\langle \widehat{F}, f \rangle = \langle F, \widehat{f} \rangle.$$

What is the Fourier transform of the distribution defined by χ ?

The multiplicative group acts on $\mathcal{S}(\mathbb{R})$ by the right regular representation:

$$[\mu_c f](x) = f(cx).$$

It acts on distributions by duality:

$$\langle \mu_c F, f \rangle = \langle F, \mu_{1/c} f \rangle.$$

$$\mu_c \chi = \chi(c) \chi.$$

How do the Fourier transform and μ_c interact?

17.3. Lemma. For $c \neq 0$

$$\widehat{\mu_c F} = |c| \mu_{1/c} \widehat{F} .$$

Proof.

$$[\widehat{\mu_c F}](y) = \int_{\mathbb{R}} F(x/c) e^{-2\pi i x y} dx = |c| \int_{\mathbb{R}} F(u) e^{-2\pi i c u y} du .$$

For any multiplicative character χ the dimension of distributions F such that $\mu_c F = \chi(c)F$ for all $c \neq 0$ is one.

From now on, let $|\bullet|$ be the modulus character $x \mapsto |x|$. When χ and $|\bullet| \chi^{-1}$ are defined as distributions, $\widehat{\chi}$ is a scalar multiple of $|\bullet| \chi^{-1}$.

17.4. Theorem. Under the Fourier transform

- (a) $|\bullet|^s$ is taken to $\pi^{-s/2} \Gamma(s/2) \cdot |\bullet|^{1-s}$;
- (a) $\text{sgn} |\bullet|^s$ is taken to $\pi^{-s+1/2} \Gamma(s+1/2) \cdot \text{sgn} |\bullet|^{1-s}$;

Proposition 16.5 tells us that

$$\pi^{-s/2} \Gamma(s/2) = \int_{\mathbb{R}} |x|^{s-1} e^{-\pi x^2} dx .$$

Set

$$\chi^\vee = |\bullet| \chi^{-1} .$$

When things go wrong: the $\delta^{(n)}$ and principal value integrals. The distribution δ_0 corresponds to $\chi = 1$, while the function 1 corresponds to $|\bullet|$. These are transforms of each other. This fits since $1 = |\bullet| \cdot |\bullet|^{-1}$, and $|\bullet| = |\bullet| \cdot 1$.

Tate's local functional equation

$$\frac{\langle \chi, \widehat{f} \rangle}{L(\chi, s)} = \gamma(\chi, \psi) \cdot \frac{\langle \chi^\vee, f \rangle}{L(\chi^\vee, s)}$$

It is best to keep the dependence on ψ in mind—when ψ changes to ψ_c ?

Remark about global situation, Eisenstein series and functional equation.

18. Appendix. Invariant integrals on quotients

Suppose G to be a locally compact group, $d_\ell x$ a Haar (positive, left-invariant) measure. This measure might not be right-invariant, but for any g in G the measure $d_\ell xg$ defined by

$$\int_G f(x) d_\ell xg = \int_G f(xg^{-1}) d_\ell g$$

is also left-invariant, and hence a scalar multiple of $d_\ell x$. In other words

$$d_\ell xg = \delta_G^{-1}(g) d_\ell x$$

for some non-zero scalar $\delta_G(g)$, which defines a continuous homomorphism from G to $\mathbb{R}_{>0}^\times$. It is called the **modulus character** of G , and it also satisfies the equation $d_\ell g x g^{-1} = \delta_G(g) d_\ell x$.

One consequence of the definition is that to every left-invariant measure $d_\ell x$ is associated the right-invariant measure $d_r x = \delta_G(x) d_\ell x$.

Now suppose $H \subseteq G$ to be a closed subgroup. The quotient $H \backslash G$ will not in general possess a G -invariant measure, but this failure is easily dealt with.

Recall that a measure of compact support on a locally compact space is a continuous linear function on the space $C(G)$ of continuous \mathbb{C} -valued functions. Define $\Omega_c(H \backslash G)$ to be the space of continuous measures of compact support on $H \backslash G$. When G is a Lie group, elements of this space are also called one-densities on $H \backslash G$, and if $H \backslash G$ is oriented this is the same as the space of continuous forms of highest degree and of compact support. The assertion above means that $\Omega_c(H \backslash G)$ is not generally, as a G -space, isomorphic to $C_c(H \backslash G)$. But there exists a simple modification, which amounts to identifying the space of one-densities as sections of a line bundle on $H \backslash G$. If χ is a character of H , define the induced representation

$$\begin{aligned} \text{Ind}_c(\chi) &= \text{Ind}_c(\chi|_H, G) \\ &= \{f \in C(G) \text{ of compact support modulo } H \mid f(hg) = \chi(h)f(g) \text{ for all } h \in H, g \in G\}. \end{aligned}$$

The group G acts on this by right multiplication. If dh is a Haar measure on H , χ is a character of H , and $f \in C_c(G)$ then the integral

$$(18.1) \quad F(g) = \int_H \chi(h)f(hg) dh$$

will satisfy

$$\begin{aligned} F(xg) &= \int_H \chi(h)f(hxg) dh \\ &= \int_H \chi(yx^{-1})f(yg) d yx^{-1} \\ &= \delta_H(x)\chi^{-1}(x)F(g) \end{aligned}$$

for all x in H , and therefore lies in $\text{Ind}_c(\delta_H \chi^{-1})$. The following is found in §9 of [Weil:1965]:

18.2. Lemma. *The map from $C_c(G)$ to $\text{Ind}_c(\delta_H \chi^{-1})$ taking $f \mapsto F$ defined by (18.1) is surjective.*

Spaces of induced representations are essentially sections of line bundles. Given what happens for oriented manifolds $H \backslash G$, the following should not be too surprising. For f in $C_c(G)$ define

$$\bar{f}(g) = \int_H \delta_G(h)f(hg) dh.$$

It lies in $\text{Ind}_c(\delta_H / \delta_G)$.

18.3. Proposition. *Suppose given Haar measures dh, dg on H, G . There exists a unique G -invariant linear functional I on $Ind_c(\delta_H/\delta_G)$ such that*

$$\int_G \delta_G(g) f(g) dg = \langle I, \bar{f} \rangle.$$

As a consequence, the choices of dh, dg determine an isomorphism of $Ind_c(\delta_H/\delta_G)$ with $\Omega_c(H \backslash G)$.

The proof proceeds by applying Lemma 18.2 to find for any f in $Ind_c(\delta_H/\delta_G)$ a function φ in $C_c(G)$ such that $\bar{\varphi} = f$, then verifying that

$$\int_G \delta_G(g) \varphi(g) dg$$

depends only on f .

I shall sketch here the proof only in a special case of interest to us. Suppose G to be unimodular, which means that $\delta_G = 1$. Suppose also that there exists a compact subgroup K such that $G = HK$. In particular, $H \backslash G$ is compact. This happens, for example, if G is a reductive Lie group and H is a parabolic subgroup. I claim:

18.4. Proposition. *Integration over K is a G -invariant integral on $Ind(\delta_H)$.*

Proof. Assign K total measure 1. The integral

$$f^K(g) = \int_K f(gk) dk$$

defines a projection of $C(G)$ onto $C(G/K)$, which possesses a measure $I_{G/K}$ that is left G -invariant, unique up to scalar multiple. But $H/H \cap K = G/K$ since $G = HK$, and G/K therefore also possesses an essentially unique H -invariant measure. It must be also G -invariant. Hence on the one hand

$$\int_G f(g) dg = \int_G f^K(g) dg = \langle I_{G/K}, f^K \rangle$$

and on the other

$$\int_H f^K(h) dh = \langle I_{G/K}, f^K \rangle.$$

Therefore

$$\int_G f(g) dg = \int_{H \times K} f(hk) dh dk = \int_K \bar{f}(k) dk.$$

We deduce:

18.5. Proposition. (Iwasawa integration formula) *Suppose G to be a reductive group defined over a local field, $P = MN$ a parabolic subgroup. Then with a suitable choice of Haar measures*

$$\int_G f(g) dg = \int_K \int_M \delta_P^{-1}(m) \left(\int_N f(nmk) dn \right) dm dk.$$

The character δ_P is the modulus character of P . In terms of vector bundles, it is the factor defining the vector bundle induced by the absolute value of the action of P on the forms of highest degree on the tangent space at 1 of $P \backslash G$. Thus

$$\delta_P(m) = |\det \text{Ad}_{\mathfrak{n}^\circ}(m)|^{-1} = |\det \text{Ad}_{\mathfrak{n}}(m)|.$$

Part III. The discrete series

Initially I described the irreducible admissible representations DS_n^\pm by a formal algebraic procedure, following methods not so different from Bargmann's. Later, I showed that the representations DS_n^\pm are embedded into certain principal series representations. In this part, I want to exhibit new realizations of the representations DS_n^\pm in terms of certain holomorphic vector bundles on which $G = SL_2(\mathbb{R})$ acts. It is this feature that explains exactly why these representations are relevant to the theory of holomorphic automorphic forms.

I'll also explain how they may be embedded as discrete summands of $L^2(G)$.

19. Projective spaces and representations

I can motivate the constructions to come by giving a new characterization of the principal series. Let

$$\mathfrak{X}_{\mathbb{R}} = \mathbb{R}^2 - \{0\}.$$

I recall that the projective line $\mathbb{P}(\mathbb{R})$ is defined as the space of lines in \mathbb{R}^2 , which may be identified with the quotient $\mathfrak{X}_{\mathbb{R}}/\mathbb{R}^\times$. The group G acts on $\mathfrak{X}_{\mathbb{R}}$ by linear transformations, and this induces in the natural way its action on $\mathbb{P}(\mathbb{R})$. If χ is a character of \mathbb{R}^\times , a function f on $\mathfrak{X}_{\mathbb{R}}$ is said to be χ -homogeneous if

$$f(\lambda v) = \chi(\lambda)f(v)$$

for all v in $\mathfrak{X}_{\mathbb{R}}$ and λ in \mathbb{R}^\times .

Recall that P is the group of upper triangular matrices in G and N is its subgroup of unipotent matrices. The group N takes the point $(1, 0)$ to itself, and therefore any function f on $\mathfrak{X}_{\mathbb{R}}$ gives rise to a function F on $N \backslash G$ according to the formula

$$F(g) = f\left(g^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right).$$

I leave as an exercise:

19.1. Proposition. *The map taking f on $\mathfrak{X}_{\mathbb{R}}$ to F on $N \backslash G$ is a G -equivariant isomorphism of the smooth functions on $\mathfrak{X}_{\mathbb{R}}$ that are χ -homogeneous with the smooth principal series $\text{Ind}(\chi^{-1}\delta^{-1/2})$.*

At first this might seem an uninteresting result but, among other things, it makes it immediately clear that certain principal series representations contain the irreducible finite-dimensional representations of G , since if $\chi(x) = x^m$ for a non-negative integer m the space of homogeneous functions contains the homogeneous polynomials of degree m in \mathbb{R}^2 , a space on which G acts as it does on FD_m . This result defines the smooth principal series representations without making a choice of P —it is, in effect, a coordinate-free treatment. Proposition 19.1 will play a role in what's to come, but for the moment it will serve mostly as motivation for the next step.

Now let

$$\begin{aligned} \mathfrak{X}_{\mathbb{C}} &= \mathbb{C}^2 - \{0\} \\ \mathbb{P}(\mathbb{C}) &= \text{the space of complex lines in } \mathbb{C}^2 \\ &= \mathfrak{X}_{\mathbb{C}}/\mathbb{C}^\times. \end{aligned}$$

We shall again be interested in functions on certain subsets of $\mathfrak{X}_{\mathbb{C}}$ that are homogeneous. But there will be a new restriction—I want to look mostly at functions that are holomorphic. This requires that the character χ be holomorphic as well, which in turn means that $\chi(\lambda) = \lambda^m$ with m now an arbitrary integer.

The group G acts on $\mathfrak{X}_{\mathbb{C}}$ by linear transformations. Since $SL_2(\mathbb{R})$ is connected, it must take each component of the complement of $\mathbb{P}(\mathbb{R})$ in $\mathbb{P}(\mathbb{C})$ into itself. We can be more explicit.

19.2. Lemma. *If*

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(\mathbb{R}) \text{ then } \text{Im}(g(z)) = \frac{\det(g)}{|cz + d|^2}.$$

Proof. Straightforward calculation:

$$\begin{aligned} \operatorname{IM} \left(\frac{az + b}{cz + d} \right) &= \frac{1}{2i} \left(\frac{az + b}{cz + d} - \frac{a\bar{z} + b}{c\bar{z} + d} \right) \\ &= \frac{1}{|cz + d|^2} \operatorname{IM}(z). \end{aligned}$$

Let \mathfrak{X}^* be the inverse image in $\mathfrak{X}_{\mathbb{C}}$ of the complement \mathcal{H}^{\pm} of $\mathbb{P}(\mathbb{R})$, where $\operatorname{IM}(z/w) \neq 0$. It has two components

$$\mathfrak{X}^{\pm} = \{(z, w) \mid \operatorname{sign} \operatorname{IM}(z/w) = \pm\}.$$

As a consequence of Lemma 19.2, the group G takes each component \mathfrak{X}^{\pm} into itself.

The group G acts on the space of smooth functions on \mathfrak{X}^* :

$$F \longmapsto L_g F, \quad [L_g F](x) = F(g^{-1}x).$$

The space \mathfrak{X}^* is an open subset of $\mathfrak{X}_{\mathbb{C}}$, hence possesses an inherited complex structure. Since G preserves this complex structure, it takes the subspace of holomorphic functions on \mathfrak{X}^* into itself. Define

$$\begin{aligned} C_m^{\infty} &= \{f \in C^{\infty}(\mathfrak{X}^*) \mid f(\lambda z) = \lambda^m f(z)\} \\ \mathcal{C}_m &= \text{holomorphic functions in } C_m^{\infty}. \end{aligned}$$

I. e. the first is the space of smooth functions on \mathfrak{X}^* that are homogeneous of degree m with respect to \mathbb{C}^{\times} . In effect, the functions in C_m^{∞} (respectively \mathcal{C}_m) are smooth (holomorphic) sections of a holomorphic line bundle on the complement \mathcal{H}^* of $\mathbb{P}(\mathbb{R})$ in $\mathbb{P}(\mathbb{C})$. Since G commutes with scalar multiplication it takes both C_m^{∞} and \mathcal{C}_m to themselves. Let π_m be the representation of G on \mathcal{C}_m .

If $m \geq 0$ this space contains the finite-dimensional subspace of homogeneous polynomials of degree m . In particular, it acts on \mathcal{C}_1 , which contains the two coordinate functions

$$(z, w) \mapsto z, w.$$

But I'll usually assume from now on that $m < 0$. Since \mathfrak{X}^* is the union of its two connected components \mathfrak{X}^{\pm} , π_m is the direct sum of two components π_m^{\pm} . The representation π_m^{-} may also be interpreted as a representation of G on *anti-holomorphic* functions on \mathfrak{X}^+ .

20. Restriction to K

What is the restriction to K of the representations π_m^{\pm} ? What are its eigenfunctions?

The basic formula is that

$$[\pi_m(g)F] \left(\begin{bmatrix} z \\ w \end{bmatrix} \right) = F \left(\begin{bmatrix} az + bw \\ cz + dw \end{bmatrix} \right) \quad \text{if } g^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

From this it follows that the functions $z \pm iw$ of degree one are eigenfunctions of K with eigencharacters $\varepsilon^{\mp 1}$.

On \mathfrak{X}^+ the function $(z + iw)^{-1}$ is well defined. The function

$$(20.1) \quad \frac{(z - iw)^p}{(z + iw)^{p+m}}$$

lies in C_{-m}^+ and is an eigenfunction of K with eigencharacter ε^{2p+m} . Similarly, the function

$$(20.2) \quad \frac{(z + iw)^p}{(z - iw)^{p+m}}$$

lies in C_{-m}^- and is an eigenfunction of K with eigencharacter $\varepsilon^{-(2p+m)}$.

20.3. Theorem. *The functions in (20.1) and (20.2) exhaust the eigenfunctions of K in π_{-m} .*

I'll postpone the proof to the next section. As a consequence:

20.4. Corollary. *The representation π_{-n}^\pm for $n > 0$ is isomorphic to DS_n^\pm .*

Note that I have used only the characterization of DS_n^\pm by K -spectrum, and not specified the way in which \mathfrak{sl}_2 acts in π_m^\pm so as to obtain an explicit isomorphism.

21. In classical terms

The projection from $\mathfrak{X}_{\mathbb{C}}$ to $\mathbb{P}(\mathbb{C})$ taking $(u, v) \mapsto u/v$ specifies $\mathfrak{X}_{\mathbb{C}}$ as a fibre space over $\mathbb{P}(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$ with fibre equal to \mathbb{C}^\times .

21.1. Lemma. *The map taking z to the point $(z, 1)$ in $\mathfrak{X}_{\mathbb{C}}$ is a section of this projection over the embedded copy of \mathbb{C} .*

This section does not extend continuously to ∞ . Since the group G commutes with scalar multiplication, it preserves the fibring over \mathbb{C} . Elements of G do not take the image of the section into itself. Formally, we have

$$g \left(\begin{bmatrix} z \\ 1 \end{bmatrix} \right) = \begin{bmatrix} az + b \\ cz + d \end{bmatrix} = (cz + d) \begin{bmatrix} (az + b)/(cz + d) \\ 1 \end{bmatrix} = j(g, z) \begin{bmatrix} g(z) \\ 1 \end{bmatrix} \quad \text{if } g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Here $g(z) = (az + b)/(cz + d)$ is the traditional linear fractional action. The term $j(g, z) = cz + d$, when it is non-zero, is called the **automorphy factor**. It does not in fact vanish as long as $\text{IM}(z) \neq 0$ and g is real. This is consistent with the fact that $GL_2(\mathbb{R})$ takes the image of $\mathbb{P}(\mathbb{R})$ into itself, hence also its complement. At any rate, from this formula follows immediately the first of our elementary results:

21.2. Lemma. *Whenever both terms on the right hand side are non-zero we have*

$$j(gh, z) = j(g, h(z))j(h, z).$$

A function in C_m^∞ is determined by its restriction to the embedded copy of \mathcal{H}^* in \mathfrak{X}^* . The spaces $C_m^\infty(\mathcal{C}_m)$ may therefore be identified, respectively, with the space of all smooth functions (holomorphic) functions on \mathcal{H}^* . If F lies in C_m^∞ define

$$f(z) = F \left(\begin{bmatrix} z \\ 1 \end{bmatrix} \right).$$

Of course F is holomorphic if and only if f is. We can recover F from f :

$$F \left(\begin{bmatrix} z \\ w \end{bmatrix} \right) = w^m F \left(\begin{bmatrix} z/w \\ 1 \end{bmatrix} \right) = w^m f(z/w).$$

How does the action of G on F translate to an action of G on f ?

$$\begin{aligned} [\pi_m(g)F] \left(\begin{bmatrix} z \\ 1 \end{bmatrix} \right) &= F \left(\begin{bmatrix} az + b \\ cz + d \end{bmatrix} \right) \quad \text{where } g^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ &= (cz + d)^m F \left(\begin{bmatrix} (az + b)/(cz + d) \\ 1 \end{bmatrix} \right) \\ &= (cz + d)^m f \left(\frac{az + b}{cz + d} \right) \\ &= j(g^{-1}, z)^m f(g^{-1}(z)). \end{aligned}$$

Hence:

21.3. Proposition. For f in C_m^∞ identified with a function on $\mathbb{C} - \mathbb{R}$

$$[\pi_m(g)f](z) = j(g^{-1}, z)^m f(g^{-1}(z)).$$

I now take up the proof of Theorem 20.3. It will suffice to deal with π_n^+ .

In considering the action of G on \mathcal{H} when restricted to K , it is often a good idea to work with the conjugate action of G on the unit disk, which is obtained via the Cayley transform. We do that here. Recall that

$$C(z) = \frac{z-i}{z+i}, \quad C^{-1}(z) = i \cdot \frac{1+z}{1-z}$$

corresponding to matrices

$$\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}, \quad \frac{1}{2i} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

Thus for z in \mathbb{D} , g in $SL_2(\mathbb{R})$, g takes

$$z \mapsto (C \cdot g \cdot C^{-1})(z),$$

and in particular

$$\begin{bmatrix} c & -s \\ s & c \end{bmatrix} \mapsto \begin{bmatrix} c-is & 0 \\ 0 & c+is \end{bmatrix}.$$

We can define a representation of G on the inverse image of \mathbb{D} in $\mathfrak{X}_{\mathbb{C}}$ according to the formula

$$\rho_m(g) = (C^{-1})^* \pi_m(g) C^* \quad ([C^* f](z) = f(C(z))) j(C, z)^m.$$

The product formula equation for the automorphy factor tells us that this also has the form

$$[\rho_m(g)f](z) = f(g^{-1}(z)) j(g^{-1}, z)^m,$$

In this realization, K acts by 'twisted' rotations around the origin. More explicitly, an eigenfunction f for character ε^ℓ must satisfy the equation

$$[\rho_m(k)f](z) = \varepsilon^\ell f(z) = f(k^{-1}(z)) j(k^{-1}, z)^m$$

for all k in K . But in this realization K is represented by matrices

$$r_\theta = \begin{bmatrix} 1/u & 0 \\ 0 & u \end{bmatrix} \quad (u = e^{i\theta}).$$

Since $k^{-1}z = u^2z$ and $j(k, z) = u^{-1}$, f must satisfy the equation

$$f(u^2z)u^{-m} = u^\ell f(z), \quad f(u^2z) = u^{\ell+m} f(z).$$

Here $|z| < 1$. If we set $u = -1$ in this equation we see that $m + \ell$ must be even, say $\ell + m = 2p$. If we set $z = c$ with $0 < c < 1$ and $u = e^{i\theta/2}$ this equation gives us $f(ce^{i\theta}) = e^{ip\theta} f(c)$. The following is a basic fact about holomorphic functions:

21.4. Lemma. If $f(z)$ is holomorphic in the unit disk $|z| < 1$ and

$$f(ce^{i\theta}) = e^{ip\theta}$$

for some $0 < c < 1$ then $f(z) = (z/c)^p$.

Proof. Apply Cauchy's integral formula:

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{|\zeta|=c} \frac{f(\zeta)}{\zeta - z} d\zeta \\ &= \frac{c}{2\pi} \int_0^{2\pi} \frac{f(ce^{i\theta})}{ce^{i\theta} - z} e^{i\theta} d\theta \\ &= \frac{f(c)}{2\pi} \int_0^{2\pi} \frac{e^{ip\theta}}{1 - (z/c)e^{-i\theta}} d\theta. \end{aligned}$$

Now express the integral as a geometric series, and integrate term by term. ▮

In other words, the eigenfunctions of K in the representation on \mathbb{D} are the monomials z^p , with eigencharacter ε^{2p-m} (which is positive, since I have assumed $m < 0$). Reverting to the action on \mathcal{H} , we conclude the proof of Theorem 20.3. ▮

22. Relation with functions on the group

Suppose F to be in $C_m^\infty(\mathfrak{X}^+)$. Let $f(z)$ be the corresponding function on \mathcal{H} , the restriction of F to the image of \mathcal{H} in \mathfrak{X} . Define the function $\Phi = \Phi_F$ on G by the formula

$$(22.1) \quad \Phi_F(g) = F\left(g \begin{bmatrix} i \\ 1 \end{bmatrix}\right).$$

This section will be devoted to proving:

22.2. Theorem. *The map $F \mapsto \Phi_F$ is a G -equivariant isomorphism of C_m^+ with the space of all smooth functions Φ in $C^\infty(G)$ such that*

- (a) $\Phi(gk) = \varepsilon^m(k)\Phi(g)$ for all k in K ;
- (b) $R_{x_+}\Phi = 0$.

Something similar holds for C_m^- .

The first step:

22.3. Lemma. *The map taking F to the function Φ_F defined in (22.1) is an isomorphism of $C_m^\infty(\mathfrak{X}^+)$ with the space of all smooth functions Φ on G satisfying*

$$(22.4) \quad \Phi(gk) = \varepsilon^m(k)\Phi(g).$$

If

$$k = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}$$

then

$$k \begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} ci - s \\ si + c \end{bmatrix} = (c + is) \begin{bmatrix} i \\ 1 \end{bmatrix} = \varepsilon(k) \begin{bmatrix} i \\ 1 \end{bmatrix}.$$

Therefore (22.4) holds for all k in K and all g in G .

Conversely, suppose given Φ in $C^\infty(G)$ such that (22.4) holds. Then equation (22.1) and the condition of homogeneity defines F in $C_m^\infty(\mathfrak{X}^+)$ uniquely. ▮

Next, we must see how to characterize the holomorphicity of F (or f) in terms of Φ .

THE CAUCHY EQUATIONS. I'll recall here some elementary facts of complex analysis. A smooth \mathbb{C} -valued function $f = u(x, y) + iv(x, y)$ on an open subset of \mathbb{C} is holomorphic if and only if the real Jacobian matrix of f

$$\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}$$

considered as a map from \mathbb{R}^2 to itself lies in the image of \mathbb{C} in $M_2(\mathbb{R})$. (Since this image generically coincides with the group of orientation-preserving similitudes, this means precisely that it is conformal.) This condition is equivalent to the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Holomorphicity may also be expressed by the single equation

$$\frac{\partial f}{\partial \bar{z}} = 0$$

where

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

When f is holomorphic, its complex derivative is

$$\frac{\partial f}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right).$$

The notation is designed so that for an arbitrary smooth function

$$df = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z}$$

where $dz = dx + i dy$.

HOLOMORPHICITY AND THE LIE ALGEBRA. The approach I take requires a slight digression. Let GL_2^+ be the subgroup of $GL_2(\mathbb{R})$ consisting of g such that $\det(g) > 0$. According to Lemma 19.2, it takes each \mathfrak{X}^\pm to itself. It also commutes with the scalar action on \mathfrak{X} , hence acts on \mathcal{C}_m^\pm , extending the representation of $SL_2(\mathbb{R})$. The point of shifting from SL_2 to GL_2^+ is that if

$$(22.5) \quad p = \begin{bmatrix} y & x \\ 0 & 1 \end{bmatrix} \quad (y > 0)$$

then $j(p, z) = 1$ for all z , and p takes the copy of \mathcal{H} in \mathfrak{X} to itself.

22.6. Lemma. Every g in $GL_2^+(\mathbb{R})$ may be expressed uniquely as $p\lambda$ with p in P , λ of the form

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

I leave this as an exercise. Keep in mind that $GL_2(\mathbb{R})$ acts on the right on row vectors, and that P is the stabilizer of $(0, 1)$.

Given F in $C_m^\infty(\mathfrak{X}^+)$, now define Φ_F to be a function on GL_2^+ by the same formula as before:

$$\Phi_F(g) = F\left(g \begin{bmatrix} i \\ 1 \end{bmatrix}\right).$$

If

$$(22.7) \quad \lambda = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

then

$$\lambda \begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} ai - b \\ bi + a \end{bmatrix} = (a + ib) \begin{bmatrix} i \\ 1 \end{bmatrix} = \varepsilon(\lambda) \begin{bmatrix} i \\ 1 \end{bmatrix} \quad (\varepsilon(\lambda) = a + ib).$$

Hence for all g

$$\Phi(g\lambda) = \varepsilon^m(\lambda)\Phi(g).$$

22.8. Proposition. For F in $C_m^\infty(\mathfrak{X}^+)$ we have

$$[R_{x_+}\Phi_F](g) = -4iy\varepsilon^2(\lambda/|\lambda|)\varepsilon^m(\lambda)\frac{\partial f(z)}{\partial \bar{z}}$$

if

$$g = p\lambda, \quad p = \begin{bmatrix} y & x \\ 0 & 1 \end{bmatrix} \quad (p(i) = z = x + iy).$$

Proof. We have

$$\begin{aligned} [R_{x_+}\Phi_F](p\lambda) &= [R_k R_{x_+}\Phi_F](p) = [R_{\text{Ad}(\lambda)x_+} R_\lambda \Phi_F](p) \\ &= \varepsilon^2(\lambda/|\lambda|)[R_{x_+} R_k \Phi_F](p) \\ &= \varepsilon^2(\lambda/|\lambda|)\varepsilon^m(\lambda)[R_{x_+}\Phi_F](p), \end{aligned}$$

so it suffices to prove the claim for $g = p$.

Introduce temporarily as basis of the Lie algebra of GL_2^+ :

$$\eta = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \nu_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \zeta = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \kappa = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

The first two span the Lie algebra of matrices (22.5), the second those of (22.7), and

$$\begin{aligned} x_+ &= \begin{bmatrix} 1 & -i \\ -i & -1 \end{bmatrix} \\ &= 2(\eta + i\nu_+) - (\zeta + i\kappa). \end{aligned}$$

But now

$$\begin{aligned} R_{x_+}F(p) &= (R_\alpha - 2iR_{\nu_+} - iR_\kappa)\Phi_F(p) \\ &= (2R_\eta - R_\zeta - 2iR_{\nu_+} - iR_\kappa)\Phi_F(p) \\ &= (2R_\eta - 2iR_{\nu_+} + m - i(-mi))\Phi_F(p) \\ &= (2R_\eta - 2iR_{\nu_+})\Phi_F(p) \end{aligned}$$

Apply the basic formula $R_X f(g) = [\Lambda_{gXg^{-1}}f](g)$ (where $\Lambda = -L$) to get

$$(2R_\eta - 2iR_{\nu_+})\Phi_F(p) = (2\Lambda_{p\eta p^{-1}} - 2i\Lambda_{p\nu_+ p^{-1}})\Phi_F(p).$$

But

$$\Lambda_\eta = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}, \quad p\eta p^{-1} = y\eta - x\nu_+$$

so

$$\begin{aligned} (2\Lambda_{p\eta p^{-1}} - 2i\Lambda_{p\nu_+ p^{-1}})\Phi_F(p) &= 2y \frac{\partial f}{\partial y} - 2iy \frac{\partial f}{\partial x} \\ &= -2iy \left(\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right) \\ &= -4iy \frac{\partial f}{\partial \bar{z}}. \end{aligned}$$

22.9. Corollary. *The function F in C_m^∞ is holomorphic if and only if $R_{x_+} \Phi_F = 0$.*

This concludes the proof of the Theorem.

23. Holomorphic automorphic forms

Define the norm on G :

$$\|g\| = \text{trace } g^t g = a^2 + b^2 + c^2 + d^2 \quad \left(g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right).$$

It is left and right invariant under K , and hence determines a norm on \mathcal{H} . A function f on \mathcal{H} will be said to be of moderate growth if $|f(z)| \ll \|z\|^N$ for some N .

Suppose Γ to be a discrete subgroup of G such that $\Gamma \backslash G$ has finite volume. For $m > 0$ the space $\mathcal{A}_m(\Gamma \backslash G)$ of automorphic forms of weight m is that of all functions f on $\Gamma \backslash \mathcal{H}$ of moderate growth such that

$$f(\gamma(z)) = f(z)j(\gamma, z)^m$$

for all γ in Γ .

Set

$$\|g\| = \inf_{\gamma \in \Gamma} \|\gamma g\|.$$

It is right invariant under K , hence a function on $\Gamma \backslash \mathcal{H}$. Define $A(\Gamma \backslash G)$ to be the space of functions of moderate growth on $\Gamma \backslash G$.

Theorem 22.2

$\text{Hom}(DS_n^-, A(\Gamma \backslash G))$ in bijection with Φ in $\mathcal{A}_n(\Gamma \backslash \mathcal{H})$. That is to say, embedding the *anti-holomorphic* discrete series into $A(\Gamma \backslash G)$ comes from *holomorphic* automorphic forms.

24. Square-integrability

Assume $n \geq 2$. In this section, I'll show that DS_n^\pm can be embedded equivariantly into $L^2(G)$. It will be shown that the K -finite functions Φ given by (22.1) and Theorem 22.2 are square-integrable.

INVARIANT MEASURES AND HOLOMORPHIC REALIZATIONS. First, I generalize the definition of C_m^∞ . For every multiplicative character χ of \mathbb{C}^\times define $C_\chi^\infty(\mathfrak{X}^+)$ to be that of all smooth functions F on \mathfrak{X}^+ such that

$$F(\lambda v) = \chi(\lambda)F(v).$$

This space is taken into itself by G . As before, such a function is determined by its restriction f to the embedded copy of \mathcal{H} in \mathfrak{X} , and the effect of G on such restrictions is defined by the formula

$$[\pi_\chi f](z) = \chi(j(g^{-1}, z))f(g^{-1}(z)).$$

In particular, if $\chi(\lambda) = |\lambda|^n$ then

$$[\pi_\chi f](z) = |j(g^{-1}, z)|^n f(g^{-1}(z)).$$

Lemma 19.2 tells us that if $n = 2$ then $\text{IM}(z)$ is invariant under this action, which suggests defining the function

$$\text{IM} \left(\begin{bmatrix} z \\ w \end{bmatrix} \right) = |w|^2 \text{IM}(z/w).$$

Now suppose F to be in $C_{-n}^\infty(\mathfrak{X}^+)$. Then $|F|^2 = F\bar{F}$ satisfies the functional equation

$$|F|^2(x\lambda) = |\lambda|^{-2n} |F|^2(x).$$

for all x in \mathfrak{X} . The product

$$F(v)\text{IM}^n(v)$$

is therefore invariant under scalar multiplication by λ , hence a function on \mathcal{H} . On \mathcal{H} the Riemannian metric $(dx^2 + dy^2)/y^2$ is G -invariant, and defines on \mathcal{H} a non-Euclidean geometry. The associated G -invariant measure is

$$dx dy / y^2.$$

We may therefore integrate $|F|^2$ against $dx dy / y^2$, at least formally, What we deduce is this:

24.1. Proposition. *The measure $y^{n-2} dx dy$ is G -invariant on π_{-n} .*

A concrete interpretation of this is that for f in $C_c^\infty(\mathcal{H})$ and $f_* = \pi_{-n}(g^{-1})f$ we have

$$\int_{\mathcal{H}} |f_*(z)|^2 y^n \frac{dx dy}{y^2} = \int_{\mathcal{H}} |f(z)|^2 y^n \frac{dx dy}{y^2}.$$

Proof. We can prove this directly. The first integrand is

$$|f(g(z))|^2 |j(g, z)|^{-2n} y^n(z) = |f(g(z))|^2 y^n(g(z)),$$

so the result follows from the invariance of $dx dy / y^2$. ▣

Now comes the interesting part. Take

$$F = \frac{(z - i)^p}{(z + i)^{p+n}},$$

which is an eigenfunction of K in π_{-n}^+ . Since $(x - i)^p/(x + ui)^p$ is bounded on \mathcal{H} and $1/(z + i)^n$ is square-integrable with respect to the measure $y^{n-2} dx dy$ for $n > 1$, integration defines a G -invariant Hilbert norm on \mathcal{C}_{-n}^+ . If we take

$$\varphi(z) = \frac{1}{(z + i)^n}$$

then each function

(24.2)
$$\Phi_\varphi(g) = \pi(g^{-1})f \bullet \bar{\varphi}$$

satisfies conditions (a) and (b) of Theorem 22.2.

THE CARTAN INTEGRATION FORMULA. It remains to be seen that the function (24.2) is square-integrable on G . This will be straightforward, once I recall an integral formula on G .

The group K acts by non-Euclidean rotation on \mathcal{H} , fixing the point i . Under these rotations, the vertical ray $[i, i\infty)$ sweep out all of \mathcal{H} . Since $\mathcal{H} = G/K$, this tells us:

24.3. Lemma. *Every g in G may be factored as $g = k_1 a k_2$ with each k_i in K and*

$$a = \begin{bmatrix} t & 0 \\ 0 & 1/t \end{bmatrix} \quad (t > 0).$$

The factorization can be found explicitly. If such an expression is valid, then

$${}^t g g = k_2^{-1} a^2 k_2.$$

Thus a^2 is the diagonal matrix of eigenvalues of the positive definite matrix ${}^t g g$, and k_2 is its eigenvector matrix. After finding these, set

$$k_1 = g a^{-1} k_2^{-1}.$$

In non-Euclidean radial coordinates this measure is

$$2\pi \sinh(r) dr d\theta,$$

or, replacing r by coordinate on A , and taking into account that $\{\pm 1\}$ in K acts trivially:

24.4. Proposition. (Cartan integration formula)

(24.5)
$$\int_G f(g) dg = \frac{1}{2} \int_{K \times A^+ \times K} f(k_1 a k_2) (\alpha(a) - 1/\alpha(a)) dk_1 da dk_2.$$

Here A^+ is the set of diagonal matrices

$$\begin{bmatrix} t & 0 \\ 0 & 1/t \end{bmatrix}$$

with $t > 1$, and

$$\alpha; \begin{bmatrix} t & 0 \\ 0 & 1/t \end{bmatrix} \mapsto t^2.$$

This is the same as δ , but for various reasons I prefer new notation here. Very roughly speaking, this formula is true because at iy we have

$$\frac{\partial}{\partial \theta} = (y^2 - 1) \frac{\partial}{\partial x},$$

so that (also at iy)

$$\frac{dx dy}{y^2} = \frac{(y^2 - 1) d\theta dy}{y^2}.$$

I now leave as an exercise the verification that for $n \geq 2$ the function defined by (24.2) lies in $L^2(G)$.

Part IV. Matrix coefficients and differential equations

25. Matrix coefficients

Suppose (π, V) to be any continuous admissible representation of G and \widehat{V} the space of continuous linear functions on V . The group G acts on \widehat{V} by the formula

$$\langle \widehat{\pi}(g)\widehat{v}, v \rangle = \langle \widehat{v}, \pi(g)^{-1}v \rangle,$$

so that the pairing $\widehat{V} \otimes V \rightarrow \mathbb{C}$ is G -invariant.

Suppose v to be in V , \widehat{v} in \widehat{V} . The associated **matrix coefficient** is the continuous functions on G :

$$\Phi_{\widehat{v}, v}(g) = \langle \widehat{v}, \pi(g)v \rangle.$$

The terminology comes about because if π is a finite-dimensional representation on a space V with basis (e_i) and dual basis (f_i) then $\Phi_{f_i, e_j}(g)$ is the (i, j) entry of the matrix $\pi(g)$.

The group G acts on both right and left on functions in $C^\infty(G)$:

$$[R_g f](x) = f(xg), \quad [L_g f](x) = f(g^{-1}x).$$

These are both left representations of G :

$$R_{g_1 g_2} = R_{g_1} R_{g_2}, \quad L_{g_1 g_2} = L_{g_1} L_{g_2}.$$

Hence the group $G \times G$ acts.

If π is a representation of G then

$$R_g \Phi_{\widehat{v}, v} = \Phi_{\widehat{v}, \pi(g)v}, \quad L_g \Phi_{\widehat{v}, v} = \Phi_{\widehat{\pi}(g)\widehat{v}, v}.$$

25.1. Lemma. *The map from $\widehat{V} \otimes V$ to $C(G)$ is $G \times G$ -equivariant.*

In particular, for a fixed \widehat{v} the map from V to $C^\infty(G)$ taking v to $\Phi_{\widehat{v}, v}$ is R_G -equivariant.

Associated to the left and right regular representations are right and left actions of \mathfrak{g} on smooth functions:

$$\begin{aligned} [R_X f](y) &= \left. \frac{d}{dt} \right|_{t=0} f(y \exp(tX)) \\ [L_X f](y) &= \left. \frac{d}{dt} \right|_{t=0} f(\exp(-tX)y). \end{aligned}$$

These equations translate directly to facts about the Lie algebra:

25.2. Lemma. *For f in $C^\infty(G)$, X in \mathfrak{g} , v differentiable:*

$$R_X \Phi_{\widehat{v}, v} = \Phi_{\widehat{v}, \pi(X)v}, \quad L_X \Phi_{\widehat{v}, v} = \Phi_{\widehat{\pi}(X)\widehat{v}, v}.$$

We know that if V is any irreducible admissible representation of (\mathfrak{g}, K) there exists at least one smooth representation of G for which V is the subspace of K -finite vectors. There may be several such extensions to G , but the corresponding matrix coefficients are the same:

25.3. Lemma. For K -finite vectors \hat{v}, v the matrix coefficient $\Phi_{\hat{v}, v}$ does not depend on which smooth extension to G is chosen.

I'll say more along this line later on.

Proof. The point is that the matrix coefficient may be characterized by certain properties that depend only on the representation of (\mathfrak{g}, K) . \circ The derivative $R_X \Phi$ depends only on $\pi(X)v$. Thus the Taylor series at 1 is determined. \circ Every isotypic K -component is finite-dimensional and stable under the center $Z(\mathfrak{g})$ of the enveloping algebra. Since this includes the Casimir operator, it is a sum of eigenvectors for the Casimir, which acts on the matrix coefficients of this component as an elliptic differential operator. The function Φ is therefore analytic. It is therefore determined on the connected component of G . \circ But K meets all connected components of G , so the function is determined everywhere on G . ▣

26. Differential equations

There exist explicit formulas for matrix coefficients in case π is irreducible, but these are not as important as a less precise qualitative description. The point is that matrix coefficients of an admissible representation of (\mathfrak{g}, K) are solutions to certain ordinary differential equations. According to Lemma 24.3, $G = KAK$. If \hat{v} and v are chosen to be eigenfunctions of K , then the corresponding matrix coefficient is determined by its restriction to A . But if π is irreducible, it is taken into a scalar multiple of itself by the Casimir operator $\pi(\Omega)$. Because of this, the restriction to A satisfies an ordinary differential equation of second order, which I shall exhibit explicitly.

THE EUCLIDEAN LAPLACIAN. Soon we'll need to know something about a certain class of differential equations, and I'll motivate what is to come by looking first at a somewhat familiar example that is not directly in our line but illustrates well what we are going to see later.

Consider the Laplacian in the Euclidean plane:

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2},$$

and let's suppose we want to find solutions of the eigenvalue equation

$$\Delta f = \lambda f$$

in a region with circular symmetry—for example, want to solve the wave equation describing a vibrating drum. The first step is separation of variables in polar coordinates. To do this one first expresses Δ in those coordinates:

$$\Delta f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \frac{\partial f}{\partial r} + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2}.$$

Solutions we are looking for can be expressed in terms of Fourier series

$$f(r, \theta) = \sum_n f_n(r) e^{in\theta}$$

so we are led to look for solutions of the form

$$f(r, \theta) = \varphi(r) e^{in\theta}.$$

This leads to the ordinary differential equation

$$(26.1) \quad \varphi''(r) + \frac{1}{r} \cdot \varphi'(r) - \frac{n^2}{r^2} \cdot \varphi(r) = \lambda \varphi$$

For $\lambda \neq 0$ solutions of this equation are **Bessel functions**. The equation is evidently singular at $r = 0$. If one changes variables $t = 1/r$ to see what happens at $r = \infty$ one also sees that it has a singularity at infinity. In fact, it has a **regular singularity** at the origin and an **irregular singularity** at infinity. It is the first that I am most interested in now.

REGULAR SINGULARITIES. A differential equation

$$y'' + A(x)y' + B(x)y = 0$$

is said to have a regular singularity at $x = 0$ if $A(x)$ is a meromorphic function of the form $a(x)/x$ with $a(x)$ analytic at $x = 0$, and similarly $B(x) = b(x)/x^2$, so the equation can be rewritten as

$$x^2y'' + xa(x)y' + b(x)y = 0.$$

This can also be rewritten more intelligibly. Set $\partial = d/dx$ and then let $D = xd/dx = x\partial$ be the multiplicatively invariant derivative. Then

$$\begin{aligned} Dy &= x \partial y \\ D^2y &= x(\partial y + x \partial^2 y) \\ &= x \partial y + x^2 \partial^2 y \\ xy' &= Dy \\ x^2y'' &= D^2y - Dy, \end{aligned}$$

so the equation becomes

$$D^2y + (a(x) - 1)Dy + b(x)y = 0.$$

Thus differential equations with regular singularities at 0 are those that can be written as

$$a(x)D^2y + b(x)Dy + c(x)y = 0$$

in which a, b, c are analytic at $x = 0$ and $a(0) \neq 0$. Special cases are those of the form

$$aD^2y + bDy + c = 0$$

with $a \neq 0, b, c$ constants. These are **Euler equations**. They can be solved in terms more familiar (at least to engineering instructors) by a change of independent variable $x = e^t$ to arrive at the equivalent equation

$$ay'' + by' + c = 0.$$

This is solved by exponential functions, leading to solutions of the form $y = x^s$ for Euler's equation itself. The exponent r must be a root of the **indicial equation**

$$ar^2 + br + c = 0.$$

The basic fact about Euler's equations is this:

26.2. Lemma. *Suppose given an Euler's equation*

$$aD^2y + bDy + c = 0$$

in which $a \neq 0, b, c$ are constants. Then

- (a) if the indicial equation has two roots r_i then x^{r_1} and x^{r_2} are a basis of solutions;
- (b) if there is just one root r (with multiplicity two) then $x^r, x^r \log |x|$ are a basis.

Now suppose given the linear differential operator

$$L: y \mapsto a(x)D^2 + b(x)Dy + c(x)$$

with $a(x), b(x), c(x)$ all analytic at 0 and $a(0) \neq 0$. Define the **associated Euler operator**

$$L_0: y \mapsto a(0)D^2 + b(0)Dy + c(0)y.$$

The solutions of the original equation are closely related to those of the associated Euler's equation $L_0y = 0$.

Let \mathcal{O} be the ring of convergent series at 0.

26.3. Proposition. *In the circumstances above, suppose the roots of the associated indicial equation to be r_1, r_2 . Then:*

- (a) if $r_1 - r_2 \notin \mathbb{Z}$ there exists a basis of solutions of the form $y_i = x^{r_i} f_i$;
- (b) if $r_1 = r_2 =$ (say) r there exists a basis of solutions of the forms $y_1 = x^r f_1, y_2 = x^r f_2(x) + y_1 \log |x|$;
- (c) if $r_1 - r_2$ is a positive integer n , there exists a basis of solutions of the forms $y_1 = x^{r_1} f_1, y_2 = x^{r_2} f_2(x) + \mu y_1 \log |x|$ for some constant μ .

In this, all f_i are in \mathcal{O} .

To find the solutions explicitly, one looks for the candidate series through recursion relations among the coefficients. It is known that any formal power series that solves the equation will actually converge. In most cases this will be evident.

Let's look at an example based on (26.1), written now as

$$D^2y - n^2y = \lambda x^2y.$$

To avoid trivial cases, assume $\lambda \neq 0$, and just for simplicity take $n = 0$. The equation to be solved is

$$D^2y = \lambda x^2y.$$

The indicial root is $r = 0$ with multiplicity two. The calculation of the solutions will be simpler if I note that the solution is even, hence odd terms in the series vanish. So we proceed:

$$\begin{aligned} y_1 &= 1 + c_2x^2 + c_4x^4 + \dots + c_{2n}x^{2n} + \dots \\ D^2y_1 &= 4c_2x^2 + 16c_4x^4 + \dots + 4n^2c_{2n}x^{2n} + \dots \\ x^2\lambda y_1 &= \lambda x^2 + \lambda c_2x^4 + \dots + \lambda c_{2n-2}x^{2n} + \dots \end{aligned}$$

getting the recursion relations

$$c_0 = 1, \quad c_{2n} = \frac{\lambda}{4n^2} \cdot c_{2n-2}.$$

This defines an entire function analytically varying with λ . For the second solution we proceed:

$$\begin{aligned} y_2 &= z_1 + y_1 \log x \\ Dy_2 &= Dz_1 + Dy_1 \log x + y_1 \\ D^2y_2 &= D^2z_1 + D^2y_1 \log x + 2Dy_1 \\ \lambda x^2y_2 &= \lambda x^2z_1 + \lambda x^2y_1 \log x \\ (D^2 - \lambda x^2)y_2 &= (D^2 - \lambda x^2)z_1 + 2Dy_1, \end{aligned}$$

and then solve for z_1 .

In all cases—i.e. all n —there is just a one-dimensional family of solutions that are not singular at 0, defining Bessel functions.

Near infinity the equation looks more or less like the equation with constant coefficients

$$(26.4) \quad \varphi''(r) = \lambda\varphi(r).$$

The solutions of the original equation possess solutions whose asymptotic behaviour is suggested by the approximating equation (26.4). Note that λ plays a role in this asymptotic behaviour but not in the behaviour of solutions near 0.

Differential equations with regular singularities are the ones that occur in analyzing the behaviour of matrix coefficients. Irregular singularities occur in representation theory when looking at Whittaker models.

The standard reference on ordinary differential equations with singularities is [Coddington-Levinson:1955]. It is clear if dense. A more readable account, but without all details, is [Brauer-Nohel:1967].

Justification of the previous result comes down to some relatively simple algebra. Let \mathcal{A}_r be the space spanned by $x^r F \log^n x$. How does \mathcal{D} act on $\mathcal{A}^r/\mathcal{A}^{r+1}$? Let $\mathcal{L} = \log x$, $P_0 = P_0(r)$, the indicial polynomial of the Euler operator

$$L_0 = a_k(0)D^k + a_{k-1}(0)D^{k-1} + \cdots + a_0(0).$$

Let \bar{L} be the graded operator. Then

$$Dx^r \mathcal{L}^n = rx^r \mathcal{L}^n + x^r \cdot \mathcal{L}^{n-1}$$

leading to

$$\bar{L}x^r \mathcal{L}^n = P(r)\mathcal{L}^n + P'(r) \cdot n\mathcal{L}^{n-1} + \frac{P''(r)}{2} \cdot n(n-1)\mathcal{L}^{n-2} + \cdots$$

which explains the traditional formulas.

27. The non-Euclidean Laplacian

In non-Euclidean geometry, the Laplacian is

$$\Delta_{\mathcal{H}} = \frac{1}{y^2} \cdot \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

on the upper half plane \mathcal{H} . In non-Euclidean polar coordinates this becomes

$$(27.1) \quad \frac{\partial^2}{\partial r^2} + \frac{1}{\tanh r} \cdot \frac{\partial}{\partial r} + \frac{1}{\sinh^2 r} \cdot \frac{\partial^2}{\partial \theta^2}.$$

As a rough check, note that for very small values of r this looks approximately like the Euclidean Laplacian. An eigenfunction of $\Delta_{\mathcal{H}}$ and the rotation group satisfies

$$f''(r) + \frac{1}{\tanh r} f'(r) - \frac{n^2}{\sinh^2 r} f(r) = \lambda f(r).$$

This has a regular singularity at 0 and an irregular one at ∞ . But there is a major difference between this case and the Euclidean one—if we change variables $x = e^r$ we obtain the equation

$$D^2f - \left(\frac{1+x^2}{1-x^2} \right) Df - \frac{n^2 x^2}{(1-x^2)^2} f = \lambda f.$$

This has a regular singularity at both 1 and ∞ . While both Euclidean and non-Euclidean Laplacian equations have irregular singularities at infinity, the ‘asymptotic’ series of the second are actually convergent.

Let’s look more closely at what happens. The point $x = 0$ is at effectively at infinity on G , and At 1 the associated Euler’s equation is

$$D^2 f - Df = \lambda f$$

with solutions $x^{\pm r}$ with

$$r = 1/2 \pm \sqrt{\lambda + 1/4}$$

as long as $\lambda \neq -1/4$, and

$$x^{1/2}, x^{1/2} \log x$$

otherwise. Now $\lambda = -1/4 - t^2$ is the eigenvalue of a unitary principal series parametrized by it , and in this case we have series solutions

$$f_+(x) x^{1/2+it} + f_-(x) x^{1/2-it}$$

as long as $t \neq 0$, and otherwise

$$f_0(x) x^{1/2} + f_1(x) x^{1/2} \log x.$$

According to the formula (24.5) these just fail to be in $L^2(G)$.

At $x = 1$, we need to make a change of variables $t = y - 1$. I leave it as an exercise to see that the solutions here behave exactly as for the Euclidean Laplacian.

DERIVING RADIAL COMPONENTS. How does one derive (27.1)? Suppose π to be an irreducible admissible representation of (\mathfrak{g}, K) . I now want to describe a matrix coefficient

$$\Phi(g) = \langle \widehat{\pi}(g)\widehat{v}, v \rangle$$

with \widehat{v} and v eigenfunctions of K —say

$$\pi(k)\widehat{v} = \varepsilon^m(k)\widehat{v}, \quad \pi(k)v = \varepsilon^n(k)v.$$

Since $G = KAK$, the function Φ is determined by its restriction to A . Since π is irreducible, the Casimir operator acts on \widehat{V} and V by some scalar γ , and this means that Φ is an eigenfunction of Ω . In short, the situation looks much like it did when searching for radial solutions of Laplace’s equation, and we expect to find an ordinary differential equation satisfied by the restriction of Φ to A .

What is the ordinary differential equation satisfied by the restriction of Φ to A ? What we know about Φ can be put in three partial differential equations satisfied by Φ :

$$\begin{aligned} (27.2) \quad R_\Omega \Phi &= \gamma \Phi \\ L_\kappa \Phi &= mi \Phi \\ R_\kappa \Phi &= ni \Phi. \end{aligned}$$

The Cartan decomposition Lemma 24.3 says that the product map from $K \times A \times K$ to G is surjective. looking at what happens on the unit disk, it is apparently a non-singular map except near K . That is to say, for $a \neq 1$ in $|A|$, we are going to verify that $L_\kappa, R_h,$ and R_κ for a basis of the tangent space at a (recall the notation of §2).

The basic fact comes from a trivial computation. For any g in G and X in \mathfrak{g}

$$\begin{aligned} [L_{-X} f](g) &= \frac{d}{dt} f(\exp(tX)g) \\ &= \frac{d}{dt} f(g \cdot g^{-1} \exp(tX)g) \\ &= [R_{Xg} f](g). \end{aligned}$$

(We have seen this before in (10.5) .) Thus our remark about the three vectors spanning the tangent space at a reduce to this:

27.3. Lemma. (Infinitesimal Cartan decomposition) For $a \neq \pm 1$ in A

$$\mathfrak{g} = \mathfrak{k}^a \oplus \mathfrak{a} \oplus \mathfrak{k}.$$

Proof. Since we know that

$$\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{k},$$

it suffices to see that ν_+ can be expressed as a linear combination of κ^a and κ . Let α be the character of A taking

$$\begin{bmatrix} t & 0 \\ 0 & 1/t \end{bmatrix} \mapsto t^2.$$

We shall need both of the following formulas:

$$\begin{aligned} \nu_+ &= \frac{\alpha(a)}{1 - \alpha^2(a)} \cdot (\kappa^a - \alpha(a)\kappa) \\ \nu_- &= \frac{1}{1 - \alpha^2(a)} \cdot (\kappa - \alpha(a)\kappa^a). \end{aligned}$$

These can be easily deduced (I write α for $\alpha(a)$, ν for ν_+):

$$\begin{aligned} \kappa &= \nu_- - \nu \\ \kappa^a &= \alpha\nu_- - \alpha^{-1}\nu \\ \alpha\kappa &= \alpha\nu_- - \alpha\nu \\ \kappa^a - \alpha\kappa &= (\alpha - \alpha^{-1})\nu \\ \nu &= \frac{\kappa^a - \alpha\kappa}{\alpha - \alpha^{-1}} \\ \alpha^{-1}\kappa &= \alpha^{-1}\nu_- - \alpha^{-1}\nu \\ \nu_- &= \frac{\kappa^a - \alpha^{-1}\kappa}{\alpha^{-1} - \alpha}. \end{aligned}$$

If X is any element of \mathfrak{g} we can write

$$\begin{aligned} \nu X &= \frac{\alpha(a)}{1 - \alpha^2(a)} \cdot \kappa^a X - \frac{\alpha^2(a)}{1 - \alpha^2(a)} \cdot \kappa X \\ &= \frac{\alpha(a)}{1 - \alpha^2(a)} \cdot \kappa^a X - \frac{\alpha^2(a)}{1 - \alpha^2(a)} \cdot X\kappa - \frac{\alpha^2(a)}{1 - \alpha^2(a)} \cdot [\kappa, X]. \end{aligned}$$

Since

$$\Omega = \frac{h^2}{4} - \frac{h}{2} + \nu\nu_-,$$

this leads to:

27.4. Proposition. If the equations (27.2) are satisfied, then the restriction φ of Φ to A satisfies

$$\varphi''(x) - \left(\frac{1+x^2}{1-x^2} \right) \varphi'(x) - \left(\frac{x^2(m^2+n^2) - x(1+x^2)mn}{(1-x^2)^2} \right) \varphi(x) = \lambda\varphi(x).$$

Here $x = t^2$ and $\partial/\partial x = (1/2)\partial/\partial t$.

28. Realizations of admissible modules

We have seen that every irreducible (\mathfrak{g}, K) module for $G = SL_2(\mathbb{R})$ is the subspace of K -finite vectors in a continuous representation of G , by first classifying all irreducible (\mathfrak{g}, K) -modules and then explicitly realizing each of them in a representation of G . This still leaves open the question, can every finitely generated admissible (\mathfrak{g}, K) -module be realized in a representation of G ? In this section I'll prove that this is so, by applying a result about ordinary differential equations with regular singularities.

If V is an admissible (\mathfrak{gothg}, K) module, I'll call it realizable if it is the subspace of K -finite vectors in a continuous representation of G .

28.1. Theorem. *Every finitely generated admissible (\mathfrak{g}, K) -module is realizable.*

Part V. Langlands' classification

29. The Weil group

In a later section I'll explain how the **Weil group** $W_{\mathbb{R}}$ plays a role in the representation theory of $G = SL_2(\mathbb{R})$. In this one I'll recall some of its properties.

DEFINITION OF THE WEIL GROUPS. If F is any local field and E/F a finite Galois extension, the Weil group $W_{E/F}$ fits into a short exact sequence

$$1 \longrightarrow E^\times \longrightarrow W_{E/F} \longrightarrow \mathcal{G}al(E/F) \longrightarrow 1.$$

It is defined by a certain non-trivial cohomology class in the Brauer group $H^2(\mathcal{G}al(E/F), E^\times)$ determined by local class field theory. If F is \mathfrak{p} -adic and E_{ab} is a maximal abelian extension of E , then local class field theory asserts that $W_{E/F}$ may be identified with the subgroup of the Galois group of E_{ab}/F projecting modulo \mathfrak{p} onto powers of the Frobenius. If $F = \mathbb{C}$ or \mathbb{R} there is no such interpretation, and I think it is fair to say that, although the groups $W_{\mathbb{C}} = W_{\mathbb{C}/\mathbb{C}}$ and $W_{\mathbb{R}} = W_{\mathbb{C}/\mathbb{R}}$ are very simple, there is some mystery about their significance.

The group $W_{\mathbb{C}}$ is just \mathbb{C}^\times . The group $W_{\mathbb{R}}$ is an extension of \mathbb{C}^\times by $\mathcal{G} = \mathcal{G}al(\mathbb{C}/\mathbb{R}) = \{1, \sigma\}$, fitting into an exact sequence

$$1 \longrightarrow \mathbb{C}^\times \longrightarrow W_{\mathbb{R}} \longrightarrow \mathcal{G} \longrightarrow 1.$$

It is generated by the copy of \mathbb{C}^\times and an element σ mapping onto the non-trivial element of \mathcal{G} , with relations

$$z \cdot \sigma = \sigma \cdot \bar{z}, \quad \sigma^2 = -1$$

Since 1 is not the norm of a complex number, the extension does not split. It is not a coincidence that it is isomorphic to the normalizer of a copy of \mathbb{C}^\times in the unit group of the Hamilton quaternions \mathbb{H} . If E/F is any finite Galois extension of local fields, then $W_{E/F}$ may be embedded into the normalizer of a copy of E^\times in the multiplicative group of the division algebra over F defined by the class in the Brauer group mentioned above.

The **norm map** NM from $W_{\mathbb{R}}$ to \mathbb{R}^\times extends the norm on its subgroup \mathbb{C} , mapping

$$\begin{aligned} z &\longmapsto |z|^2 \\ \sigma &\longmapsto -1. \end{aligned}$$

It is a surjective homomorphism.

CHARACTERS OF THE WEIL GROUP. What are the characters of $W_{\mathbb{R}}$? They factor through the maximal abelian quotient of $W_{\mathbb{R}}$, and this is simple to describe.

29.1. Proposition. *The norm map NM identifies \mathbb{R}^\times with the maximal abelian quotient of $W_{\mathbb{R}}$.*

Proof. Since $|z|^2 > 0$ for every z in \mathbb{C}^\times , the kernel of the norm map is the subgroup \mathbb{S} of s in \mathbb{C}^\times of norm 1. Every one of these can be expressed as z/\bar{z} , and since

$$z\sigma z^{-1}\sigma^{-1} = z/\bar{z},$$

it is a commutator. ▮

As one consequence:

29.2. Corollary. *The characters of $W_{\mathbb{R}}$ are those of the form $\chi(NM(w))$ for χ a character of \mathbb{R}^\times .*

Recall that in this essay a character (elsewhere in the literature sometimes called a quasi-character) of any locally compact group is a homomorphism from it to \mathbb{C}^\times , and that characters of \mathbb{R}^\times are all of the form $x \mapsto |x|^s(x/|x|)^n$ (depending only on the parity of n). The characters of the compact torus \mathbb{S} are of the form

$$\varepsilon^n: z \mapsto z^n$$

for some integer n , and this implies that all characters of \mathbb{C}^\times are of the form

(29.3)
$$z \mapsto |z|^s(z/|z|)^n.$$

Together with Corollary 29.2, this tells us explicitly what the characters of $W_{\mathbb{C}}$ and $W_{\mathbb{R}}$ are.

It will be useful later on to know that another, more symmetrical, way to specify a character of \mathbb{C}^\times is as

(29.4)
$$z \mapsto z^\lambda \bar{z}^\mu,$$

where λ and μ are complex numbers such that $\lambda - \mu$ lies in \mathbb{Z} . The two forms (29.3) and (29.4) are related by the formulas

$$\begin{aligned} \lambda &= (s + n)/2 \\ \mu &= (s - n)/2. \end{aligned}$$

IRREDUCIBLE REPRESENTATIONS OF THE WEIL GROUP. Since \mathbb{C}^\times is commutative, any continuous finite-dimensional representation of \mathbb{C}^\times must contain an eigenspace with respect to some character. This tells us that all irreducible continuous representations of $W_{\mathbb{C}}$ are characters.

Now suppose (ρ, U) to be an irreducible continuous representation of $W_{\mathbb{R}}$ that is not a character. The space U must contain an eigenspace for \mathbb{C}^\times , say with character χ . If the restriction of χ to \mathbb{S} is trivial, this eigenspace must be taken into itself by σ . It decomposes into eigenspaces of σ , and each of these becomes an eigenspace for a character of all of $W_{\mathbb{R}}$. The irreducible representation ρ must then be one of the two possible characters.

Otherwise, suppose the restriction of χ to \mathbb{S} is not trivial, and that $V \subseteq U$ is an eigenspace for χ . Then $\sigma(V)$ is an eigenspace for $\bar{\chi}$, and U must be the direct sum of V and $\sigma(V)$.

We can describe one of these representations of dimension 2 in a simple manner. Consider the two-dimensional representation of $W_{\mathbb{R}}$ induced by χ from \mathbb{C}^\times . One conclusion of my recent remarks is:

29.5. Proposition. *Every irreducible representation of $W_{\mathbb{R}}$ that is not a character is of the form $\text{Ind}(\chi | \mathbb{C}^\times, W_{\mathbb{R}})$ for some character χ of \mathbb{C}^\times that is not trivial on \mathbb{S} .*

Only χ and $\bar{\chi}$ give rise to isomorphic representations of $W_{\mathbb{R}}$.

If χ is trivial on \mathbb{S} then the induced representation decomposes into the direct sum of the two characters of $W_{\mathbb{R}}$ extending those of \mathbb{C}^\times .

Proof. Frobenius reciprocity gives a G -equivariant map from ρ to $\text{Ind}(\chi | \mathbb{C}^\times, W_{\mathbb{R}})$. ▮

There is a natural basis $\{f_1, f_\sigma\}$ of the space $\text{Ind}(\chi)$. Here f_x has support on $\mathbb{C}^\times x$ and $f_x(zx) = \chi(z)$. For this basis

$$(29.6) \quad \begin{aligned} R_z &= \begin{bmatrix} \chi(z) & 0 \\ 0 & \bar{\chi}(z) \end{bmatrix} \\ R_\sigma &= \begin{bmatrix} 0 & \chi(-1) \\ 1 & 0 \end{bmatrix}. \end{aligned}$$

PROJECTIVE REPRESENTATIONS. Later on we shall be interested in continuous maps from $W_{\mathbb{R}}$ to $\text{PGL}_2(\mathbb{C})$. If ρ is a two-dimensional representation of $W_{\mathbb{R}}$, it determines also a continuous homomorphism into $\text{PGL}_2(\mathbb{R})$. There are two types:

(1) Direct sums $\chi_1 \oplus \chi_2$ of two characters. One of these gives rise, modulo scalars, to the homomorphism

$$w \mapsto \begin{bmatrix} [\chi_1/\chi_2](w) & 0 \\ 0 & 1 \end{bmatrix}.$$

That is to say, these correspond to the homomorphisms (also modulo scalars)

$$w \mapsto \begin{bmatrix} \chi(w) & 0 \\ 0 & 1 \end{bmatrix}$$

for characters χ of $W_{\mathbb{R}}$.

(2) Irreducible representations $\text{Ind}(\chi | \mathbb{C}^\times, W_{\mathbb{R}})$ with $\bar{\chi} \neq \chi$. In this case, we are looking at the image modulo scalar matrices of the representation defined by (29.6). But modulo scalars all complex matrices

$$\begin{bmatrix} 0 & b \\ a & 0 \end{bmatrix}$$

are conjugate by diagonal matrices. Hence if $\chi(z) = |z|^s (z/|z|)^n$, this becomes the representation (modulo scalars)

$$\begin{aligned} z &\mapsto \begin{bmatrix} (z/\bar{z})^n & 0 \\ 0 & 1 \end{bmatrix} \\ \sigma &\mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \end{aligned}$$

In fact:

29.7. Proposition. *Any continuous homomorphism from $W_{\mathbb{R}}$ to $\text{PGL}_2(\mathbb{C})$ is the image of one to $\text{GL}_2(\mathbb{C})$.*

Given what we know about the classification of representations of $W_{\mathbb{R}}$, this will classify all homomorphisms from $W_{\mathbb{R}}$ to $\text{PGL}_2(\mathbb{R})$. In fact, the proof will do this explicitly.

Proof. Suppose given a homomorphism φ from $W_{\mathbb{R}}$ to $\text{PGL}_2(\mathbb{C})$. We embed $\text{PGL}_2(\mathbb{C})$ into $\text{GL}_3(\mathbb{C})$ by means of the unique 3-dimensional representation induced by that of $\text{GL}_2(\mathbb{C})$ taking

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \mapsto \begin{bmatrix} a/b & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & b/a \end{bmatrix}.$$

The image of $\text{PGL}_2(\mathbb{C})$ in $\text{GL}_3(\mathbb{C})$ can be identified with the special orthogonal group of the matrix

$$Q = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

This can be seen most easily by considering conjugation of the 2×2 matrices of trace 0, which leaves invariant the quadratic form \det .

It is elementary to see by eigenvalue arguments that every commuting set of semi-simple elements in any GL_n may be simultaneously diagonalized. This is true of the image of $\varphi(\mathbb{C}^\times)$ in $SL_3(\mathbb{C})$. The diagonal matrices on $SO(Q)$ are those of the form

$$\begin{bmatrix} t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/t \end{bmatrix},$$

which is the image of the group of diagonal matrices in $PGL_2(\mathbb{C})$. Hence $\varphi(\mathbb{C}^\times)$ itself can be conjugated into the diagonal subgroup of $PGL_2(\mathbb{C})$. It is then (modulo scalars) of the form

$$z \mapsto \begin{bmatrix} \rho(z) & 0 \\ 0 & 1 \end{bmatrix}$$

for some character ρ . There are now two cases to look at: (a) the character ρ is trivial, or (b) it is not. In the first case, $\varphi(\sigma)$ can be any element of $PGL_2(\mathbb{C})$; they are all conjugate. In this case, lifting φ to $GL_2(\mathbb{C})$ is immediate. In the second case, $\varphi(\sigma)$ must lie in the normalizer of the diagonal matrices. Either it lies in the torus itself, in which case φ is essentially a character of $W_{\mathbb{R}}$, and again lifting is immediate; or it is of the form

$$\begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} \text{ which is conjugate by a diagonal matrix to } \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

In this case $\varphi(\sigma)$ has order two. But since $\sigma^2 = -1$, this means that the restriction of φ to \mathbb{C}^\times has -1 in its kernel. So

$$\rho(z) = (z/\bar{z})^n.$$

Again we can lift. ▣

Remark. This is a very special case of a more general result first proved in [Langlands:1974] (Lemma 2.10) for real Weil groups, and extended to other local fields by [Labesse:1984]. That \mathbb{C}^\times has image in a torus is a special case of Proposition 8.4 of [Borel:1991]. That the image of $W_{\mathbb{R}}$ is in the normalizer of a torus follows from Theorem 5.16 of [Springer-Steinberg:1970].

30. Langlands' classification for tori

Quasi-split reductive groups defined over \mathbb{R} are those containing a Borel subgroup rational over \mathbb{R} . They are classified (up to isomorphism) by pairs (\mathcal{L}, Φ) where \mathcal{L} is a based root datum $(L, \Delta, L^\vee, \Delta^\vee)$ and Φ an involution of Φ of \mathcal{L} . The associated **dual group** is the complex group \widehat{G} associated to the dual root datum \mathcal{L}^\vee , and its L -group (at least in one incarnation) is the semi-direct product $\widehat{G} \rtimes \mathcal{G}$. More precisely, \widehat{G} is assigned an épinglage, and σ acts on \widehat{G} by an automorphism of the épinglage in which σ acts according to how Φ acts on \mathcal{L} .

If G is split, then ${}^L G$ is just the direct product $\widehat{G} \times \mathcal{G}$. For a non-split example, let G be the special unitary group of the Hermitian matrix

$$H = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

the group of 3×3 matrices g such that $g = {}^t \bar{g}^{-1}$. Here \widehat{G} is PGL_3 , and σ acts on it by the involution $g \mapsto w_\ell g^\theta w_\ell^{-1}$, where

$$g^\theta = {}^t g^{-1}, \quad w_\ell = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

A homomorphism from $W_{\mathbb{R}}$ into ${}^L G$ is called **admissible** if (a) its image lies in the set of semi-simple elements of ${}^L G$ and (b) the map is compatible with the canonical projection from ${}^L G$ onto \mathcal{G} . [Langlands:1989] classifies admissible representations of (\mathfrak{g}, K) by such admissible homomorphisms. That is to say, each admissible homomorphism corresponds to a finite set of irreducible (\mathfrak{g}, K) -modules, called an L-packet, and all representations of G arise in such packets. We shall see later what packets look like for $\mathrm{SL}_2(\mathbb{R})$. One point of this correspondence is **functoriality**—whenever we are given an algebraic homomorphism from ${}^L H$ to ${}^L G$ compatible with projections onto \mathcal{G} , each L-packet of representations of H will give rise to one of G .

In particular, characters of a real torus ${}^L T$ are classified by conjugacy classes of admissible homomorphisms from $W_{\mathbb{R}}$ to ${}^L T$. I'll not explain the most general result, but just exhibit how it works for the three tori of relevance to representations $\mathrm{SL}_2(\mathbb{R})$.

If T is a torus defined over \mathbb{R} , let $X^*(T)$ be its lattice of algebraic characters over \mathbb{C} . The involution Φ that defines T as a real torus may be identified with an involution of $X^*(T)$.

I first state what happens for complex tori. Here $W_{\mathbb{C}} = \mathbb{C}^{\times}$ and every torus is T just a product of copies of \mathbb{C} . Let $X_*(T)$ be its lattice of algebraic homomorphisms from \mathbb{C}^{\times} to T (its algebraic cocharacters) and $X^*(T)$ the lattice of algebraic characters. These are dual lattices. If λ and μ are in $\mathbb{C} \otimes X^*(T)$ and $\lambda - \mu \in X^T$, then one can specify a character of T by the formula

$$\chi_{\lambda, \mu}: t \mapsto t^{\lambda} \bar{t}^{\mu}$$

It is characterized and even defined by the property that if α is any algebraic cocharacter then

$$\chi_{\lambda, \mu}(\alpha(z)) = z^{\langle \lambda, \alpha \rangle} \bar{z}^{\langle \mu, \alpha \rangle}.$$

This gives an isomorphism of the group of characters of T with such pairs (λ, μ) . The effect of this on \mathbb{S} is to take s to $s^{\lambda - \mu}$. I have already pointed this out for T of dimension one, and the general result follows easily from that. But $X^*(T)$ is $X_*(\widehat{T})$, and continuous homomorphisms from \mathbb{C}^{\times} are parametrized by exactly the same data, so Langlands' classification is verified in this case.

Now I look at some simple real tori.

Case 1. Let $T = \mathbb{G}_m$ over \mathbb{R} . Its group of rational points is \mathbb{R}^{\times} . The lattice L is just \mathbb{Z} , and $\Phi = I$. Its dual group is just \mathbb{C}^{\times} . Admissible homomorphisms from $W_{\mathbb{R}}$ to ${}^L T$ may be identified with characters of $W_{\mathbb{R}}$, which may be identified with characters of $\mathbb{R}^{\times} = T(\mathbb{R})$ according to Corollary 29.2.

Case 2. Let $T = \mathbb{S}$. Again $X^*(T) = \mathbb{Z}$, but now $\Phi = -I$. The L-group of T is the semi-direct product of $\widehat{T} = \mathbb{C}^{\times}$, with σ acting on it as $z \mapsto z^{-1}$. Suppose φ to be an admissible homomorphism from $W_{\mathbb{R}}$ to ${}^L T$. It must take

$$\begin{aligned} z &\mapsto z^{\lambda} \bar{z}^{\mu} \\ \sigma &\mapsto c \times \sigma \end{aligned}$$

for some λ, μ in \mathbb{C} with $\lambda - \mu$ in \mathbb{Z} , some c in \mathbb{C}^{\times} .

Conjugating $c \times \sigma$ changes c by a square, so that one may as well assume $c = 1$.

The condition $\varphi(\sigma^2) = \varphi(-1)$ implies that $\varphi(-1) = 1$, or that $\lambda - \mu$ must be an even integer. That $\varphi(z\sigma) = \varphi(\sigma\bar{z})$ means that $\mu = -\lambda$. Therefore λ itself must be an integer n . This map corresponds to the character $z \mapsto z^n$ of \mathbb{S} .

Case 3. Now let T be the real torus obtained from the multiplicative group over \mathbb{C} by restriction of scalars. Its character group, as we have seen, may be identified with all maps $z^{\lambda} \bar{z}^{\mu}$ with $\lambda - \mu \in \mathbb{Z}$. In effect I am going to check in this case that Langlands classification is compatible with restriction of scalars.

The dual group \widehat{T} is now $\mathbb{C}^{\times} \times \mathbb{C}^{\times}$, and Φ swaps factors. Suppose φ to be an admissible homomorphism from $W_{\mathbb{R}}$ to ${}^L T$. Since $z\sigma = \sigma\bar{z}$ in $W_{\mathbb{R}}$, the restriction of φ to \mathbb{C} takes

$$(30.1) \quad z \mapsto (z^{\lambda} \bar{z}^{\mu}, z^{\lambda} \bar{z}^{\mu}) \quad (\lambda - \mu = n \in \mathbb{Z})$$

for some λ, μ with $\lambda - \mu \in \mathbb{Z}$. Then $\varphi(\sigma) = (x, y) \times \sigma$ for some $(x, y) \in \widehat{T}$. The equation $\sigma^2 = -1$ translates to the condition $xy = (-1)^n$. Up to conjugation we may take $(x, y) = ((-1)^n, 1)$, and then

$$(30.2) \quad \sigma \mapsto ((-1)^n, 1) \times \sigma.$$

31. Langlands' classification for $\mathrm{SL}(2)$

Following a phenomenon occurring for unramified representations of p -adic groups, Langlands has offered a classification of irreducible representations of real reductive groups that plays an important role in the global theory of automorphic forms. It is particularly simple, but nonetheless instructive, in the case of $G = \mathrm{SL}_2(\mathbb{R})$. The point is to associate to every admissible homomorphism from $W_{\mathbb{R}}$ to $\mathrm{PGL}_2(\mathbb{C})$ a set of irreducible admissible (\mathfrak{sl}_2, K) modules. Furthermore, the structure of the finite set concerned will be described.

The best way to see the way things work is to relate things to the embeddings of the L -groups of tori into ${}^L G$. There are two conjugacy classes of tori in G , split and compact.

Case 1. Suppose T is split. Then ${}^L T$ is just the direct product of \mathbb{C}^\times and \mathcal{G} , and we may map it into the L -group of G :

$$z \mapsto \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma \mapsto I \times \sigma.$$

Case 2. Suppose T is compact. Then ${}^L T$ is the semi-direct product of \mathbb{C}^\times by \mathcal{G} , with σ acting as $z \mapsto 1/z$. We may embed this also into the L -group of G :

$$\begin{aligned} z &\mapsto \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix} \\ \sigma &\mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \end{aligned}$$

But now if we look back at the classification of admissible homomorphisms from $W_{\mathbb{R}}$ to $\mathrm{PGL}_2(\mathbb{C})$, we see that *all admissible homomorphisms from $W_{\mathbb{R}}$ into \widehat{G} factor through some ${}^L T$* . The principal series for the character χ of A corresponds to the admissible homomorphism into ${}^L A$ that parametrizes χ , and the representations $\mathrm{DS}_{n, \pm}$ make up the L -packet parametrized by the admissible homomorphisms that parametrize the character $z^{\pm n}$ of the compact torus.

One feature of Langlands' classification that I have not yet mentioned is evident here—tempered representation of $\mathrm{SL}_2(\mathbb{R})$ are those for which the image of $W_{\mathbb{R}}$ in \widehat{G} is bounded.

This parametrization might seem somewhat arbitrary, but it can be completely justified by seeing what happens globally, where the choices I have made are forced by a matching of L -functions associated to automorphic forms.

Part VI. Verma modules and representations of \mathfrak{G}

32. Verma modules

Let \mathbb{C}_λ be the one dimensional representation of the Lie algebra \mathfrak{t} on which \mathfrak{h} acts by the constant λ . It lifts to a unique representation of \mathfrak{b} annihilated by \mathfrak{n} . The **Verma module** associated to it is $V_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$. Because of the decomposition $\mathfrak{g} = \mathfrak{n}^\circ \oplus \mathfrak{t} \oplus \mathfrak{n}$, this is a free module over $U(\mathfrak{n}^\circ)$. Choosing as basis the $v_k = f^k v_0$ ($k \geq 0$), we have immediately

$$\begin{aligned} \mathfrak{h} \cdot v_0 &= \lambda v_0 \\ \mathfrak{e} \cdot v_0 &= 0 \\ \mathfrak{f} \cdot v_k &= v_{k+1} \\ \mathfrak{h} \cdot v_k &= (\lambda - 2k)v_k. \end{aligned}$$

The Casimir operator multiplies v_0 by $\gamma = \lambda^2/4 + \lambda/2$, and since it commutes with \mathfrak{f} it multiplies all of the v_k by this scalar.

It remains to say, how does \mathfrak{e} act? For each k there exists c_k with

$$\mathfrak{e} \cdot v_k = c_k v_{k-1}.$$

What is an explicit formula for c_k ? By we also know that

$$\gamma v_k = \frac{(\lambda - 2k)^2}{4} v_k + \frac{(\lambda - 2k)}{2} v_k + c_k v_k.$$

Solving

$$\frac{(\lambda - 2k)^2}{4} + \frac{(\lambda - 2k)}{2} + c_k = \frac{\lambda^2}{4} + \frac{\lambda}{2}$$

gives us:

32.1. Proposition. *We have*

$$\mathfrak{e} \cdot v_k = k(\lambda + 1 - k)v_k.$$

Thus $\mathfrak{e} \cdot v_k = 0$ if and only if $k = 0$ or $k = \lambda + 1$. This leads to:

32.2. Corollary. *We have*

$$H^0(\mathfrak{n}, V_\lambda) = \begin{cases} \mathbb{C}v_0 \oplus \mathbb{C}v_{k+1} & \text{if } \lambda = k \geq 0 \\ \mathbb{C}v_0 & \text{otherwise.} \end{cases}$$

As \mathfrak{t} -modules this gives us identities

$$H^0(\mathfrak{n}, V_\lambda) = \begin{cases} \mathbb{C}_k \oplus \mathbb{C}_{-k-2} & \text{if } \lambda = k \geq 0 \\ \mathbb{C}_\lambda & \text{otherwise.} \end{cases}$$

In the exceptional case, you can deduce that the finite-dimensional representation of \mathfrak{sl}_2 of dimension $k + 1$ is a quotient of V_k . It is trivial to see that any \mathfrak{g} -submodule of V_λ must possess a non-zero \mathfrak{n} -invariant vector, so in the second case V_λ is irreducible. Note also that in the exceptional case we have an embedding of V_{-k-1} into V_k .

33. Verma modules and intertwining operators

Let $I = I_\chi^\infty$ for $\chi = \chi_{\lambda, \mu}$. We have seen that if $V = I_w^\infty$ then $V/\mathfrak{n}V$ has dimension one. The isomorphism with \mathbb{C} is given by the map

$$f \mapsto \int_N f(wn) dn.$$

But now let $V = I_1^\infty$. The space I_1^∞ is defined to be the quotient I^∞/I_w^∞ , but it can be described precisely.

Each f in I^∞ determines a map from $U(\mathfrak{g})$ to \mathbb{C} , taking X to $R_X f(1)$. We get therefore a map Ω from I^∞ to $\text{Hom}(U(\mathfrak{g}), \mathbb{C})$. For Y in $U(\mathfrak{b})$ we have

$$\langle \Omega, R_Y f \rangle = \langle d\chi + \rho, Y \rangle \langle \Omega, f \rangle.$$

The image of f therefore lies in $\text{Hom}_{U(\mathfrak{b})}(U(\mathfrak{g}), \mathbb{C}(\chi + \rho))$ (with $U(\mathfrak{b})$ acting on $U(\mathfrak{g})$ on the right). By definition of I_w^∞ this map vanishes on I_w^∞ , hence factors through I_1^∞ . A theorem of Émile Borel says that it induces an isomorphism

$$I_1^\infty \cong \text{Hom}_{U(\mathfrak{b})}(U(\mathfrak{g}), \mathbb{C}(\chi + \rho)).$$

This is a representation of \mathfrak{g} through the left regular action, and it is also a representation of the group B .

The linear functional $f \mapsto f(1)$ from I^∞ is \mathfrak{n} -invariant. It is in effect a kind of distribution on I^∞ with support at 1. It will turn out that there are sometimes other \mathfrak{n} -invariant functionals of a similar kind. They are essentially differential operators. In some cases they amount to de Rham differentials.

Let \widehat{I} be the dual of I^∞ , and let $\mathfrak{g}_\mathbb{C}$ be $\mathbb{C} \otimes \mathfrak{g}$. The dual \widehat{I}_1 is the space of all functionals on I^∞ of the form

$$f \mapsto [R_X f](1)$$

with X in the universal enveloping algebra $U(\mathfrak{g}_\mathbb{C})$. It is a representation of $U(\mathfrak{g}_\mathbb{C})$, by means of right multiplication. It is the dual of I_1^∞ .

The functional Ω_1 is in it, and $R_b \Omega_1 = \chi^{-1} \delta^{-1/2} \Omega_1$. There hence exists a map

$$U(\mathfrak{g}_\mathbb{C}) \otimes_{U(\mathfrak{b}_\mathbb{C})} \chi^{-1} \delta^{-1/2} \rightarrow \widehat{I}_1,$$

and it turns out to be an isomorphism. I have mentioned this before as a consequence of a theorem of E. Borel. This is because G contains $N \cdot N^\circ$ as an open subset, and also $\mathfrak{g} = \mathfrak{n}^\circ \oplus \mathfrak{b}$.

For certain χ , the space \widehat{I}_1 contains other \mathfrak{n} -invariant functionals. Since $\mathfrak{g}_\mathbb{C}$ is isomorphic to $\mathfrak{g} \oplus \mathfrak{g}$, $U(\mathfrak{g}_\mathbb{C})$ is the tensor product $U(\mathfrak{g}) \otimes U(\mathfrak{g})$. To see this, we need to examine the structure. Therefore the space we are examining is the tensor product of two Verma modules. Let ρ be the differential of $\delta^{1/2}$. Our space is

$$U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_{\lambda+\rho}, \quad U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_{\mu+\rho}.$$

We can describe very explicitly its structure. Of course this Verma module is essentially a tensor product of two of them, each for \mathfrak{sl}_2 . Results from the previous section tell us that I/\mathfrak{n} is not obvious only when I^∞ contains some finite-dimensional representation $\rho_k \otimes \rho_\ell$, and then gives rise to some embedding of the quotient by V embeds into some principal series

The differential operators are to $\text{Ind}(z^{-k}\bar{z}^{-\ell})$ (the identity), $\text{Ind}(z^k\bar{z}^{-\ell})$, $\text{Ind}(z^{-k}\bar{z}^\ell)$, $\text{Ind}(z^k\bar{z}^\ell)$ (the residue of T_w). If $k = \ell$, the middle two are unitary principal series. If $k = 0$ or $\ell = 0$, the number decreases. If $k = \ell = 0$, there is only one. This is the residue of T_w at χ . This implies that I_χ in this case is irreducible.

What happens in general is illustrated by the diagram of the real principal series.

Why does this work? The basic idea is to look, not just at n -invariant distributions, but all those annihilated by some power of n . If V is any smooth representation of G , let $\widehat{V}^{[n]}$ be the space of n -torsion in its dual. The basic theorem is that if $V = I^\infty, V_w, V_1$ defined accordingly then the filtration of V gives rise to this filtration:

$$0 \longrightarrow \widehat{V}_1^{[n]} \longrightarrow \widehat{V}^{[n]} \longrightarrow \widehat{V}_w^{[n]}$$

So the problem of finding V/nV reduces to two problems in algebra—finding the n -invariants in each of the outer spaces, and understanding this extension.

I avoid for now explaining in detail why this is true. It is not trivial.

34. The Bruhat filtration

Let $\chi = \chi_{s,n}$. We know that the restriction of $\text{Ind}(\chi|P, G)$ to K is isomorphic to the space of smooth functions f on K such that $f(\pm k) = (-1)^n f(k)$ for all k . In this section, we investigate the restriction to N and N° .

The space $\mathbb{P} = \mathbb{P}^1(\mathbb{R})$ is the union of two copies of \mathbb{R} , one the lines $\langle\langle x, 1 \rangle\rangle$, the other the $\langle\langle 1, x \rangle\rangle$. Fix a function ϕ on \mathbb{P} with compact support on the second, and lift it back to a function on G with support on PN° . The $1 - \varphi$ has support on PwN . For every function f on K define $f_{1,s} = \varphi f_s, f_{w,s} = (1 - \varphi)f_s$, so that $f_s = f_{1,s} + f_{w,s}$. The restriction of the first to N° has compact support, and the restriction of the second to wN . Both vary analytically with s .

Recall that

$$\langle \lambda_w, f \rangle = \int_{\mathbb{R}} f(w n_x) dx \quad \left(n_x = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \right).$$

This converges without difficulty for $f = f_{w,s}$, since the integrand has compact support on \mathbb{R} . It varies analytically with s . What about for $f = f_{1,s}$?

Suppose for the moment that f is an arbitrary smooth function of compact support on N° . It becomes a function in $\text{Ind}^\infty(\chi_{s,n})$ by the specification

$$f(bn^\circ) = \chi_{s,n} \delta^{1/2}(b) f(n^\circ).$$

The reason this defines a smooth function on G is that f has compact support on N° . To see what $\langle \Lambda_w, f \rangle$ is, we must evaluate $f(w n_x)$ —i.e. factor $w n_x$ as $b n^\circ$:

$$\begin{aligned} w n_x &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 \\ 1 & x \end{bmatrix} \\ &= \begin{bmatrix} 1/x & -1 \\ 0 & x \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1/x & 1 \end{bmatrix}. \end{aligned}$$

Thus

$$\langle \Lambda_w, f \rangle = \int_{\mathbb{R}} |x|^{-(1+s)} \text{sgn}^n(x) f(1/x) dx.$$

This makes sense and converges for $\text{RE}(s) > 0$ because $f(1/x) = 0$ for x small, and is bounded. We can write the integral as

$$\int_0^\infty x^{-(1+s)} f(1/x) dx + (-1)^n \int_0^\infty x^{-(1+s)} f(-1/x) dx$$

and then make a change of variables $y = 1/x$, giving

$$\int_0^\infty y^{s-1} f(y) dy + (-1)^n \int_0^\infty y^{s-1} f(-y) dy.$$

In this last integral, $f(y)$ is smooth of compact support. In effect, we are reduced to understanding integration against $y^{s-1} dy$ as a distribution on $[0, \infty)$. The integral converges for $\operatorname{Re}(s) > 0$. We can now integrate by parts:

$$\int_0^\infty y^{s-1} f(y) dy = \frac{-1}{s} \int_0^\infty y^s f'(y) dy,$$

to analytically continue for $\operatorname{Re}(s) > -1$. Continuing, we see that the integral may be defined meromorphically over all of \mathbb{C} with possible poles in $-\mathbb{N}$. The actual location of poles for Λ_w itself depends on n .



35. The Jacquet module



See [Casselman:2011].

36. The Paley-Wiener theorem for spherical functions

Follow Gangolli, prove that what Serge Lang calls the Harish transform is an isomorphism. Note that part of the discussion shows that principal series characters and hyperbolic orbital integrals are dual through a Mellin transform.



37. The canonical pairing



Refer to [Casselman:2011].

38. Whittaker models

Suppose G to be a real reductive Lie group of Harish-Chandra class [HC:], P a minimal parabolic subgroup of G and N the unipotent radical of P . Let K be a maximal compact subgroup of G . Denote by \mathfrak{g}_\circ , \mathfrak{n}_\circ and \mathfrak{k}_\circ the Lie algebras of G , N and K respectively. Let $\mathfrak{g}_\circ = \mathfrak{k}_\circ \oplus \mathfrak{q}_\circ$ be the Cartan decomposition of \mathfrak{g}_\circ . Denote by \mathfrak{a}_\circ a maximal abelian Lie subalgebra of \mathfrak{q}_\circ . Let \mathfrak{g} , \mathfrak{k} , \mathfrak{n} and \mathfrak{a} be the complexifications of \mathfrak{g}_\circ , \mathfrak{k}_\circ , \mathfrak{n}_\circ and \mathfrak{a}_\circ respectively. Let R be the (restricted) root system in \mathfrak{a}^* of the pair $(\mathfrak{g}, \mathfrak{a})$. Then \mathfrak{n} is the direct sum of root subspaces \mathfrak{g}_α for roots α in a set R^+ of positive roots in R . Denote by B the corresponding set of simple roots in R^+ . Let ψ be a unitary character of N , and η its differential. Then η is completely determined by its restrictions to the root subspaces \mathfrak{g}_α corresponding to the roots $\alpha \in B$. The character ψ is said to be **non-degenerate** if all of these restrictions are non-trivial. Denote by \mathbb{C}_η the one-dimensional representation of N with action given by ψ . If (π, V) is a representation of \mathfrak{g} , a **Whittaker functional** on π is an \mathfrak{n} -covariant map from V to \mathbb{C}_η . If π is the restriction to \mathfrak{g} of a smooth representation of G , then a Whittaker functional corresponds to an N -covariant map to \mathbb{C}_η as well, therefore a G -covariant map from V into the space of smooth functions f on G such that

$$f(ng) = \psi(n)f(g)$$

for all $g \in G$, $n \in N$. Functions in the image are called Whittaker functions, because in simple cases they essentially coincide with confluent hypergeometric functions treated extensively in Whittaker's classic text. They, or rather their analogues for certain finite groups, were apparently introduced into representation theory by I. M. Gelfand and colleagues. The most important role they play is probably in constructing archimedean factors of L -functions associated to certain representations of adèle groups, and in particular to automorphic forms. There have been many interesting papers about them, notably Jacquet's thesis [jacq:.] an early paper by Shalika [shalika:.] a paper of Kostant [wh:.] and the thesis of Tze-Ming To [to:.] There remain so far unanswered, however, a few interesting technical but important questions which we propose to answer here. We will also offer new proofs of known results, and along the way explain a fundamental result in the representation theory which as so far been only weakly exploited, what we call the **Bruhat filtration** of the smooth principal series. This is a refinement of much earlier work of Bruhat [bruhat:].

The main results of this paper depend strongly on the Bruhat filtration, and include (1) a proof of the exactness of what we call the Whittaker functor, (2) uniqueness of Whittaker functionals for suitable smooth irreducible representations of quasisplit groups, and (3) holomorphicity of Whittaker functionals for the smooth principal series and other analytic families of induced representations. Many results in this paper have appeared before in various work of others, but not in as useful or as general a form as we would like (see Matumoto [matumoto:.] Wallach [wallach:]). The Bruhat filtration can be used to give more elegant proofs of many other results in the representation theory of G . It might serve, in fact, as the cornerstone of a reasonably elegant exposition of the subject from its beginnings. We hope to return to these ideas in a subsequent paper.

It is well known that Harish-Chandra was much concerned with these matters right up to his untimely death, and we therefore think it is not inappropriate to dedicate this paper to his memory. We would like to thank Hervé Jacquet, who pointed out to us a long time ago that there is some truth to the claim that nearly all of analysis reduces to integration by parts. One of us (D.M.) would also like to thank the Department of Mathematics, Harvard University, for their support during the period when the final draft of the paper was written.

The simplest case

The main part of this paper is tough and abstract, and we want to explain the themes here in the simplest case, when the arguments can be laid out in very classical terms. Let G be the group $SL(2, \mathbb{R})$ of unimodular two-by-two real matrices, P the subgroup of upper triangular matrices and N the unipotent radical of P .

Further, let

$$\psi: N \rightarrow \mathbb{C}, \quad \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \mapsto e^{\eta x}$$

be a unitary character of N , with η purely imaginary. At first we shall make no assumption on η , but eventually we shall assume $\eta \neq 0$.

For $s \in \mathbb{C}$ we define the smooth principal series representation

$$I_s = \{f \in C^\infty(G) \mid f(nag) = |y|^{s+1} f(g)\}$$

for $g \in G, n \in N$, and

$$a = \begin{bmatrix} y & 0 \\ 0 & y^{-1} \end{bmatrix}.$$

The group G is the disjoint union of two subsets PwN and P , where

$$w = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

The space I_s is that of smooth sections of a suitable real analytic line bundle on $\mathbb{P}^1(\mathbb{R})$. It contains the subspace $I_{s,w}$ of all sections vanishing of infinite order at ∞ , and has as quotient the space $I_{s,1}$ of formal sections (Taylor series) of the analytic line bundle at ∞ . For example, if $s = -1$ the space I_s is just $C^\infty(\mathbb{P}^1(\mathbb{R}))$, which is non-canonically the same as the space of smooth functions on the circle, $I_{s,w}$ is the subspace of functions vanishing of infinite order at ∞ , and the quotient may be identified with $\mathbb{C}[1/x]$. The complement of ∞ on $\mathbb{P}^1(\mathbb{R})$ is a single orbit under N , so that functions in $I_{s,w}$ may be identified with functions on N . It is elementary in this case that this allows us to identify $I_{s,w}$ as an N -module with $\mathcal{S}(\mathcal{N})$, the Schwartz space of the additive group. This is essentially a special case of the well known observation of Laurent Schwartz that the Schwartz space of \mathbb{R}^n may be identified with the space of all smooth functions on the unit n -dimensional sphere S^n vanishing of infinite order at the north pole, **via** stereographic projection. The exact sequence

$$0 \rightarrow I_{s,w} \rightarrow I_s \rightarrow I_{s,1} \rightarrow 0$$

is what we call the Bruhat filtration of I_s . The last map is surjective because of the well known result of Émile Borel. Note that these spaces are all stable under N and \mathfrak{g} . The dual of the space $I_{s,1}$, a space of distributions with support at ∞ , may be identified with a certain Verma module over \mathfrak{g} .

For a function $f \in I_{s,w}$, the integral

$$\Omega_{\psi,s}(f) = \int_N \psi^{-1}(n) f(wn) dn$$

converges absolutely, and defines a ψ -covariant functional on $I_{s,w}$ holomorphic in s . The basic question we are going to investigate here is this: to what extent can this functional be extended covariantly to one on all of I_s ? It turns out that the integral defining $\Omega_{\psi,s}$ is convergent for s in a certain half-plane of \mathbb{C} , and that it possesses a meromorphic continuation for all s , holomorphic in the case $\eta \neq 0$. It is the holomorphicity that we are really concerned with here.

These results can be formulated in entirely classical terms.

Some elementary functional analysis

In this section we shall investigate some problems which have no obvious relevance to representation theory, but in the next we shall explain the connection.

Suppose f to be in $\mathcal{S}(\mathbb{R})$, $s \in \mathbb{C}$, $\eta \in i\mathbb{R}$. The integral

$$\int_{-\infty}^{\infty} e^{-\eta/x} |x|^s f(x) dx$$

converges for $\operatorname{RE}(s) > -1$ and defines in this region a tempered distribution on \mathbb{R} . As long as f vanishes of infinite order at 0 this integral is defined and analytic for all values of s . Does it have for all s a natural extension to all of $\mathcal{S}(\mathbb{R})$?

If $\eta = 0$ the answer is negative, but the situation is perfectly well understood. Integration by parts gives us

$$\int_{-\infty}^{\infty} |x|^s f(x) dx = -1/s + 1 \int_{-\infty}^{\infty} |x|^s x f'(x) dx$$

and this allows us to continue the distribution at least into the region where $\operatorname{RE}(s) > -2$. If we multiply by $s + 1$ and take the limit as $s \rightarrow -1$ we get

$$- \int_{-\infty}^{\infty} \operatorname{sgn}(x) f'(x) dx = \int_{-\infty}^0 f'(x) dx - \int_0^{\infty} f'(x) dx = 2f(0).$$

In other words, integration by parts allows to extend the distribution meromorphically to the region $\operatorname{RE}(s) > -2$, with a residue at $s = -1$ equal to $2\delta_0$. Of course we can again integrate by parts and eventually we see that we have a meromorphic map from \mathbb{C} to the space of tempered distributions on \mathbb{R} , with simple poles at $-1, -3, -5, \dots$ and residues that are proportional to derivatives of δ_0 .

Now suppose that $\eta \neq 0$. Since η is purely imaginary, the function $e^{-\eta/x}$ oscillates wildly near 0, and it is reasonable to expect that the integral

$$\int_{-\infty}^{\infty} e^{-\eta/x} |x|^s f(x) dx$$

can be holomorphically extended. A different kind of integration by parts will do the trick. The function

$$\varphi_s(x) = |x|^s e^{-\eta/x}$$

satisfies the differential equation

$$x^2 \varphi'_s(x) - (sx + \eta) \varphi_s(x) = 0$$

Therefore the integral becomes

$$\begin{aligned} \int_{-\infty}^{\infty} \varphi_s(x) f(x) dx &= 1/\eta \int_{-\infty}^{\infty} x^2 \varphi'_s(x) f(x) dx - 1/\eta \int_{-\infty}^{\infty} sx \varphi_s(x) f(x) dx \\ &= -1/\eta \int_{-\infty}^{\infty} x^2 \varphi_s(x) f'(x) dx - 1/\eta \int_{-\infty}^{\infty} (s+2)x \varphi_s(x) f(x) dx \\ &= 1/\eta \int_{-\infty}^{\infty} \varphi_s(x) (-x^2 f'(x) - (s+2)x f(x)) dx. \end{aligned}$$

This converges for $\operatorname{RE}(s) > -2$, since the function $x \mapsto -x^2 f'(x) - (s+2)x f(x)$ vanishes at 0. As before, we can repeat this procedure, eventually obtaining a distribution holomorphic as a function of s .

What is relevant for the rest of this paper is that the second argument reduces in essence to this observation:

The differential operator

$$f(x) \mapsto -x^2 f'(x) - (s+2)xf(x)$$

induces a topologically nilpotent map of the space of formal power series $\mathbb{C}[[x]]$ into itself.

RELEVANCE TO REPRESENTATION THEORY. Return to the notation of the earlier section on $\mathrm{SL}(2, \mathbb{R})$. The group G is the disjoint union of PwN and P , and the overlapping union of open subsets PwN and $P\bar{N}$, where \bar{N} is the subgroup of lower unipotent matrices. A function in I_s may therefore be expressed as a sum of functions f_w and f_1 , where f_w has compact support modulo P on PwN and f_1 has compact support modulo P on $P\bar{N}$. In considering how to extend the functional $\Omega_{\psi,s}$, we may assume $f = f_1$.

Therefore suppose now that f has support on $P\bar{N}$. How do we evaluate $\Omega_{\psi,s}(f)$?

We have

$$\Omega_{\psi,s}(f) = \int_N \psi^{-1}(n) f(wn) dn$$

and also $f(p\bar{n}) = |c|^{s+1} f(\bar{n})$ if $p \in P$, $\bar{n} \in \bar{N}$, where the restriction of f to \bar{N} may be identified with a function $\varphi(y)$ of compact support. Here

$$p = \begin{bmatrix} c & x \\ 0 & c^{-1} \end{bmatrix}, \quad \bar{n} = \begin{bmatrix} 1 & 0 \\ y & 1 \end{bmatrix}.$$

We can write

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & x \end{bmatrix} = \begin{bmatrix} 1 & -x^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x^{-1} & 0 \\ 0 & x \end{bmatrix} \begin{bmatrix} 1 & 0 \\ x^{-1} & 1 \end{bmatrix}$$

and therefore the integral becomes

$$\int_{-\infty}^{\infty} |x|^{-s-1} \varphi(x^{-1}) e^{-\eta x} dx = - \int_{-\infty}^{\infty} |z|^{s-1} e^{-\eta/z} \varphi(z) dz$$

which of course leads us back to the calculation in the previous section.

This is admittedly elementary, but not entirely satisfactory. The explicit calculation does not really tell us what is going on in terms of representations of $\mathrm{SL}(2, \mathbb{R})$. So we shall now reformulate it. Consider the exact sequence

$$0 \rightarrow I_{s,w} \rightarrow I_s \rightarrow I_{s,1} \rightarrow 0$$

as a sequence of modules over \mathfrak{n} , where \mathfrak{n} is the Lie algebra of N . The integral defining the functional $\Omega_{\psi,s}$ induces, as we have seen, an \mathfrak{n} -covariant map from $I_{s,w}$ to \mathbb{C}_η , or equivalently an \mathfrak{n} -covariant map from $\mathbb{C}_{-\eta}$ to the strong dual $I_{s,w}'$ of $I_{s,w}$, or again equivalently an \mathfrak{n} -invariant element of $I_{s,w}' \otimes \mathbb{C}_\eta$. We can now, however, look at the long Lie algebra cohomology sequence derived from the exact sequence above:

$$\begin{aligned} 0 &\rightarrow H^0(\mathfrak{n}, \mathcal{I}_{s,1}' \otimes \mathbb{C}_\eta) \rightarrow \mathfrak{H}^0(\mathfrak{n}, \mathcal{I}_s' \otimes \mathbb{C}_\eta) \rightarrow \mathfrak{H}^0(\mathfrak{n}, \mathcal{I}_{s,w}' \otimes \mathbb{C}_\eta) \\ &\rightarrow H^1(\mathfrak{n}, \mathcal{I}_{s,1}' \otimes \mathbb{C}_\eta) \rightarrow \mathfrak{H}^1(\mathfrak{n}, \mathcal{I}_s' \otimes \mathbb{C}_\eta) \rightarrow \mathfrak{H}^1(\mathfrak{n}, \mathcal{I}_{s,w}' \otimes \mathbb{C}_\eta) \rightarrow \mathfrak{o}. \end{aligned}$$

We have already remarked that $I_{s,1}'$ is a Verma module, which means that \mathfrak{n} acts locally nilpotently on it, and this implies in turn that each nonzero element of \mathfrak{n} acts bijectively on $I_{s,1}' \otimes \mathbb{C}_\eta$, as long as ψ is not trivial. Therefore the \mathfrak{n} -cohomology of $I_{s,1}'$ vanishes completely, and the map

$$H^0(\mathfrak{n}, \mathcal{I}_s' \otimes \mathbb{C}_\eta) \rightarrow \mathfrak{H}^0(\mathfrak{n}, \mathcal{I}_{s,w}' \otimes \mathbb{C}_\eta)$$

is an isomorphism. This guarantees the canonical extension of $\Omega_{\psi,s}$ from $I_{s,w}$ to all of I_s . Holomorphicity in s follows, among other reasons, from a slightly more technical argument about various topological spaces of holomorphic functions, as will be shown in the main part of this paper.

How does this argument relate to the simple one in the previous section? The Lie algebra \mathfrak{n} of N is spanned by the matrix

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

The action of the corresponding differential operator on functions in I_s restricted to \overline{N} is

$$-y^2(d/dy) - (s+1)y$$

if we identify \overline{N} with \mathbb{R} as before. The Verma module $I_{s,1}'$ corresponds to the module of distributions with the support at 0. Therefore, the local nilpotency we mentioned before is just the statement dual to the observation from the end of last subsection.

39. The module

Recall the factorization

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1/c & a \\ 0 & c \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & d/c \\ 0 & 1 \end{bmatrix}.$$

Suppose F to lie in I_s . What I want to know is how \mathfrak{g} acts on the restriction of F to wN , and more particularly how it acts on the dual of I_w , which is the \mathfrak{g} -module defined by Kostant associated to the infinitesimal character by which $Z(\mathfrak{g})$ acts in the dual of I_s .

Let $f(x) = f_F(x) = F(wn_x)$. What are $f_{R_\nu f}$, $f_{R_{\overline{\nu}} f}$, $f_{R_\alpha f}$? The first is very easy.

$$f_{R_\nu f} = f'. \quad (1)$$

As for the second, let

$$e(t) = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}$$

I must calculate the restriction to wn of

$$\left. \frac{d}{dt} \right|_{t=0} R_{e(t)} F.$$

I start with

$$\begin{aligned} wn_x e(t) &= we(t) \cdot e(-t) n_x e(t) \\ &= e(-t) wn_{e^{-2t} x} \end{aligned}$$

from which we deduce that

$$R_\alpha F \mapsto -(s+1)f - 2x f'. \quad (2)$$

Finally, for the third, if we apply the Bruhat factorization we see that

$$wn_x \overline{n}_t = \begin{bmatrix} -t & -1 \\ 1+xt & x \end{bmatrix} = \begin{bmatrix} 1/(1+xt) & -t \\ 0 & 1+xt \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & x/(1+xt) \\ 0 & 1 \end{bmatrix}.$$

And from this it follows that

$$R_{\overline{\nu}} F \mapsto -(s+1)xf - x^2 f'. \quad (3)$$

Suppose Ψ to be a non-trivial unitary character of N . Let $\psi = d\Psi(\nu)$. Then $\psi \neq 0$ in $i\mathbb{R}$, say

$$n_x \mapsto e^{2\pi i x} \quad (\psi = 2\pi i).$$

Kostant's Whittaker module is the subspace of $(\nu - \psi)$ -torsion in the dual of $I_{w,s}$. It is spanned by the functionals

$$\langle \Omega_{\psi, P}, F \rangle = \int_N P(n) \psi^{-1}(n) F(wn) dn = \int_{\mathbb{R}} p(x) e^{-2\pi i x} f(x) dx.$$

in which P is an affine function on N and, equivalently, p is the polynomial on \mathbb{R} such that $P(n_x) = p(x)$. The function $f(x)$ now lies in the Schwartz space $\mathcal{S}(\mathbb{R})$. I want now to find explicit formulas for how \mathfrak{g} acts on this.

The Whittaker dual of I_s is the same as that of $I_{w,s}$ because of unique extension. If we apply the Whittaker integral with $p = 1$ to various K -eigenfunctions, we get various forms of the classical Whittaker function. There are two such functions, and the one that arises here is that with rapidly decreasing behaviour at infinity (because of uniqueness). If I can evaluate its behaviour near 0, I should recover a classical formula.

40. Whittaker models for $SL(2)$

Suppose ψ to be a non-trivial unitary character of N . A Whittaker function on $SL_2(\mathbb{R})$ is an eigenfunction of Ωf satisfying the equation $f(ng) = \psi(n)f(g)$ for all n in N , g in G . We shall be interested in the ones that are also eigenfunctions with respect to the right action of K — $f(gk) = \varepsilon^m(k)f(g)$ for some m , all k in K . Such functions are determined by their restriction to A , on which they satisfy a **Whittaker differential equation**. *What is the equation?*

This involves a calculation much like one we have already seen. Every X in \mathfrak{g} may be expressed as a sum

$$X = X_{\mathfrak{n}} + X_{\mathfrak{a}} + X_{\mathfrak{k}}$$

with $X_{\mathfrak{n}}$ in \mathfrak{n} , etc. For every a in A , we may express $X_{\mathfrak{n}} = a^{-1}\alpha(a)X_{\mathfrak{n}}a$, so that

$$[R_X f](a) = \alpha(a)[\Lambda_{a^{-1}X_{\mathfrak{n}}a} f](a) + [R_{X_{\mathfrak{a}}} f](a) + [R_{X_{\mathfrak{k}}} f](a)$$

and then that any element X of $U(\mathfrak{g})$ may be expressed as a sum of products

$$a^{-1}X_{\mathfrak{n}}a \cdot X_{\mathfrak{a}}X_{\mathfrak{k}}.$$

In these circumstances the restriction of $R_X f$ to A will be the corresponding sum of terms

$$[\Lambda_{aX_{\mathfrak{n}}a^{-1}R_{X_{\mathfrak{a}}}R_{X_{\mathfrak{k}}} f](a).$$

Applied to a Whittaker function f such that $R_{\Omega}f = \gamma f$ this gives

$$y^2 \frac{\partial^2 f}{\partial y^2} + \dots$$

Part VII. References

1. Jeffrey Adams and David Vogan, 'L-groups, projective representations, and the Langlands classification', *American Journal of Mathematics* **114** (1992), 45–138.
2. Michael Atiyah, 'Characters of semi-simple groups', 399–557 in volume 4 of **Collected Works**, Oxford Press, 1988.
3. Valentine Bargmann, 'Irreducible representations of the Lorentz group', *Annals of Mathematics* **48** (1947), 568—640.
4. Alexander Beilinson and Joseph Bernstein, 'A generalization of Casselman's submodule theorem', pp. 35–52 in **Representation theory of reductive groups**, edited by Peter C. Trombi, Birkhäuser, Boston, 1982.
5. Joseph Bernstein and Bernhard Krötz, 'Smooth Fréchet globalizations of Harish-Chandra modules', preprint [arXiv.org:0812.1684](https://arxiv.org/abs/0812.1684), 2010.
6. Armand Borel, **Linear algebraic groups**, Springer, 1995.
7. ——— and Bill Casselman (editors), **Automorphic forms, representations, and L-functions**, *Proceedings of Symposia in Pure Mathematics XXXIII*, American Mathematical Society, 1979.
8. Émile Borel, 'Sur quelques points de la théorie des fonctions', *Annales Scientifiques de École Normale Supérieure*, **12** (1895), 9–55.
9. Fred Brauer and John A. Nohel, **Ordinary differential equations - a first course**, W. A. Benjamin, 1967

10. Bill Casselman, 'Canonical extensions of Harish-Chandra modules to representations of G' ', *Canadian Journal of Mathematics* **41** (1989), 385–438.
11. —, 'Computations in real tori', pp. 137–151 in **Representation theory of real reductive Lie groups**, *Contemporary Mathematics* **472**, American Mathematical Society, 2008.
12. —, 'Jacquet modules and the asymptotic behaviour of matrix coefficients', pp. 81–97 in **On certain L -functions**, American Mathematical Society, 2011.
13. —, 'Unitary representations', last updated in 2014. Available at
<http://www.math.ubc.ca/~cass/research/pdf/Unitary.pdf>
14. —, Henryk Hecht, and Dragan Miličić, article Bruhat filtrations and Whittaker vectors for real groups, pp. 151–190 in **The mathematical legacy of Harish-Chandra**, *Proceedings of Symposia in Pure Mathematics* **68**, American Mathematical Society, 1998.
15. J. W. S. Cassels and A. Fröhlich, **Algebraic Number Theory**, Academic Press, 1967.
16. Augustin Louis Cauchy, 'Sur les intégrales définies prises entre les limites imaginaires', *Bulletin de Férussac III* (1825), 214–221.
17. Earl A. Coddington and Norman Levinson, **Theory of ordinary differential equations**, McGraw-Hill, 1955.
18. Pierre Deligne, 'Les constantes des équations fonctionnelles des fonctions L ', pp. 501–95 in **Antwerp II**, *Lecture Notes in Mathematics* **349**, Springer, 1973.
19. Paul Garrett, 'Irreducibles as kernels of intertwinings among principal', preprint, 2009. Available from
<http://www.math.umn.edu/~garrett/m/v/>
20. Anthony Knapp, **Representation theory of semisimple groups—an overview based on examples**, Princeton University Press, 1986.
21. Jean-Pierre Labesse, 'Cohomologie, L -groupes et functorialité', *Compositio Mathematica* **55** (1984), 163–184.
22. Robert P. Langlands, 'Representations of abelian algebraic groups', *Pacific Journal of Mathematics* (special issue, 1997), 231–250.
23. —, 'On the classification of irreducible representations of real algebraic groups', pp. 101–170 in **Representation theory and harmonic analysis on semisimple Lie groups**, *Mathematics Surveys Monographs* **31**, American Mathematical Society, 1989.
24. Laurent Schwartz, **Méthodes mathématiques pour les sciences physiques**, Hermann, 1965.
25. Alok Shukla, 'A useful integration formula', at
<http://www2.math.ou.edu/~ashukla/Integral1.pdf>
26. Tonny Springer and Robert Steinberg, 'Conjugacy classes', pp. 167–266 in **Seminar in algebraic groups and related finite groups**, *Lecture notes in Mathematics* **131**, 1970.
27. John Tate, 'Fourier analysis in number fields and Hecke's zeta functions', pp. 305–47 in [Cassels-Fröhlich:1967].
28. —, 'Number theoretic background', pp. 3–26 in [Borel-Casselman:1979].
29. D. Topping, **Lectures on von Neumann algebras**, van Nostrand, 1968.
30. David Vogan, **Representations of real reductive groups**, Birkhäuser, 1981.
31. Garth Warner, **Harmonic analysis on semi-simple Lie groups I**, Springer-Verlag, 1972.

- 32.** André Weil, **L'intégration dans les groupes topologiques**, Hermann, 1965.