Introduction to admissible representations of p-adic groups

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Chapter III. Induced representations and the Jacquet module

In this chapter, let G be the $\mathfrak{k}$-rational points on a Zariski-connected reductive group defined over $\mathfrak{k}$. I'll introduce here representations induced from parabolic subgroups to G, as well as a related adjoint construction going from representations of G to those of parabolic subgroups.

These constructions will lead eventually to a rough classification of irreducible admissible representations of G. Parabolic induction is a classic technique in representation theory, but the adjoint construction has its origins in [Jacquet-Langlands:1970] and [Jacquet:1971].

Contents

SUMMARY. If $P = MN$ is a parabolic subgroup of G and V a smooth representation of G, its Jacquet module is its maximal N-trivial quotient V_N. This is a smooth representation of $M = P/N$, and if V is an admissible representation of G defined over a field the Jacquet module is an admissible representation of M. It is almost by definition that the Jacquet module controls equivariant maps from V into representations induced from P to G, but it also controls the asymptotic behaviour of matrix coefficients of V. This dual role originates in work of Harish-Chandra on representations of real reductive groups, and is the basis for harmonic analysis on G.

1. Representations induced from parabolic subgroups

Suppose P to be a parabolic subgroup of G, and (σ, U) an admissible representation of P. The (normalized) representation induced from σ is the right regular representation of G on the space

$$\mathrm{Ind}(\sigma|P,G) = \left\{ f \in C^\infty(G,U) \,\middle|\, f(pg) = \delta_P^{1/2}(p)\sigma(p)f(g) \text{ for all } p \in P, g \in G \right\}.$$

Here

$$\delta_P(p) = |\det_{\mathfrak{n}}(p)|$$

is the modulus character of P. Since $P\backslash G$ is compact, according to his representation is admissible.

Bruhat and Tits define a **good** compact subgroup of G to be a compact open K such that $G = PK$ for all parabolic subgroups P. They prove that such exist. If K is good then the restriction of $\mathrm{Ind}(\sigma|P,G)$ to K is a K-isomorphism of $\mathrm{Ind}(\sigma|P,G)$ with $\mathrm{Ind}(\sigma|K \cap P, K)$. From now on, assume K to be a good compact.

An **unramified character** of P is one that is trivial on $K \cap P$. It follows from the observation above that if χ is unramified then $\mathrm{Ind}(\sigma|P,G)$ and $\mathrm{Ind}(\sigma\chi|P,G)$ are canonically isomorphic as K-representations. Another way of putting this is that these representations may be defined on the same space.

If φ lies in $\mathrm{Ind}(\delta_P^{1/2} \mid P, G)$ the integral

$$\int_K \varphi(k)\, dk$$

defines a G-invariant linear functional. Since G/P is compact, mplies:

III.1.1. Proposition. *The contragredient of* $\mathrm{Ind}(\sigma|P,G)$ *is isomorphic to* $\mathrm{Ind}(\widetilde{\sigma}|P,G)$.

If f lies in the first and φ the second, then the product-pairing $\langle f(x), \varphi(x)\rangle$ lies in $\mathrm{Ind}(\delta^{1/2}|P,G)$. The pairing can then be chosen to be

$$\int_K \langle f(k), \varphi(k)\rangle\, dk\,.$$

III.1.2. Proposition. *The representation* $\mathrm{Ind}(\sigma|P,G)$ *is unitary if* σ *is .*

This choice of invariant integral is by no means canonical, and in other contexts other choices are natural.

There is a great deal more to be said about these representations, but first we need to investigate admissible representations of P.

2. Admissible representations of parabolic subgroups

Let $P = M_P N_P = MN$ be a parabolic subgroup of G, $A = A_P$ the split centre of M_P. There exists a basis of neighbourhoods of P of the form $U_M U_N$ where U_M is a compact open subgroup of M, U_N is one of N, and U_M conjugates U_N to itself.

The group M may be identified with a quotient of P, and therefore the admissible representations of M may be identified with those of P trivial on N. It happens that there are no others:

III.2.1. Proposition. *Every admissible representation of* P *is trivial on* N.

Proof. Let (π, V) be an admissible representation of P over the (Noetherian) ring $\mathcal{R}$, and suppose v in V. We want to show that $\pi(n)v = v$ for all n in N.

So suppose n in N, and suppose $U = U_M U_N$ fixes v. We can find a in A such that $a^{-1}na \in U_N$ as well as $a^{-1}Ua \subseteq U$. Then $\pi(a)$ takes V^U to itself, since for u in U we have

$$\pi(u)\pi(a)v = \pi(a)\pi(a^{-1}ua)v = \pi(a)v\,.$$

The operator $\pi(a)$ is certainly injective. I shall prove in a moment that it is bijective on V^U. Assuming this, we can find v_* such that $\pi(a)v_* = v$, and then:

$$\begin{aligned} \pi(n)v &= \pi(n)\pi(a)\pi(a^{-1})v \\ &= \pi(n)\pi(a)v_* \\ &= \pi(a)\pi(a^{-1}na)v_* \\ &= \pi(a)v_* \\ &= v\,. \end{aligned}$$

If $\mathcal{R}$ is a field, the claim that $\pi(a)$ is surjective is is trivial, since V^U is finite-dimensional and $\pi(a)$ is injective. But if $\mathcal{R}$ is an arbitrary commutative Noetherian ring, one has to be a bit more careful. The following will tell us what is needed:

III.2.2. Lemma. *Suppose B to be a finitely generated module over the Noetherian ring R. If $f\colon B \to B$ is an R-injection with the property that for each maximal ideal $\mathfrak{m}$ of R the induced map $f_{\mathfrak{m}}\colon B/\mathfrak{m}B \to B/\mathfrak{m}B$ is also injective, then f is itself an isomorphism.*

Proof. Let C be the quotient $B/f(B)$. The exact sequence

$$0 \longrightarrow B \xrightarrow{f} B \longrightarrow C \longrightarrow 0$$

induces for each $\mathfrak{m}$ an exact sequence

$$0 \longrightarrow B/\mathfrak{m}B \xrightarrow{f_{\mathfrak{m}}} B/\mathfrak{m}B \longrightarrow C/\mathfrak{m}C \longrightarrow 0\ .$$

It is by assumption that the left hand map is injective. Since $F = R/\mathfrak{m}$ is a field and B is finitely generated, the space $B/\mathfrak{m}B$ is a finite-dimensional vector space over F, and therefore $f_{\mathfrak{m}}$ an isomorphism. Hence $C/\mathfrak{m}C = 0$ for all $\mathfrak{m}$. The module C is Noetherian, which means that if $C \neq 0$, it possesses at least one maximal proper submodule D. The quotient C/D must be isomorphic to $R/\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$. But then $C/\mathfrak{m}C \neq 0$, a contradiction. Therefore $C = 0$ and f an isomorphism.

3. The Jacquet module

Suppose $P = MN$ to be a parabolic subgroup of G. (π, V) an admissible representation of G, and (σ, U) one of P. Frobenius reciprocity (tells us that

$$\mathrm{Hom}_G\big(V, \mathrm{Ind}(\sigma|P, G)\big) \cong \mathrm{Hom}_P(V, U)\,,$$

while the results of the previous section tell us that σ is trivial on N and factors through the canonical projection $P \to M$. In this section we explore the consequences of joining these two facts.

III.3.1. Lemma. *If N is a $\mathfrak{p}$-adic unipotent group, it possesses arbitrarily large compact open subgroups.*

Proof. It is certainly true for the group of unipotent upper triangular matrices in GL_n. Here, if a is the diagonal matrix with $a_{i,i} = \varpi^i$ then conjugation by powers of a will scale any given compact open subgroup to an arbitrarily large one. But any unipotent group can be embedded as a closed subgroup in one of these.

Fix the parabolic subgroup $P = MN$. If (π, V) is any smooth representation of N, define $V(N)$ to be the subspace of V generated by vectors of the form

$$\pi(n)v - v$$

as n ranges over N. The group N acts trivially on the quotient

$$V_N = V/V(N)$$

It is universal with respect to this property:

III.3.2. Proposition. *The projection from V to V_N induces for every smooth $\mathcal{R}$-representation (σ, U) on which N acts trivially an isomorphism*

$$\mathrm{Hom}_N(V, U) \cong \mathrm{Hom}_{\mathcal{R}}(V_N, U)\ .$$

III.3.3. Lemma. *For v in V the following are equivalent:*

(a) *v lies in $V(N)$;*
(b) *v lies in $V(U)$ for some compact open subgroup of N;*
(c) *we have*

$$\int_U \pi(u)v\,du = 0$$

for some compact open subgroup U of N.

Proof. The equivalence follows immediately from . That of (b) and (c) follows from

III.3.4. Proposition. *If*

$$0 \to U \to V \to W \to 0$$

is an exact sequence of smooth representations of N, then the sequence

$$0 \to U_N \to V_N \to W_N \to 0$$

is also exact.

Proof. That the sequence

$$U_N \to V_N \to W_N \to 0$$

is exact follows immediately from the definition of $V(N)$. The only non-trivial point is the injectivity of $U_N \to V_N$. If u in U lies in $V(N)$ then it lies in $V(S)$ for some compact open subgroup S of N. According to the space V has a canonical decomposition

$$V = V^S \oplus V(S)\ ,$$

and v lies in $V(S)$ if and only if

$$\int_S \pi(s)v\,ds = 0$$

But this last equation holds in U as well, since U is stable under S, so v must lie in $U(S)$.

If σ is trivial on N, any P-map from V to U factors through V_N. The space $V(N)$ is stable under P, and there is hence a natural representation of M on V_N. The **Jacquet module** of π is this representation twisted by the character $\delta_P^{-1/2}$. In other words, if u lies in $V/V(N)$ and v in V has image u, then

$$\pi_N(m)u \text{ is the image of } \delta_P^{-1/2}(m)\pi(m)v$$

for m in M. This is designed exactly to allow the simplest formulation of this:

III.3.5. Proposition. *If (π, V) is any smooth representation of G and (σ, U) one of M that is trivial on N, then evaluation at 1 induces an isomorphism*

$$\mathrm{Hom}_G\big(\pi, \mathrm{Ind}(\sigma|P,G)\big) \cong \mathrm{Hom}_M(\pi_N, \sigma)$$

According to the hypothesis on σ holds in particular if it is admissible.

Proof. From

4. Iwahori factorizations

Suppose $P = MN$ to be a parabolic subgroup of G, $\overline{P}$ the corresponding opposite one, with $\overline{P} \cap P = M$. If K is a compact open subgroup, it is said to have an **Iwahori factorization** with respect to P if (a) the product map from $\overline{N}_K \times M_K \times N_K$ to K is a bijection, where $M_K = M \cap K$ etc. and (b) $aN_K a^{-1} \subset N_K$, $a^{-1}\overline{N}_K a \subset \overline{N}_K$ for every a in A_P^{--}.

III.4.1. Lemma. *Let $P_\emptyset$ be a minimal parabolic subgroup of G. There exists a basis $\{K_n\}$ of neighbourhoods of $\{1\}$ in G such that*

- *(a) each K_n is a normal subgroup of K_0;*
- *(b) If P is a parabolic subgroup of G containing $P_\emptyset$ then K_n has an Iwahori factorization with respect to P;*
- *(c) if $P = MN$ is a parabolic subgroup containing $P_\emptyset$ then $M \cap K_n$ has an Iwahori factorization with respect to $M \cap P_\emptyset$.*

Proof. Keep in mind that the statement depends only on the conjugacy class of the parabolic group. Assume first that G is split over $\mathfrak{k}$. According to [Iwahori-Matsumoto:1967], there exists a split group scheme $G_\mathfrak{o}$ defining G over $\mathfrak{k}$ by base extension. We may choose $P_\emptyset$ also to be obtained by base extension from a smooth parabolic subgroup defined over $\mathfrak{o}$. The sequence of congruence subgroups $G(\mathfrak{p}^n)$ for $n \geq 1$ satisfies the conditions of the Lemma.

Now suppose G arbitrary. Let $\mathfrak{l}/\mathfrak{k}$ be a finite Galois extension $\mathfrak{l}/\mathfrak{k}$ over which G splits. Let $K_{\mathfrak{l},n}$ be a sequence satisfying the Proposition for a minimal parabolic subgroup contained in $P_\emptyset \times_\mathfrak{k} \mathfrak{l}$, and let $K_n = K_{\mathfrak{l},n} \cap G$. Galois theory together with uniqueness of the Iwahori factorizations allows us to conclude.

5. Admissibility of the Jacquet module

Now fix an admissible representation (π, V) of G. Let P, $\overline{P}$ be an opposing pair of parabolic subgroups, K_0 to be a compact open subgroup possessing an Iwahori factorization $K_0 = N_0 M_0 \overline{N}_0$ with respect to this pair. For each a in A_P^- let T_a be the smooth distribution $\mu_{K_0 a K_0/K_0}$ on G. For any smooth representation (π, V) and v in V^{K_0} let τ_a be the restriction of $\pi(T_a)$ to V^{K_0}.

III.5.1. Lemma. *For v in V^{K_0}*

$$\tau_a v = \pi(a) \sum_{a^{-1}N_0 a/N_0} \pi(n) v\,.$$

Proof. By definition

$$\begin{aligned}\tau_a(v) &= \pi(T_a)\, v \\ &= \sum_{K_0 a K_0/K_0} \pi(g)\, v \\ &= \sum_{K_0/K_0 \cap a K_0 a^{-1}} \pi(k)\pi(a)\, v\,.\end{aligned}$$

This is valid since the isotropy subgroup of a in the action of K_0 acting on $K_0 a K_0/K_0$ is $aK_0 a^{-1} \cap K_0$, hence

$$k \mapsto kaK_0$$

is a bijection of $K_0/K_0 \cap aK_0 a^{-1}$ with $K_0 a K_0/K_0$.

We also have $K_0 = N_0 M_0 \overline{N}_0$ and $aK_0 a^{-1} = (aN_0 a^{-1})M_0(aN_0 a^{-1})$. Since $\overline{N} \subseteq a\overline{N}a^{-1}$, the inclusion of $N_0/aN_0 a^{-1}$ into $K_0/(aK_0 a^{-1} \cap K_0)$ is in turn a bijection. Since the index of $aN_0 a^{-1}$ in N_0 or, equivalently, that of N_0 in $a^{-1}N_0a$ is $\delta_P^{-1}(a)$:

$$\begin{aligned}\tau_a v &= \sum_{K_0/aK_0a^{-1}\cap K_0} \pi(k)\pi(a)v \\ &= \sum_{N_0/aN_0a^{-1}} \pi(n)\pi(a)v \\ &= \pi(a) \sum_{a^{-1}N_0a/N_0} \pi(n)v\ .\end{aligned}$$

Since $\pi(n)v$ and v have the same image in V_N, and $[a^{-1}N_0a\colon N_0] = \delta_P^{-1}(a)$:

III.5.2. Lemma. *If v lies in V^{K_0} with image u in V_N, the image of $\tau_a v$ in V_N is equal to $\delta_P^{-1/2}(a)\pi_N(a)u$.*

III.5.3. Lemma. *For every a, b in A_P^{--},*

$$\tau_{ab} = \tau_a\tau_b$$

Proof. We have

$$\begin{aligned}\tau_a\tau_b v &= \sum_{N_0/aN_0a^{-1}} \sum_{N_0/bN_0b^{-1}} \pi(n_1)\pi(a)\pi(n_2)\pi(b)v \\ &= \sum_{N_0/aN_0a^{-1}} \sum_{N_0/bN_0b^{-1}} \pi(n_1)\pi(an_2a^{-1})\pi(ab)v \\ &= \sum_{N_0/abN_0b^{-1}a^{-1}} \pi(n)\pi(ab)v \\ &= \tau_{ab}v\end{aligned}$$

since as n_1 ranges over representatives of N_0/aN_0a^{-1} and and n_2 over representatives of N_0/bN_0b^{-1}, the products $n_1\, an_2a^{-1}$ range over representatives of $N_0/abN_0b^{-1}a^{-1}$.

III.5.4. Lemma. *For any a in A_P^{--} the subspace of V^{K_0} on which τ_a acts nilpotently coincides with $V^{K_0} \cap V(N)$.*

Proof. Since $\mathcal{R}$ is Noetherian and V^{K_0} finitely generated, the increasing sequence

$$\ker(\tau_a) \subseteq \ker(\tau_{a^2}) \subseteq \ker(\tau_{a^3}) \subseteq \dots$$

is eventually stationary. It must be shown that it is the same as $V^{K_0} \cap V(N)$.

Choose n large enough so that $V^{K_0} \cap V(N) = V^{K_0} \cap V(a^{-n}N_0a^n)$. Let $b = a^n$. Since

$$\tau_b v = \pi(b) \sum_{b^{-1}N_0b/N_0} \pi(n)v\ ,$$

and $\tau_b v = 0$ if and only if $\sum_{b^{-1}N_0b/N_0} \pi(n)v = 0$, and again if and only if v lies in $V(N)$.

The canonical map from V to V_N takes V^{K_0} to $V_N^{M_0}$. The kernel of this map is $V \cap V(N)$, which by is equal to the kernel of τ_{a^n} for $n \gg 0$. If $\mathcal{R}$ is a field, the Jordan decomposition asserts that there is a unique τ_a-stable complement $V_N^{K_0}$ on which τ_a is invertible. It is also the image of τ_{a^n} if $n \gg 0$.

From now on in this chapter, I assume $\mathcal{R}$ to be a field.

III.5.5. Proposition. *The canonical projection from $V_N^{K_0}$ to $V_N^{M_0}$ is an isomorphism.*

Proof. It suffices to show that it is surjective. Suppose given u in $V_N^{M_0}$. Since M_0 is compact, we can find v in V^{M_0} whose image in V_N is u. Suppose that v is fixed also by $\overline{N}_*$ for some small $\overline{N}_*$. If we choose b in A_P^{--} such that $b\overline{N}_0 b^{-1} \subseteq \overline{N}_*$, then $v_* = \delta^{1/2}(b)\pi(b)v$ is fixed by $M_0\overline{N}_0$. Because $K_0 = N_0 M_0 \overline{N}_0$, the average of $\pi(n)v_*$ over N_0 is the same as the average of $\pi(k)v_*$ over K_0. This average lies in V^{K_0} and has image $\pi_N(b)u$ in V_N. But then $\tau_a v_*$ has image $\delta^{1/2}(a)\pi_N(ab)u$ in V_N and also lies in $V_N^{K_0}$. Since τ_{ab} acts invertibly on $V_N^{K_0}$, we can find v_{**} in $V_N^{K_0}$ such that $\tau_{ab}v_{**} = \tau_a\tau_b v_{**} = \tau_a v_*$, and whose image in V_N is u.

As a consequence:

III.5.6. Theorem. *If (π, V) is an admissible representation of G then (π_N, V_N) is an admissible representation of M.*

Thus whenever K_0 is a subgroup possessing an Iwahori factorization with respect to P, we have a canonical subspace of V^{K_0} projecting isomorphically onto V^{M_0}. For a given M_0 there may be many different K_0 suitable; how does the space $V_N^{K_0}$ vary with K_0?

III.5.7. Lemma. *Let $K_1 \subseteq K_0$ be two compact open subgroups of G possessing an Iwahori factorization with respect to P. If v_1 in $V_N^{K_1}$ and v_0 in $V_N^{K_0}$ have the same image in V_N, then $\pi(\mu_{K_0/K_0})v_1 = v_0$.*

Remark. *Does the Theorem remain true if $\mathcal{R}$ is assumed only to be a Noetherian ring?* An earlier version of this chapter had an incorrect proof of this claim. The error was caught by Guy Henniart.

6. The canonical pairing

Continue to let K_0 be a compact open subgroup of G possessing an Iwahori factorization $\overline{N}_0 M_0 N_0$ with respect to the parabolic subgroup P, (π, V) an admissible representation of G.

III.6.1. Lemma. *For v in $V_N^{K_0}$, $\widetilde{v}$ in $\widetilde{V}^{K_0} \cap \widetilde{V}(\overline{N})$, $\langle \widetilde{v}, v\rangle = 0$.*

Proof. Fix a in A^{--} for the moment and choose v_0 in $V_N^{K_0}$ with $\tau_a v_0 = v$. Then

$$\langle v, \tilde{v}\rangle = \langle \tau_a v_0, \tilde{v}\rangle = \langle v_0, \tau_{a^{-1}}\tilde{v}\rangle \,.$$

According to τ_a is nilpotent on $\widetilde{V}^{K_0} \cap \widetilde{V}(\overline{N})$, so if we choose a suitably the right-hand side is 0.

III.6.2. Theorem. *If $(\pi, V$ is an admissible representation of G, then there exists a unique pairing between V_N and $\widetilde{V}_{\overline{N}}$ with the property that whenever v has image u in V_N and $\widetilde{v}$ has image $\widetilde{u}$ in $\widetilde{V}_{\overline{N}}$, then for all a in A_P^{--} near enough to 0*

$$\langle \widetilde{v}, \pi(a)v\rangle = \delta_P^{1/2}(a)\langle \widetilde{u}, \pi_N(a)u\rangle \,.$$

Similarly with the roles of V and $\widetilde{V}$ reversed.

Proof. Let u in V_N and $\widetilde{u}$ in $\widetilde{V}_{\overline{N}}$ be given. Suppose that u and $\widetilde{u}$ are both fixed by elements of M_0. Let v be a vector in $V_N^{K_0}$ with image u, and similarly for $\widetilde{v}$ and $\widetilde{u}$. Define the pairing by the formula

$$\langle \widetilde{u}, u\rangle_{\mathrm{can}} = \langle \widetilde{v}, v\rangle \,.$$

It follows from and that this definition depends only on u and $\widetilde{u}$, and not on the choices of v and $\widetilde{v}$. That

$$\langle \widetilde{v}, \pi(a)v\rangle = \delta_P^{1/2}(a)\langle \widetilde{u}, \pi_N(a)u\rangle_{\mathrm{can}}$$

also follows from and . That this property characterizes the pairing follows from the invertibility of τ_a on $V_N^{K_0}$.

This pairing is called the **canonical pairing**.

III.6.3. Corollary. *If $\mathcal{R}$ is a field, the canonical pairing defines an isomorphism of $(\pi_{\overline{N}}, \widetilde{V}_{\overline{N}})$ with the contragredient of (π_N, V_N), as a representation of M_P.*

Proof. The canonical pairing is invariant under M because for m in M the pairing $\langle \pi_N(m)u, \pi_{\overline{N}}(m)\tilde{u}\rangle_{\mathrm{can}}$ also satisfies the conditions characterizing the canonical pairing.

For non-degeneracy, suppose u in V_N such that $\langle u, \tilde{u}\rangle_{\mathrm{can}} = 0$ for all $\tilde{u} \in \widetilde{V}_{\overline{N}}$. Let v be a canonical lift of u. Let $\tilde{v}$ be arbitrary in $\widetilde{V}^{K_0}$. Suppose $v = \tau_a v_0$ for v_0 also in $V_N^{K_0}$. Then

$$\langle v, \tilde{v}\rangle = \langle \tau(a) v_0, \tilde{v}\rangle = \langle v_0, \tau(a)^{-1}\tilde{v}\rangle \,.$$

But if we choose a suitably then $\tau_{a^{-1}} v$ lies in $\widetilde{V}_{\overline{N}}$, so this last is a canonical a pairing, hence 0. Therefore $\langle v, \tilde{v}\rangle = 0$ for all $\tilde{v}$ in $\widetilde{V}^{K_0}$, and $v = 0$.

Remark. What I call the 'canonical pairing' has been extended in a remarkable fashion in what [Bernstein:1987] calls 'second adjointness'.

7. References

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