Jordan decompositions

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If F is an algebraically closed field, any element in $\mathrm{M}_n(F)$ is similar to a sum of a diagonal matrix and a nilpotent matrix whose non-zero entries are all 1, just above the diagonal. Something similar is true for elements of an arbitrary affine algebraic group as well as its Lie algebra. That's what this essay will attempt to explain.

I begin with a very elementary account of what happens for M_n, then go on to use what might be called Tannakian methods to deal with the general case. My main references, in addition to the items by Newton, have been [Serre:1965] and [Springer:1998]. Section 3 was suggested by §2.5 of Springer's book.

Contents

1. Interpolation

In the next section we shall require a strong form of polynomial interpolation. **Hermite interpolation** solves the following:

> **Problem.** *Suppose given distinct numbers α_i, integers $m_i \ge 1$, and polynomials $Q_i(x)$. Find a polynomial $P(x)$ such that*
> $$P(x) \equiv Q_i(x) \mod (x - \alpha_i)^{m_i}$$
> *for all i.*

Equivalently, one can prescribe values of the polynomial $P(x)$ and other Taylor series coefficients $P^{(k)}(\alpha_i)/k! = \beta_i^{(k)}$ $(k < m_i)$ at the points α_i. There are $m = \sum m_i$ conditions, and consistent with this, there exists a unique suitable polynomial $P(x)$ of degree $m - 1$.

The construction that is conceptually simplest treats this as an extension of the Chinese Remainder Theorem for the polynomial ring.

1.1. Lemma. (Chinese Remainder Theorem) *Given relatively prime polynomials $a(x)$, $b(x)$ and arbitrary polynomials $q(x)$, $r(x)$, there exists a polynomial $p(x)$ such that*

$$\begin{aligned} p(x) &\equiv q(x) \mod a(x) \\ p(x) &\equiv r(x) \mod b(x) \,. \end{aligned}$$

Proof of the Lemma. We want to find polynomials $A(x)$ and $B(x)$ auch that

$$q(x) + A(x)a(x) = r(x) + B(x)b(x) \quad \text{or} \quad A(x)a(x) - B(x)b(x) = r(x) - q(x)\,.$$

Since $a(x)$, $b(x)$ are relatively prime we can write

$$\begin{aligned} \alpha(x)a(x) - \beta(x)b(x) &= 1 \\ \big(r(x) - q(x)\big)\big(\alpha(x)\,a(x) - \beta(x)\,b(x)\big) &= r(x) - q(x) \\ &= A(x)a(x) - B(x)b(x) \end{aligned}$$

with $A(x) = (r(x) - q(x))\alpha(x)$, $B(x) = (r(x) - q(x))\beta(x)$.

Induction on the number of α_i will then solve the original problem.

There are other ways to solve the problem in special cases. Suppose for the moment that all $m_i = 1$ for $i = 1, \ldots, m$. In this case, we are simply requiring that $f(\alpha_i) = \beta_i$ for all i. There are in this special case several methods for constructing f. The one that occurs immediately is to set for $i = 1, \ldots, m$

$$\begin{aligned} N_1(x) &= 1 \\ N_i(x) &= (x - \alpha_1) \ldots (x - \alpha_{i-1}) \end{aligned}$$

The polynomial $N_i(x)$ has degree $i-1$. These span the space of polynomials of degree $< m$, and $N_i(\alpha_j) = 0$ for $j < i$. If

$$\begin{aligned} P(x) &= \sum c_i N_i(x) \\ &= c_0 + c_1(x - \alpha_1) + c_1(x - \alpha_0)(x - \alpha_1) + \cdots + c_{m-1}(x - \alpha_1) \ldots (x - \alpha_{m-1}), \end{aligned}$$

we want to solve the linear system

$$\begin{aligned} c_0 &= \beta_1 \\ c_0 + c_1(\alpha_2 - \alpha_1) &= \beta_2 \\ &\ldots \\ c_0 + c_1(\alpha_m - \alpha_1) + \cdots + c_{m-1}(\alpha_m - \alpha_1) \ldots (\alpha_m - \alpha_{m-1}) &= \beta_m \end{aligned}$$

This is certainly possible, since in the i-th equation the coefficient of c_i is not equal to zero under the assumption that $\alpha_i \neq \alpha_j$ for $i \neq j$.

But my favourite solution is due to Isaac Newton. One of its advantages is that only a slight modification will be necessary to solve the original interpolation problem. We first lay out the numbers β_i in a row and under them, for convenience, the α_i. Above the β_i put the first order **divided differences**:

$$\begin{array}{ccccc} \dfrac{\beta_2 - \beta_1}{\alpha_2 - \alpha_1} & \dfrac{\beta_3 - \beta_2}{\alpha_3 - \alpha_2} & \dfrac{\beta_4 - \beta_3}{\alpha_4 - \alpha_3} & \dfrac{\beta_5 - \beta_4}{\alpha_5 - \alpha_4} & \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 \end{array}$$

which I write more succinctly as

$$\begin{array}{ccccc} \beta_1^{(1)} & \beta_2^{(1)} & \beta_3^{(1)} & \beta_4^{(1)} & \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 \end{array}$$

where

$$\beta_i^{(1)} = \frac{\beta_{i+1} - \beta_i}{\alpha_{i+1} - \alpha_i}.$$

Then we expand the tableau upwards to

$$\begin{array}{cccccc} \Delta^4\beta & \beta_1^{(4)} & & & & \\ \Delta^3\beta & \beta_1^{(3)} & \beta_2^{(3)} & & & \\ \Delta^2\beta & \beta_1^{(2)} & \beta_2^{(2)} & \beta_3^{(2)} & & \\ \Delta\beta & \beta_1^{(1)} & \beta_2^{(1)} & \beta_3^{(1)} & \beta_4^{(1)} & \\ \beta & \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 \\ \alpha & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 \end{array}$$

where in row k we have the divided differences of order k:

$$\beta_i^{(k)} = \frac{\beta_{i+1}^{(k-1)} - \beta_i^{(k-1)}}{\alpha_{i+k} - \alpha_i} .$$

Here is the point of this construction:

1.2. Theorem. *If the α_i for $i = 0$ to $n-1$ are all distinct and we set*

$$\begin{aligned} f(x) &= \beta_0 + \beta_0^{(1)}(x-\alpha_0) + \beta_0^{(2)}(x-\alpha_0)(x-\alpha_1) \\ &\quad + \beta_0^{(3)}(x-\alpha_0)(x-\alpha_1)(x-\alpha_2) + \cdots + +\beta_0^{(n)}(x-\alpha_0)(x-\alpha_1)\ldots(x-\alpha_{n-1}) \end{aligned}$$

then $f(\alpha_i) = \beta_i$ for all i.

I'll postpone the proof for a moment.

The case most commonly encountered is that in which the α_i are equally spaced, say by $\Delta\alpha$. In this case, the theorem is a straightforward consequence of the binomial theorem applied to the operator $(I+\Delta)^n$ where Δ is the operator $f(\alpha+\Delta\alpha) - f(\alpha)$. But in the *Principia Mathematica*, in which the formula was first set forth (Book III, Lemma V), Newton applied his formula to the estimation of comet orbits from scattered sightings, in which this condition is not satisfied. The general case seems to me much more subtle, although by now familiar enough that the surprise has worn off. Newton himself wrote in a letter to Henry Oldenbourg, "Perhaps indeed it is one of the prettiest problems that I can ever hope to solve."

There is one important qualitative difference between the cases of equal and unequal spacing. We know that there exists a polynomial $f(x)$ such that $f(\alpha_i) = \beta_i$, we just need to find it. If $\alpha_i = \alpha + ih$ and f is a polynomial of degree n, then the first differences are

$$\frac{f(\alpha_i + h) - f(\alpha_i)}{h} = f'(\alpha_i) + \frac{h}{2} f''(\alpha_i) + \cdots + \frac{h^{n-1}}{n!} f^{(n)}(\alpha_i) ,$$

which is in turn a polynomial function of α_i, of degree $n-1$. This continues as we calculate higher and higher differences, but with an extra factor $1/k$ at level k. In other words, the calculation of divided differences with equal spacing is recursive. If the spacing is unequal, this is no longer true, the differences are no longer a function of a single variable:

$$\begin{array}{ccc} \alpha_1+\alpha_2 & \alpha_2+\alpha_3 & \\ \alpha_1^2 & \alpha_2^2 & \alpha_3^2 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{array}$$

In general, the k-differences involve $k+1$ values of the α_i. It is this feature that makes it hard to figure out exactly how Newton came to the method. I do not know if he ever gave any hint of its origin.

Proof of the theorem. The standard reference for this is Chapter 1 of [Milne-Thompson:1933]. Newton himself does not seem to have given more than hints of a proof, but it is these hints that I'll follow. As I have already remarked, we already know that there exists a polynomial $f(x)$ such that $f(\alpha_i) = \beta_i$, and we just need to find it.

To understand why Newton's formula is valid, it will help to one example in mind, in which the function $f(x)$ is known. Here is what the difference table looks like for $f(x) = x^3$:

$$\begin{array}{cccccc} \Delta^4 f & 0 & & & & \\ \Delta^3 f & 1 & 1 & & & \\ \Delta^2 f & \alpha_1+\alpha_2+\alpha_3 & \alpha_2+\alpha_3+\alpha_4 & \alpha_3+\alpha_4+\alpha_5 & & \\ \Delta^1 f & \alpha_1^2+\alpha_1\alpha_2+\alpha_2^2 & \alpha_2^2+\alpha_2\alpha_3+\alpha_3^2 & \alpha_3^2+\alpha_3\alpha_4+\alpha_4^2 & \alpha_4^2+\alpha_4\alpha_5+\alpha_5^2 & \\ \Delta^0 f & \alpha_1^3 & \alpha_2^3 & \alpha_3^3 & \alpha_4^3 & \alpha_5^3 \\ & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 \end{array} .$$

It is easy to guess the general formula for the difference table when $f(x) = x^k$. The row $\Delta^m f$ is calculated from functions $f[x_1, \ldots, x_{m+1}]$ where for $m \le k$

$$f[x_1, \ldots , x_{m+1}] = \sum_{1 \le i_1 \le i_2 \le i_{k-m} \le m+1} x_{i_1} x_{i_2} \ldots x_{i_{k-m}} \,.$$

For example, when $m = 0$ all the i_j must be equal to 1 and we get $f[x_1] = x_1^k$. This formula can then be proved by induction on m. In particular

$$\begin{aligned} f[x_1, \ldots , x_k] &= x_1 + \cdots + x_k \\ f[x_1, \ldots , x_{k+1}] &= 1 \,. \end{aligned}$$

The immediate consequence is that

1.3. Lemma. *If F is a polynomial $f(x)$ of degree k then $\Delta^m f \equiv 0$ for $m > k$.*

Now to conclude the proof of Newton's formula. We know that any polynomial of degree less than n can be expressed as a linear combination of the Newton polynomials $f(x) = (x - \alpha_1) \ldots (x - \alpha_m)$, where $0 \le m < n$, so it suffices to prove the formula for one of these. But simple calculation tells us that

$$f[\alpha_1, \ldots, \alpha_i] = \begin{cases} 0 & \text{if } i < m \\ 1 & \text{if } i = m \end{cases}$$

while the Lemma tells us that $f[\alpha_1, \ldots, \alpha_i] = 0$ for $i > m$. The Proposition is therefore true for $f(x)$. ▯

The reason why Newton's formula is so good in our situation is what happens when $\beta_i = f(\alpha_i)$ for some smooth function, and we consider the limit as some of the α_i coalesce, say $\alpha_i = \alpha$ for $1 \le i \le m$. The divided difference $f[\alpha_1, \ldots , \alpha_m]$ then has limit $f^{(n)}(\alpha)/n!$. In the case where all the α_i coalesce, Newton's formula becomes Taylor's. In case only some smaller subsets coalesce, we obtain Hermite interpolation which allows us to find a polynomial of degree $n-1$ matching n conditions on values and derivatives at various points.

2. The Jordan decomposition in M(n)

Suppose F to be an arbitrary field. In characteristic 0 set $F_\bullet = F$, and in characteristic $p > 0$ set $F_\bullet = F^{p^{-\infty}}$. Let $\overline{F}$ be an algebraic closure of F containing $F_\bullet$.

2.1. Proposition. *If T is any matrix in $M_n(F)$, there exist unique matrices S and N in $M_n(F_\bullet)$ such that (a) $T = S + N$; (b) S is diagonalizable and N is nilpotent; (c) S and N commute. There exist polynomials P and Q with $P(0) = Q(0) = 0$, $P(T) = S$ and $Q(T) = N$.*

The decomposition $T = S + N$ is the **Jordan decomposition** of T.

Proof. The first part is a standard argument in linear algebra. If A is any matrix, the chain

$$\mathrm{Ker}(A) \subseteq \mathrm{Ker}(A^2) \subseteq \ldots$$

is eventually stationary, so there exists n such that A^n annihilates all A-torsion. If U is the subspace of such vectors then A is invertible on V/U. The exact sequence of A-modules

$$0 \to U \to V \xrightarrow{A^n} \mathrm{Im}(A^n) \to 0$$

tells us that the image of A^n is isomorphic to V/U. So V is the direct sum of two unique A-stable components, with A nilpotent on one and invertible on the other.

Applying this to $A = T - \lambda I$, where λ is an eigenvalue of T in $\overline{F}$, we see that $\overline{V} = V \otimes_F \overline{F}$ is the direct sum of a subspace $\overline{V}_\lambda$ on which $T - \lambda I$ is nilpotent, and a subspace V^λ on which it is invertible. Considering this last space in turn, we may express $\overline{V}$ as a direct sum of spaces $\overline{V}_\lambda$ on which $T - \lambda I$ is nilpotent. Let S be the endomorphism of $\overline{V}$ which is λI in $\overline{V}_\lambda$. Then S is diagonalizable and commutes with T since it is scalar multiplication on each V_λ The transformation $N = T - S$ is nilpotent, and N commutes with both S and T. It is straightforward to see that S and N are unique with these properties. Hence it is invariant with respect to F-automorphisms of $\overline{F}$, and therefore S and N may be taken to be endomorphisms of $V_\bullet = V \otimes_F F_\bullet$.

It remains to find the polynomial P. Since $T = S + N$ and S and N commute, the binomial theorem tells us that

$$P(T) = P(S) + N\,R(S,N)$$

where R is a polynomial in two variables. Hence $N\,R(S,N)$ is nilpotent, and therefore the semi-simple component of $P(T)$ is $P(S)$.

The vector space $\overline{V}$ is the sum of spaces $\overline{V}_i$ on which $(T - \lambda_i)^{m_i} = 0$ and S multiplies by λ_i. We are therefore looking for a polynomial $P(X)$ such that

$$P(X) = \lambda_i \bmod (X - \lambda_i)^{m_i}, \quad P(X) \equiv 0 \bmod X\,,$$

for all eigenvalues i. This is possible according to the results on interpolation in the previous section. Uniqueness implies that its coefficients lie in $F_\bullet$. $\square$

2.2. Corollary. *Any g in $\mathrm{GL}_n(F)$ may be expressed as $g_s g_u$ where $g_u \in GL_n(F_\bullet)$ is diagonalizable, $g_u \in GL_n(F_\bullet)$ is unipotent, and $g_u g_s = g_s g_u$. These factors are unique.*

Proof. If $g = S + N$ then S is invertible and $g = S(I + S^{-1}N)$. Since S and N commute, $S^{-1}N$ is nilpotent and $I + S^{-1}N$ is unipotent. $\square$

2.3. Proposition. *Suppose A, B to be endomorphisms of the finite-dimensional vector spaces U, V and $f\colon U \to V$ a linear map. If this diagram commutes:*

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "A"'] & V \arrow[d, "B"] \\ U \arrow[r, "f"] & V \end{tikzcd}$$

then so do these:

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "A_s"'] & V \arrow[d, "B_s"] \\ U \arrow[r, "f"] & V \end{tikzcd} \qquad \begin{tikzcd} U \arrow[r, "f"] \arrow[d, "A_n"'] & V \arrow[d, "B_n"] \\ U \arrow[r, "f"] & V \end{tikzcd}\,.$$

In other words, Jordan components are functorial.

Proof. We may as well assume F is algebraically closed. The space U is a direct sum of A-primary components on which $(A - \lambda I)$ is nilpotent. Commutativity implies that the image under f of one is also a primary component for B. But A_s is characterized by the property that $A = \lambda I$ on this component, and then $B_s = \lambda I$ on its image, which proves that the first diagram commutes. Commutativity of the second follows immediately. $\square$

3. The Jordan decomposition in affine groups

Suppose in this section that G is an affine algebraic group defined over F whose affine ring is $A_F[G]$. Continue to let $F_\bullet$ be $F^{p^{-\infty}}$.

The G group G acts on the F-variety V if the transformation map $\tau\colon G \times V \to V$ is algebraic. This means that there exists a map of rings

$$A_F[V] \longrightarrow A_F[G \times V] = A_F[G] \otimes_F A_F[V]$$

satisfying certain conditions that make this a real group action. For any given g we have the linear map λ_g on $A_F[V]$:

$$[\lambda_g f](v) = f(g^{-1}(v)) .$$

3.1. Proposition. *The action of G on $A_F[V]$ is locally finite.*

This means that any function in $A_F[V]$ is contained in a finite-dimensional G-stable subspace.

Proof. Given f in $A_F[V]$, the function $f(g(v))$ is an affine function on $G \times V$, hence

$$f(g(v)) = \sum_i \varphi_i(g) f_i(v)$$

for some finite set of affine functions φ_i on G, f_i on V. But then all G-translates of f are contained in the space spanned by the f_i.

The group G acts on itself by left and right translations:

$$L_g\colon\ x \longmapsto gx, \quad R_g\colon\ x \longmapsto xg .$$

These commute with each other, and either can be characterized in terms of this property:

3.2. Lemma. *If α is an algebraic automorphism of G that commutes with the left action of G then it is right multiplication by some element of G.*

Proof of the Lemma. Suppose α takes 1 to g. Then for any h in G we have

$$[\alpha{\circ}L_h](1) = \alpha(h) = [L_h{\circ}\alpha](1) = hg .$$

There is an analogue for the Lie algebra of G. Any X in $\mathfrak{g}$ may be interpreted as a derivation of $A_F[G]$ commuting with the left action of G (see §4.4 of [Springer:1998]). Conversely:

3.3. Lemma. *If α is a derivation of $A_F[G]$ that commutes with the left action of G then it is the right derivation by some element of the Lie algebra $\mathfrak{g}$ of G.*

By Proposition 3.1 we know that with respect to left translation, the ring $A_F[G]$ is a locally finite representation of G. If $\{f_k\}$ are generators of $A_F[G]$, then the space generated by their left translations will be a faithful representation of G. Hence one consequence of the result above is that such a representation always exists.

There is one way to construct G-stable subspace of $A_F[G]$. Suppose (π, V) to be any algebraic F-rational representation of G, $\widehat{V}$ its dual. For v in V, $\widehat{v}$ in $\widehat{V}$ the function

$$\Phi_{v,\hat{v}}(g) = \langle \widehat{\pi}(g)\widehat{v}, v\rangle$$

is the corresponding **matrix coefficient**. It lies in $A_F[G]$ and the map from $V \otimes \widehat{V}$ to $A_F[G]$ is $G \times G$-equivariant.

Given g in G, then to every F-representation (π, V) of G we may associate the endomorphism $\alpha_\pi = \pi(g)$ of V. The following conditions are satisfied:

(a) $\alpha_I = I$;
(b) given a G-map f from V_π to V_ρ, the following diagram commutes:

$$\begin{array}{ccc} V_\pi & \xrightarrow{f} & V_\rho \\ \alpha_\pi \downarrow & & \downarrow \alpha_\rho \\ V_\pi & \xrightarrow{f} & V_\rho \end{array}$$

(c) $\alpha_{\pi\otimes\rho} = \alpha_\pi \otimes \alpha_\rho$.

3.4. Proposition. *Suppose we are given for every F-representation π of G an invertible linear map α_π of V satisfying the conditions (a)–(c) just above. There exists a unique g in G such that $\alpha_\pi = \pi(g)$.*

Proof. The definition of α may be extended compatibly to any locally finite representation, in particular λ on $A_F[G]$ and another on $A_F[G] \otimes_F A_F[G]$. Apply Lemma 3.2.

Given X in $\mathfrak{g}$, then to every F-representation (π, V) of G we may associate the endomorphism $\alpha_\pi = d\pi(X)$ of V. The following conditions are satisfied:

(a) $\alpha_I = 0$;
(b) given a G-map f from V_π to V_ρ, the following diagram commutes:

$$\begin{array}{ccc} V_\pi & \xrightarrow{f} & V_\rho \\ \alpha_\pi \downarrow & & \downarrow \alpha_\rho \\ V_\pi & \xrightarrow{f} & V_\rho \end{array}$$

(c) $\alpha_{\pi\otimes\rho} = \alpha_\pi \otimes I + I \otimes \alpha_\rho$.

3.5. Proposition. *Suppose we are given for every F-representation π of G an invertible linear map α_π of V satisfying the conditions (a)–(c) just above. There exists a unique X in G such that $\alpha_\pi = d\pi(X)$.*

Proof. The definition of α may be extended compatibly to any locally finite representation, for example λ on $A_F[G]$ and another on $A_F[G] \otimes_F A_F[G]$. The map λ corresponds to an algebraic automorphism of G, and assumptions on α imply that it commutes with the left action of G. Apply Lemma 3.3.

This allows the most natural way of deducing the Jordan decompositions of elements of $G(F)$ as a product of unique semi-simple and unipotent parts, both in $G(F_\bullet)$, and of elements of $\mathfrak{g}$ into a sum of unique semi-simple and nilpotent parts, both in $\mathfrak{g}(F_\bullet)$. Start with the classical case of g in $\mathrm{GL}_n(F)$ or X in $M_n(F)$.

3.6. Lemma. *Suppose g to be in $G(F)$. The following are equivalent:*

(a) there exists a faithful representation (π, V) of G such that $\pi(g)$ is semi-simple (resp. unipotent);
(b) $\pi(g)$ is semi-simple (resp. unipotent) for any representation of G.

If either of these holds, the element g is said to be semi-simple (resp. unipotent).

Proof. If π is faithful and $\pi(g)$ is semi-simple, then the corresponding automorphisms of the tensor algebra of V is also semi-simple. But the affine ring of G is a quotient of the symmetric tensors of $V \otimes \widehat{V}$, and by Proposition 2.3 this implies that the image of g in any representation of G is semi-simple.

3.7. Theorem. *Every g in G may be expressed as $g_s g_u$ where $g_s \in G(F_\bullet)$ is semi-simple, $g_u \in G(F_\bullet)$ is unipotent, and $g_s g_u = g_u g_s$.*

3.8. Theorem. *Every X in $\mathfrak{g}$ may be expressed as $X_s + X_n$ where $X_s \in \mathfrak{g}(F_\bullet)$ is semi-simple, $X_n \in \mathfrak{g}(F_\bullet)$ is nilpotent, and $X_s X_n = X_n X_s$.*

The following is a simple consequence of Proposition 2.3:

3.9. Proposition. *The factorization of g in G and the sum decomposition of X in $\mathfrak{g}$ behave compatibly with respect to homomorphisms of algebraic groups.*

4. Final remarks

The 'Third Theorem of Lie' (see LG §5.8 of [Serre:1965]) states that if $F = \mathbb{R}$ or $\mathbb{C}$ then every Lie algebra is the Lie algebra of an analytic group over F. I do not know what, if anything, distinguishes the Lie algebras of algebraic groups. I do not know if Jordan decompositions exist for arbitrary Lie algebras. The following is proved in LA §6 of [Serre:1965] without any reference to a group. From it he derives a Jordan decomposition in arbitrary semi-simple Lie algebras in characteristic 0, but it has independent interest.

4.1. Theorem. *Suppose F to be algebraically closed, of characteristic 0. If $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ is semi-simple over F, it is same as the subalgebra of $\mathfrak{gl}(V)$ leaving invariant all the tensor invariants of $\mathfrak{g}$.*

5. References

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