Essays on mathematical astronomy

Planetary motion and Kepler's equation

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The curves defined by conic sections were apparently first analyzed in generality by the Greeks of classical times, particularly Apollonius. The theory only became of great significance when it was discovered by Kepler that

(First Law) within any error he could measure, the path of a planet is an ellipse with the Sun at its focus;
(Second Law) there is a rule, at once simple theoretically and reasonably practical, for determining how a planet moves along its ellipse, given its period;
(Third Law) the period of a planet is proportional to $a^{3/2}$ if a is the semi-major axis of the orbit.

Half a century after Kepler, Newton discovered simple laws of dynamics that at once explained Kepler's laws as well as the deviations from it. He formulated the inverse-square law of gravitational force and the rules that determined motion from this force. This implied that the orbit of essentially any planet-sized object in the Solar system followed a path which was, again up to a very small error, a conic section. That is to say, parabolas and hyperbolas were also possible in addition to ellipses, at least for objects whose origins lay outside the Solar system itself. In this essay, I'll look only at elliptical orbits.

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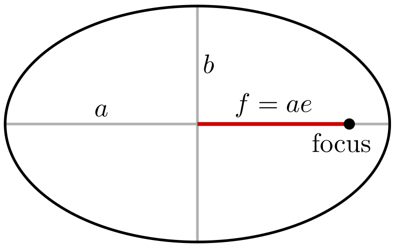
1. The geometry of ellipses

This is just a quick summary. An ellipse is obtained from the unit circle by scaling horizontally and vertically. If we have semi-major axis a and semi-minor axis b its equation is

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1\,.$$

Its area A is πab.

There are two **foci** $(\pm f, 0)$ wuth the property that the piecewise linear path from one to the other, bouncing off the ellipse on the way, has a constant length. The **eccentricity** of the ellipse is the ratio $e = f/a$. Also $f = \sqrt{a^2 - b^2}$. Since the orbit is an ellipse, e lies in the range $[0, 1)$. It is 0 for circles, and tends to 1 as the ellipse flattens out.



Given a and e we have

$$b = a\sqrt{1 - e^2}\,.$$

Since $\sqrt{1-e^2} \sim 1 - e^2/2$, the semi-axes b and a are very close for even moderately small e. Thus the shape of the ellipse is close to a circle unless e is close to 1.

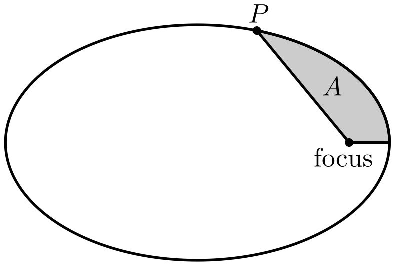
2. Conics and planetary motion

In a system of two massive bodies, the two bodies travel in similar elliptical orbits around their centre of mass. The ratio of axes is inversely proportional to their masses. If one body is much larger than the other, and the centre of mass is essentially fixed, and the smaller object travels around the larger one in an apparent ellipse. This is in effect what happens in the solar system.

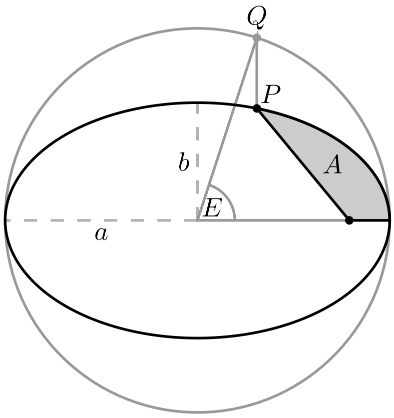
We shall begin by applying the second law to see how a planet's motion varies with time. How the second law tells us this turns out to be simple geometry. How the geometry determines the motion is, however, somewhat complicated.

Consider an orbit with semi-major axis a and semi-minor axis b. The focus is $(f, 0)$ with $f > 0$. Suppose that at time $t = 0$ the planet is at **perihelion**, that is to say the point in its orbit nearest to the Sun. Suppose that at time t it is at location P on the ellipse. Let A be the area of the region traversed by the radius vector of the planet from the Sun. Kepler's Second Law tells us that

$$A = \alpha t, \quad (\alpha = \pi ab/t_{\mathrm{period}})\,.$$



We shall use a coordinate system with origin at the centre of the ellipse, and we shall describe points on the orbit in terms of points in the circle one can inscribe around the ellipse with radius a. The ellipse is obtained from this circle by compressing along the y axis by a factor b/a. Thus if $Q = (a\cos E, a\sin E)$ is a point on the circle, it corresponds to the point $P = (a\cos E, b\sin E)$ on the ellipse.

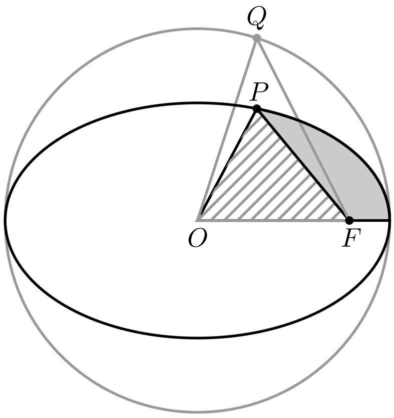


We now want to relate the area A to the point P.

2.1. Lemma. *In the above diagram*

$$A = \frac{ab}{2} \cdot (E - e\sin(E)\,.$$

Proof. We can easily relate the analogous things on the circle, and we obtain the ones on the ellipse by scaling vertically by the factor b/a.



The area A is the difference between the area of the elliptic sector with angle E, and the triangle OFP, which has height $b\sin E$ and base $f = \sqrt{a^2 - b^2}$. Therefore

$$A = \frac{b}{a}\left(\pi a^2 \frac{E}{2\pi} - \frac{a^2 e \sin E}{2}\right) = \frac{ab}{2}E - \frac{b\sqrt{a^2 - b^2}}{2}\sin E$$

which we can rewrite as

$$\frac{A}{ab/2} = E - e\sin E\,.$$

The quantity $M = A/(ab/2)$ is also, by Kepler's Law, equal to $2\pi(t/t_{\mathrm{period}})$. Hence:

2.2. Proposition. *At time t the angle E is the unique solution to the equation*

$$M = 2\pi \cdot \frac{t}{t_{\mathrm{period}}} = E - e\sin E\,.$$

This is called **Kepler's equation**. Remember that knowing E is equivalent to knowing P, since

$$P = (a\cos E, b\sin E) = a(\cos E, \sqrt{1-e^2}\sin E)\,.$$

while knowing M is equivalent to knowing t. Thus Kepler's equation asserts directly that *if we know the position P we can tell what t is*. This is usually opposite to what we usually want to know, which is how to determine P in terms of t. *In order to do this we must solve Kepler's equation for E in terms of M.* This is not so straightforward. In fact, there is no simple formula for E in terms of M (or P in terms of t), and we must solve the equation numerically.

Sometimes the polar coordinates of P with respect to the focus are convenient. For radius ρ:

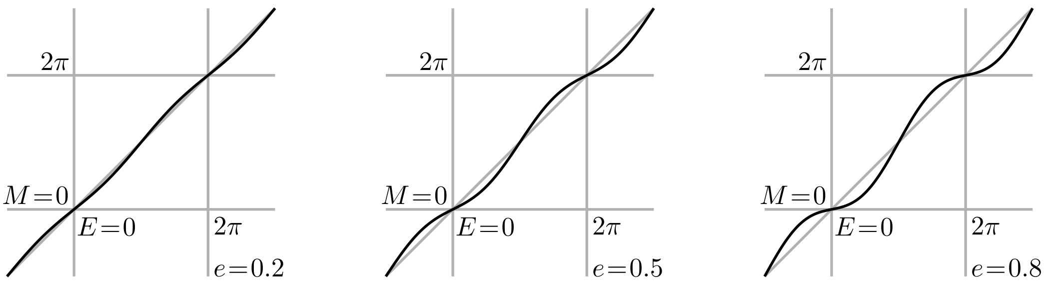
$$\begin{aligned}
\rho^2 &= (a\cos E - f)^2 + (b\sin E)^2 \\
&= a^2\cos^2 E - 2af\cos E + f^2 + b^2\sin^2 E \\
&= (a^2 - b^2)\cos^2 E - 2af\cos E + f^2 + b^2(\cos^2 E + \sin^2 E) \\
&= f^2\cos^2 E - 2af\cos E + a^2 \\
&= a^2(1 - e\cos E)^2 \\
\rho &= a(1 - e\cos E)\,.
\end{aligned}$$

As we'll see in a moment, this has an approximation

$$\rho = a(1 - e\cos M) + O(e^2)\,.$$

3. Solving Kepler's equation

The first thing to do is to get a rough idea of what the problems are. This we can do by graphing M as a function of E for a few values of e.



It is seen in these figures, and verified easily by taking derivatives, that the function taking E to $E - e\sin E$ is **monotonic increasing**, which means that as E increases so does M. This means that for any given value of M there is a unique value of E for which $M = E - e\sin E$. This is certainly reassuring. This remains true as long as $0 \le e < 1$, or in other words for all elliptical orbits, which is all we can expect. So the problem we are trying to solve is at least well posed.

Therefore we now consider directly the question of how to find E given M. If e is small, the following trick works well. We rewrite the equation

$$x = f(x) = M + e\sin x$$

so that we are looking for a number x taken to itself by the transformation $x \mapsto M + e\sin x$. We start with some initial approximation x_0 for x. As long as e is small, $E_0 = M$ will be close enough, since $M + e\sin M$ will not be too far from M. Then we calculate in succession

$$x_{n+1} = M + e\sin x_n$$

I recall quickly how fixed point iteration performs. If

$$x_{n+1} = f(x_n)$$

then

$$h_n = x_{n+1} - x_n = f(x_n) - x_n$$

and

$$h_{n+1} = f(x_{n+1}) - f_{n+1} = f(x_n + h_n) - f(x_n) \sim h_n f'(x_n)$$

so that if initial values are reasonably close to a fixed point x with $|f('(x)| < 1$ we can expect geometric convergence. In our case

$$f'(x) = -e \cos x\,.$$

which is always of absolute value $\le e$.

Example. Set $e = 0.1$, say, and $M = 1$. We get values

$$\begin{aligned}
E_0 &= 1 \\
E_1 &= 1 + 0.1\ \sin 1 \\
&= 1.084147 \\
E_2 &= 1.088390 \\
E_3 &= 1.088588 \\
E_4 &= 1.088597 \\
E_5 &= 1.088598 \\
E_6 &= 1.088598 \\
&\ldots
\end{aligned}$$

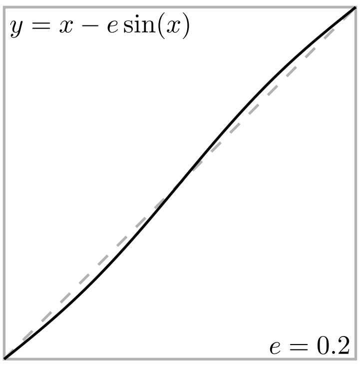
so it does in fact converge quickly in this case. In order to calculate planet positions as a function of time it is important to be able to solve Kepler's equation efficiently, and this is promising.

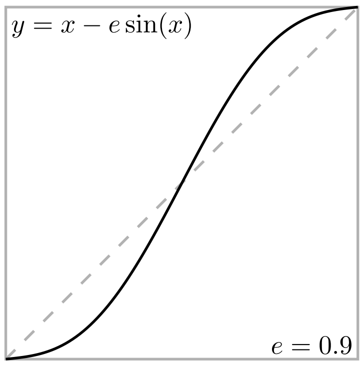
4. Finding a better initial guess

The method I have suggested for solving Kepler's equation proceeds by picking an initial guess E_0 and then repeating

$$E_{n+1} = M + e \sin E_n$$

until convergence. The initial guess I have suggested is $E_0 = M$. Using this initial guess amounts to approximating the function $E - e \sin E$ by E. But as these pictures show, this is not a very good guess except for small values of e.





So the method of solution we'll use in practice is the fixed point iteration when $e < 0.01$, and otherwise Newton's method combined with a more sophisticated initial guess.

A better idea for the initial guess is to approximate the function $E - e\sin E$ near the three points 0, π, 2π and use as an initial guess a solution for these approximations. Near $E = 0$ we have

$$E - e\sin E \sim E - eE + e\,\frac{E^3}{3!}$$

because of the series

$$\sin x = x - \frac{x^3}{3!} + \cdots .$$

So as an approximate root of Kepler's equation near $y = 0$ we have the root of

$$y = x(1-e) + \frac{e}{6}\,x^3 .$$

Near $E = \pi$ we can write

$$\begin{aligned} E &= \pi + h \\ E - e\sin E &= (\pi + h) - e\sin(\pi + h) \\ &= \pi + h + e\sin h \\ &\sim \pi + h + eh - e\,\frac{h^3}{3!} . \end{aligned}$$

So as an approximate root of Kepler's equation near $y = \pi$ we have $x = \pi + h$ where h is the root of

$$(y - \pi) = h(1+e) - \frac{e}{6}\,h^3 .$$

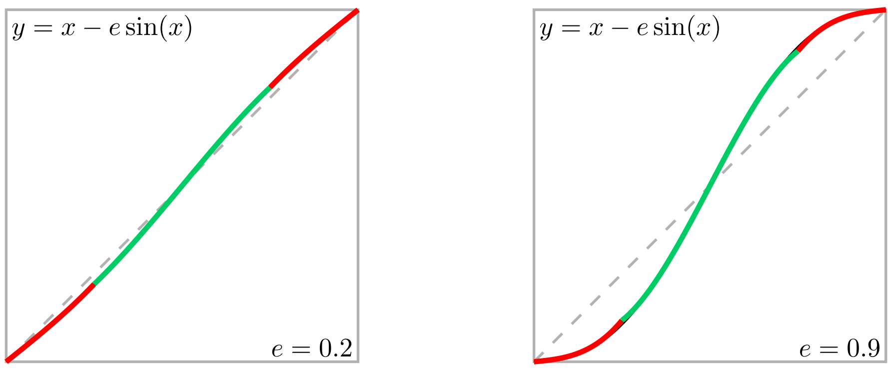
Near $E = 2\pi$ we have

$$\begin{aligned} E &= 2\pi + h \\ E - e\sin E &\sim 2\pi + h(1-e) + \frac{e}{6}\,h^3 . \end{aligned}$$

So as an approximate root of Kepler's equation near $y = 2\pi$ we have $x = 2\pi + h$ where h is the root of

$$(y - 2\pi) = h(1-e) + \frac{e}{6}\,h^3 .$$

How good are these initial guesses? These pictures give an idea of how good, and also in what ranges to choose the different approximations:



So we divide up the initial approximation cases into the x-ranges $[0, 1/4)$, $[1/4, 3/4)$, $[3/4, 1]$. Equivalently, we make the breaks at $x - e\sin(x)$ where $x = 1/4, 3/4$.

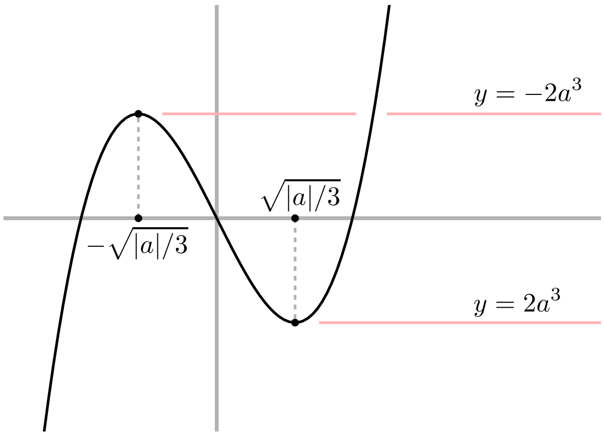
In all cases, as a good initial guess we can take a root of a certain cubic equation. To find such a root is not trivial, but it is not so difficult either.

5. Solving cubic equations

Finding roots of an arbitrary cubic polynomial reduces, through a change of variables, to solving an equation

$$x^3 + ax = y\,.$$

There are two basic cases. The simplest is where $a \geq 0$, in which case there is exactly one root for all y, since $x^3 + ax$ has derivative $3x^2 + a$ and is monotonic. If $a < 0$ then the slope at 0 is negative and there is a further breakdown, illustrated in the following figure:



If $|y| > 2|a|^3$ there is one root; if $|y| = 2|a|^3$ there are two roots; and if $|y| < 2|a|^3$ there are three.

The solution starts with the observations

$$(\alpha + \alpha^{-1})^3 = \alpha^3 + \alpha^{-3} + 3\alpha + 3\alpha^{-1}$$
$$(\alpha - \alpha^{-1})^3 = \alpha^3 - \alpha^{-3} - 3\alpha + 3\alpha^{-1}$$

so that if $X = c(\alpha + \alpha^{-1})$

$$X^3 - 3c^2X = c^3(\alpha^3 + \alpha^{-3})$$

and if $X = c(\alpha - \alpha^{-1})$ then

$$X^3 + 3c^2X = c^3(\alpha^3 - \alpha^{-3})\,.$$

This leads to the following. Suppose we want to solve

$$x^3 + ax = y$$

for x.

• Suppose $a \geq 0$. We first solve

$$3c^2 = a, \quad (\bullet)\ c = \sqrt{a/3}\,.$$

Then we find α such that

$$y = c^3\left(\alpha^3 - \alpha^{-3}\right)\,.$$

If $\beta = \alpha^3$ this reads

$$y = c^3\left(\beta - \beta^{-1}\right)$$

or

$$\begin{aligned}\beta - \beta^{-1} &= y/c^3 \\ \beta^2 - 1 &= (y/c^3)\beta \\ &= 2z\beta \quad \text{where } (\bullet)\ z = y/2c^3 \\ (\beta - z)^2 &= 1 + z^2 \\ (\bullet)\ \beta &= z + \sqrt{1+z^2}\end{aligned}$$

and then write

$$(\bullet)\ \alpha = \beta^{1/3}\ .$$

Finally we get the unique solution

$$(\bullet)\ x = c\left(\alpha - \alpha^{-1}\right)\ .$$

• Now suppose $a < 0$. We first solve

$$-3c^2 = a, \quad (\bullet)\ c = \sqrt{|a|/3}\,.$$

Then we find α such that

$$y = c^3\left(\alpha^3 + \alpha^{-3}\right)\ .$$

If $\beta = \alpha^3$ this reads

$$y = c^3\left(\beta + \beta^{-1}\right)$$

or

$$\begin{aligned}\beta + \beta^{-1} &= y/c^3 \\ \beta^2 + 1 &= (y/c^3)\beta \\ &= 2z\beta \quad \text{where } (\bullet)\ z = y/2c^3 \\ (\beta + z)^2 &= z^2 - 1 \\ (\bullet)\ \beta &= z + \sqrt{z^2-1}\end{aligned}$$

There are now essentially three different possibilities to be dealt with. (a) If $|z| > 1$, then we just take the positive square root to get the real number β. We write

$$(\bullet)\ \alpha = \beta^{1/3}\ .$$

and finally get the unique solution

$$(\bullet)\ x = c\left(\alpha + \alpha^{-1}\right)\ .$$

(b) If $|z| = 1$, we have two roots $x = \pm c$.

(c) Finally, suppose $|z| < 1$. In this case β is a complex number of absolute value 1, hence of the form $\beta = e^{i\theta}$. We may as well start immediately with trigonometry for a solution. We have for any θ

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

or

$$2c^3\cos 3\theta = T^3 - 3c^2\,T \quad \text{where } T = 2c\cos\theta\,.$$

So we must solve

$$2c^3\cos 3\theta = y$$

for θ. This is possible because

$$\left|\frac{y}{2c^3}\right| < 1\,.$$

There are three possible values of θ. For each one we get the root $x = 2c\cos\theta$. They are sorted according to whether $\cos\theta$ lies in $(-1, -c)$, $(-c, c)$, or $(c, 1)$. In our case, looking for a solution of Kepler's equation, we want the middle one.

6. Newton's method

The most efficient way to solve

$$y = x - e\sin(x)$$

is by Newton's method. Here

$$x_{n+1} = x_n + h_n$$

where

$$h_n = -\frac{f(x_n)}{f'(x_n)}$$

with

$$f(x) = x - y - e\sin(x), \quad f'(x) = 1 - e\cos(x)\,.$$

At the next step we can calculate

$$f(x_{n+1}) = f(x_n + h_n)\,.$$

Here we have

$$\begin{aligned} f(x+h) &= (x+h) - y - e\sin(x+h) \\ &= x - y + h - e\sin(x)\cos(h) - e\cos(x)\sin(h) \\ &= \big(x - y - e\sin(x)\big) + h + e\sin(x)\big(1-\cos(h)\big) - e\cos(x)\sin(h) \\ &= f(x) + h + e\sin(x)\big(1-\cos(h)\big) - e\cos(x)\sin(h) \end{aligned}$$

and the significance of this formula is that the incremental term

$$h + e\sin(x)\big(1-\cos(h)\big) - e\cos(x)\sin(h)$$

is small if h is. This eliminates some cancellation problems in practical calculations, and at least reduces these to the relatively simple problem of computing $1-\cos h$ for small h, which can be well estimated by a Taylor series.

7. Angular velocity

One natural question: what is the angular velocity of an orbiting body at time t? This is easily reduced to the case where time τ progresses from 0 to 2π in one revolution, which I'll called the **normalized parametrization**.

We start with a well known result in calculus:

7.1. Lemma. *If*

$$t \longmapsto (x(t), y(t))$$

is any parametrized path in $\mathbb{R}^2$, then

$$\arg'(t) = \frac{-yx'(t) + xy'(t)}{x^2+y^2}\,.$$

For example, if the path is a circle of radius r given the parametrization $(r\cos(t), r\sin(t))$ then $\arg'(t) = 1$.

This implies immediately:

7.2. Lemma. *In an elliptical orbit given the normalized parametrization*

$$\arg'(\tau) = E' \cdot ab \cdot \frac{1 - e\cos(E)}{r^2} \qquad (r^2 = a^2(\cos(E)-e)^2 + b^2\sin^2(E)\,.$$

Proof. We have

$$
\begin{aligned}
x &= a\cos(E) - f \\
x' &= -a\sin(E)\cdot E' \\
y &= b\sin(E) \\
y' &= b\cos(E)\cdot E' \\
-yx' + xy' &= E'\cdot((-b\sin(E))(-a\sin(E)) + (a\cos(E) - f)(b\cos(E)) \\
&= E'(ab\sin^2(E) + ab\cos^2(E) - bf\cos(E)) \\
&= E'\cdot b\cdot(a - f\cos(E)) \\
&= E'\cdot ab\cdot(1 - e\cos(E))\,.
\end{aligned}
$$

In our case $E(\tau)$ is given implicitly by the equation

$$\tau = E - e\sin(E)\,.$$

This tells us that

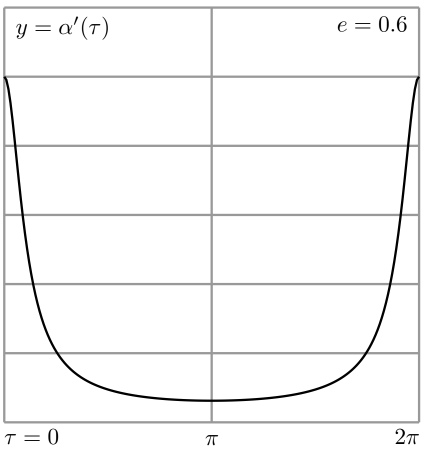
$$E'(\tau) = \frac{1}{1 - e\cos(E)}\,,$$

which we can plug into the formula of Lemma 7.2. This gives us finally

$$\alpha'(\tau) = \frac{ab}{r^2}\,. \tag{7.3}$$

In the language of physics, this is a formula for the (constant) angular velocity of the system.

For example, if $a = 1$, $e = 0.6$ here is the graph of $\arg'(\tau)$:



8. A curious application

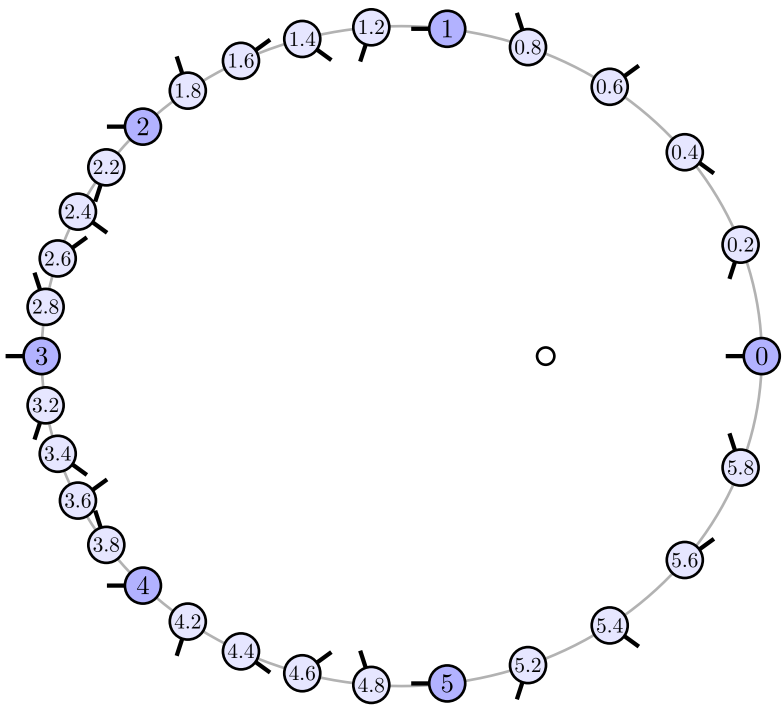
The Earth rotates around the Sun in an orbit of ellipticity 0.0167. As it goes around it rotates on its own axis, essentially at a constant rate–at a constant rate, that is to say, relative to the (relatively) fixed stars. The length of time it takes to do this is a **sidereal day**. One earth orbit is known to take $T = 366.2422$ sidereal days.

But for those living on the planet, a day—a **solar day**—is the length of time from one noon time to the next. Now at any point on Earth, the meridian is an imaginary arc running overhead in the sky from south to north, and noon is when the Sun crosses this arc—when it makes a meridian transit. Because the Earth is rotating around the Sun as it is rotating around its axis, the number of solar days in a year is exactly one less than the number of sidereal days, or $T - 1 = 365.2422$. A solar day is a bit longer than a sidereal day. To be precise, an average solar day is equal to $T/(T - 1)$ sidereal days.

But in fact the length of a true solar day varies during the year. The principal reason for this is that the orbital speed of every planet varies, as we have seen, in its orbit—it goes faster at perihelion, slower at aphelion. I want to investigate here the effect of orbital speed on the length of solar day.

To do this, in order to make things slightly simpler, I'll look at an artificial Earth, one whose equatorial plane is the same as the ecliptic. There is a basic technique that tells how to compute the length of solar days, which I'll explain in a moment.

But in order to make things even simpler, I'll make things slightly more abstract. Suppose we are looking at some 'planet' that moves in a Kepler orbit, rotating at a sidereal rate of θ radians per unit time. I'll assume that both the planet in its orbit and the planet around its axis rotate counter-clockwise. In the following diagram, the position is plotted at equal time intervals. The planet undergoes six sidereal rotations in the course of traversing the orbit. The numbers keep track of the elapsed number of sidereal days from the start at perihelion.



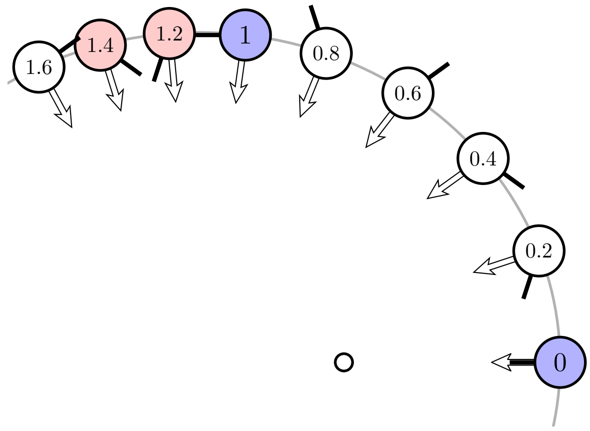
Suppose that at time t_0 the focus (i.e. 'sun') of the orbit is directly overhead at a particular location on the body. An **apparent day** is the length of time $\Delta t = t_1 - t_0$ it takes for that to occur again for the first time.

In the diagram above, 'noons' (i.e. aka 'meridian transits') occur when the little radius points towards the focus, and there are (as there should be) five apparent days in the cycle, as you can surmise from the diagram. But they do not occur at regular intervals—the length of apparent days is not constant.

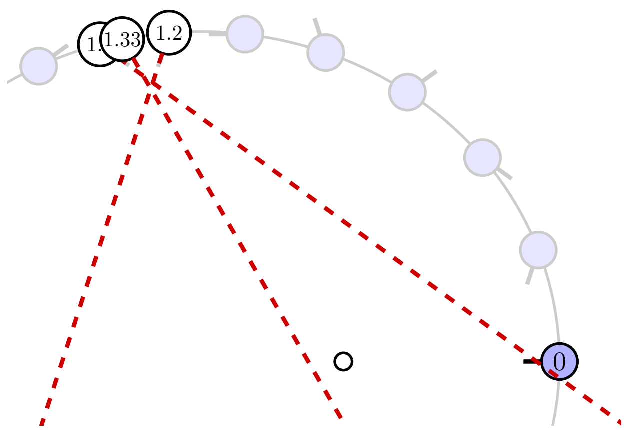
How can we find these times of apparent noon? This reduces to a slightly more precise question: *Given a particular time, how can we find the next apparent noon time?*

This presupposes that we know how to calculate the location of the overhead meridian at that given time.

There is a very simple method that the diagram above suggests. The idea is illustrated more exactly in the following:



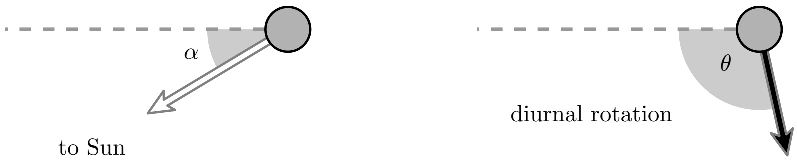
What this shows is that the first 'noon' occurs between sidereal times 1.2 and 1.4. A closer look suggests a more exact guess at a time around $2/3$ of the way from 1.2 to 1.4, or about $t = 1.33$. We can now check how things are at this time, with a different mode of illustration:



So now we see that this first noon occurs between times 1.33 and 1.4, about $1/10$ of the way along. So we can repeat with a new guess between 1.33 and 1.40, say roughly 1.34 . . .

But now let me set up the problem more precisely. Fix a ray in the orbital plane, that running from the planet at perihelion towards the Sun. Given this base line, associate to every time t two angles. One is the direction $\alpha(t)$ of the Sun with respect to that ray. This is the same as the direction of the planet as seen from the Sun, the **argument** of the planet's position.

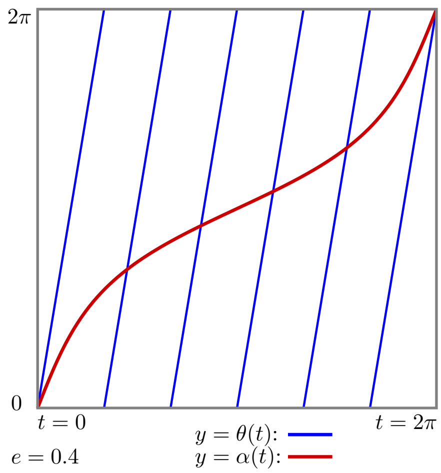
The other is the angle $\theta(t)$ between this ray and the meridian at time t. If θ_0 is the value of this angle at time $t = 0$, then $\theta(t) = \theta_0 + \omega t$, in which ω is the rate of rotation of the planet around its axis. This is the same as $2\pi n$, if there are n sidereal days in a year. (This n need not be an integer.) I'll take both these angles to lie in the interval $[0, 2\pi)$.



I'll assume time normalized, so that one orbit takes 2π units of time. Suppose we start at time $t = 0$, with the planet at perihelion. As t runs from 0 to 2π the planet rotates, say n times, so that θ runs through an accumulated angle of $2\pi n$. At time t

$$\theta(t) = \theta_0 + nt \quad (\text{ modulo } 2\pi)\,.$$

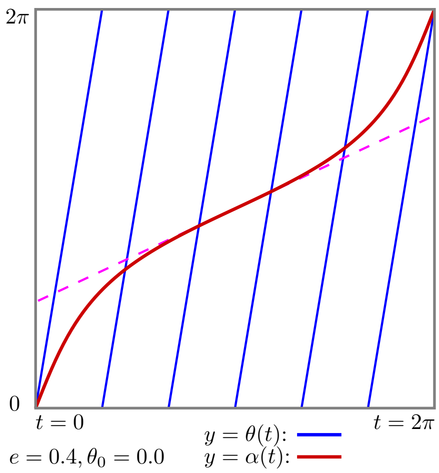
In the example above $n = 6$, while for Earth $n = 366.2422$. In the same period, $\alpha(t)$ runs from 0 to 2π. Here are the relevant graphs for the example above:



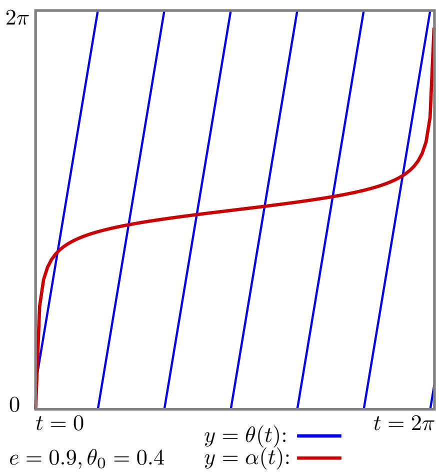
The graph of α is straightforward to plot from Kepler's equation. The graph of θ is made up of a number of line segments, each of slope n starting at points $\theta_0 + 2\pi k/n$ on the t-axis with $0 \le k \le n$. More explicitly:

$$\theta(t) = (\theta_0 + 2\pi k/n) + n(t - 2\pi k/n) \quad (2\pi k/n \le t < 2\pi(k+1)/n,\ 0 \le k \le n)\,.$$

The following figure suggests how to locate the moments of noon time:



First locate the intersections of the tangent line to the graph of α at the mid-point (π, π) with the line segments making up the graph of θ. The slope of this tangent is $ab/(a+f)^2$, and its equation is $y = \pi + ab(t-\pi)/(a+f)^2$. To the left of the mid-point, the noon marks will be below these intersections, and to the right they will be above. The method of secants will locate them easily.



There will generally be either $n-1$ or n noons in one orbital revolution. However, there will be sometimes $n+1$. When the derivative of α near 0 is greater than n—which happens when e is large—and θ_0 is small enough, there will be an extra noon close to perihelion to find. This happens in the figure above.