A simple way to compute structure constants of semi-simple Lie algebras

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Suppose $\mathfrak{g}$ to be a complex Lie algebra with basis (x_i). Then

$$[x_i, x_j] = \sum\nolimits_k c_{i,j}^k x_k$$

for some numbers $c_{i,j}^k$, called *structure constants*.

A standard technique for computing structure constants of semi-simple Lie algebras, which has been used in the computer program MAGMA, is described well by [Cohen-Murray-Taylor:2005]. It relies on the additive structure of roots. Another method, that works only for simply-laced root systems and relies on associated affine root systems, is explained in [Frenkel-Kac:1980]. A version of this for the remaining root systems can be found in [Rylands:2000]. It uses the identification of these others as folded quotients of simply laced ones.

In [Casselman:2015b] I explained yet another way to compute these structure constants by implementing an idea originally found in [Tits:1966a]. Tits' idea was to replace the additive structure by features of the normalizer of a maximal torus. This introduced some mathematical structure to the problem of computing structure constants that was missing in the standard approach. In practice, computation based on this method went fairly rapidly and seemed at least roughly comparable in efficiency to reported runs of the standard computation. There were, however, a number of rather ugly and presumably inefficient formulas involved in this new algorithm. A while ago, it was suggested by Robert Kottwitz (in May of 2014, with a supplementary remark later that year), that an observation of his about choosing bases of semi-simple Lie algebras might make it possible to bypass the nastiest parts in a more elegant manner. In this paper, with Kottwitz' permission, I'll explain how this goes.

Kottwitz' observations can be briefly summarized. Suppose

$$\begin{aligned}
G &= \text{a simple, connected, simply connected, complex group} \\
\mathfrak{g} &= \text{Lie algebra of } G \\
B &= \text{Borel subgroup} \\
T &= \text{maximal torus in } B \\
\Sigma &= \text{associated root system} \\
\Delta &= \text{associated simple roots} \\
W &= \text{Weyl group.}
\end{aligned}$$

Because G is simply connected, the coroot lattice $X_*(T)$ may be identified with the lattice spanned by the simple coroots $\alpha^\vee$.

The root spaces $\mathfrak{g}_\gamma$ all have dimension one. Fix for each α in Δ an element $e_\alpha \neq 0$ in $\mathfrak{g}_\alpha$. The triple $(B, T, \{e_\alpha\})$ make up a **frame** for G. The set of all frames is a principal homogeneous space for the adjoint quotient of G. (This notion originated in work of French mathematicians. In French the term is 'épinglage', which some translate literally into the noun 'pinning'. But 'frame' is the term adopted in the English translation of Bourbaki's treatise on Lie algebras.)

As I'll recall later, Chevalley has defined integral structures on $\mathfrak{g}$ and G. The map

$$\{\pm 1\}^\Delta \longrightarrow T, \quad (c_\alpha) \longmapsto \prod_{\alpha\in\Delta} \alpha^\vee(c_\alpha)$$

identifies $T(\mathbb{Z})$ with a two-torsion group. If $\mathcal{N}(\mathbb{Z})$ is the group of integral points in the normalizer $\mathcal{N} = N_G(T)$, it fits into a well understood extension

$$1 \longrightarrow T(\mathbb{Z}) \longrightarrow \mathcal{N}(\mathbb{Z}) \longrightarrow W \longrightarrow 1\,.$$

[Tits:1966b] defined a certain convenient section $w \mapsto \overset{\bullet}{w}$ of the last quotient map, and described this extension precisely enough to enable computations in it. This extension certainly does not generally split (as it does, perhaps deceptively, for GL_n). But now let $V_{\mathbb{Z}}$ be the direct sum of non-trivial root spaces in $\mathfrak{g}_{\mathbb{Z}}$. Let $S(\mathbb{Z})$ be the subgroup of transformations in $\mathrm{GL}(V_{\mathbb{Z}})$ that act as ± 1 on each root space. It may be identified with $\mathrm{Hom}(\Sigma, \pm 1) = (-1)^{\Sigma}$. The adjoint action of T defines a canonical homomorphism from $T(\mathbb{Z})$ to $S(\mathbb{Z})$: $\alpha^\vee(x)$ goes to $(x^{\langle \gamma, \alpha^\vee \rangle})_{\gamma \in \Sigma}$. The kernel is Z_G. The homomorphism from T to S gives rise to an extension

$$1 \longrightarrow S(\mathbb{Z}) = \{\pm 1\}^{\Sigma} \longrightarrow \mathcal{N}_{\mathrm{ext}}(\mathbb{Z}) \longrightarrow W \longrightarrow 1\,.$$

Although it does not act as automorphisms of $\mathfrak{g}$, the extension does act on $V_{\mathbb{Z}}$, compatibly with the adjoint action of $T(\mathbb{Z})$. Kottwitz' notable observation is that *this new extension splits*, and he gives an explicit splitting $w \mapsto \overset{\triangle}{w}$. It has the property that if $w\lambda = \lambda$ then $\overset{\triangle}{w}$ acts as the identity on $\mathfrak{g}_\lambda$. This allows one to specify a natural choice of Chevalley basis invariant under this action. One consequence of the new method is a very simple description of the action of $\mathcal{N}(\mathbb{Z})$ on $\mathfrak{g}$. This is especially important in applications to computation in the group G rather than just its Lie algebra.

Some of the previous methods known have the virtue that they may be extended to all Kac-Moody root systems (see [Casselman:2015b]). Some variant of the method I describe here will work for a large class of these. I do not see how it can be extended to all of them, but one might hope that some variation of Kottwitz' idea will work, taking into account some explicit obstruction. One promising prerequisite for extending the method to Kac-Moody algebras can be found in [Carbone et al.:2015], which classifies conjugacy classes of simple roots.

Curiously, it was in [Langlands-Shelstad:1987] that an explicit formula for a defining 2-cocycle of Tits' sections $w \mapsto \overset{\bullet}{w}$ first appeared. Recently Tasho Kaletha has found other applications of Tits' construction and results of this paper to related problems. I wish to thank him for comments on an earlier version.

Contents

For g in G, x in $\mathfrak{g}$, I'll write

$$g \diamond x = \mathrm{Ad}(g)x\,.$$

Even though $\mathcal{N}_{\mathrm{ext}}(\mathbb{Z})$ does not act by automorphisms of $\mathfrak{g}$, I'll use this notation for its action on $V_{\mathbb{Z}}$ as well.

I'll usually refer to [Tits:1966a] as [T].

1. Chevalley bases

Fix once and for all a maximal torus T in G, with Lie algebra $\mathfrak{t}$. The associated roots are the non-trivial characters by which T acts on eigenspaces, each of which has dimension one. For the moment, suppose γ to be a root. If $e \ne 0$ lies in $\mathfrak{g}_\gamma$, then for every f in $\mathfrak{g}_{-\gamma}$ the bracket $h = [f, e]$ will lie in $\mathfrak{t}$. There will exist exactly one f such that $[h, e] = 2e$. In these circumstances I'll call (e, h, f) an ***SL₂* triple**. It is completely determined by the choices of T and of e in $\mathfrak{g}_\gamma$.

Given such a triple, there exists a unique embedding ι_e of SL_2 into G whose differential takes

$$\begin{bmatrix} \circ & 1 \\ \circ & \circ \end{bmatrix} \longmapsto e$$
$$\begin{bmatrix} \circ & \circ \\ -1 & \circ \end{bmatrix} \longmapsto f$$
$$\begin{bmatrix} 1 & \circ \\ \circ & -1 \end{bmatrix} \longmapsto h\,.$$

If we change e to xe with $x \ne 0$, then f changes to $x^{-1}f$, and ι_e changes to its conjugate by

$$\begin{bmatrix} \sqrt{x} & \circ \\ \circ & 1/\sqrt{x} \end{bmatrix}.$$

The associated embedding of $\mathbb{C}^\times$ is the coroot $\gamma^\vee$, and is independent of the choice of e.

Now fix in addition a Borel subgroup B containing T. Let Δ be the corresponding set of simple roots and for each α in Δ fix an element $e_\alpha \ne 0$ in $\mathfrak{g}_\alpha$. The triple $(B, T, \{e_\alpha\})$ makes up a frame for G. The set of all frames is a principal homogeneous space for the group of inner automorphisms of G.

The frame determines embeddings ι_α of SL_2 into G, one for each simple root. Let h_α be the image under $d\iota_\alpha$ of

$$\begin{bmatrix} 1 & \circ \\ \circ & -1 \end{bmatrix}.$$

The image $e_{-\alpha}$ of

$$\begin{bmatrix} \circ & \circ \\ -1 & \circ \end{bmatrix}$$

is the unique element of $\mathfrak{g}_{-\alpha}$ such that

$$[e_{-\alpha}, e_\alpha] = h_\alpha\,.$$

This choice of sign is Tits'. It is not the common one, but it is exactly what is needed to make his analysis of structure constants work. The point is that there exists an automorphism θ of $\mathfrak{g}$ acting as $-I$ on $\mathfrak{t}$ and taking each e_α ($\alpha \in \Delta$) to $e_{-\alpha}$. It is uniquely determined by the choice of frame.

TITS' SECTION. The group $\mathcal{N}(\mathbb{Z})$ fits into a short exact sequence

$$1 \longrightarrow T(\mathbb{Z}) = (\pm 1)^\Delta \longrightarrow \mathcal{N}(\mathbb{Z}) \longrightarrow W \longrightarrow 1\,,$$

and [Tits:1966b] shows how to define a particularly convenient section. Define

$$\overset{\bullet}{s}_\alpha = \iota_\alpha\left(\begin{bmatrix} \circ & 1 \\ -1 & \circ \end{bmatrix}\right).$$

It lies in the normalizer of T. Suppose w in W to have the reduced expression $w = s_1 \ldots s_n$. Then the product

$$\overset{\bullet}{w} = \overset{\bullet}{s}_1 \ldots \overset{\bullet}{s}_n$$

depends only on w, not the particular product expression. The defining relations for this group, given those for T and W, are

$$\begin{aligned} (xy)^{\bullet} &= \overset{\bullet}{x}\overset{\bullet}{y} \quad (\text{when } \ell(xy) = \ell(x) + \ell(y)) \\ \overset{\bullet}{s}{}_{\alpha}^{2} &= \alpha^{\vee}(-1) \quad (\alpha \in \Delta)\,. \end{aligned}$$

CHEVALLEY'S FORMULA. Suppose $(e_\gamma, h_\gamma, e_{-\gamma})$ to form an $\boldsymbol{SL_2}$ triple, and suppose that $e_\gamma^\theta = ce_{-\gamma}$. If $f_\gamma = e_\gamma/\sqrt{c}$ and $f_{-\gamma} = \sqrt{c}e_\gamma$, then $(f_\gamma, h_\gamma, f_{-\gamma})$ also make up an $\boldsymbol{SL_2}$ triple, with $f_\gamma^\theta = f_{-\gamma}$. Up to sign—but only up to sign—f_γ is unique with this invariance condition.

Any complete set $\{e_\gamma\}$ invariant under θ up to sign is often called a Chevalley basis (with respect to the given frame). It determines an integral structure on the Lie algebra $\mathfrak{g}$.

1.1. Definition. *I'll call such a basis an* **integral basis**. *If it is actually invariant under θ, as it is here, I'll call it an* **invariant** *basis.*

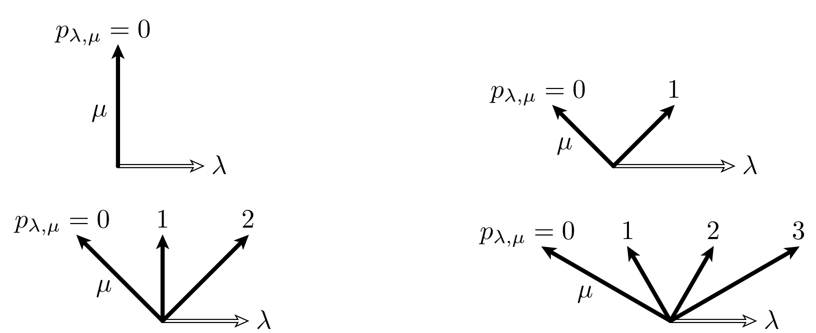
Remark. The integral structure on $\mathfrak{g}$ is determined by the frame, and more directly from the involution θ it defines. It is curious that θ also determines a maximal compact subgroup of G. Of course for p-adic groups, there is a more immediate relation between integral structure and compact subgroups.

Given any integral basis (e_λ)F, Chevalley proved that if λ, μ, ν are roots with $\lambda + \mu + \nu = 0$ then

(1.2) $$[e_\lambda, e_\mu] = \pm(p_{\lambda,\mu} + 1)e_{-\nu}\,.$$

Here $p_{\lambda,\mu}$ is the least p such that $\mu - p\lambda$ is a root. This was the crucial result used to construct the Chevalley groups over arbitrary fields.

The possible values for the string constants $p_{\lambda,\mu}$ (associated to finite root systems) are shown in the following figures:

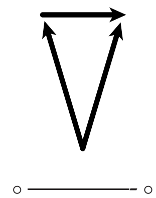


The fourth figure occurs only in type G_2. In practice, we shall be interested in computing $p_{\lambda,\mu}$ only when $\langle \mu, \lambda^\vee \rangle \le 0$. Under this assumption, as the figures illustrate:

(1.3) $$p_{\lambda,\mu} = \begin{cases} 0 & \text{if } \mu - \lambda \text{ is not a root} \\ 1 & \text{otherwise} \end{cases} \qquad (\text{assuming } \langle \mu, \lambda^\vee \rangle \le 0)\,.$$

I refer to [Chevalley:1955] or [Carter:1972] for the original proof of (1.2) and to [Casselman:2015a] for a proof extracted from [T], which works uniformly for all Kac-Moody groups. Tits' choice of the $e_{-\alpha}$ (as opposed to the more common choice with the opposite sign) introduces an elegant symmetry that greatly simplifies both proofs and formulas.

Remark. Ultimately, Chevalley's formula depends on the simple fact that for strings of length 2, as in the second figure above, one always has $\|\lambda\| \ge \|\mu\|$. That is to say, the following configuration never occurs.



Determining the sign in (1.2) has always seemed rather mysterious. Of course there can be no simple formula, since the choice of an integral basis is not canonical. But I don't think it has ever been very clear what is going on. Changing even one e_γ to $-e_\gamma$ forces a lot of other sign changes without apparent pattern. The situation has now been cleared up somewhat by Kottwitz, who has explained to me how to choose an almost canonical integral basis. I'll discuss that in §3.

In §4 I'll show how Kottwitz' basis simplifies the computation of the signs in Chevalley's formula. The starting point, at least, is the same as it was in [Casselman:2015b], in which I have already outlined the principal ingredients of a recipe for the constants. One of the troublesome points in the earlier approach was a somewhat arbitrary choice of integral basis. Kottwitz' basis eliminates this inconvenience.

2. Tits' idea

In order to understand how Kottwitz' basis makes calculation of structure constants simple, I must explain how it fits into the scheme covered in [Casselman:2015b] for computing structure constants. I'll do that in this section and the next.

In this one I shall recall results of Tits alluded to at the beginning of §1. We have seen there that a choice of root vector e determines an embedding ι_e of SL_2 into G, and in particular determines the element

$$\sigma_e = \iota_e\left(\begin{bmatrix} \circ & 1 \\ -1 & \circ \end{bmatrix}\right) .$$

Tits starts with a variation on this fact, an elementary observation about $G = \mathrm{SL}_2$. Let T be the subgroup of diagonal matrices. Its normalizer in G is the union of T itself and the subset M of matrices of the form

$$\begin{bmatrix} \circ & x \\ -1/x & \circ \end{bmatrix} .$$

Let $\mathfrak{g}_+$ be the Lie algebra of upper nilpotent matrices

$$\begin{bmatrix} \circ & x \\ \circ & \circ \end{bmatrix} ,$$

$\mathfrak{g}_-$ that of lower nilpotent ones. The following is Proposition 1 of §1.1 of [Tits:1966a]:

2.1. Lemma. *Suppose e in $\mathfrak{g}_+$, f in $\mathfrak{g}_-$, σ in M. The following are equivalent:*

$$\begin{aligned} &(a) \qquad \exp(e)\exp(f)\exp(e) = \sigma\,; \\ &(b) \qquad \exp(f)\exp(e)\exp(f) = \sigma\,. \end{aligned}$$

If any one of these three matrices is specified, conditions (a) or (b) determine the other two uniquely.

Proof. An easy matrix calculation shows that if

$$\begin{bmatrix} 1 & x \\ \circ & 1 \end{bmatrix}\begin{bmatrix} 1 & \circ \\ y & 1 \end{bmatrix}\begin{bmatrix} 1 & x \\ \circ & 1 \end{bmatrix}$$

lies in the normalizer of T, then $y = -1/x$, in which case the product is

$$\begin{bmatrix} \circ & x \\ -1/x & \circ \end{bmatrix} .$$

This proves the last claim. The equivalence of (a) and (b) follows from the equation

$$\begin{bmatrix} \circ & x \\ -1/x & \circ \end{bmatrix} \begin{bmatrix} 1 & x \\ \circ & 1 \end{bmatrix} \begin{bmatrix} \circ & -x \\ 1/x & \circ \end{bmatrix} = \begin{bmatrix} 1 & \circ \\ -1/x & 1 \end{bmatrix} .$$

I'll call the triplet (e, σ, f) **compatible**, and sometimes express the element in the normalizer as σ_e, which is the same as σ_f. This is one place where Tits' choice of f, rather then $-f$, is significant.

Now let G, T be arbitrary, as earlier. Suppose given some f_γ generating $\mathfrak{g}_\gamma \cap \mathfrak{g}_\mathbb{Z}$. It is unique up to sign. As we have seen, it determines an embedding of $\mathrm{SL}_2(\mathbb{Z})$ into $G(\mathbb{Z})$ and elements h_γ, $f_{-\gamma}$ spanning a copy of $\mathfrak{sl}_2$. We also get then an element σ_γ in $\mathcal{N}(\mathbb{Z})$, the image of

$$\begin{bmatrix} \circ & 1 \\ -1 & \circ \end{bmatrix} .$$

These also satisfy the equation

$$\exp(f_\gamma) \exp(f_{-\gamma}) \exp(f_\gamma) = \sigma_\gamma .$$

I'll also call the triplet $(f_\gamma, \sigma_\gamma, f_{-\gamma})$ compatible. If $\gamma = \alpha$ and $f_\alpha = e_\alpha$ for α in Δ, then $\sigma_\alpha = \dot{s}_\alpha$, but I do not assume this to hold. In any case the image of σ_γ in W will be s_γ, the reflection corresponding to γ. The basic observation of Tits ([T], Proposition 1) is that each of the objects $f_{\pm\lambda}$, σ_λ determines the other two. The choice of sign for any one of these determines a change of sign in the others.

In other words, the choice of an invariant basis is equivalent to a certain choice of elements in the normalizer $\mathcal{N}(\mathbb{Z}) = N_G(T) \cap G(\mathbb{Z})$.

- *For the indefinite future, fix an invariant Chevalley basis* (f_γ).

I repeat that I do *not* assume that $f_\alpha = e_\alpha$ for simple roots α. This determines also for each γ an element σ_γ, subject to the equations

$$\begin{aligned} \sigma_\gamma^{-1} &= \gamma^\vee(-1)\sigma_\gamma \\ \sigma_{-\gamma} &= \sigma_\gamma . \end{aligned}$$

Let $M_\gamma(\mathbb{Z})$ be the subset of $N_G(\mathbb{Z})$ in the image of $M \subset \mathrm{SL}_2$ determined by γ. It has two elements, and contains precisely the $\gamma^\vee(\pm 1)\, \sigma_\gamma$.

One practical consequence of Tits' observation is this:

2.2. Lemma. *Suppose ω to be in $\mathcal{N}(\mathbb{Z})$. Let w be its image in W, and assume that $w\lambda = \mu$.*

$$\omega \diamond f_\lambda = \varepsilon\, f_\mu$$

if and only if

$$\omega\, \sigma_\lambda \omega^{-1} = \mu^\vee(\varepsilon)\, \sigma_\mu .$$

Here ε is necessarily ± 1.

Proof. Since

$$\exp(\varepsilon f_\mu) \exp(\varepsilon f_{-\mu}) \exp(\varepsilon f_\mu) = \mu^\vee(\varepsilon)\sigma_\mu .$$

I remind you that the problem we are considering is this:

Given the integral basis (f_γ), we want to figure out how to calculate the sign in Chevalley's formula

$$[f_\lambda, f_\mu] = \pm(p_{\lambda,\mu} + 1) f_{\lambda+\mu} .$$

Tits has introduced a convenient symmetry into this problem by his choice of $f_{-\gamma}$. For example, since this basis is invariant under θ, the constants are now the same for $-\lambda$, $-\mu$ and λ, μ. Tits has introduced a second symmetry by another simple notion. I define a **Tits triple** to be a set of roots λ, μ, ν whose sum is 0. He makes this choice instead of taking, more conventionally, $\lambda + \mu = \nu$.

In any finite irreducible root system there are at most two lengths. Hence if $\lambda + \mu + \nu = 0$, two of them must be of the same length. As I have already mentioned, the common length cannot be greater than the third. Therefore any Tits triple can be cyclically permuted to satisfy the condition

$$\|\lambda\| \geq \|\mu\| = \|\nu\| \, .$$

In this case, I shall call it an **ordered** triple.

2.3. Proposition. ([T], Lemme 1 of §2.5) *Suppose* (λ, μ, ν) *to be a Tits triple. The following are equivalent:*

(a) it is an ordered triple;
(b) $s_\lambda \mu = -\nu$*;*
(c) $\langle \mu, \lambda^\vee \rangle = -1$.

The upshot of the discussion so far is that there exists a function $\varepsilon(\sigma_\lambda, \sigma_\mu, \sigma_\nu)$, defined on all products $M_\lambda(\mathbb{Z}) \times M_\mu(\mathbb{Z}) \times M_\nu(\mathbb{Z})$ whenever (λ, μ, ν) is a Tits triple, such that

$$[f_\lambda, f_\mu] = \varepsilon(\sigma_\lambda, \sigma_\mu, \sigma_\nu)\,(p_{\lambda,\mu} + 1) f_{-\nu} \, .$$

Of course I am assuming that the σ and f are compatible. The following is the basis of computation of structure constants by Tits' method.

2.4. Proposition. ([T], §2.9) *The function* $\varepsilon(\sigma_\lambda, \sigma_\mu, \sigma_\nu)$ *satisfies these basic properties*

(εa) replacing σ_λ *by* σ_λ^{-1} *changes its sign;*
(εb) it is skew-symmetric in any pair;
(εc) it is invariant under cyclic rotation of the arguments;
(εd) if λ, μ, ν *are an ordered triple with* $\sigma_\lambda \sigma_\mu \sigma_\lambda^{-1} = \sigma_\nu$ *then*

$$\varepsilon(\sigma_\lambda, \sigma_\mu, \sigma_\nu) = (-1)^{p_{\lambda,\mu}} \, .$$

The first two are immediate, but the third is not quite so. Together, these mean that we can apply a permutation to any triple to reduce to a special case, but what is now needed is one explicit formula in that special case—i.e. to pin down signs. That is what the last does. It follows from an analysis (in [T], §1.3) of the action of copies of $\mathrm{SL}_2(\mathbb{Z})$ on the spaces in $\mathfrak{g}$ determined by root strings in $\mathfrak{g}$.

For an ordered triple, because of Lemma 2.2 and the equality of σ_γ and $\sigma_{-\gamma}$:

$$\sigma_\lambda \sigma_\mu \sigma_\lambda^{-1} = \nu^\vee(\pm 1) \sigma_\nu \, .$$

2.5. Theorem. *Suppose* (λ, μ, ν) *to be an ordered Tits triple,* $\varepsilon = \pm 1$*. The following are equivalent:*

(a) $\sigma_\lambda \sigma_\mu \sigma_\lambda^{-1} = \nu^\vee(\varepsilon)\,\sigma_\nu$*;*
(b) $\sigma_\lambda \diamond f_\mu = \varepsilon\, f_{-\nu}$*;*
(c) $[f_\lambda, f_\mu] = \varepsilon\,(-1)^{p_{\lambda,\mu}}\,(p_{\lambda,\mu} + 1)\, f_{-\nu}$.

Combining these two propositions:

2.6. Corollary. *Suppose* (λ, μ, ν) *to be an ordered triple,* $\varepsilon = \pm 1$*. Assume that*

$$\sigma_\lambda \diamond \mathfrak{f}_\mu = \varepsilon\, f_{-\nu} \, .$$

If

$$c = \varepsilon\,(-1)^{p_{\lambda,\mu}}$$

then

$$\begin{aligned} [f_\lambda, f_\mu] &= c\,(p_{\lambda,\mu} + 1) f_{-\nu} \\ [f_\mu, f_\nu] &= c\,(p_{\mu,\nu} + 1) f_{-\lambda} \\ [f_\nu, f_\lambda] &= c\,(p_{\nu,\lambda} + 1) f_{-\mu} \, . \end{aligned}$$

This will be the basis of computations, once we have figured out how to calculate $\sigma_\lambda \diamond \mathfrak{f}_\mu$ for ordered triples.

One more thing we'll find useful is due to Tits, but formulated more explicitly in Chapter 4 (Theorem 4.1.2 (ii)) of [Carter:1972] and as Lemma 2.5 in [Casselman:2015]:

2.7. Proposition. *If (λ, μ, ν) is a Tits triple then*

$$\frac{p_{\lambda,\mu}+1}{\|\nu\|^2} = \frac{p_{\mu,\nu}+1}{\|\lambda\|^2} = \frac{p_{\nu,\lambda}+1}{\|\mu\|^2}$$

In other words, $p_{\lambda,\mu}$ satisfies a twisted cyclic symmetry.

3. Computation I

How do results in the previous section apply to practical computation of structure constants?

The ultimate goal is to come up with a procedure to determine brackets $[\mathfrak{f}_\lambda, \mathfrak{f}_\mu]$ easily, given an invariant basis $(\mathfrak{f}_\lambda)$. There are three possibilities. (1) If $\lambda = -\mu$, the bracket is h_μ. We can express it as a linear combination of basis elements h_α:

$$h_\mu = \sum_\alpha c_\alpha h_\alpha \, ,$$

in which the coefficients c_α are found in the course of constructing the roots, since this equation is equivalent to

$$\mu^\vee = \sum_\alpha c_\alpha \alpha^\vee \, .$$

(2) The sum $\lambda + \mu$ is not a root, and the bracket is 0. (3) We have an equation

$$[\mathfrak{f}_\lambda, \mathfrak{f}_\mu] = N_{\lambda,\mu} \mathfrak{f}_{\lambda+\mu}$$

for some constant $N_{\lambda,\mu}$ of the form $\pm(p_{\lambda,\mu}+1)$. So we would be given a Tits triple (λ, μ, ν). We can rotate it to make it an ordered triple. According to Corollary 2.6 and , our problem is thus reduced to finding just the values $N_{\lambda,\mu}$ when (λ, μ, ν) is an ordered triple. Since $N_{-\lambda,-\mu} = N_{\lambda,\mu}$, we may restrict to the case $\lambda > 0$.

It is quite reasonmable to store all values of $N_{\lambda,\mu}$ for ordered Tits triples. The amount of storage required is roughly proportional to the number of Tits triples. As reported in [Cohen-Murray-Taylor:2005], this is of order r^3, where r is the rank of the system, so this procedure is entirely feasible, and noticeably better in storage use than storing all the $N_{\lambda,\mu}$, since there are roughly r^4 such pairs. (Of course using the smaller table involves more computation. The trade-off of time versus memory that we see here is a basic problem in all programming.)

There are three steps to this computation.

Step 1. In the first, we construct the root system, without reference to a Lie algebra. This includes (i) root lengths $\|\lambda\|$, (ii) values of $\langle \lambda, \alpha^\vee \rangle$, (iii) root reflection tables $s_\alpha \lambda$, (iv) an expression for each root as a linear combination of the α in Δ, and (v) a corresponding expression for each $\lambda^\vee$ as a sum of $\alpha^\vee$. We can also construct a table recording whether or not a given array of coordinates is that of a root or not.

Step 2. In some way specified in the next sections, we then find an invariant basis $(\mathfrak{f}_\lambda)$. It is here where Kottwitz' contribution appears. It will give us also the associated Tits section $\mathring{w}$ from W to $N_G(T)$, in which $\mathring{s}_\alpha$ for α in Δ. Miraculously:

> *Constructing the invariant basis $(\mathfrak{f}_\lambda)$ will give at the same time formulas for the constants $c(s_\alpha, \lambda)$ (with α simple) such that*

(3.1) $$\mathring{s}_\alpha \diamond \mathfrak{f}_\mu = c(s_\alpha, \lambda) \mathfrak{f}_{s_\alpha \lambda} \, .$$

I repeat: we start with a frame (e_α), but the new basis elements $\mathfrak{f}_\alpha)$ will be different, and the elements $\overset{\bullet}{s}_\alpha$ will be different from the $\overset{\circ}{s}_\alpha$.

We shall now have a simple recipe for computing any $\overset{\circ}{w} \diamond \mathfrak{f}_\lambda$, since if

$$\overset{\circ}{w} \diamond f_\lambda = c(w,\lambda)\mathfrak{f}_{w\lambda}$$

then

$$c(xy,\lambda) = c(x,y\lambda)c(y,\lambda)\,.$$

Remark. This can be somewhat inefficient, since the element w can have length up to the number of positive roots. There is a possible improvement, however, offering a trade of memory for time. Choose an ordering of Δ, and let W_i be the subgroup of W generated by the s_{α_j} for $j \le i$. As Fokko du Cloux pointed out, every w in W can be expressed as a unique product

$$w = w_1 w_2 \ldots w_r$$

with each w_i a distinguished representative of $W_{i-1}\backslash W_i$. The sizes of these cosets are relatively small, and it is perhaps not infeasible to store values of the $w\lambda$ and the $c(w,\lambda)$ for w a distinguished element in one of them.

Step 3. Given the results of the previous step, we want now to tell how to compute the constants $N_{\lambda,\mu}$ when (λ,μ,ν) make up an ordered Tits triple with $\lambda > 0$.

We can do this by a kind of induction on λ. Every positive root $\lambda = w\alpha$ for w in W and α simple. The **depth** n of λ is the minimal length of a chain

$$\alpha = \lambda_0 - \lambda_1 - \cdots - \lambda_n = \lambda$$

in which each $\lambda_{i+1} = s_{\alpha_i}\lambda_i$ for some simple α_i. Finding such chains for all positive roots is part of the natural process for constructing the set of roots in the first place. If

$$[\mathfrak{f}_\lambda, \mathfrak{f}_\mu] = N_{\lambda,\mu}\mathfrak{f}_{-\nu}$$

then

$$[\overset{\circ}{s}_\alpha \diamond \mathfrak{f}_\lambda, \overset{\circ}{s}_\alpha \diamond \mathfrak{f}_\mu] = N_{\lambda,\mu}(\overset{\circ}{s}_\alpha \diamond \mathfrak{f}_{-\nu})\,,$$

and hence

$$N_{s_\alpha\lambda, s_\alpha\mu} = c(s_\alpha,\lambda)c(s_\alpha,\mu)c(s_\alpha,-\mu)N_{\lambda,\mu}\,.$$

Reflections transform ordered triples to ordered triples. Hence if we know how to deal with the case in which $\lambda = \alpha$ is simple we can compute all the constants for ordered triples in which $\lambda > 0$ by following up the chain. Furthermore, according to Proposition 2.3 it is very easy to list ordered triples (α,μ,ν).

Now according to Theorem 2.5 we have

$$[\mathfrak{f}_\alpha, \mathfrak{f}_\mu] = c(s_\alpha,\mu)(-1)^{p_{\alpha,\mu}}(p_{\alpha,\mu}+1)\mathfrak{f}_{-\nu}\,.$$

Since $\langle \mu, \alpha^\vee \rangle = -1$ we know that $p_{\alpha,\mu}$ is 0 if $\mu - \alpha$ is not a root, and is 1 otherwise (in which case we are dealing with G_2).

At the end we have the structure constants for all ordered Tits triples with λ positive.

4. Kottwitz' splittings

It remains to explain how to construct an invariant basis $(\mathfrak{f}_\lambda)$ and give formulas for the constants $c(s_\alpha, \lambda)$ appearing in (3.1) .

In any method of computation in Lie algebras, the first—and perhaps most important—step is to specify an integral basis of the algebra. [Cohen-Murray-Taylor:2005] specifies such a basis in terms of an ordered decomposition of a given root as a sum of simple ones. First of all, they assign an order to the simple roots. Every positive root may be expressed uniquely as $\mu = \alpha + \lambda$ in which the height of λ is less than that of μ, and α is least with this property. One then defines the elements e_μ by induction:

$$[e_\alpha, e_\lambda] = (p_{\alpha,\lambda} + 1)e_\mu \,.$$

In effect, such a basis is determined by a choice of spanning tree in a graph whose nodes are the positive roots, with a link between each pair λ and $\alpha + \lambda$.

The method I described in [Casselman:2015a] and [Casselman:2015b] chooses a basis in terms of paths in a spanning tree in a different graph whose nodes are again the positive roots. The simplest implementation starts also with an ordering of simple roots. Every positive root may be expressed as $\mu = s_\alpha\lambda$, with λ of smaller height and α minimal. Then define by induction

$$e_\mu = \overset{\bullet}{s}_\alpha e_\lambda \,.$$

There is a great deal of arbitrariness in both methods, since they depend on a somewhat arbitrary choice of spanning tree in a graph. Kottwitz' contribution is to remove nearly all this annoying ambiguity. A basis chosen directly by his method will not be invariant under θ, but it will be easy to determine from it one that is.

The original choice of frame gives us Tits' map $w \mapsto \overset{\bullet}{w}$ from W back to $\mathcal{N}(\mathbb{Z})$, and then to the extended group $\mathcal{N}_{\mathrm{ext}}(\mathbb{Z})$. How can it be modified to become a homomorphism?

We are looking for a splitting of the sequence

$$1 \longrightarrow S(\mathbb{Z}) \longrightarrow \mathcal{N}_{\mathrm{ext}}(\mathbb{Z}) \longrightarrow W \longrightarrow 1 \,.$$

This will be of the form

$$w \longmapsto \overset{\vartriangle}{w} = \overset{\bullet}{w}\cdot \tau_w \,,$$

with each τ_w in $S(\mathbb{Z})$. Thus for each root β we are looking for a factor $\tau_w(\beta) = \pm 1$. The map $w \mapsto \overset{\vartriangle}{w}$ will be a homomorphism if and only if (for α in Δ)

$$\begin{array}{ll} (a) & \overset{\vartriangle}{1} = 1 \\ (b) & \overset{\vartriangle}{s}_\alpha \overset{\vartriangle}{x} = (s_\alpha x)^{\vartriangle} \ \text{ if } s_\alpha x > x \\ (c) & \overset{\vartriangle}{s}_\alpha \overset{\vartriangle}{s}_\alpha = 1 \,. \end{array}$$

These translate directly to properties of τ_w:

$$\begin{array}{ll} (a') & \tau_1 = 1 \\ (b') & \tau_{s_\alpha}(y\beta)\tau_y(\beta) = \tau_{s_\alpha y}(\beta) \ \text{ for all } \beta \ \text{ if } \ s_\alpha y > y \\ (c') & (-1)^{\langle \beta, \alpha^\vee \rangle} = \tau_{s_\alpha}(s_\alpha\beta)\cdot \tau_{s_\alpha}(\beta) \,. \end{array}$$

We shall see a bit later a fourth useful condition on $\overset{\vartriangle}{w}$ and hence also on τ_w.

At any rate, here is Kottwitz' solution of the problem. For w in W set

$$R_w = \{\lambda > 0 \,|\, w\lambda < 0\} \,.$$

Thus $\ell(xy) = \ell(x) + \ell(y)$ if and only if

(4.1) $$R_{xy} = R_y \sqcup y^{-1} R_x \,,$$

and in particular

$$\begin{aligned} R_1 &= \emptyset \\ R_{s_\alpha} &= \{\alpha\} \\ R_{s_\alpha w} &= R_w \sqcup \{w^{-1}\alpha\} \quad (w^{-1}\alpha > 0)\,. \end{aligned}$$

According to Kottwitz' recipe, we have

(4.2) $$\tau_w(\beta) = (-1)^{F(w,\beta)} \quad \text{with} \quad F(w,\beta) = \sum_{\gamma \in R_w} \langle\!\langle \beta, \gamma \rangle\!\rangle \,.$$

The summands are yet to be specified, and everything in this formula is to be taken modulo 2.

• Since $R_1 = \emptyset$ and an empty sum is 0, condition (a) above is immediate.

• What about condition (b)? Suppose $x = s_\alpha y > y$. It must be shown that the cocycle condition

$$F(s_\alpha y, \beta) = F(s_\alpha, y\beta) + F(y, \beta)$$

holds. First of all, note that

$$F(s_\alpha, \beta) = \langle\!\langle \beta, \alpha \rangle\!\rangle$$

since $R_{s_\alpha} = \{\alpha\}$. Also

$$F(x,\beta) = \sum_{\gamma \in R_x} \langle\!\langle \beta, \gamma \rangle\!\rangle = \langle\!\langle \beta, y^{-1}\alpha \rangle\!\rangle + \sum_{\gamma \in R_y} \langle\!\langle \beta, \gamma \rangle\!\rangle$$

whereas

$$F(s_\alpha, y\beta) + F(x,\beta) \; = \langle\!\langle y\beta, \alpha \rangle\!\rangle + \sum_{\gamma \in R_y} \langle\!\langle \beta, \gamma \rangle\!\rangle \,.$$

Therefore (b) will be satisfied if W-invariance holds:

$$\langle\!\langle w\beta, w\gamma \rangle\!\rangle = \langle\!\langle \beta, \gamma \rangle\!\rangle \text{ for all } w \text{ in } W.$$

• Condition (c)? We have

$$\overset{\vartriangle}{s}_\alpha \diamond e_\beta = (-1)^{\langle\!\langle \beta, \alpha \rangle\!\rangle} \overset{\bullet}{s}_\alpha \diamond e_\beta \,.$$

Since $\overset{\bullet}{s}{}_\alpha^2 = \alpha^\vee(-1)$ we thus require that

$$\langle\!\langle s_\alpha\beta, \alpha \rangle\!\rangle + \langle\!\langle \beta, \alpha \rangle\!\rangle = \langle \beta, \alpha^\vee \rangle \,.$$

This last condition suggests what comes now. If $\langle \beta, \alpha^\vee \rangle = 0$ and hence $s_\alpha\beta = \beta$ this imposes no condition (since everything is modulo 2). Otherwise $\langle \beta, \alpha^\vee \rangle$ and $\langle s_\alpha\beta, \alpha^\vee \rangle$ will be of different signs. It is therefore natural to set

(4.3) $$\langle\!\langle \beta, \gamma \rangle\!\rangle = \begin{cases} \langle \beta, \gamma^\vee \rangle & \text{if } \langle \beta, \gamma^\vee \rangle > 0 \\ 0 & \text{if } \langle \beta, \gamma^\vee \rangle < 0. \end{cases}$$

One good sign:

4.4. Lemma. *The function $\langle\!\langle \beta, \gamma \rangle\!\rangle$ is Weyl-invariant.*

Proof. Since the pairing $\langle \beta, \gamma^\vee \rangle$ is W-invariant.

The requirement that $w \mapsto \overset{\vartriangle}{w}$ be a homomorphism imposes no extra condition in the case that $\langle \beta, \gamma^\vee \rangle = 0$, but one more requirement will do so. I ask now, for reasons that will become apparent in a moment, that

$$\overset{\vartriangle}{w} \diamond e_\beta = e_\beta$$

if $w\beta = \beta$. To guarantee that this occurs, it suffices to assume that β lies in the closed positive Weyl chamber. Then the w fixing β are generated by simple root reflections, so we need to require only that $\overset{\vartriangle}{s}_\alpha v_\beta = v_\beta$ ($v_\beta \in \mathfrak{g}_\beta$) for simple roots α with $\langle \beta, \alpha^\vee \rangle = 0$. Consideration of the representation of SL_2 corresponding to the root string tells us that

$$\overset{\bullet}{s}_\alpha \diamond e_\beta = (-1)^{p_{\alpha,\beta}} e_\beta \,.$$

Therefore

$$\overset{\vartriangle}{s}_\alpha \diamond e_\beta = (-1)^{\langle\!\langle \beta, \alpha \rangle\!\rangle} (-1)^{p_{\alpha,\beta}} e_\beta$$

and so we set

(4.5) $$\langle\!\langle \beta, \gamma \rangle\!\rangle = p_{\gamma,\beta} \quad \text{if } \langle \beta, \gamma^\vee \rangle = 0 \,.$$

Equations (4.3) and (4.5) define the terms $\langle\!\langle \beta, \gamma \rangle\!\rangle$ completely. In summary:

4.6. Theorem. (Kottwitz) *Let*

$$\begin{aligned} \langle\!\langle \beta, \gamma \rangle\!\rangle &= \begin{cases} \langle \beta, \gamma^\vee \rangle & \text{if this is positive} \\ p_{\gamma,\beta} & \text{if } \langle \beta, \gamma^\vee \rangle = 0 \\ 0 & \text{otherwise.} \end{cases} \\ F(w, \beta) &= \sum_{\gamma \in R_w} \langle\!\langle \beta, \gamma \rangle\!\rangle \\ \tau_w &= \big((-1)^{F(w,\beta)} \big)_{\beta \in \Sigma} \,. \end{aligned}$$

Then

$$\overset{\vartriangle}{w} = \overset{\bullet}{w} \cdot \tau_w$$

is a splitting homomorphism of $\mathcal{N}_{\mathrm{ext}}(\mathbb{Z})$. In addition, if $w\gamma = \gamma$ then $\mathrm{Ad}(\overset{\vartriangle}{w})$ is the identity on $\mathfrak{g}_\gamma$.

If the root system is simply laced or equal to G_2 then $s_\lambda \beta = \beta$ implies that $p_{\lambda,\beta} = 0$. Therefore the non-trivial case occurs only for systems B_n, C_n, or F_4.

Remark. *Lemma 2.1A of [Langlands-Shelstad:1987] exhibits the 2-cocycle defining the extension $\mathcal{N}(\mathbb{Z})$ determined by Tits's splitting $w \mapsto \overset{\bullet}{w}$. Explicitly,*

$$\overset{\bullet}{x}\overset{\bullet}{y} = \kappa(x, y) (xy)^\bullet \ \textit{ with } \ \kappa(x, y) = \prod_{\substack{\gamma > 0 \\ x^{-1}\gamma < 0 \\ y^{-1}x^{-1}\gamma < 0}} \gamma^\vee(-1) \,.$$

Does Kottwitz' splitting allow arguments of Langlands and Shelstad to be simpler?

$\circ$ ———— $\circ$

The W-orbits in Σ are the sets of all roots of the same length. Pick one simple root α in each orbit, and let $\mathfrak{e}_\alpha = e_\alpha$ be the corresponding element in the frame chosen at the beginning. If $\lambda = w\alpha$ is root with α equal to one of these distinguished choices, define

$$\mathfrak{e}_\lambda = \overset{\vartriangle}{w} \diamond \mathfrak{e}_\alpha \,.$$

The definition of $F(s_\alpha, \beta)$ in the case when $\langle \beta, \alpha^\vee \rangle = 0$ insures that this is a valid definition. As a consequence of Theorem 4.6:

4.7. Corollary. *The integral basis $(\mathfrak{e}_\gamma)$ of $V_\mathbb{Z}$ is such that $\overset{\triangle}{w} \diamond \mathfrak{e}_\gamma = \mathfrak{e}_{w\gamma}$ for all roots γ and w in W.*

4.8. Definition. *I'll call such a basis* **semi-canonical**.

There are several possibilities, two for each W-orbit in Σ.

In practice, we shall want to compute τ_w explicitly only when $w = s_\alpha$ for α in Δ. In this case, there is a simplification, since R_{s_α} is a singleton.

$$F(s_\alpha, \lambda) = \begin{cases} \langle \lambda, \alpha^\vee \rangle & \text{if it is positive} \\ p_{\alpha,\beta} & \text{if } \langle \lambda, \alpha^\vee \rangle = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Example. For a simply laced root system, if $\langle \beta, \alpha^\vee \rangle = 0$ then $p_{\alpha,\beta} = 0$. Therefore

$$\tau_{s_\alpha}(\lambda) = \begin{cases} (-1)^{\langle \beta, \alpha^\vee \rangle} & \text{if } \langle \lambda, \alpha^\vee \rangle > 0 \\ \quad 1 & \text{otherwise.} \end{cases}$$

This applies in particular to $G = \mathrm{SL}_3$. Take α, β as the standard simple roots, and let $\gamma = \alpha + \beta$. Recall that $e_{i,j}$ is the matrix with a single non-zero entry 1 at (i, j). Choose $e_{1,2}$ and $e_{2,3}$ to define the frame, spanning the root spaces for α, β. The corresponding elements of $\mathcal{N}(\mathbb{Z})$ are

$$\overset{\bullet}{s}_\alpha = \begin{bmatrix} \circ & 1 & \circ \\ -1 & \circ & \circ \\ \circ & \circ & 1 \end{bmatrix}, \quad \overset{\bullet}{s}_\beta = \begin{bmatrix} 1 & \circ & \circ \\ \circ & \circ & 1 \\ \circ & -1 & \circ \end{bmatrix}.$$

And here is a table of the $\diamond$ actions:

λ	e_λ	$\overset{\bullet}{s}_\alpha \diamond e_\lambda$	$\langle \lambda, \alpha^\vee \rangle$	$\langle\langle \lambda, \alpha \rangle\rangle$	$\overset{\bullet}{s}_\beta \diamond e_\lambda$	$\langle \lambda, \beta^\vee \rangle$	$\langle\langle \lambda, \beta \rangle\rangle$
α	$e_{1,2}$	$-e_{2,1}$	2	0	$-e_{1,3}$	-1	0
β	$e_{2,3}$	$e_{1,3}$	-1	0	$-e_{3,2}$	2	0
γ	$e_{1,3}$	$-e_{2,3}$	1	1	$e_{1,2}$	1	1
$-\alpha$	$e_{2,1}$	$-e_{1,2}$	-2	0	$-e_{3,1}$	1	1
$-\beta$	$e_{3,2}$	$e_{3,1}$	1	1	$-e_{2,3}$	2	0
$-\gamma$	$e_{3,1}$	$-e_{3,2}$	-1	0	$e_{2,1}$	-1	0

Recall that $\langle \alpha, \beta^\vee \rangle < 0$ while $\langle \gamma, \alpha^\vee \rangle > 0$. If we start with $\mathfrak{e}_\alpha = e_{1,2}$ we get

$$\begin{aligned} \mathfrak{e}_\alpha &= e_{1,2} = e_\alpha \\ \mathfrak{e}_\gamma &= \overset{\triangle}{s}_\beta \mathfrak{e}_\alpha \\ &= (-1)^0 \overset{\bullet}{s}_\beta \diamond e_{1,2} \\ &= -e_{1,3} \\ \mathfrak{e}_\beta &= \overset{\triangle}{s}_\alpha \mathfrak{e}_\gamma \\ &= (-1)^1 \overset{\bullet}{s}_\alpha \diamond (-e_{1,3}) \\ &= -e_{2,3} = -e_\beta\,. \end{aligned}$$

Thus:

4.9. Proposition. *If $G = \mathrm{SL}_3$ and $\mathfrak{e}_\alpha = e_\alpha$, then $\mathfrak{e}_\beta = -e_\beta$.*

This example has consequences for arbitrary root systems.

Something similar is true for SL_n. Here, choose the base point of the Dynkin diagram to be the end corresponding to the simple root $\varepsilon_1 - \varepsilon_2$. Then

$$\mathfrak{e}_{i,j} = (-1)^j e_{i,j}\,.$$

$\circ$ ——— $\circ$

A semi-canonical basis will not be invariant under θ, but it is easy to see how it fails, and then how to modify it to be so. Recall that the height of a root is defined by the formula

$$\mathrm{ht}\Big(\sum\nolimits_{\Delta} \lambda_\alpha \alpha\Big) = \sum\nolimits_{\Delta} \lambda_\alpha \,.$$

4.10. Theorem. *For any root γ and semi-canonical basis $(\mathfrak{e}_\gamma)$*

$$\mathfrak{e}_\gamma^\theta = (-1)^{\mathrm{ht}(\gamma)-1}\mathfrak{e}_{-\gamma} \,.$$

In particular, if α is simple then

$$\mathfrak{e}_\alpha^\theta = \mathfrak{e}_{-\alpha} \,.$$

This particularly simple formulation is due to Kottwitz.

Proof. In a number of short steps.

Step 1. The following is straightforward:

• For all β, γ

$$\langle\!\langle \beta, \gamma \rangle\!\rangle + \langle\!\langle -\beta, \gamma \rangle\!\rangle = \langle \beta, \gamma^\vee \rangle$$

This is to be interpreted modulo 2, of course.

Step 2. Now let

$$h(w, \beta) = \sum_{\gamma \in R_w} \langle \beta, \gamma^\vee \rangle \,.$$

• *For v in $\mathfrak{g}_\beta$*

(4.11)
$$(\overset{\vartriangle}{w} \diamond v)^\theta = (-1)^{h(w,\beta)}\, \overset{\vartriangle}{w} \diamond v^\theta \,.$$

This is because $\overset{\bullet}{s}{}_\alpha^\theta = \overset{\bullet}{s}_\alpha$.

Step 3. Induction on the length of w together with (4.1) will prove:

• *For w in W and root λ*

$$\mathrm{ht}(w\lambda) - \mathrm{ht}(\lambda) = h(w, \lambda) \,.$$

In order to specify the $\mathfrak{e}_\gamma$, given a frame (e_α), we fix one simple root α in each W-orbit, and set $\mathfrak{e}_\alpha = e_\alpha$. Fixing the $\mathfrak{e}_\beta$ for other simple roots β is then very easy. For finite-dimensional Lie algebras, W-orbits of roots are in correspondence with possible root lengths. For irreducible systems, there are at most two possible lengths, and the simple roots of a given length make up a connected segment Ξ in the Dynkin diagram. It is only systems B, C, F, and G that there are two lengths, and only for system F is there more than one simple root of each length.

Choose, somewhat arbirarily, one **special** root α_Ξ on each segment Ξ. For every simple root α, let

$$d(\alpha) = \text{ the distance from } \alpha \text{ to the special root } \alpha_\Xi \text{ in its segment.}$$

Any two neighbours in the Dynkin diagram of the same length lie in the simple root system of a copy of SL_3. The choice of $\mathfrak{e}_\alpha$ determines an element σ_α. The following is a consequence of Proposition 4.9:

4.12. Corollary. *For α in Δ let $c_\alpha = (-1)^{d(\alpha)}$. Then*

$$
\begin{aligned}
\mathfrak{e}_\alpha &= c_\alpha \, e_\alpha \\
\sigma_\alpha &= \alpha^\vee(c_\alpha) \, \overset{\bullet}{s}_\alpha \, .
\end{aligned}
$$

Here, I recall, σ_α is the element of $N_{\mathbb{Z}}(T)$ associated by Tits' scheme to the choice of $\mathfrak{e}_\alpha$ as basis of $\mathfrak{g}_\alpha$ (or of $\mathfrak{e}_{-\alpha}$ for $\mathfrak{g}_{-\alpha}$).

Remark. I have mentioned the 'root graph' without being precise, and I should say something more about it. It is a graph whose nodes are the positive roots, and its base is made up of the simple roots. There is an oriented edge from λ to $s_\alpha\lambda$ if and only if $s_\alpha\lambda$ has greater height than λ, or equivalently if and only if $\langle \lambda, \alpha^\vee \rangle < 0$. This is very useful, since in these circumstances $\langle\!\langle \lambda, \alpha \rangle\!\rangle$ is always 0. One consequence is an easy construction of the basis $(\mathfrak{e}_\lambda)$. Following upward links in the root graph, one represents every root as an increasing chain

$$
\alpha = \lambda_0 \prec \ldots \prec \lambda_n = \lambda \quad (\lambda_{i+1} = s_{\alpha_i}\lambda_i)
$$

and then

$$
\mathfrak{e}_\lambda = \overset{\bullet}{s}_{n-1} \ldots \overset{\bullet}{s}_0 \diamond \mathfrak{e}_\alpha \, .
$$

This is very useful for debugging programs, since for the classical root systems one can construct Kottwitz' basis in terms of explicit matrices, for which one can calculate Lie brackets in terns of matrix products.

5. Computation II

Define

$$
\gamma(\lambda) = \begin{cases} 1 & \text{if } \lambda > 0 \\ (-1)^{\mathrm{ht}(-\gamma)-1} & \text{if } \lambda < 0. \end{cases}
$$

As an immediate consequence of Theorem 4.10:

5.1. Proposition. *Given the Kottwitz basis $(\mathfrak{e}_\lambda)$, the elements*

$$
\mathfrak{f}_\lambda = \gamma(\lambda)\mathfrak{e}_\lambda
$$

form an invariant integral basis.

Remark. I emphasize: we start with a given frame, then find a new frame that is rarely the same as the original. It is this new frame that we extend to an integral basis in a uniquely determined way.

Let $\overset{\circ}{w}$ be the corresponding Tits section.

The following result encapsulates the basic reason why Kottwitz' basis makes computation simple. Recall that

$$
c_\alpha = (-1)^{d(\alpha)} \, ,
$$

where $d(\alpha)$ measures distance along the Dynkin diagram from the nearest simple root α_Ξ.

5.2. Theorem. *For α in Δ and $\lambda > 0$ let*

$$
m_{\alpha,\lambda} = (-1)^{\langle\!\langle \lambda,\alpha \rangle\!\rangle} c_\alpha^{\langle \lambda, \alpha^\vee \rangle} \, .
$$

Then for every root λ

$$
\overset{\circ}{s}_\alpha \diamond \mathfrak{f}_\lambda = c(s_\alpha, \lambda)\mathfrak{f}_{s_\alpha\lambda}
$$

with

$$
c(s_\alpha, \lambda) = \begin{cases} m_{\alpha,\lambda} & \text{if } \lambda > 0 \\ m_{\alpha,-\lambda} & \text{if } \lambda < 0 \, . \end{cases}
$$

Proof. Since $\lambda(\alpha^\vee(x)) = x^{\langle \lambda, \alpha^\vee \rangle}$:

$$\begin{aligned} \overset{\triangle}{s}_\alpha \diamond \mathfrak{e}_\lambda &= \mathfrak{e}_\mu \\ &= (-1)^{\langle\!\langle \lambda, \alpha \rangle\!\rangle}\, \overset{\bullet}{s}_\alpha \diamond \mathfrak{e}_\lambda && \text{(definition)} \\ \overset{\bullet}{s}_\alpha \diamond \mathfrak{e}_\lambda &= c_\alpha^{\langle \lambda, \alpha^\vee \rangle}\, \overset{\circ}{s}_\alpha \diamond \mathfrak{e}_\lambda && \text{(Corollary 4.12)} \\ \overset{\circ}{s}_\alpha \diamond \mathfrak{e}_\lambda &= c_\alpha^{\langle \lambda, \alpha^\vee \rangle} (-1)^{\langle\!\langle \lambda, \alpha \rangle\!\rangle} \mathfrak{e}_\mu \\ &= m_{\alpha,\lambda} \cdot \mathfrak{e}_\mu \,. \end{aligned}$$

This concludes when $\lambda > 0$, even if $\lambda = \alpha$ and $\sigma_\alpha \diamond \mathfrak{f}_\alpha = \mathfrak{f}_{-\alpha}$. When not, apply the involution θ to this equation, noting that $\overset{\circ}{s}_\alpha$ commutes with it.

Example. Look at SL_3 again. What is $[\mathfrak{e}_\alpha, \mathfrak{e}_\beta]$?

$$\begin{aligned} c_\alpha &= 1 \\ \langle \beta, \alpha^\vee \rangle &= -1 \\ \langle\!\langle \alpha, \beta \rangle\!\rangle &= 0 \\ p_{\alpha,\beta} &= 0 \\ c(s_\alpha, \beta) &= 1 \end{aligned}$$

Hence $\overset{\circ}{s}_\alpha \mathfrak{e} = \mathfrak{e}_\gamma$.

Example. Say $G = \mathrm{Sp}(4)$. Let $\alpha = \varepsilon_0 - \varepsilon_1$ and $\beta = 2\varepsilon_1$ be the simple roots. Since there are two lengths of roots, we may set as frame

$$\mathfrak{e}_\alpha = e_\alpha = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathfrak{e}_\beta = e_\beta = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then

$$\overset{\bullet}{s}_\beta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Since $\langle \alpha, \beta^\vee \rangle = -1$, $s_\beta \alpha = \alpha + \beta =$ (say) γ. Also, $\langle\!\langle \beta, \alpha \rangle\!\rangle = 0$ and hence

$$\mathfrak{e}_\gamma = \overset{\bullet}{s}_\beta \mathfrak{e}_\alpha \overset{\bullet}{s}_\beta^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

One calculates directly that

$$[\mathfrak{e}_\alpha, \mathfrak{e}_\beta] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = -\mathfrak{e}_\gamma \,.$$

But it is instructive to trace how the computations in this paper would go. We are looking at the triple $(\alpha, \beta, -\gamma)$. Since $\|\beta\| = 2$ while $\|\alpha\| = 1$, the associated ordered triple is $(\beta, -\gamma, \alpha)$. Since $-s_\beta \gamma = -\alpha$, we must next compute the constant ε such that

$$\overset{\bullet}{s}_\beta \mathfrak{f}_{-\gamma} = \varepsilon \mathfrak{f}_{-\alpha} \,.$$

This is $c(s_\beta, -\gamma)$, which according to Theorem 5.2 is

$$m_{\beta,\gamma} = (-1)^{\langle\!\langle \gamma,\beta \rangle\!\rangle} = -1\,.$$

Remark. It is not difficult to compute any $p_{\lambda,\mu}$ by finding directly the maximum value of n such that $\mu - n\lambda$ is a root. But this is more expensive in time than necessary. The circumstances in which we have to compute $p_{\lambda,\mu}$ are in fact somewhat limited: (1) when $\lambda = \alpha$ is simple and $\langle \mu, \alpha^\vee \rangle = -1$; (2) when α is simple and $\langle \mu, \alpha^\vee \rangle = 0$; (3) (λ, μ, ν) form a Tits triple. In case (1) or (2), we just have to check whether $\mu - \alpha$ is a root. But if α is simple, computing $\mu - \alpha$ is trivial, a matter of decrementing one coordinate. In case (3), we can apply Proposition 2.7 in order to reduce to the case in which (λ, μ, ν) is an ordered triple. These are dealt with in the process of ascending the root graph that is mentioned at the end of §3, since $p_{s_\alpha\lambda, s_\alpha\mu} = p_{\lambda,\mu}$.

6. References

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