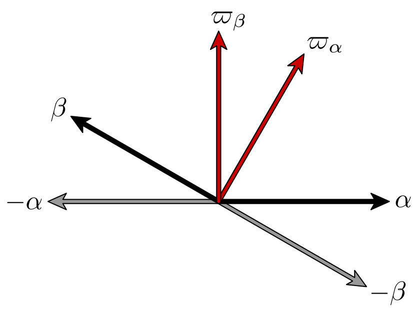
A geometric proof of Langlands' Combinatorial Lemma

Bill Casselman
University of British Columbia
cass@math.ubc.ca

Much of Arthur's work on automorphic forms depends on a simple result in geometrical combinatorics known as 'Langlands' Combinatorial Lemma'. It is not quite trivial to perceive, and in this note I'll give a short treatment explaining the geometry behind it.

Suppose Δ to be a basis of a Euclidean space V, with $\alpha \bullet \beta \le 0$ for $\alpha \ne \beta$ in Δ. Let $\Pi = \{\varpi_\alpha\}$ be the dual basis. The elements of Δ span an obtuse cone that contains the acute cone spanned by the ϖ_α. We shall be interested in various cones spanned by elements of Δ, $-\Delta$, and Π.



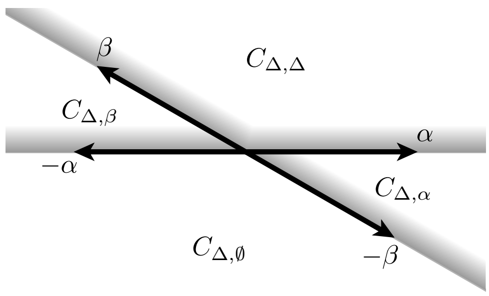
Most important are two partitions of V. The first is the **coordinate partition**. For each $S \subseteq \Delta$ let $C_{\Delta,S}$ be the vectors

$$v = \sum c_\alpha \alpha$$

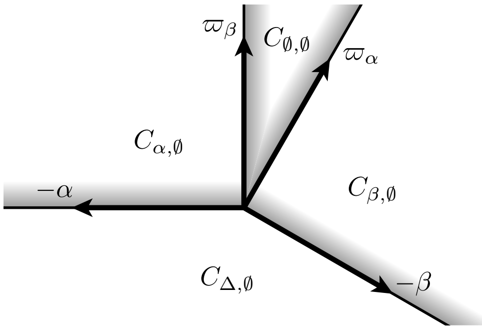
with

$$c_\alpha > 0 \text{ for } \alpha \in S, \quad c_\alpha \le 0 \text{ for } \alpha \notin S\,.$$

In two dimensions, for example, these just partition the plane into quadrants.



Another partition is associated to the open acute cone $V_\emptyset^{++} = C_{\emptyset,\emptyset}$ spanned by Π. To each subset $S \subseteq \Delta$ is associated the cone $C_{S,\emptyset}$ of points in V for which the nearest face of $C_{\emptyset,\emptyset}$ is the face V_S^{++} spanned by the ϖ_α with α in the complement of S.



Explicitly

$$C_{S,\emptyset} = \Big\{ \sum_{\Delta - S} c_\alpha \varpi_\alpha + \sum_S c_\alpha \alpha \,\Big|\, c_\alpha > 0 \text{ for } \alpha \notin S,\ c_\alpha \le 0 \text{ for } \alpha \in S \Big\}.$$

This partition first appeared in representation theory in [Langlands:1972/1989]. Following the tradition of attaching a mathematician's name to his most trivial observation, it is often called the **Langlands partition**.

There is a large family of partitions of V interpolating these two extremes. I extend the notation I have used so far—for $T \subseteq S \subseteq \Delta$ define

$$C_{S,T} = \Big\{ \sum_{\Delta - S} c_\alpha \varpi_\alpha + \sum_S c_\alpha \alpha \,\Big|\, c_\alpha > 0 \text{ for } \alpha \notin S,\ c_\alpha \le 0 \text{ for } \alpha \in S - T,\ c_\alpha > 0 \text{ for } \alpha \in T \Big\}.$$

Keep in mind that the subspace spanned by the ϖ in this definition is perpendicular to that spanned by the α.

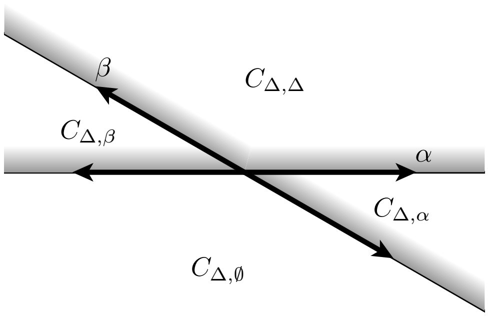
For $\Theta \subseteq \Delta$, let V_Θ be the subspace perpendicular to the α in Θ. It is spanned by the ϖ_α for α in the complement of Θ. To the cone V_Θ^{++} spanned by these ϖ_α is associated a Langlands partition of V_Θ. Corresponding to this in turn is a partition of V itself obtained by perpendicular projection onto V_Θ. It is parametrized by sets S with $\Theta \subseteq S \subseteq \Delta$.

1. Lemma. *Suppose $\Theta \subseteq S$ to be subsets of Δ. The inverse image in V of the subset of the Langlands partition of V_Θ parametrized by S is partitioned by the $C_{S,T}$ with $T \subseteq \Theta$.*

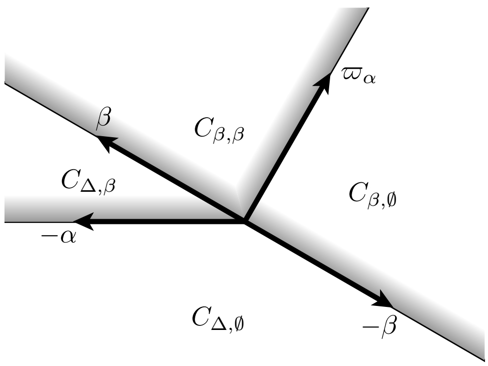
I'll prove this in a moment. As an immediate consequence of this result and the Langlands partition of V_Θ:

2. Proposition. *For $\Theta \subseteq \Delta$ the sets $C_{S,T}$ with $T \subseteq \Theta \subseteq S$ partition V.*

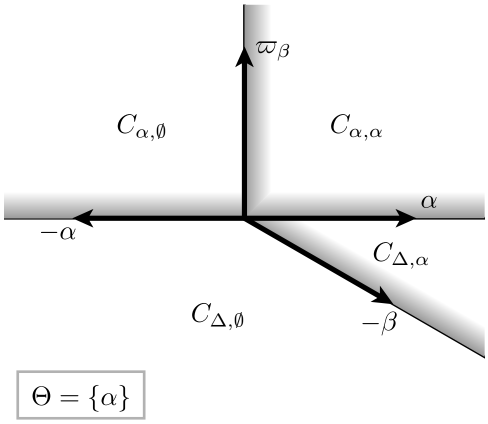
If $\Theta = \Delta$ we get the coordinate partition, and if $\Theta = \emptyset$ we get the Langlands partition. Since the Langlands partition of V_Θ is parametrized by subsets of $\Delta - \Theta$, each of these new partitions also has $2^{|\Delta|}$ components.

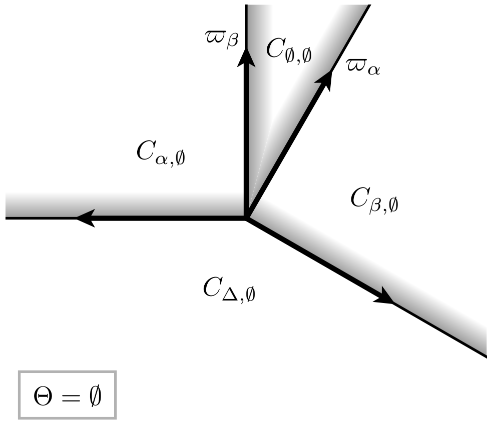


$\Theta = \Delta$



$\Theta = \{\beta\}$





Proof of Lemma 1. Let $|$ stand for orthogonal projection $v \mapsto v|V_\Theta$ onto V_Θ. The kernel of this projection is spanned by the α in Θ. The subset of V_Θ in its Langlands partition corresponding to S is therefore the set

$$\sum_{\alpha \notin S} c_\alpha \varpi_\alpha + \sum_{\alpha \in S - \Theta} c_\alpha \, (\alpha \mid V_\Theta)$$

where $c_\alpha > 0$ for $\alpha \notin S$, $c_\alpha \le 0$ for $\alpha \in S - \Theta$. The inverse image of this in V is the set of

$$\sum_{\alpha \notin S} c_\alpha \varpi_\alpha + \sum_{\alpha \in S} c_\alpha$$

where now $c_\alpha > 0$ for $\alpha \notin S$, $c_\alpha \le 0$ for $\alpha \in S - \Theta$, and the c_α are arbitrary for $\alpha \in \Theta$. Partition this according to which of the c_α with $\alpha \in \Theta$ are positive.

The Proposition usually referred to as Langlands' combinatorial lemma is now an easy consequence. For $T \subseteq S \subseteq \Delta$ let

$$\mathfrak{char}_{S,T} = \text{ the characteristic function of } C_{S,T}\,.$$

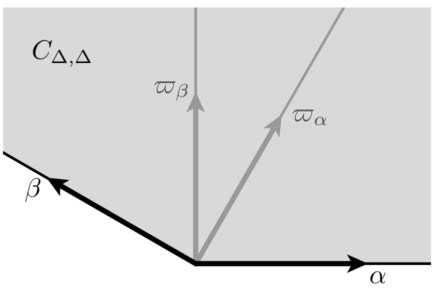
3. Corollary. (Langlands' combinatorial lemma) *We have*

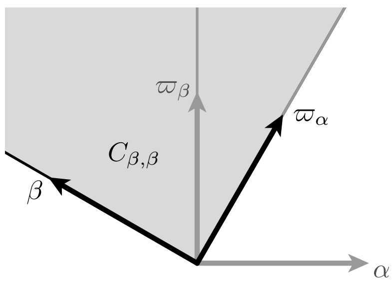
$$\sum_{S \subseteq \Delta} (-1)^{|S|} \mathfrak{char}_{S,S} = 0\,.$$

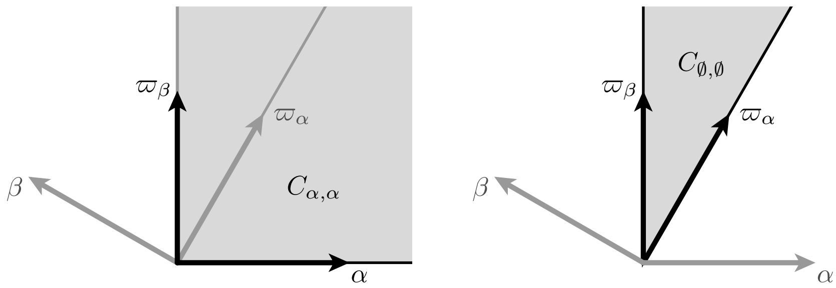
The assertion is deceptively simple when the dimension is two, since in that case

$$C_{S \cap T, S \cap T} = C_{S,S} \cap C_{T,T}$$

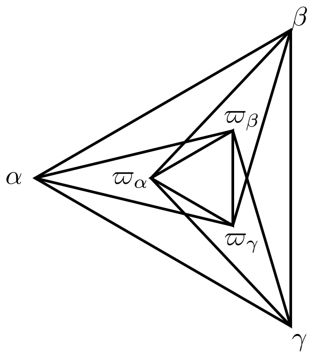
for all S, T.







This ceases to be true in higher dimensions, and indeed already for dimension three it is not easy to see why the result holds. The following figure illustrates a slice through the relevant 3D diagram:



Proof of the Corollary. Its proof follows a standard template for applying Proposition 2. The basic fact is that $(1-1)^n = 0$ if $n > 0$, but in the guise of the equation

$$\sum_{\Theta\subseteq\Delta} (-1)^{|\Theta|} = \begin{cases} 1 & \text{if } \Delta = \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2 tells us that for a fixed Θ

$$\sum_{\substack{S,T \\ T\subseteq\Theta\subseteq S}} \mathfrak{char}_{S,T} = 1 \ .$$

If we then take an alternating sum over all Θ we obtain

$$\sum_{\Theta} (-1)^{|\Theta|} \sum_{\substack{S,T \\ T\subseteq\Theta\subseteq S}} \mathfrak{char}_{S,T} = 0 \ .$$

This can be rearranged to give

$$\sum_{\substack{S,T \\ T\subseteq S}} \sum_{\substack{\Theta \\ T\subseteq\Theta\subseteq S}} (-1)^{|\Theta|} \mathfrak{char}_{S,T} = \sum_{\substack{S,T \\ T\subseteq S}} \mathfrak{char}_{S,T} \sum_{\substack{\Theta \\ T\subseteq\Theta\subseteq S}} (-1)^{|\Theta|} = 0 \ .$$

But the inner alternating sum vanishes unless $T = S$, and in that case gives

$$\sum_{\Theta} (-1)^{|\Theta|} \mathfrak{char}_{\Theta,\Theta} = 0 \ .$$ ▯

This is a special case of a more general result we shall need in a later section:

4. Proposition. *Suppose* $\Theta \subseteq \Delta$. *Then*

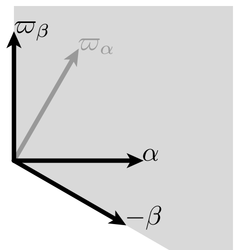
$$\sum_{S\subseteq\Delta} (-1)^{|S\cap\Theta|} \mathfrak{char}_{S,S\cap\Theta} = \begin{cases} 1 & \textit{if } \Theta = \emptyset \\ 0 & \textit{otherwise.} \end{cases}$$

Langlands' combinatorial lemma is the case $\Theta = \Delta$.

Proof. This follows by rearrangement from the formula

$$\sum_{S\subseteq\Theta} (-1)^{|S|} \sum_{T\subseteq\Theta\subseteq S} \mathfrak{char}_{S,T} = \begin{cases} 1 & \text{if } \Theta = \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

For example, when $\Theta = \{\alpha\}$:



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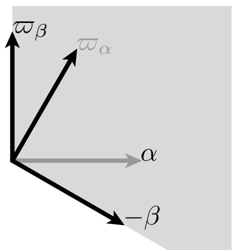

$= 0$

This seems to be equivalent to Lemma 6.3 of [Arthur:1978].

There appear to be two distinct roles for Langlands' combinatorial lemma, and two different proofs. Another proof is given in [Labesse:1984–85], and a generalization is given in [Casselman:2003]. The implicit starting point is that the characteristic function of a simplicial cone is the alternating sum of the characteristic functions of its faces' exteriors. This is a special case of a similar result about arbitrary convex polyhedra. I do not see how to fit Proposition 4 into this framework.

Interesting applications of the Lemma can be found in [Laumon-Rapoport:1996] and [Goresky-Kottwitz-MacPherson:1997].

1. References

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