Maass forms

Let

$$
\begin{aligned}
G &= \mathrm{SL}_2(\mathbb{R}) \\
K &= \mathrm{SO}(2) \\
P &= \text{ group of upper triangular matrices} \\
N &= \text{ unipotent radical of } P \\
\mathcal{H} &= \{z \,|\, \mathrm{IM}(z) > 0\} \\
\Gamma &= \text{ a proper discrete subgroup of } G\,.
\end{aligned}
$$

Let $N(\mathbb{Z})$ be the subgroup of integral matrices in N.

The group G acts by fractional linear transformations on $\mathcal{H}$. The isotropy subgroup of i is K, so that $\mathcal{H}$ may be identified with G/K. The condition on Γ means that $\Gamma\backslash\mathcal{H}$ is the union of a compact subset and a finite number of parabolic domains.

What is a parabolic domain? Suppose q to be either ∞ or a real number, and let P_q be its stabilizer in G. It will be a conjugate in G of $P = P_\infty$. Let N_q be its unipotent radical, which will be a conjugate of N. The point q is called a **cusp** if $\Gamma \cap N_q$ is an infinite cyclic group, and in this case its stabilizer P_q is called a cuspidal parabolic subgroup. For example, ∞ is a cusp of the group $\mathrm{SL}_2(\mathbb{Z})$. Some conjugate of $\Gamma \cap N_q$ in $\mathrm{SL}_2(\mathbb{R})$ will be exactly $N(\mathbb{Z})$, and the pull backs x_q, y_q of the functions x, y I call **parabolic coordinates** on $\mathcal{H}$ associated to q. A parabolic domain associated to q is one of the regions $\mathcal{H}_q^Y$ where $y_q > Y$, or its image in $\Gamma\backslash\mathcal{H}$. If $Y \gg 0$ the projection from $(\Gamma \cap P_q)\backslash\mathcal{H}_q^Y$ to $\Gamma\backslash\mathcal{H}$ is an embedding.

The standard example is $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. In this case the region

$$\{z \,|\, \mathrm{IM}(z) > 0, |z| \ge 1, \mathrm{RE}(z) \le 1/2\}$$

is a fundamental domain for Γ. It is the union of the parabolic domain where $\mathrm{IM}(z) > 1$ and a small compact subset. Suppose F to be a holomorphic automorphic form, say of weight k, with respect to Γ. Then it is invariant under translation by $N(\mathbb{Z})$, so may be expressed as a convergent series

$$F(z) = F(x+iy) = \sum_{n\ge 0} F_n e^{2\pi i n z} = \sum_{n \ge 0} F_n e^{-2\pi n y}\, e^{2\pi i n x}\,.$$

The difference between $F(z)$ and its constant term F_0 is evidently exponentially decreasing as a function of y.

This essay will be concerned with analogous properties for certain other smooth functions on arithmetic quotients $\Gamma\backslash\mathcal{H}$ and to some extent related functions on $\Gamma\backslash G$, for an arbitrary proper subgroup Γ. The basic fact is that on a parabolic domain $F(z)$ is asymptotic to its constant term at the corresponding cusp. But in some applications the fine structure of this behaviour is important. In analyzing it, one may as well assume that the cusp is ∞ with stabilizer $P = AN$, and that $\Gamma \cap N = N(\mathbb{Z})$. The function $F(z)$ may be expanded in a Fourier series

$$F(x+iy) = \sum\nolimits_{\mathbb{Z}} F_n(y) e^{2\pi i n x}$$

with Fourier coefficients

$$F_n(y) = \int_0^1 F(x+iy) e^{-2\pi i n x}\, dx$$

that depend on y. There are several variations on the theme that $F(z) \sim F_0(y)$ as $y \to \infty$. One is that in which F is an eigenfunction of the Laplacian Δ of moderate growth as $y \to \infty$ (i.e. it is a **Maass form**), another in which F satisfies a certain somewhat technical condition of *uniform* moderate growth. Yet another concerns the Laplacian as an unbounded operator on $\mathrm{L}^2(\Gamma\backslash\mathcal{H})$. In this essay, I'll be mostly concerned with Maass forms.

The relationship between constant terms and asymptotic behaviour is fundamental in the theory of automorphic forms.

Contents

1. Maass forms

The Laplacian on $\mathcal{H}$ has the formula

$$\Delta = \Delta_{\mathcal{H}} = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

and is invariant under G. A **Maass form** on $\Gamma\backslash\mathcal{H}$ is a smooth function F on $\Gamma\backslash\mathcal{H}$ such that

- $\Delta F = \gamma F$ for some scalar γ;
- F is of moderate growth at every cusp q:

$$F(x_q + iy_q) = O(y_q^M)$$

  for some M, as $y \to \infty$.

The non-Euclidean Laplacian Δ is an elliptic differential operator. Solutions of an elliptic differential equation with locally smooth coefficients are smooth, so F is necessarily a smooth function of z. Even better, solutions of an elliptic differential equation with analytic coefficients are real analytic, so that F is in fact real analytic. It is also true, if not immediately apparent, that an equivalent definition of a Maass form is as an eigendistribution of Δ that is tempered in some sense.

The operator Δ with domain $C_c^\infty(\Gamma\backslash\mathcal{H})$ is essentially self-adjoint, which means that it possesses a unique extension to an unbounded self-adjoint operator on $\mathrm{L}^2(\Gamma\backslash\mathcal{H})$. It is non-positive. Its spectrum contains the interval $(-\infty, -1/4)$ contributed by Eisenstein series and a discrete part in $(-\infty, 0]$. The eigenfunctions of the discrete part comprise the constants and the cusp forms. The latter are the cuspidal Maass forms, and are characterized by the vanishing of their constant terms at all cusps. As a consequence, as we shall see, they are rapidly decreasing at infinity at all cusps. The cuspidal spectrum breaks up into two parts, the intervals $(-\infty, -1/4]$ and $(-1/4, 0]$. For congruence subgroups the latter is conjectured to be empty, and in any case it is only a finite set.

What can one say about the asymptotic behaviour of Maass forms near the cusps of Γ? As I have already mentioned in the Introduction, by conjugating Γ in G if necessary, I may assume the cusp at hand to be ∞, and $\Gamma \cap N_\infty$ to be $N(\mathbb{Z})$. In addition, that F is invariant under all of Γ will play no role in the discussion to come. Therefore:

> *From now on, I assume only that F is an eigenfunction of Δ on the quotient $N(\mathbb{Z})\backslash\mathcal{H}$ with $F(x+iy) = O(y^M)$ for some M, as $y \to \infty$.*

This hypothesis, for example, applies to holomorphic forms F of even weight $k > 0$. What I am going to say is trivial in this case, but it can serve as a simple model. Since F is smooth, it may be expanded in a Fourier series

$$F(x+iy) = \sum_{-\infty}^{\infty} F_n(y) e^{2\pi i n x}$$

with smooth Fourier coefficients

$$F_n(y) = \int_0^1 F(x+iy) e^{-2\pi i n x}\, dx\,.$$

If F is holomorphic, the condition of moderate growth implies that the expansion has no terms of negative index and is therefore

$$F(z) = \sum_{n \geq 0} F_n e^{2\pi i n z} .$$

Since $e^{2\pi i n z} = e^{2\pi i n x} e^{-2\pi n y}$ we have

$$\big| F(z) - F_0 \big| = O(e^{-2\pi y}) \quad (y \to \infty) .$$

In this section and the next I'll explain how this conclusion remains valid in general.

In general, since the Laplacian and N commute, the Fourier terms $F_n(y)$ are also eigenfunctions of Δ. This means that the coefficients $F_n(y)$ satisfy an ordinary differential equation. The two cases in which $n = 0$ and $n \neq 0$ are very different.

• ***n*** $=$ ***0***. For the constant term F_0 we get the differential equation

$$y^2 F_0'' = \gamma F_0 .$$

It is an Euler equation

$$D^2 F_0 - D F_0 - \gamma F_0 = 0$$

in which D is the multiplicative derivative $y\, d/dy$. This differential equation has a **regular singularity** at ∞. The operator $y\, d/dy$ is invariant on the multiplicative group of real numbers, which is isomorphic to the additive group of real numbers via the exponential map $y = e^x$. If I set $\Phi(x) = F(e^x)$ then Φ now satisfies the equation

$$\Phi'' - \Phi' - \gamma \Phi = 0 .$$

This equation has constant coefficients, and for all but one value of γ it will have as basis of solutions $e^{s_1 x}$ and $e^{s_2 x}$ where the s_i are solutions of the equation

$$s^2 - s - \gamma = 0, \qquad \text{hence } \ s = \frac{1 \pm \sqrt{1 + 4\gamma}}{2} .$$

The exception is when $\gamma = -1/4$, when a basis of solutions is made up of $e^{x/2}$ and $xe^{x/2}$. Thus, the solutions of the original equations are the linear combinations of y^{s_1} and y^{s_2} as long as $\gamma \neq -1/4$. If $\gamma = -1/4$, on the other hand, the solutions are linear combinations of $y^{1/2}$ and $y^{1/2} \log y$.

• ***n*** $\neq$ ***0***. For the Fourier coefficient $F_n(y)$ we get the differential equation

$$y^2 \big(F_n'' - 4\pi^2 n^2 F_n \big) = \gamma F_n, \quad F_n'' - \big(4\pi^2 n^2 + \gamma / y^2 \big) F_n = 0 ,$$

which has an **irregular singularity** at ∞. As $y \to \infty$ this differential equation has as limit the constant coefficient equation

$$F'' - 4\pi^2 n^2 F = 0 .$$

with solutions $F(y) = e^{\pm 2\pi n y}$, so one might expect some similarity between the behaviour of F_n and of the functions $e^{\pm 2\pi n y}$. Since one of these grows exponentially and the other decreases, the following is plausible:

1.1. Proposition. *The space of solutions of the equation*

$$F'' - 4\pi^2 n^2 F = (\gamma / y^2) F$$

that are of moderate growth on $(1, \infty)$ *has dimension one, and it has as basis the unique solution which is asymptotic to* $e^{-2\pi n y}$ *as* $y \to \infty$.

This will take some work to explain. First we simplify by a scale change. Let $\lambda = 2\pi|n|$, and then set $F(y) = \mathcal{W}(\lambda y) = \mathcal{W}(2\pi|n|y)$. This gives us

$$F''(y) - 4\pi^2 n^2 F(y) = \lambda^2 \mathcal{W}''(\lambda y) - \lambda^2 \mathcal{W}(2\lambda y)\,,$$

so that if we set $x = \lambda y$ we see that $\mathcal{W}$ satisfies the differential equation

$$\mathcal{W}''(x) - \mathcal{W}(x) = (\gamma/x^2)\mathcal{W}\,,$$

which is now independent of λ. According to the standard formula found in Theorem 4.7 of [Brauer-Nohel:1967], there exist solutions of this equation with **asymptotic series expansions**

$$\mathcal{W}(x) = e^{\pm x}\left(1 + \frac{c_1}{x} + \frac{c_2}{x^2} + \cdots\right)\,,$$

which means that

$$\mathcal{W}(x)/e^{\pm x} - \left(1 + \frac{c_1}{x} + \frac{c_2}{x^2} + \cdots + \frac{c_n}{x^n}\right) = O(x^{-n-1})$$

for each n as $x \to \infty$, and that something similar remains true for all derivatives of $\mathcal{W}$.

Because of the condition of moderate growth on F as $y \to \infty$, only the solution with leading term e^{-x} is relevant here. The coefficients c_i of the formal series can be calculated by a recursion, but before doing that it is probably easiest to make a slight change, setting $\mathcal{W} = e^{-x}G$. Then

$$\begin{aligned}
\mathcal{W} &= e^{-x}G \\
\mathcal{W}' &= -e^{-x}G + e^{-x}G' \\
\mathcal{W}'' &= G - e^{-x}G' + e^{-x}G'' \\
\mathcal{W}'' - \mathcal{W} &= e^{-x}G'' - e^{-x}G'\,,
\end{aligned}$$

thus getting for G the differential equation

$$G'' - G' = \frac{\gamma}{x^2}G\,.$$

Setting formally

$$G = 1 + \frac{c_1}{x} + \frac{c_2}{x^2} + \frac{c_3}{x^3} + \cdots + \frac{c_n}{x^n} + \cdots$$

leads us to expansions

$$\begin{array}{rcccccccc}
\dfrac{\gamma G}{x^2} & = & \dfrac{\gamma}{x^2} & + \dfrac{\gamma c_1}{x^3} & + \dfrac{\gamma c_2}{x^4} & + \cdots + & \dfrac{\gamma c_{n-1}}{x^{n+1}} & + \cdots \\
-G' & = & \dfrac{c_1}{x^2} & + \dfrac{2c_2}{x^3} & + \dfrac{3c_3}{x^4} & + \cdots + & \dfrac{nc_n}{x^{n+1}} & + \cdots \\
G'' & = & & \dfrac{2c_1}{x^3} & + \dfrac{3\cdot 2c_2}{x^4} & + \cdots + & \dfrac{n(n-1)c_{n-1}}{x^{n+1}} & + \cdots\,.
\end{array}$$

giving us $n(n-1)c_{n-1} + nc_n = \gamma c_{n-1}$ and then recursion formulas

(1.2)
$$\begin{aligned}
c_1 &= \gamma \\
c_n &= \frac{\gamma - n(n-1)}{n}\,c_{n-1} \quad (n \ge 2)\,.
\end{aligned}$$

The numerator here grows more rapidly than the denominator, so the series certainly does not converge. The functions $\mathcal{W}$ are a variant of Bessel functions called **Whittaker functions**.

In the next section I'll follow Chapter 5 of [Coddington-Levinson:1955] in sketching the proof of the asymptotic expansion. In the rest of this one I'll just assume this to be so and prove:

1.3. Proposition. *Suppose F to be an eigenfunction of Δ on $(\Gamma \cap P)\backslash\mathcal{H}$ of moderate growth. Then as $y \to \infty$*

$$\big|F(x+iy) - F_0(y)\big| \ll_F e^{-2\pi y}\,.$$

Here I use Serge Lang's generalization of O-notation—$\ll_F X$ means $\le CX$ where C depends on F.

Proof. We start with

$$\begin{aligned} F(x+iy) - F_0(y) &= \sum_{n\ne 0} e^{2\pi nix} F_n(y) \\ &= \sum_{n\ne 0} c_n e^{2\pi inx}\mathcal{W}(2\pi|n|y) \\ \big|F(x+iy) - F_0(y)\big| &\le \sum_{n\ne 0} |c_n|\big|\mathcal{W}(2\pi|n|y)\big|\,. \end{aligned}$$

Since for $k > 0$

$$\frac{\partial^k F(x+iy)}{\partial x^k} = \sum_{n\ne 0}(2\pi in)^k c_n \mathcal{W}(2\pi|n|y)e^{2\pi inx}$$

we have

$$\begin{aligned} c_n(2\pi in)^k\mathcal{W}(2\pi|n|y) &= \int_0^1 \frac{\partial^k F(x+iy)}{\partial x^k}\, e^{-2\pi nix}\, dx \\ c_n &= \frac{1}{(2\pi in)^k}\frac{1}{\mathcal{W}(2\pi|n|y)}\int_0^1 \frac{\partial^k F(x+iy)}{\partial x^k}\, e^{-2\pi nix}\, dx \end{aligned}$$

for every k and y, as long as $\mathcal{W}(2\pi|n|y) \ne 0$. But we also know that $\mathcal{W}(t) \sim e^{-t}$. Choose t_0 large enough so $1/2 < \mathcal{W}(t)/e^{-t} < 2$ for $t > t_0$. Let $y_0 = t_0/2\pi$. Thus for $y \ge y_0$

$$\begin{aligned} c_n &= \frac{1}{(2\pi in)^k}\frac{1}{\mathcal{W}(2\pi|n|y_0)}\int_0^1 \frac{\partial^k F(x+iy_0)}{\partial x^k}\, e^{-2\pi nix}\, dx \\ |c_n| &\le \frac{2e^{2\pi|n|y_0}}{|2\pi n|^k}\int_0^1 \left|\frac{\partial^k F(x+iy_0)}{\partial x^k}\right| dx \\ \sum_{n\ne 0}|c_n|\big|\mathcal{W}(2\pi|n|y)\big| &\le C_k \sum_{n\ne 0}\frac{e^{-2\pi|n|(y-y_0)}}{|n|^k} \\ &\le C_k^*\, e^{-2\pi y}\Big(\sum_{n>0}\frac{1}{n^k}\Big)\,. \end{aligned}$$

Remark. In general, an automorphic form is a function on $\Gamma\backslash G$ satisfying these conditions:

(a) it is of moderate growth on G;
(b) it is an eigenfunction of the Casimir operator Ω;
(c) it is an eigenfunction of K with respect to some character.

Choose for the basis of the Lie algebra $\mathfrak{sl}_2$ the matrices

$$\begin{aligned} h &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ \nu_+ &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ \kappa &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \end{aligned}$$

The Casimir element of $U(\mathfrak{g})$ may be expressed as

$$\frac{h^2}{4} - \frac{h}{2} + \nu_+^2 - \nu_+\kappa\,.$$

If F is an automorphic form, the third condition above says that $R_\kappa f = ki\,F$ for some integer k. Also because of the third condition F is determined by its restriction to the connected component $|P|$ of P. But this may be identified with $\mathcal{H}$, with coordinates x, y. If

$$R_\Omega F = \gamma F$$

then in this coordinate system the restriction satisfies

$$y^2\left(\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2}\right) - (ki)\,y\left(\frac{\partial F}{\partial x}\right) = \gamma F\,.$$

This restriction will again be a sum of functions $F_n(y)e^{2\pi nix}$, and F_n will satisfy the equation

$$y^2 F_n^{''} - 4\pi^2 n^2 y^2 F_n + 2\pi kny\,F_n = \gamma F_n\,.$$

Setting $F_n(y) = \Phi(2\pi ny)$ we see that Φ satisfies the differential equation

(1.4) $$y^2\Phi'' - y^2\Phi + ky\,\Phi = \gamma\Phi\,.$$

2. Asymptotic expansions for irregular singularities

I now introduce standard notation. If $\mathcal{W}'' - \mathcal{W} = (\gamma/x^2)\mathcal{W}$ and $W(x) = \mathcal{W}(x/2)$ then

$$W'' - \frac{W}{4} = \left(\frac{\gamma}{x^2}\right) W\,.$$

The differential equation

$$W'' + \left(-\frac{1}{4} + \frac{k}{x} + \frac{1/4 - m^2}{x^2}\right) W = 0\,.$$

is called the **Whittaker equation** with parameters k, m. We are looking at the case $k = 0$, $\gamma = -1/4 + m^2$, but we can see from(1.4) that $k \neq 0$ arises when we look at more general automorphic forms.

The parameter m may be any complex number, but in interesting cases (related to the spectrum of Δ) its values are restricted. Since Δ on $\Gamma\backslash\mathcal{H}$ is a negative operator, $-1/4 + m^2 \leq 0$, and m consequently must lie either in $[-1/2, 1/2]$ or in $i\mathbb{R}$.

In general, there is exactly one solution of moderate growth at infinity, and in fact it decreases exponentially. It has the asymptotic expansion

$$W \sim x^k e^{-x/2}\left(1 + \frac{C_1}{x} + \frac{C_2}{x^2} + \cdots\right)\,.$$

What we have seen in the last section is that with $k = 0$, after some minor adjustments to take into account the scale change in x, the C_i here satisfy the recursion relations

$$\begin{aligned} C_1 &= 2\gamma \\ C_n &= \frac{2\,(\gamma - n(n-1))}{n}\,C_{n-1} \quad (n \geq 2) \end{aligned}$$

This function is designated $W_{k,m}$, the **Whittaker function** with parameters k, m. As explained in §16.2 of [Whittaker-Watson:1952], Whittaker functions occur frequently in applied mathematics—the error function, the incomplete Gamma function, the logarithmic integral, and Bessel functions all have simple expressions in terms of certain Whittaker functions. What is important for our purposes is that they also occur as Fourier coefficients of automorphic forms on arithmetic quotients and in the **Whittaker models** of representations of real groups of rank one. Because of this, they play an important role in the zeta functions of automorphic representations.

In the rest of this section I'll take the opportunity to explain what happens more generally for irregular singularities at ∞.

The Whittaker equation may be transformed into a system of first order equations by the usual trick of introducing a new dependent variable $V = W'$. The system we get is

$$\begin{bmatrix} W \\ V \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ 1/4 - k/x - (1/4 - m^2)/x^2 & 0 \end{bmatrix} \begin{bmatrix} W \\ V \end{bmatrix} .$$

The matrix can be expressed as

$$\begin{bmatrix} 0 & 1 \\ 1/4 & 0 \end{bmatrix} + \frac{1}{x} \begin{bmatrix} 0 & 0 \\ -k & 0 \end{bmatrix} + \frac{1}{x^2} \begin{bmatrix} 0 & 0 \\ m^2 - 1/4 & 0 \end{bmatrix} .$$

This is therefore a special case of a system

$$y'' = A(x)y$$

in which $A(z)$ has a convergent expansion

$$A(x) = A_0 + \frac{A_1}{x} + \frac{A_2}{x^2} + \cdots$$

satisfying the condition that the eigenvalues of A_0 are distinct. We shall now look at this more general situation.

2.1. Proposition. *Suppose*

$$A(x) = A_0 + \frac{A_1}{x} + \frac{A_2}{x^2} + \cdots$$

to be a convergent expansion near ∞, and that the eigenvalues of A_0 are distinct. Then there exists a matrix solution of

$$F' = A(x)F$$

which has an asymptotic expansion of the form

$$F(x) \sim \widehat{F}(x) = P(x)x^R e^{\Lambda x} ,$$

where R and Λ are complex matrices that commute with each other and with A_0, and P is an asymptotic series in non-negative powers of $1/x$.

Proof. I'll follow Chapter 5 of [Coddington-Levinson:1955], in which a more general result about systems

$$F' = x^r A(x) \quad (r \in \mathbb{N})$$

is treated. The proof comes in two steps, the first explaining how to find the components $P(x)$, R, and Λ of the formal solution, and the second explaining how to relate the formal solution to an asymptotic expansion.

First of all, we may reduce to the case where A_0 is diagonal, replacing $F(x)$ by $EF(x)$ if EA_0E^{-1} is diagonal, since from $F' = AF$ we deduce $EF' = EAE^{-1} \cdot EF$. If $F(x) = P(x)x^R e^{\Lambda x}$ then

$$F'(x) = P'(x)x^R e^{\Lambda x} + (1/x)P(x)Rx^R e^{\Lambda x} + P(x)x^R \Lambda\, e^{\Lambda x}$$

so we must solve

$$P'(z)z^R e^{\Lambda x} + (1/x)P(x)Rz^R e^{\Lambda x} + P(x)z^R \Lambda e^{\Lambda x} = A(x)P(x)z^R e^{\Lambda x} \,.$$

It will turn out that the matrices R and Λ can be chosen to be diagonal. Therefore they all commute, so we may cancel $z^R e^{\Lambda x}$, leading to the equation

$$P'(x) + (1/x)P(x)R + P(x)\Lambda = A(x)P(x) \,.$$

We now equate successively coefficients of the powers of $1/x$.

Let the diagonal entries of A_0 now be (a_i).

Step 1. ***n* = *0*.** The constant term gives us $P_0\Lambda = A_0$ so we set

$$P_0 = I, \quad \Lambda = A_0 \,.$$

Step 2. ***n* = *1*.** Equating coefficients of $1/x$ gives us

(2.2) $$R + P_1 A_0 = A_0 P_1 + A_1, \quad R = (A_0 P_1 - P_1 A_0) + A_1 \,.$$

The following trivial observations will be applied again and again:

2.3. Lemma. *(a) If D is a diagonal matrix and B an arbitrary one then $DB - BD$ is a matrix with entries*

$$B_{i,j}(d_i - d_j) \,.$$

(b) If A is any matrix whose diagonal entries do not vanish and $AB = C$, then the diagonal entries of B may be found in terms of the entries of A, the diagonal entries of C, and the off-diagonal entries of B.

Proof. The first claim is immediate. For the second:

$$c_{i,i} = a_{i,i}b_{i,i} + \sum_{j \neq i} a_{i,j} b_{j,i} \,.$$

Because of (a), equation (2.2) tells us that we may take $R = \mathfrak{d}(A_1)$. Here I write

$$A = \mathfrak{d}(A) + A^*$$

with $\mathfrak{d}(A)$ the diagonal submatrix of A, A^* its complement.

Equation (2.2) also tells us that

$$(P_1)_{i,j} = \frac{(A_1)_{i,j}}{a_i - a_j}$$

for $i \neq j$. We therefore know P_1^*. We can deduce nothing, however, about the diagonal $\mathfrak{d}(P_1)$. It will be determined in the next step.

Step 3. ***n*** $= 2$. Equating coefficients of $1/x^2$, keeping in mind the assignments of R and Λ, we get

$$\begin{aligned} -P_1 + P_1 R + P_2 A_0 &= A_0 P_2 + A_1 P_1 + A_2 \\ &= A_0 P_2 + R P_1 + A_1^* P_1 + A_2 \\ -P_1 - (A_0 P_2 - P_2 A_0) &= (R P_1 - P_1 R) + A_1^* P_1 + A_2 \\ (I + A_1^*) P_1 - (A_0 P_2 - P_2 A_0) &= -(R P_1 - P_1 R) - A_2\,. \end{aligned}$$

Equating diagonal and off-diagonal components, we deduce first of all that

$$\mathfrak{d}((I + A_1^*) P_1) = -\mathfrak{d}(A_2)\,,$$

from which, according to (b) of the Lemma, we can find the diagonal entries of P_1. Given this, we can then apply (a) to find the off-diagonal component of P_2.

Step 4. ***n*** ≥ 3. Suppose that we are given inductively all of the P_m with $m < n-1$ and the off-diagonal matrix P_{n-1}^* of P_{n-1}. Equating coefficients of $1/x^n$ we get

$$\begin{aligned} -(n-1)P_{n-1} + P_{n-1}R + (P_n A_0 - A_0 P_n) &= A_1 P_{n-1} + \cdots + A_n \\ &= (R + A_1^*) P_{n-1} + \cdots + A_n \\ &= R P_{n-1} + A_1^* P_{n-1} + \cdots + A_n \\ (-(n-1)I - A_1^*) P_{n-1} + (P_n A_0 - A_0 P_n) &= (R P_{n-1} - P_{n-1} R) + A_2 P_{n-1} + \cdots + A_n\,. \end{aligned}$$

As in the previous step, this equation determines at once the diagonal of P_{n-1} and the off-diagonal of P_n. This concludes the construction of a formal solution $\widehat{F}(x)$ of the Whittaker equation.

I'll not include here the proof that $\widehat{F}(x)$ is an asymptotic approximation to a fundamental solution to the differential equation, except in the special case of Whittaker's equation with $k = 0$. Details are to be found in §5.4 of [Coddington-Levinson:1955]. The example I shall look at closely has at least a few of the features to be found in the general case.

2.4. Proposition. *The differential equation*

$$W'' - \left(\frac{1}{4}\right) W = \left(\frac{\gamma}{x^2}\right) W$$

has solutions with asymptotic expansions

$$W \sim e^{\pm x/2} \left(1 + \frac{c_1}{x} + \frac{c_2}{x^2} + \cdots\right)$$

The solution asymptotic to $e^{-x/2}$ is unique, but that asymptotic to $e^{x/2}$ is not, since the sum of it and any exponentially decreasing solution will have the same asymptotic behaviour.

Proof. The proof will make a couple of applications of the technique called 'variation of constants' in elementary courses on differential equations. Suppose φ to be a function which (a) is on each of the positive and negative axes a solution to the differential equation

$$W'' - (1/4)W = 0\,;$$

(b) is continuous at 0, but at 0 its first derivative jumps by 1. For example:

$$\varphi(x) = \begin{cases} 0 & \text{if } x < 0; \\ 2\sinh(x/2) & \text{otherwise.} \end{cases}$$

Such a function is a solution of the distributional equation

$$\varphi'' - \varphi/4 = \delta_0\,,$$

as can be easily verified applying integration by parts. Hence the function

$$F(x) = Ae^{-x/2} + Be^{x/2} + \int_a^b \varphi(x-s)G(s)\,ds = Ae^{-x/2} + Be^{x/2} + 2\int_a^x \sinh\left(\frac{x-s}{2}\right) G(s)\,ds$$

is a solution of $F'' - F/4 = G$ in the interval (a, b). This can also be seen by applying the formula

$$H'(x) = h(x, x) + \int_a^x \frac{\partial h}{\partial x}(x, s)\,ds$$

if

$$H(x) = \int_a^x h(x, s)\,ds\,.$$

The first application of this idea, or at least a mild modification of it, will be to construct a solution asymptotic to $e^{-x/2}$. More explicitly, it will construct a solution of the integral equation

$$F(x) = e^{-x/2} + 2\int_x^\infty \sinh\left(\frac{s-x}{2}\right) \frac{F(s)}{s^2}\,ds$$

for $x > 0$ by a sequence of approximate solutions. We set $F_0(x) = 0$, and then in succession

$$F_{n+1}(x) = e^{-x/2} + 2\int_x^\infty \sinh\left(\frac{s-x}{2}\right) \frac{F_n(s)}{s^2}\,ds\,.$$

Thus $F_1(x) = e^{-x/2}$. It will turn out that

$$|F_n(x)| \le e^{-x/2} e^{1/x} \,,$$

which guarantees that every integral converge absolutely. I shall now show that the sequence $F_n(x)$ converges to a solution of the integral equation $F(x)\ e^{-x/2}$. By induction, I'll assume $F_n(x) \le e^{-x/2}$. This is an argument standard at the very beginning of the theory of differential equations. We have

$$F_{n+1}(x) - F_n(x) = +2 \int_x^\infty \sinh\left(\frac{s-x}{2}\right) \frac{F_n(s) - F_{n-1}(s)}{s^2}\, ds$$

which implies that

$$\big|F_{n+1}(x) - F_n(x)\big| \le \int_x^\infty e^{\frac{s-x}{2}} \cdot \frac{|F_n(s) - F_{n-1}(s)|}{s^2}\, ds \,.$$

Thus

$$|F_2(x) - F_1(x)| \le \int_x^\infty e^{\frac{s-x}{2}} \cdot \frac{e^{-s/2}}{s^2}\, ds = e^{-x/2} \int_x^\infty \frac{1}{s^2}\, ds = \frac{e^{-x/2}}{x}$$

and by induction one can prove that

$$\big|F_{n+1} - F_n(x)\big| \le \frac{e^{-x/2}}{n!\, x^n} \,,$$

leading to

$$\big|F_n(x)\big| \le e^{-x/2} e^{1/x}, \quad F(x) \sim e^{-x/2} \,.$$

Integration by parts will lead to a proof that the entire asymptotic series is valid.

The second application of the idea will use

$$\varphi(x) = \begin{cases} -e^{x/2} & \text{if } x < 0; \\ -e^{-x/2} & \text{otherwise.} \end{cases}$$

Again we start off with $F_0(x) = 0$ and set

$$F_{n+1}(x) = e^{x/2} + \int_1^\infty \varphi(x-s) \frac{F_n(s)}{s^2}\, ds \,.$$

Thus $F_1(x) = e^{x/2}$, and $F_n(x) = O(e^{-x/2})$, as a similar argument will show. The sequence converges to a solution of the integral equation

$$F(x) = e^{x/2} + \int_1^\infty \varphi(x-s) \frac{F(s)}{s^2}\, ds$$

which is asymptotic to $e^{x/2}$.

We shall see later that all partial derivatives of F also decrease exponentially.

3. Computing Whittaker functions

We know what the asymptotic behaviour of W near ∞ is, but what does it look like on all of $(0,\infty)$? This is not at all a trivial question, and the problems involved in answering it arise often when discussing the classic 'special functions' It turns out that there are roughly three regions in which it behaves very differently. One of the three is near ∞, in which the asymptotic exponential decay decribes it in a satisfactory manner. Another is near 0. The Whittaker differential equation has a regular singularity there. This provides us with a converging series that oscillates wildly near 0. The most interesting region is in between the other two, where it switches from oscillation to exponential decay.

WHAT HAPPENS NEAR 0. Whittaker's equation (with $k=0$) is

$$W'' + \left(-\frac{1}{4} + \frac{1/4 - m^2}{x^2}\right) W = 0\,.$$

Let $\gamma = 1/4 - m^2$. Multiplying through by x^2, keeping in mind that $W'' = D^2W - DW$, I can rewrite this as

$$D^2W - DW + (\gamma - x^2/4)W = 0 \quad (D = x d/dx)\,.$$

Near $x=0$ this looks like the Euler's equation

$$D^2W - DW + \gamma W = 0\,,$$

A basis of solutions is made up of functions x^r where r is a root of

$$r^2 - r + \gamma = 0, \qquad \text{hence } r = 1/2 \pm \sqrt{1/4 - \gamma}\,,$$

and in this case

$$x^{1/2 \pm m}\,,$$

as long as $m \neq 0$. The singularity is regular, and for the original equation, as long as $m - (-m) \notin \mathbb{N}$, we have solutions in converging series, expressed conventionally as

$$M_m(x) = x^{1/2 \pm m}(1 + c_1 x + c_2 x^2 + \cdots)\,.$$

In the exceptional cases (including $m=0$, after all) there are some $\log x$ factors present. I shall ignore these cases.

Coefficients in this series can be found by recursion. For simplicity I set $W = x^{1/2} w$, so that

$$\begin{aligned} DW &= (1/2)x^{1/2}w + x^{1/2}w' \\ D^2W &= (1/4)x^{1/2}w + x^{1/2}w' + x^{1/2}w'' \\ D^2W - DW &= x^{1/2}w'' - (1/4)x^{1/2}w\,. \end{aligned}$$

We obtain the equation

$$D^2w = m^2w + (x^2/4)w = 0\,.$$

Set

$$w = x^r + c_1 x^{r+1} + c_2 x^{r+2} + \cdots \quad (r = \pm m)\,.$$

$$\begin{array}{ccccccccccc}
w & = & x^r & + & c_1 x^{r+1} & + & c_2 x^{r+2} & + & c_3 x^{r+3} & + \cdots \\
m^2 w & = & r^2 x^r & + & r^2 c_1 x^{r+1} & + & r^2 c_2 x^{r+2} & + & r^2 c_3 x^{r+3} & + \cdots \\
(x^2/4)w & = & & & & & (1/4)x^{r+2} & + & (c_1/4)x^{r+3} & + \cdots \\
Dw & = & r x^r & + & (r+1)c_1 x^{r+1} & + & (r+1)c_2 x^{r+2} & + & (r+3)c_3 x^{r+3} & + \cdots \\
D^2 w & = & r^2 x^r & + & (r+1)^2 c_1 x^{r+2} & + & (r+2)^2 c_2 x^{r+2} & + & (r+3)^2 c_3 x^{r+3} & + \cdots
\end{array}$$

This gives equations

$$\begin{aligned}(2r+1)c_1 &= 0\\ (2r+k)k\, c_k &= \frac{c_{k-2}}{4}\\ c_k &= \frac{c_{k-2}}{4k(2r+k)} \text{ for } k>1\,.\end{aligned}$$

As long as $2r$ is not a negative integer, we see that all $c_{2m+1}=0$, and that all c_{2m} are uniquely determined. In computation, it is slightly more convenient to work with the series

$$\sum_0 a_\ell (x/4)^{2\ell}\,.$$

Since

$$c_{2\ell} = \frac{c_{2(\ell-1)}}{16\,\ell(r+\ell)} \text{ for } k>1\,,$$

we have

$$a_0 = 1, \qquad a_\ell = \frac{a_{\ell-1}}{\ell(r+\ell)}\,.$$

The series converges for all $x>0$, but it is not practical to work it with over all of $(0,\infty)$. Besides, it does tell fairly well how the function behaves near 0, but gives no clue as to what happens away from it. In this, of course, it is much like the series

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \cdots$$

which certainly does not suggest periodicity, or that for e^{-x}, which does not suggest rapid decay as $x\to\infty$. It is in fact impractical to use for explicit calculation when $|x|\gg 0$.

There is one interesting question that arises naturally. We have a unique solution W_m of Whittaker's differential equation that decreases exponentially, and two solutions $M_{\pm m}$ well defined near 0, with leading terms $x^{1/2\pm m}$. How can $W(x)$ be expressed as a linear combination of M_m and M_{-m}?

This is not at all an easy question to answer. For the moment, I just quote [Whittaker-Watson:1941]:

(3.1)
$$W_m = \frac{\Gamma(-2m)}{\Gamma(1/2-m)}\cdot M_m(x) + \frac{\Gamma(2m)}{\Gamma(1/2+m)}\cdot M_{-m}(x)\,.$$

I will come back to this matter in a later version. We shall be most interested in the case that $m=i\nu$ with ν real, $0<x<\infty$, in which case $W_m(x)$ is real. The function $x^{-1/2}M_m$ doesn't vary much in size. Stirling's formula tells us that for large t and fixed σ

$$|\Gamma(\sigma+it)| \sim \sqrt{2\pi}e^{-\pi t/2}\,.$$

so a rough estimate of $|W_m(x)|$ is $\sqrt{x}\,2e^{-3\pi t/4}$.

What does this function look like?

THE TURNING POINT. The differential equation

$$W'' + \left(-\frac{1}{4} + \frac{1/4-m^2}{x^2}\right) W = 0$$

behaves curiously when the coefficient of W vanishes, which happens when

$$1-4m^2-x^2=0, \quad x=\sqrt{1-4m^2}\,.$$

Thsi value of x is called the **turning point**. I'll consider only the case that $m = i\nu$ is pure imaginary, in which case $1 - 4m^2 = 1 + 4\nu^2$ is positive. The turning point is then approximately 2ν for $\nu \gg 0$. In the theory of automorphic forms, this is the interesting case. I am about to discuss it, but in order to match notation in the literature I'll now make yet another change of variables.

Define

$$K_m(x) = \sqrt{\pi/2x}\, W_m(2x)\,.$$

It is a kind of Bessel function. Dividing W by the factor $\sqrt{z}$ will turn out to have one great advantage in understanding behaviour—we shall be most interested in the case $m = i\nu$, in which case the function $K_{i\nu}$ behaves in a rather intelligible way, whereas the factor $x^{1/2}$ causes various phenomena to be masked.

To start things off, I'll exhibit in a moment the graphs of K_{4i}, K_{12i}, and K_{20i}. There is one thing about these graphs that has to be explained. As we have seen, the rough size of $K_{i\nu}$ is

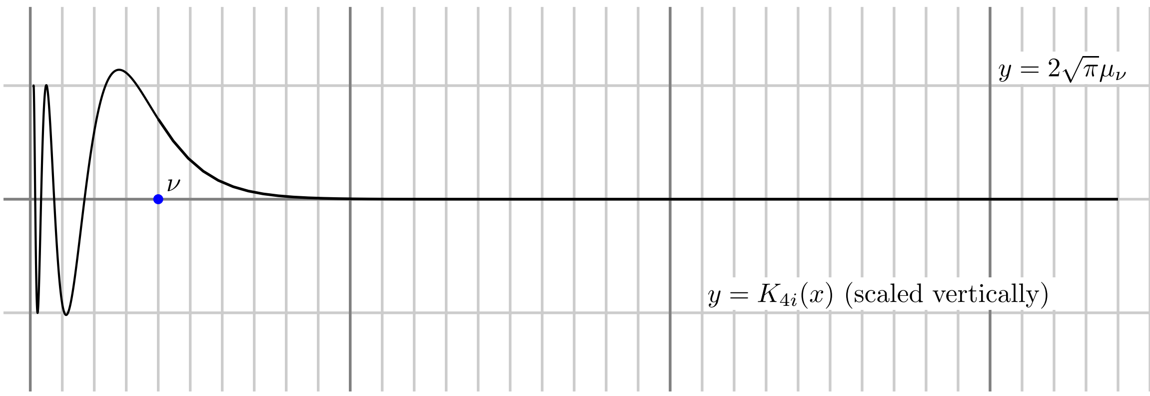
$$\mu_\nu = e^{-3\pi\nu/4}\,.$$

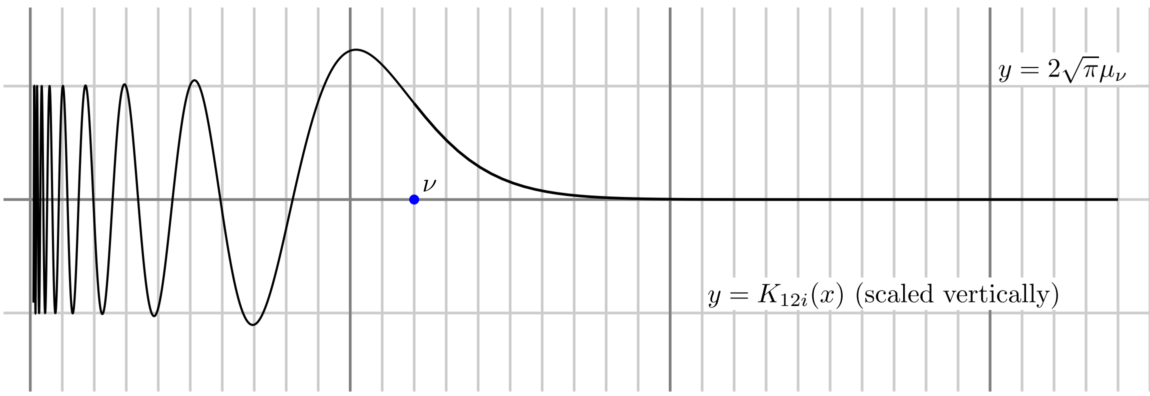
This decreases rapidly as ν grows, so I shall scale the vertical axis to take this into account. Before I show the graphs, let me summarize what we already know about $K_{i\nu}$.

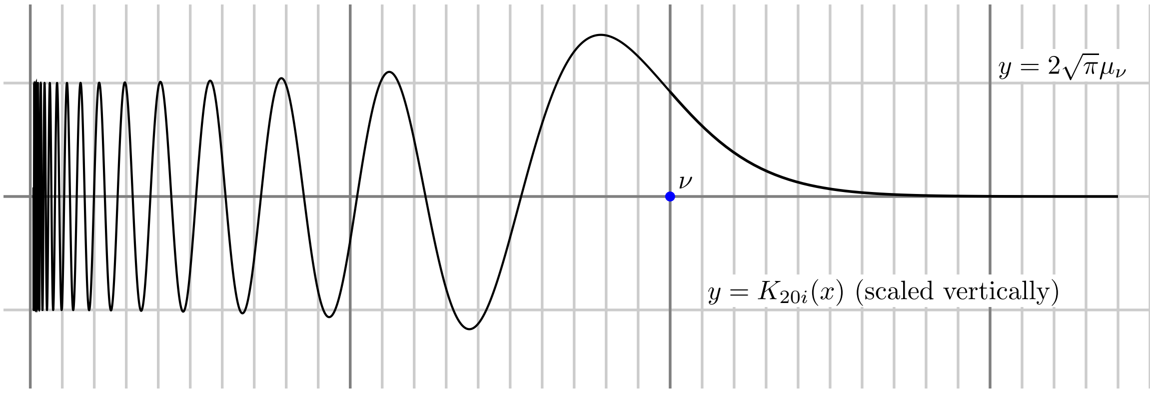
(3.2) $$K_{i\mu}(x) = C_{-i\nu}x^{i\nu} + C_{i\nu}x^{-i\nu} + O(x^2) \quad (C_m = \sqrt{\pi/2}\cdot\Gamma(2m)/\Gamma(1/2+m), \text{ for } x \text{ near } 0.$$

(3.3) $$K_{i\nu}(x) \sim x^{-1/2}e^{-x/2} \text{ as } x \to \infty.$$

Since $x^{i\nu} = e^{i\nu\log x}$, the first tells us that $K_{i\nu}$ oscillates wildly near 0. The second says it vanishes rapidly at ∞. The interesting question is, how do these two features merge?







The answer lies in the approximation formula

$$K_{\subset\nu}(x) \sim \pi e^{-\pi\nu/2}(x/2)^{-1/3}\mathrm{Ai}(-(x/2)^{-1/3}(\nu - x)\,.$$

(From [Magnus-Oberhettinger-Soni:1966], §3.14.3.) Here Ai is the Airy function, which I recall in the next subsection. It is this that explains the similarities among the graphs in the regions clustering around ν.

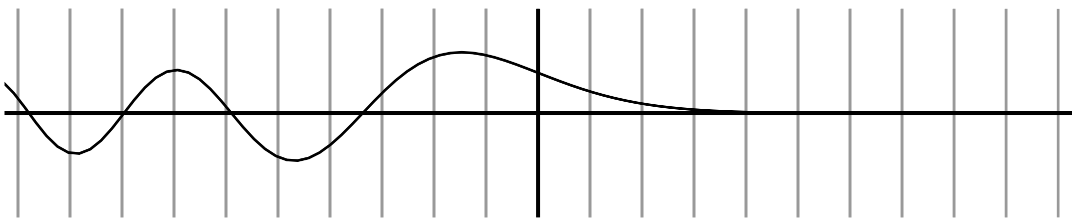
AIRY FUNCTIONS. The Airy function $\mathrm{Ai}(x)$ is defined by the formula

$$\mathrm{Ai}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\cos(t^3/3 + xt)\,dt\,.$$

It is also a solution of the differential equation

$$y'' = xy\,,$$

which is surely the simplest equation that exhibits a turning point (here, $x = 0$), and serves as a generic model for all differential equations with turning points of order 1. It was discovered by the English astronomer and mathematician George Biddell Airy, in connection with diffraction phenomena occurring in rainbows. He computed tables of $\mathrm{Ai}(x)$, but only with an extraordinary amount of trouble, by numerical integration. A bit later, Augustus De Morgan suggested to Airy, albeit against some resistance, how he could use Taylor series instead of his laborious numerical method. This was certainly an improvement, but this technique failed for $|x|$ larger than about 4 or 5, because of increasingly large oscillations in the series that tended to cancel one another. Even today this remains a difficulty, because computers operate with limited precision (whereas Airy operated with limited time and energy). This problem was eliminated by George Gabriel Stokes' introduction of asymptotic but non-converging series, which had hitherto been scorned. This was not only quite practical, but explained perfectly the asymptotic behaviour one could guess at, but without rigourous justification, from Airy's calculations.



Because of new interest in computing Maass forms, the computation of Whittaker functions has had a revival. [Booker-Strömbersson-Then:2011] explains a good algorithm to compute $K_{i\nu}$, and [Templier-Brumley:2014] discusses what happens for analogues of Maass forms for the groups GL_n.

4. References

1. Andrew R. Booker, Andreas Strömbergsson, and Holger Then, 'Bounds and algorithms for the K-Bessel function of imaginary order',*LMS Journal of Computation and Mathematics* **16** (2013), 78–108.

2. F. Brauer and J. Nohel, **Ordinary differential equations**, Benjamin, 1967.

3. Farrell Brumley and Nicolas Templier, 'Large values of cusp forms on GL(n)', preprint

`arxiv:1411.4317`

4. E. A. Coddington and N. Levinson, **Theory of ordinary differential equations**, McGraw-Hill, 1955.

5. W. Magnus, F. Oberhettinger, and R. P. Soni, **Formulas and theorems for the special functions of mathematical physics**, Springer, 11966.

6. George Gabriel Stokes, 'On the numerical calculation of a class of definite integrals and infinite series', *Transactions of the Cambridge Philosophical Society* **IX**, Part I (1850). P. 329 of volume II of Stokes' collected works.

7. E. T. Whittaker and G. N. Watson, **Modern Analysis**, Cambridge, 1952. Chapter XVI is about what we now call Whittaker functions.