Essays in analysis

Nuclear spaces

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This essay is intended to be an introduction to the theory of nuclear spaces adapted to the needs of non-experts. As the Wikipedia entry on nuclear spaces says, "Although important, nuclear spaces are not widely used, possibly because the definition is notoriously difficult to understand."

There are basically two ways to begin the subject, one using nuclear linear maps of Banach spaces, the other using tensor products. I choose the second. This defines two different ways to construct topological tensor products and, roughly speaking, characterizes nuclear spaces as ones for which these constructions are the same. There are many interesting theorems about nuclear spaces, and even several equivalent ways to define them, but I'll prove just a few that are useful useful in representation theory. One of them identifies certain function spaces as nuclear spaces. A second identifies certain function spaces on products as topological tensor products. A third asserts—again, speaking roughly—that for nuclear spaces U, the functor $V \rightsquigarrow U \widehat{\otimes} V$ is an exact functor. All of these are important at some point in harmonic analysis of relevance to number theory.

The useful literature is not extensive. By far the most readable account of the subject is part III of [Treves:1967], particularly the introduction to that part and Chapters 42 ff. It is, however, a bit verbose. Most of the original work is due to [Grothendieck:1955]. It is very technical, but he has written a somewhat more intelligible account in [Grothendieck:1952]. Another standard reference is the exposés in the seminar [Schwartz:1953–4], which was entirely devoted to Grothendieck's results about tensor products and nuclear space.

Contents

1. Bilinear functions

Suppose U, V, and W to be arbitrary complex vector spaces. A bilinear function $f: U \times V \to W$ is one that is separately linear—i.e. linear on each factor:

$$\begin{aligned} f(au, v) &= f(u, av) \\ &= a f(u, v) \\ f(u+v, w) &= f(u, w) + f(v, w) \\ f(u, v+w) &= f(u, v) + f(u, w) \,. \end{aligned}$$

⟪$\mathfrak{B}$⟫ It is jointly bilinear as well. The set of all such bilinear functions from $U \times V$ to W is a vector space $\mathfrak{B}(U \times V, W)$.
⟪$\mathfrak{B}$⟫ Now assume U, V, W to be TVS. Here are some natural subspaces of $\mathfrak{B}(U \times V, W)$:

$$\begin{aligned} B(U \times V, W) &= \text{ all continuous bilinear functions } U \times V \to W \\ \mathcal{B}(U \times V, W) &= \text{ all separately continuous bilinear functions } U \times V \to W \,. \end{aligned}$$

⟪𝔅⟫ For finite-dimensional spaces, these are all the same. If $W = \mathbb{C}$, I'll just write $\mathfrak{B}(U \times V)$, $\mathcal{B}(U \times V)$, and $B(U \times V)$.

⟪B⟫ • The subspace $B(U \times V, W)$ is that of functions f satisfying this condition: given a neighbourhood Z of W there exist neighbourhoods X if 0 in U and Y in V such that $f(X, Y) \subseteq Z$. This means that if τ is any semi-norm on W then there exist semi-norms ρ and σ such that

[bilinear-cont] **(1.1)**
$$\|f(u,v)\|_\tau \le \|u\|_\rho \, \|v\|_\sigma \,.$$

For $W = \mathbb{C}$ the topology is therefore defined by semi-norms

[rhoxsigma] **(1.2)**
$$\|f\|_{\rho\times\sigma} = \sup \frac{|f(u,v)|}{\|u\|_\rho \|v\|_\sigma} \,.$$

⟪$\mathcal{B}$⟫ • The subspace $\mathcal{B}(U \times V, W)$ is made up of those f such that for each u_0 in U and v_0 in V the maps

$$\begin{aligned} v &\longmapsto f(u_0, v) \\ u &\longmapsto f(u, v_0) \end{aligned}$$

are continuous. This means that for each semi-norm τ of W and u in U there exists a semi-norm $\sigma = \sigma_u$ of V such that

$$\|f(u,v)\|_\tau \le \|v\|_{\sigma_u} \,,$$

and for each v in V there exists a semi-norm $\rho = \rho_v$ of U such that

$$\|f(u,v)\|_\tau \le \|u\|_{\rho_v} \,.$$

I'll now look at two examples. Suppose first that U and V are finite-dimensional, with bases $\{u_i\}$, $\{v_j\}$. Let $\{\widehat{u}_i\}$ and $\{\widehat{v}_j\}$ be dual bases. If a bilinear map from $U \times V$ to $\mathbb{C}$ takes (u_i, v_j) to $c^{i,j}$ then it takes $(\sum a_i u_i, \sum b_j v_j)$ to $\sum a_i b_j c^{i,j}$. If $m = \dim U$, $n = \dim V$ then the map taking β to the matrix $(\beta(u_i, v_j))$ is a linear isomorphism of ⟪B⟫ $B(U \times V)$ with the space of $m \times n$ matrices $(c^{i,j})$. I shall rephrase this. To each pair $\widehat{u}$ in $\widehat{U}$, $\widehat{v}$ in $\widehat{V}$ is associated a bilinear map which I'll call $\widehat{u} \otimes \widehat{v}$, for reasons that will appear soon, from $U \times V$ to $\mathbb{C}$:

$$\widehat{u} \otimes \widehat{v}\colon\ (u, v) \longmapsto \langle \widehat{u}, u \rangle \langle \widehat{v}, v \rangle \,.$$

For example, $\widehat{u}_i \otimes \widehat{v}_j$ takes

$$\Big(\sum a_k u_k, \sum b_\ell v_\ell\Big) \longmapsto a_i b_j \,.$$

⟪B⟫ The new formulation is that the $\widehat{u}_i \otimes \widehat{v}_j$ form a basis of $B(U, V)$. But the version we shall want later swaps spaces and their duals, which is entirely legitimate since these spaces are finite-dimensional:

[fd-bilinear] ⟪B⟫ **1.3. Lemma.** *In these circumstances, the $u_i \otimes v_j$ form a basis of $B(\widehat{U}, \widehat{V})$.*

Now the second example. If V is a TVS, its weak dual V_ω is the TVS $\widehat{V}$ of all continuous linear functions on V, assigned semi-norms defined by finite subsets S of E:

[dual-finite] **(1.4)**
$$\|\widehat{v}\|_S = \sup_{v \in S} |\langle \widehat{v}, v \rangle| \,.$$

Any finite-dimensional subspace E of V is necessarily closed. The Hahn-Banach theorem asserts that the inclusion of E in V gives rise to a surjection

$$\widehat{V} \longrightarrow \widehat{E} \,,$$

so that we get an embedding of E into the continuous linear dual of V_ω. On the other hand, suppose Φ to be a continuous linear function on V_ω. By definition of continuity there exists some finite subset S of V such that

$$|\langle \Phi, \widehat{v} \rangle| \ll \sup_{v \in S} |\langle \widehat{v}, v \rangle| \, .$$

If E is the finite-dimensional subspace of V spanned by the $v \in S$ then Φ factors through $\widehat{E}$. Combining these two observations, we see that the space of continuous linear functions on V_ω may be identified with those linear functions that factor through some $\widehat{E}$. Equivalently, the continuous linear dual of V_ω is the union of the images of all the finite-dimensional E in V. This is just another way of saying that V may be identified with the continuous linear dual of V_ω, as is well known, but we'll see the point in a moment.

⟪B⟫ Now suppose U and V both to be TVS, with weak duals $\widehat{U}_\omega$ and $\widehat{V}_\omega$. *What is* $B(\widehat{U}_\omega, \widehat{V}_\omega)$*?* If E is a finite-dimensional subspace of U and F one of V, the embeddings of E into U and F into V give rise to a surjection

$$\widehat{U} \times \widehat{V} \to \widehat{E} \times \widehat{F} \, .$$

⟪B⟫ We get in this way an embedding of $B(\widehat{E}, \widehat{F})$ into $B(\widehat{U}_\omega, \widehat{V}_\omega)$.

[b-uhat-vhat] ⟪B⟫ **1.5. Lemma.** *The space* $B(\widehat{U}_\omega, \widehat{V}_\omega)$ *is the union of these finite-dimensional subspaces* $B(\widehat{E}, \widehat{F})$.

♣ [bilinear-cont] *Proof.* An immediate consequence of (1.1). ∎

2. The algebraic tensor product

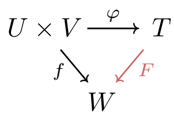
The construction of topological tensor products is motivated by the construction of algebraic tensor products, which I'll review here from scratch.

The point of tensor products is to replace bilinear maps (and eventually multi-linear maps) by linear ones—to associate to every pair U, V of complex vector spaces a third space $U \otimes V$, called their tensor product, such that ⟪$\mathfrak{B}$⟫ the space $\mathfrak{B}(U \times V)$ is canonically equal to the linear dual of $U \otimes V$.

To begin the definition, suppose U and V to be arbitrary complex vector spaces. Following the above idea, a(n algebraic) **tensor product** of U and V is a pair (T, φ) where T is a complex vector space and φ is a bilinear map from $U \times V$ to T satisfying the condition of **universality**—*if* $f\colon U \times V \to W$ *is a bilinear map there exists a unique linear map*

$$F\colon\ T \longrightarrow W$$

such that $F\big(\varphi(u, v)\big) = f(u, v)$. In other words, all bilinear maps factor through the one into T. We can interpret this in a diagram:



es/tensor-diagram.eps

It is not apparent from this definition that there exists any tensor product at all, but it is straightforward to see that if it exists it is unique up to canonical isomorphism:

[tensor-uniqueness] **2.1. Lemma.** *If* (φ_1, T_1) *and* (φ_2, T_2) *are both tensor products of* U *and* V*, then there exists a unique isomorphism of* T_1 *with* T_2 *compatible with the maps* φ_1 *and* φ_2.

Proof. Immediate from the definition. ∎

In order to avoid repeating arguments later on, I'll introduce topological structures right now. Suppose U and V to be TVS, and let $\widehat{U}$ and $\widehat{V}$ be their continuous duals. This is no restriction, because any complex vector space U

can be assigned the linear topology according to which all linear functions are continuous, in which case $\widehat{U}$ is its algebraic linear dual.

Suppose (u, v) in $U \times V$. The pair (u, v) defines the bilinear function

$$u \otimes v\colon\ (\widehat{u}, \widehat{v}) \longmapsto \langle \widehat{u}, u \rangle \langle \widehat{v}, v \rangle$$

⟨⟨B⟩⟩ on $\widehat{U} \times \widehat{V}$. The map from $U \times V$ into $B(\widehat{U}, \widehat{V})$ is bilinear. If I set $E = \mathbb{R}{\cdot}u$ and $F = \mathbb{R}{\cdot}v$, the embeddings of E into U, F into V determine dual maps $\widehat{U}$ to $\widehat{E}$, $\widehat{V}$ to $\widehat{F}$. The bilinear map from $\widehat{U} \times \widehat{V}$ to $\mathbb{C}$ factors through the ♣ [b-uhat-vhat] canonical projection onto $\widehat{E} \times \widehat{F}$, so according to Lemma 1.5 it is a continuous bilinear map from $\widehat{U}_\omega \times \widehat{V}_\omega$ to $\mathbb{C}$. I define

⟨⟨B⟩⟩
$$U \otimes_\omega V = B(\widehat{U}_\omega \times \widehat{V}_\omega)\,.$$

We have therefore a map

$$U \times V \longrightarrow U \otimes_\omega V, \quad (u, v) \longmapsto u \otimes v\,.$$

This generalizes what we saw earlier for finite-dimensional spaces.

If $f\colon U \otimes V \to W$ is a linear map, then $f^\times(u, v) = f(u \otimes v)$ is a bilinear function on $U \times V$ with values in W. This gives us a map from

⟨⟨B⟩⟩
$$\mathrm{Hom}(U \otimes V, W) \longrightarrow B(U \times V, W), \quad f \longmapsto f^\otimes\,.$$

ic-tensor-universal] **2.2. Lemma.** *The map $f \mapsto f^\times$ is an isomorphism.*

Proof. In several steps.

Step 1. Suppose at first that U and V are finite-dimensional. I need only show that $f^\times$ is injective. Fix bases $\{u_i\}$, $\{v_j\}$ of U, V. If $f^\times = 0$ then

$$f(u_i \otimes v_j) = f^\times(u_i, v_j) = 0$$

♣ [fd-bilinear] for all i, j. But according to Lemma 1.3 f is then identically 0. So $f^\times$ is an isomorphism if U and V are finite-dimensional.

Step 2. If E, F are finite-dimensional subspaces of U, V there is a canonical map from $E \otimes_\omega F$ to $U \otimes_\omega V$. ♣ [b-uhat-vhat] According to Lemma 1.5 it is an embedding, and $U \otimes_\omega$ is the union of these finite-dimensional subspaces. We therefore have for each E, F a commutative diagram

$$\begin{array}{ccc} ⟨⟨B⟩⟩\ \mathrm{Hom}(U \otimes_\omega V, \mathbb{C}) & \xrightarrow{f \mapsto f^\times} & B(U, V) \\ \downarrow & & \downarrow \\ ⟨⟨B⟩⟩\ \mathrm{Hom}(E \otimes_\omega F, \mathbb{C}) & \xrightarrow{f \mapsto f^\times} & B(E, F) \end{array}\,.$$

Step 3. We can now construct an inverse to $f \mapsto f^\times$. Given F in $B(U, V)$, suppose w in $U \otimes V$. By the previous remark, it will be in some copy of $E \otimes_\omega F$. But the lower map in the diagram is an isomorphism, so we can define an inverse on $E \otimes_\omega F$. This is independent of the choice of E and F. ▯

This leads immediately to:

roduct-verification] **2.3. Theorem.** *The vector space $U \otimes_\omega V$ together with the map $(u, v) \mapsto u \otimes v$ is a tensor product of U and V.*

Any other is canonically isomorphic to it, where by 'canonically' I mean that the embeddings of $U \times V$ correspond. Because of this result, the tensor product space $U \otimes_\omega V$ is independent of topologies, so I'll drop the subscript from now on.

♣ [fd-bilinear] Lemma 1.3 suggests another way to identify a tensor product.

endent-embedding] **2.4. Proposition.** *Suppose* T *to be a complex vector space together with a bilinear map from* $U \times V$ *to* T *such that (a) if* $\{u_i\}$ *is a linearly independent subset of* U *and* $\{v_i\}$ *one of* V*, then the set* $\varphi(u_i, v_i)$ *is linearly independent. and (b) the* $\varphi(u, v)$ *span* T*. Then the map taking* $u \otimes v$ *to* $\varphi(u, v)$ *induces an isomorphism of* $U \otimes V$ *with* T*.*

ebraic-tensor-exact] **2.5. Proposition.** *If*

$$0 \to U \to V \to W \to 0$$

is a short exact sequence of TVS and T *is also one, then*

$$0 \to U \otimes T \to V \otimes T \to W \otimes T \to 0$$

is also exact.

Proof. The sequence

$$0 \to \widehat{W}_\omega \to \widehat{V}_\omega \to \widehat{U}_\omega \to 0$$

is also exact (which for TVS means that kernels are closed, and quotients isomorphic to images), and from this it follows easily that so is

《B》

$$0 \to B(\widehat{U}, \widehat{T}) \to B(\widehat{V}, \widehat{T}) \to B(\widehat{W}, \widehat{T}) \to 0 \, .$$

∎

3. The projective tensor product

One natural question that now arises is this:

> *Suppose both U and V to be (Hausdorff, locally convex) topological vector spaces. Can we define a reasonable topological tensor product?*

As an example of the problem to be dealt with, let $\mathbb{S}$ be the circle $\mathbb{R}/\mathbb{Z}$, and $T = \mathbb{S} \times \mathbb{S}$. Fourier theory tells us that the functions $e^{2\pi imx}$ are dense in $C^\infty(\mathbb{S})$, and that the products $e^{2\pi imx}e^{2\pi imy}$ are dense in $C^\infty(T)$, and hence that the algebraic tensor product $C^\infty(\mathbb{S}) \otimes C^\infty(\mathbb{S})$ in dense in it. It will turn out that this is a special case of a general construction of function spaces on product spaces as topological tensor products of function spaces on the factors.

There are several useful and generally distinct topological tensor product constructions, among which two are particularly important. The first is the **projective** or π-tensor product $U \widehat{\otimes}_\pi V$, and the second is the **equicontinuous** or ε-tensor product $U \widehat{\otimes}_\varepsilon V$. Each of these comes with an embedded copy of the algebraic tensor product, each is the completion of this embedded copy with respect to a suitable topology on it, and each has its own virtues. In the best of circumstances, as we shall see in the next section, the two coincide.

I first recall that there are two different roles that algebraic tensor products possess: (1) the space of linear maps from $U \otimes V$ to $\mathbb{C}$ may be identified with the space of bilinear maps from $U \times V$ to $\mathbb{C}$; (2) the tensor product $U \otimes V$ is the space of continuous bilinear maps from $\widehat{U}_\omega \times \widehat{V}_\omega$ to $\mathbb{C}$. In the case of topological tensor products, these two approaches turn out to give rise to different notions. I deal with (1), which is the simpler of the two, in this section and (2) in the next.

The π-topology on $U \otimes V$ is defined very simply: a closed, convex subset Z of $U \otimes V$ is a neighbourhood of 0 if and only if its inverse image in $U \times V$ with respect to the map $(u, v) \mapsto u \otimes v$ is a neighbourhood of 0. Suppose ρ, σ to be semi-norms on U, V, and let $X = B_\rho(1)$ in U, $Y = B_\sigma(1)$ in V. Then the inverse image of Z in $U \otimes V$ contains $X \times Y$ if and only if Z contains $\Gamma = \Gamma(X \otimes Y)$, the closed, convex, balanced hull in $U \otimes V$ of $X \otimes Y$. The set Γ is convex, balanced, absorbing, so we may define $\rho \otimes \sigma$ to be the corresponding semi-norm:

$$\|w|_{\rho\otimes\sigma} = \inf_{x\in\Gamma} \{c > 0 \mid w = cx\} .$$

Another way to define the π-topology on $U \otimes V$ is by specifying that it is the topology determined by the semi-norms $\rho \otimes \sigma$.

[pi-norm] **3.1. Proposition.** *For z in $U \otimes V$ we have*

$$\|z\|_{\rho\otimes\sigma} = \inf \sum \|u_i\|_\rho \|v_i\|_\sigma ,$$

where the infimum ranges over all expressions $z = \sum u_i \otimes v_i$.

Proof. Set $X = B_\rho(1)$ in U, $Y = B_\sigma(1)$, $\Gamma = \Gamma(X \otimes Y)$. For each z in $U \otimes V$ define two sets of positive numbers:

$$S_1(z) = \{c > 0 \mid z/c \in \Gamma\}$$
$$S_2(z) = \Big\{ \sum \|u_i\|_\rho \|v_i\|_\sigma \,\Big|\, z = \sum u_i \otimes v_i \Big\}$$

so that

$$\|z\|_{\rho\otimes\sigma} = \inf S_1(z) .$$

The set $S_1(z)$ is the infinite interval $[\,\|z\|_{\rho\otimes\sigma}, \infty)$. If $z = \sum u_i \otimes v_i$ and $0 < \|u_0\|_\rho \|v_0\|_\sigma \leq 1$ we can always add and subtract to z terms $tu_0 \otimes v_0$, so that $S_2(z)$ is also an infinite non-negative interval. To prove the Proposition, it suffices to show that $S_1(z) = S_2(z)$.

If $z/c \in \Gamma$ then

$$z = c \sum t_i(x_i \otimes y_i) \quad \Big(t_i > 0,\ \sum t_i = 1,\ x_i \in X,\ y_i \in Y\Big).$$

and

$$\sum ct_i \|x_i\|_\rho \|y_i\|_\sigma \le c\,.$$

Therefore c lies in $S_2(z)$. Hence $S_1(z) \subseteq S_2(z)$.

On the other hand, suppose

$$z = \sum u_i \otimes v_i\,.$$

Then if $\alpha_i > \|x_i\|_\rho$, $\beta_i > \|y_i\|_\sigma$, and $\gamma = \sum \alpha_i\beta_i$, we have

$$z = \sum (\alpha_i\beta_i)(u_i/\alpha_i) \otimes (v_i/\beta_i), \quad z/\gamma \in \Gamma(X \otimes Y)$$

so $\gamma \in S_1(z)$. So $S_2(z) \subseteq S_1(z)$. ∎

From these, with a small amount of work, follows:

[pi-universal] **3.2. Theorem.** *Suppose U, V to be TVS. There exists on $U \times V$ a unique topology such that for each TVS W composition with $(u, v) \mapsto u \otimes v$ induces an isomorphism of the space of continuous linear maps from $U \otimes V$ to W with the space of continuous bilinear maps from $U \times V$ to W.*

In particular, the continuous dual of $U \otimes_\pi V$ is isomorphic to the space $B(U, V)$ of continuous bilinear maps from $U \times V$ to $\mathbb{C}$.

The π-topology on $U \otimes V$ is the topology defined by the collection of semi-norms $\rho \otimes \sigma$. Implicit in this definition is the first claim of the following Proposition, which is simple to verify:

[tensor-hausdorff] **3.3. Proposition.** *The π-topology is Hausdorff if U and V are, and metrizable if U and V are.*

Define

$$U \widehat{\otimes}_\pi V = \text{ the completion of } U \otimes V \text{ with respect to the } \pi \text{ topology.}$$

[pi-functorial] **3.4. Proposition.** *The π-product is functorial.*

This means that continuous linear maps $U_1 \to U_2$, $V_1 \to V_2$ induces a continuous linear map $U_1 \otimes_\pi V_1$ to $U_2 \otimes_\pi V_2$. The proof of functoriality shows that it is consistent with the identification of the dual of π-tensor products.

Proof. Because the tensor product is symmetric, it suffices to show this for a single factor. So suppose we are given a continuous linear function $F: U \to V$, and we want to show $U \otimes_\pi W \to V \otimes_\pi W$ is continuous. This is immediate from the formula

$$\|w\|_{\rho\otimes\sigma} = \inf \sum \|u_i\|_\rho \|v_i\|_\sigma$$

for the semi-norms on $U \otimes_\pi V$. ∎

There is one property of the π-product that will be very important in applications.

[pi-right-exact] **3.5. Proposition.** *Given a surjective homomorphism $F: U \to V$ and a TVS T, the map*

$$F \otimes I\colon\ U \widehat{\otimes}_\pi T \longrightarrow V \widehat{\otimes}_\pi T$$

is a homomorphism onto a dense subspace.

(a) If W is the kernel of F then the kernel of $F \widehat{\otimes}_\pi I$ is the closure of the image of $W \widehat{\otimes}_\pi T$.
(b) If U and T are Fréchet spaces, the map $F \widehat{\otimes}_\pi I$ is surjective.

I recall that a homomorphism of TVS is a continuous linear map in which the quotient and image topologies coincide.

Proof. We have a short exact sequence

$$0 \to W \to U \to V \to 0\,.$$

I claim that the sequence of adjoint maps of tensor products

⟪B⟫ $$0 \longrightarrow B(V,T) \longrightarrow B(U,T) \longrightarrow B(W,T)$$

is exact.

⟪B⟫ The adjoint map $B(V,T) \to B(U,T)$ is clearly injective, since F is. So by a well known fact, the image of $F \widehat{\otimes}_\pi I$ is dense.

Suppose $\beta(w,t) = 0$ for all w in W. If v lies in V, let u have image v in V. Then $\alpha(v,t) = \beta(u,t)$ depends only on v, not on the choice of u, I need first to show that α is continuous. Suppose $X \subset U$ and $Z \subset T$ neighbourhoods of 0 such that $|\beta(X,Z)| < 1$. Since F is a homomorphism, we can find $Y \subset V$ such that $Y \subset F(X)$. But then for every $y \in Y$ we can find $x \in X$ with $F(x) = y$, and then $|\alpha(Y,T)| < 1$.

To show $F \widehat{\otimes}_\pi I$ a homomorphism, according to a familiar criterion, it must be shown that the image of the adjoint
⟪B⟫ map is weakly closed and that every equicontinuous set in $B(U,T)$ is the image of an equicontinuous set in
⟪B⟫ $B(V,T)$.

⟪B⟫ By construction, the image of $B(V,T)$ in $B(U,T)$ is exactly the kernel of restriction to $W \times T$), hence in particular it is closed. This proves that the kernel of $F \widehat{\otimes}_\pi I$ is the closure of the image of $W \widehat{\otimes}_\pi T$.

⟪B⟫ There remains one thing to verify. Suppose E to be an equicontinuous set in $B(U,T)$. Then there exist neighbourhoods X of 0 in U, Z one in T such that $|\beta(X,Z)| < 1$ for all $\beta \in E$. But then the argument given a moment ago
⟪B⟫ shows that E in the the image of the set of α such that $|\alpha(Y,Z)| < 1$, which is an equicontinuous set in $B(V,T)$.

The last claim about Fréchet spaces is a consequence of the Open Mapping Theorem. ∎

4. The equicontinuous tensor product

Next I define the **equicontinuous** or ε- topology on the algebraic tensor product. The tensor product $U \otimes V$ is
⟪B⟫ isomorphic to the space $B(\widehat{U}_\omega \times \widehat{V}_\omega, \mathbb{C})$ of continuous bilinear maps. This in turn is contained in the space of
⟪$\mathcal{B}$⟫ *separately* continuous bilinear functions $\mathcal{B}(\widehat{U}_\omega \times \widehat{V}_\omega, \mathbb{C})$. The ε-topology on $\mathcal{B}$ is that of uniform convergence on equicontinuous subsets $X \times Y$ of $U \times V$, with X equicontinuous in $\widehat{U}$, Y in $\widehat{V}$. Some motivation for this is offered by fact that the topology of any TVS U is that associated to equicontinuous subsets of $\widehat{U}$.

The neighbourhoods of 0 in this topology are the sets

$$\Sigma(X \times Y, \varepsilon) = \left\{ f: U \times V \to \mathbb{C} \,\middle|\, |f(X \times Y)| < \varepsilon \right\}$$

where X and Y are as above. The completion of $U \otimes V$ in this topology is the ε tensor product $U \widehat{\otimes}_\varepsilon V$.

Given semi-norms ρ and σ as before define

$$\|w\|_{\rho \otimes_\varepsilon \sigma} = \sup_{\widehat{x}, \widehat{y}} \left| \sum \langle \widehat{x}, u_i \rangle \langle \widehat{y}, v_i \rangle \right|$$

where $w = \sum u_i \otimes v_i$, $\widehat{x}$ varies over B_ρ°, and $\widehat{y}$ over B_σ°. The right hand side of this formula is independent of the expression for w.

If E, F are complete so is this. Define $U \widehat{\otimes}_\varepsilon V$ to be the closure of $E \otimes F$ in it.

The π-product embeds into the ε-product. This follows from the universal property of $U \otimes_\pi V$, and can be seen directly because the ε-semi-norms are dominated by the corresponding π-semi-norm.

The ε-product is also functorial.

[epsilon-left-exact] **4.1. Proposition.** *Given an embedding $F: U \hookrightarrow V$ of a closed subspace and a TVS T, the sequence*

$$U \widehat{\otimes}_\varepsilon T \overset{F \widehat{\otimes} I}{\longrightarrow} V \widehat{\otimes}_\varepsilon T$$

is also an injective homomorphism.

[...ebraic-tensor-exact] *Proof.* It suffices to show that $U \otimes_\varepsilon T \overset{F \otimes I}{\longrightarrow} V \otimes_\varepsilon T$ is an injective homomorphism. Proposition 2.5 implies that it is injective, so it just has to be shown that it is a homomorphism.

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Trèves399 Example of $C(K) \otimes E$, where K is a compact topological space. The space $C(K)$ is a Banach space, and not a nuclear space.

Trèves400 The series representation.

Trèves400 For Hilbert spaces, the ε-product is the space of compact operators, and the π-product is that of nuclear operators.

Trèves401 The dual of $E \widehat{\otimes}_\varepsilon F$ is called $J(E, F)$, which hence embeds into the dual of $E \widehat{\otimes}_\pi F$, which is the space of continuous bilinear forms on $E \times F$, called integral forms. An operator $u: E \to F$ is integral if the associated form on $E \times F'$ is integral. (For $J(E, F)$ look also at Schwartz.) A typical integral form is

$$\int f(x) g(x) \, dx$$

Trèves401 Barreled nuclear spaces are Montel and hence reflexive. Stability: subspaces, quotients, products, projective limits, countable direct sums, countable inductive limits of nuclear are nuclear. Fréchet is nuclear if and only if its dual is nuclear.

Kernels theorem: when the strong dual E' of E is nuclear, the space of continuous linear maps from E to F with the topology of bounded convergence is isomorphic to $E'\widehat{\otimes}F$.

So distributions on $X \times Y$ is isomorphic to continuous linear maps from $C^\infty(X)$ to distributions on Y.

5. Nuclear spaces

If f_1 and f_2 are functions in $\mathcal{S}(\mathbb{R})$ then

$$f(x,y) = f_1(x)f_2(y)$$

is in $\mathcal{S}(\mathbb{R}^2)$, and in fact $\mathcal{S}(\mathbb{R}^2)$ is the closure of the space containing all sums of these products:

$$\mathcal{S}(\mathbb{R}^2) = \mathcal{S}(\mathbb{R})\widehat{\otimes}\mathcal{S}(\mathbb{R})$$

⟪B⟫ *In order that the π-topology and the ε-topology on $U \otimes V$ coincide, it is necessary and sufficient the every equicontinuous subset of $B(U,V)$ be contained in the 'convex, balanced, closed' hull of a product $\widetilde{X} \times \widetilde{Y}$ of equicontinuous subsets of U, V.*

6. References

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