Complex analytic ordinary differential equations

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Suppose given an $n \times n$ system of differential equations

$$y' = A(z)y$$

in which

$$A(z) = \frac{1}{z^{p+1}}\left(\alpha_0 + \alpha_1 z + \alpha_2 z^2 + \cdots\right)$$

is holomorphic in a neighbourhood of 0, and $\alpha_0 \neq 0$. The system has a singularity at 0. The singularity is said to be **regular** if solutions in the neighbourhood of 0 are all of moderate growth. This happens if $p = 0$, in which case the singularity is **simple**. If $p > 0$ the singularity may or may not be regular, and it is natural to ask how one can decide whether it is or not. What is in my opinion the most valuable algorithm for answering this question is found in [Dietrich:1968(a)], although this paper seems to have been almost completely neglected in subsequent literature. I am very intrigued by Dietrich's algorithm, largely because it all fits together so well, but has no apparent conceptual motivation.

The original motivation for this essay was to to explain Dietrich's algorithm, at least to myself. But I have also taken the opportunity to cover some background.

My principal references for the background have been [Brauer-Nohel:1967], [Coddington-Levinson:1955], and [Hartman:1964]. The first is very readable, but gives few proofs. The last two are more complete.

In this preliminary version I offer very little to justify Dietrich's algorithm, but concentrate on explaining how it works. In later versions I'll say more about justification and also say something about competitors.

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1. Differential equations with analytic coefficients

Here is the most basic result, which dates from the very early days of complex analysis. Let

$$\mathbb{D}_r = \left\{ z \,\middle|\, |z| < r \right\}.$$

1.1. Lemma. *Given an $n \times n$ linear system of differential equations*

$$y' = A(z)y$$

with $A(z)$ analytic in $\mathbb{D}_r$ and y_0 in $\mathbb{C}^n$, there exists a unique analytic function $y(z)$ on $\mathbb{D}_r$ with $y(0) = y_0$.

The connection between this result and basic results about differential equations on intervals in $\mathbb{R}$ is that if $z(t)$ is a path in the disk with $z(0) = 0$ then the composite $y\big(z(t)\big)$ from $\mathbb{R}$ to $\mathbb{C}^n$ solves the (real) lifted equation $y'(t) = A\big(z(t)\big)y(t)$. Another way to phrase this is that the complex differential equation, when formulated in terms of real and imaginary components, determines an integrable connection on the disk considered as a subset of $\mathbb{R}^2$. This will be discussed elsewhere.

Proof. I'll do here just the case in which $A(z) = a(z)$ is a scalar-valued function. I may as well assume $y(0) = 1$. Set

$$\begin{aligned} a &= a_0 + a_1 z + a_2 z^2 + \cdots \\ y &= 1 + c_1 z + c_2 z^2 + \cdots \\ ay &= a_0 + (a_0 c_1 + a_1) z + (a_0 c_2 + a_1 c_1 + a_2) z^2 + \cdots \\ y' &= c_1 + 2c_2 z + 3c_3 z^2 + \cdots \end{aligned}$$

leading to

$$\begin{aligned} c_1 &= a_0 \\ c_2 &= \frac{1}{2}(a_0 c_1 + a_1) \\ c_3 &= \frac{1}{3}(a_0 c_2 + a_1 c_1 + a_2) \\ &\cdots \\ c_k &= \frac{1}{k}(a_0 c_{k-1} + \cdots + a_{k-1}) \end{aligned}$$

It must now be shown that the series $\sum c_i z^i$ defined in this way converges for $|z| < r$.

I follow [Brauer-Nohel:1967]. The method is a variant of that of **majorants** invented by Cauchy, which compares the solution of the given differential equation to one that can be solved explicitly.

It suffices to show that for every $R < r$ we can find a sequence C_k with

$$|c_k| \le C_k \tag{a}$$

$$\lim_{k \to \infty} C_{k+1}/C_k = 1/R\,. \tag{b}$$

We know that since $\sum a_k z^k$ converges for $|z| < r$, there exists some $M > 0$ such that $|a_k| \le M R^{-k}$ for all k.

Now consider the differential equation

$$y'(z) = \frac{M y(z)}{1 - (z/R)}, \quad y(0) = M\,.$$

Its solution is $\sum C_k (z/R)^k$ with

$$\begin{aligned} C_0 &= M \\ C_{k+1} &= \frac{1}{k+1} \cdot (M C_k + M R^{-1} C_{k-1} + \cdots + M R^{-k})\,. \end{aligned}$$

We may verify by induction that $|c_k| \le C_k$ for all k. But

$$\begin{aligned} (k+1) C_{k+1} &= M C_k + R^{-1} k C_k \\ &= C_k (M + k R^{-1}) \\ \frac{C_{k+1}}{C_k} &= \frac{M + k/R}{k+1} \longrightarrow 1/R \quad \text{as} \quad k \longrightarrow \infty\,. \end{aligned}$$

This result applies only to disks in the complex numbers, but it can be extended to more general regions, as we'll see in a moment. There is an important feature of solving differential equations in regions in $\mathbb{C}$, however. Consider the differential equation $y' = y/2z$, which is analytic in the complement of 0. Formally, its solution is $y = z^{1/2}$, but this does not make sense throughout that region. In general, there exist solutions of differential equations throughout a region of $\mathbb{C}$ only if that region is simply connected. For example, we can choose one of the two branches of $z^{1/2}$ in any angular sector of width less than 2π.

If

$$y' = A(z)y$$

is a linear holomorphic differential equation defined on an open subset U of $\mathbb{C}$, define $\mathcal{T}_A$ to be the space of all pairs (u, f) with u in U, f the germ of a solution of the equation. If Y is an open subset of U and F a solution on Y, define $S(Y, F)$ to be the set of (y, f) for y in Y and f the germ of F at y. A topology on $\mathcal{T}_A$ is specified by taking the $S(Y, F)$ to be a basis of open sets.

1.2. Proposition. *The canonical projection from $\mathcal{T}_A$ to U is a covering map.*

Proof. Immediate from the previous Lemma.

1.3. Corollary. *If $A(z)$ is analytic in a connected and simply connected open subset U of $\mathbb{C}$, then for each u in U and each vector y_u there exists a unique solution $y(z)$ of the system $y' = A(z)y$ with $y(u) = y_u$, defined and analytic throughout U.*

Proof. If U is simply connected, any local section of T lifts uniquely to all of U.

Given the differential equation, to any open subset $Y \subset U$ one can associate the space of solutions of the system defined on Y. The results above assert that this defines a locally constant sheaf whose stalk at every point may be identified (but not canonically) with $\mathbb{C}^n$.

2. Punctured disks

In general, suppose u and v to be any points of a region U that is not necessarily simply connected and $\gamma(t)$ to be a path in U with $\gamma(0) = u$, $\gamma(1) = v$. Let $\widetilde{U}$ be the covering space of U. The differential equation lifts to one on $\widetilde{U}$, and if $\widetilde{u}$ is any point in $\widetilde{U}$ over u the path γ lifts to a unique path starting out at $\widetilde{u}$. Since $\widetilde{U}$ is simply connected, there exists a unique solution on $\widetilde{U}$ agreeing with some initial condition at $\widetilde{u}$. If $u = v$ then the endpoint of the lifted path will lie over v, and we shall get a value of the solution at v. This linear map from $\mathbb{C}^n$ to itself is called the **monodromy** of the path. It follows from Proposition 1.2 that it depends only on its homotopy class.

For example, consider the scalar equation

$$y' = \lambda y/z\,,$$

on the punctured disk

$$D_1^\times = \{z \,|\, 0 < |z| < 1\}\,.$$

The covering map is the exponential $x \mapsto e^x$ from the region $\mathrm{RE}(x) < 0$, and the lifted differential equation is

$$y' = \lambda y\,.$$

The solution on $\widetilde{U}$ is $e^{\lambda x}$, and the monodromy associated to one positive cycle around the origin amounts to multiplication by $e^{2\pi i\lambda}$.

Conventionally, the solutions on the covering space are expressed as multi-valued functions on the punctured disk. The function $e^{\lambda x}$ is thus written as z^λ. There are several rigourous ways to justify this, but I won't go into detail.

For another example, consider the equation

$$y'' + y'/z = 0\,,$$

which lifts to

$$y'' = 0\,.$$

The functions 1, x make up a basis of solutions of the lifted equation, which on the original set gives rise to the multivalued solutions 1, $\log(z)$. The monodromy associated to one positive revolution is

$$\begin{bmatrix} 1 & 2\pi i \\ 0 & 1 \end{bmatrix}.$$

The general case looks somewhat similar to these.

2.1. Proposition. *If $A(z)$ is a holomorphic matrix-valued function on $\mathbb{D}_r^\times$, a fundamental solution to*

$$y' = A(z)y$$

is of the form $S(z)z^\Lambda$, where $S(z)$ is holomorphic on $\mathbb{D}_r^\times$ and Λ is a constant matrix.

Proof. The covering space of $\mathbb{D}_r^\times$ is $\{\mathrm{RE}(x) < \log r\}$, the covering map $x \mapsto e^x$. If F is a fundamental matrix, then $F(x+2\pi i) = F \cdot M$, where M is the monodromy transformation. The matrix M can be conjugated to one in Jordan form, and hence $M = e^{2\pi i \Lambda}$ for some Λ. The matrix-valued function $F(x) \cdot e^{-\Lambda x}$ is invariant under translation by $2\pi i$, and therefore descends to an analytic function $S(z)$ on $D_r^\times$. We may hence express the fundamental matrix as $S(z)z^\Lambda$. ∎

There are two possibilities: (i) $S(z)$ has a pole at $z = 0$ or (ii) $S(z)$ has an essential singularity at 0. In the first case, the differential equation is said to have a **regular singularity**. The following is not easy to answer:

How can we tell from the equation itself whether the equation has a regular singularity?

I need now to introduce some notation. Suppose we are looking at the differential equation

$$y' = A(z)y\,,$$

in which $A(z)$ is a matrix whose entries are meromorphic functions of z. I define the **order** of $A(z)$ to be the smallest non-negative integer p such that $z^{p+1}A(z)$ is holomorphic in a neighbourhood of 0. I may then write

$$A(z) = \frac{1}{z^{p+1}} \cdot \alpha(z) \quad \big(\alpha(z) = \alpha_0 + \alpha_1 z + \alpha_2 z^2 + \cdots\big)$$

with $\alpha_0 \neq 0$. The differential equation may then also be written as

$$Dy = \frac{1}{z^p} \cdot \alpha(z)\,.$$

Here $D = zd/dz$ is the multiplicatively invariant derivative.

FORMAL SOLUTIONS. As we shall see later, solving in explicit terms a differential equation with a regular singularity at 0 often involves finding a formal solution, that is to say a finite sum of terms

$$z^r \log^k z\big(1 + c_1 z + c_2 z^2 + \cdots\big) \tag{2.2}$$

in which the series are considered as formal expressions. The coefficients in the series can then be found by recursion, as we shall see later on. Does the formal solution one obtains represent a convergent solution? It is often possible to see directly that the power series does converge in a neighbourhood of 0, but the next result asserts that this is automatic.

2.3. Lemma. *Any finite sum of series (2.2) satisfying the differential equation represents a convergent function.*

Proof. Suppose $\Phi(z) = \varphi(z)z^\Lambda$ to be a fundamental solution of the equation, $\varphi(z)$ a formal solution. It is to be shown that $\varphi(z) = \Phi(z)\xi$ for some constant vector ξ. For this, it suffices to prove that $\Phi^{-1}(z)\varphi(z)$ is a constant vector, or in other words that its derivative vanishes. For this in turn it suffices to see that if the derivative of a sum of linearly independent terms (2.2) vanishes, then so does each term. I leave this as an exercise. (See also Theorem 3.1 in Chapter 4 of [Coddington-Levinson:1955].)

SIMPLE SINGULARITIES. There are a few practical criteria for regularity. One of them is fundamental. The singularity is called **simple** if $p = 0$, in which case the equation can be written as

$$Dy = \alpha(z)y\,.$$

2.4. Proposition. *A simple singularity is a regular singularity.*

Proof. I follow the succinct account in §11 of [Hartman:1964]. Suppose $Y(z) = S(z)z^{\Lambda}$ to be a fundamental solution. It turns out that $S(z)$ also satisfies a differential equation. Since $S(z) = Y(z)z^{-\Lambda}$:

$$DS(z) = DY \cdot z^{-\Lambda} - Y \cdot \Lambda z^{-\Lambda}\,.$$

Since Λ might not commute with Y, we cannot write this as we might wish. Instead, consider $S(z)$ as a *vector-valued function* of dimension n^2, also satisfying a differential equation with a simple singularity. Given this modification, it now suffices to prove that if $y(z)$ is a single-valued analytic function on $\mathbb{D}_r^{\times}$ satisfying a differential equation

$$y' = \frac{1}{z} \cdot \sigma(z)y$$

with a simple singularity, then it is of uniform moderate growth near 0, since then some $z^n y(z)$ with be a bounded, hence holomorphic, function on all of $\mathbb{D}_r$.

Consider the differential equation and its solution restricted to the ray $y_\theta(t) = te^{i\theta}$ (with $0 < t < r$). It satisfies the real equation

$$y_\theta'(t) = \frac{1}{t} \cdot \sigma(te^{\theta})y_\theta(t)\,.$$

But then by assumption on σ

$$\left|\frac{dy_\theta}{dt}\right| \le \frac{C\|y_\theta\|}{t}$$

for some C independent of θ. The following well known lemma allows to conclude:

2.5. Lemma. *If $\|y\|$ is the Euclidean norm, then as long as $\|y\| \neq 0$*

$$\left|\frac{d\,\|y\|}{dt}\right| \le \left\|\frac{dy}{dt}\right\|\,.$$

Proof. This is well known. With $y(t) = (y_i(t))$:

$$\begin{aligned}
\|y\| &= \Big(\sum y_i^2\Big)^{1/2} \\
\frac{d\,\|y\|}{dt} &= \frac{1}{2}\,\frac{\sum 2\,y_i\,y_i'}{\|y\|} \\
\left|\frac{d\,\|y\|}{dt}\right| &\le \frac{\|y\|\,\|y'\|}{\|y\|} \qquad \text{(Cauchy-Schwarz)} \\
&= \|y'\|\,.
\end{aligned}$$

Example. The simplest multi-dimensional equations are of the form

$$Dy = \alpha\,y$$

with α a matrix of constants. This is called an **Euler system** and has as fundamental solution

$$y = z^\alpha \,.$$

More generally, the solutions of an equation with a simple singularity have something in this form as a leading term, and the coefficients of the series of a solution can be found by recursion. We shall see an example later on.

Example. Now we look at n-th order differential equations.

2.6. Proposition. *The system associated to the n-th order linear differential equation*

$$y^{(n)} + a_{n-1}(z)y^{(n-1)} + \cdots + a_0(z)y = 0$$

has a regular singularity if and only if each $\alpha_i(z) = z^{n-i}a_i(z)$ is holomorphic at 0.

Proof. Sufficiency is easy. Multiplying through by z^n, the equation becomes

$$z^n y^{(n)} + \alpha_{n-1}(z)z^{n-1}y^{(n-1)} + \cdots + \alpha_i(z)z^i y^{(i)} + \cdots + +\alpha_0(z)y = 0 \,.$$

But

$$Dz^m y^{(m)} = mz^m y^{(m)} + z^{m+1}y^{(m+1)}, \quad z^{m+1}y^{(m+1)} = (D-m)z^m y^{(m)} \,.$$

This implies that

$$z^m y^{(m)} = P(D)y \quad \text{with} \quad P(D) = \prod_1^m (D-i) \,,$$

and hence that the equation may be written as an expression in the $D^m y$ with holomorphic coefficients. The corresponding linear system is

$$Dy = \begin{bmatrix} \circ & \circ & \ldots & & & & -\beta_0(z) \\ 1 & 1 & \circ & \ldots & & & -\beta_1(z) \\ \circ & 1 & 2 & \circ & \ldots & & -\beta_2(z) \\ \ldots & \circ & 1 & \ldots & & & \\ & & & \ldots & \circ & 1 & (n-1)-\beta_{n-1}(z) \end{bmatrix} y \,,$$

which is simple.

Necessity can be proved by induction on n. I won't include the argument here (but see pp. 85–86 in [Hartman:1964]).

In general, an equation can have a regular singularity even if it is not simple. For example, consider

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2/z^2 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} .$$

From this it follows that $u'' = 2u/z^2$ or

$$D^2u - Du - 2u = 0 \ \text{ with basis of solutions } \ u = z^2, \ 1/z \,,$$

giving fundamental matrix

$$\begin{bmatrix} z^2 & 1/z \\ 2z & -1/z^2 \end{bmatrix} .$$

MEROMORPHIC EQUIVALENCE. There is an easy way to generate differential equations with regular singularities, and that is through **meromorphic equivalence**. Start with an equation in which the $n \times n$ matrix $A(z)$ has a simple pole:

$$x' = A(z)x \,.$$

Suppose $T(z)$ to be an $n \times n$ matrix whose entries are holomorphic in $\mathbb{D}_r^\times$ with a pole of finite order at 0. These entries will all be of the form $z^k f(z)$ with $f(z)$ holomorphic in $\mathbb{D}_r$. Suppose in addition that there exists an inverse matrix of the same form—that is to say, $T(z)$ is meromorphically invertible. Define a new dependent variable $y = T^{-1}(z)x$, so that $x = T(z)y$. It too satisfies a differential equation.

2.7. Lemma. *Suppose given a differential equation of dimension* n

$$x' = A(z)x$$

and a meromorphically invertible $n \times n$ *matrix* $T(z)$*. If* $y(z) = T^{-1}(z)x(z)$*, then*

$$y' = A_T(z)y \quad \textit{with} \quad A_T(z) = T^{-1}(z)A(z)T(z) + T^{-1}(z)T'(z)\,. \tag{2.8}$$

Equally, if one starts with

$$Dy = \alpha(z)y$$

one gets

$$\alpha_T(z) = T^{-1}(z)\alpha(z)T(z) - T^{-1}(z) \cdot DT(z)\,.$$

Proof. Straightforward. On the one hand

$$x' = Ax = ATy$$

but on the other

$$x' = (Ty)' = T'y + Ty'\,.$$

If the fundamental solution of the original is $S(z)z^\Lambda$ then that of the new one is $T^{-1}(z)S(z)z^\Lambda$, so of course the new one also has a regular singularity. Furthermore, a system with fundamental solution $S(z)z^\Lambda$ is meromorphically equivalent to one with fundamental solution z^Λ. We recover a result found already in [Horn:1892]:

2.9. Theorem. *A differential equation has a regular singularity if and only if it is meromorphically equivalent to an Euler system with the same monodromy.*

A SIMPLIFICATION. Horn's theorem suggests a new form of the problem of deciding whether or not a differential equation has a regular singularity or not—can we find a meromorphic matrix $T(z)$ that converts it into a simple one? We'll see later how to answer this, but here I wish to point out a slight simplification of the problem.

Let $\mathfrak{o}$ the ring of power series converging in a neighbourhood of 0 and K its quotient field, which is made up of functions $z^m f(z)$ with f in $\mathfrak{o}$, m in $\mathbb{Z}$. In other words, K is made up of functions f on some $\mathbb{D}_r^\times$ with the property that some $z^m f$ may be extended to all of $\mathbb{D}_r$.

2.10. Lemma. (Principal divisor theorem) *Every matrix* $T(z)$ *in* $M_n(K)$ *may be expressed as*

$$P(z)z^\Lambda Q(z)$$

in which Λ *is a diagonal matrix with integer entries, and* $P(z)$ *and* $Q(z)$ *are both in* $\mathrm{GL}_n(\mathfrak{o})$.

One can require that the entries of Λ increase weakly as one runs down the diagonal, and if that is specified then Λ is unique. The expression found is known as the **Smith normal form** of $T(z)$, presumably named after the nineteenth century British mathematician H. J. Smith.

If

$$A(z) = \frac{1}{z^{p+1}} \cdot \left(\alpha_0 + \alpha_1 z + \cdots\right)$$

is an $n \times n$ meromorphic matrix with $\alpha_0 \neq 0$ then, following Moser, I set

$$m(A) = p + \frac{r}{n}$$

where r is the rank of α_0. The differential equation $y' = A(z)y$ has a regular singularity if and only if $m(A) \leq 1$.

2.11. Lemma. *If $Q(z)$ lies in $\mathrm{GL}_n(\mathfrak{o})$ then $m(A_Q) = m(A)$.*

Proof. Left as exercise.

This leads to the following simplification of our problem:

2.12. Proposition. *The differential equation $y' = A(z)y$ has a regular singularity if and only if there exists a meromorphic matrix $T(z) = P(z)z^{\Lambda}$ with $P(z)$ in $\mathrm{GL}_n(\mathfrak{o})$ and Λ diagonal integral such that $m(A_T) \leq 1$.*

THE JURKAT CRITERION. There is no trivial criterion for a regular singularity, but there are are several procedures known to decide the question. Later in this essay I'll discuss in some detail one due to Volker Dietrich, but I'll sketch another one here that's more conceptual if less elementary. It is also not very efficient.

I'll first introduce a notion from[Katz:1971]. Let $\mathfrak{p}$ be the ideal (z) of the ring $\mathfrak{o}$ I introduced a moment ago. The operator $D = zd/dz$ is a derivation of K that takes $\mathfrak{o}$ to itself and $\mathfrak{p}$ to itself.

Deviating a bit from standard usage, I define a **connection** on K^n to be an operator

$$\nabla\colon\ K^n \longmapsto K^n$$

that satisfies the equation

$$\nabla(fv) = D(f) \cdot u + f \cdot \nabla(u)$$

for all f in K, u in K^n. For example,

$$D\colon\ (y_i) \longmapsto (Dy_i)$$

is a connection, one that annihilates constant vectors. More generally, if $\alpha(z)$ is a meromorphic matrix then

$$\nabla_\alpha\colon\ y \longmapsto Dy - \alpha(z)y$$

is also a connection.

2.13. Lemma. *Every connection on K^n is ∇_α for some meromorphic matrix $\alpha(z)$.*

Proof. The difference between two connections is a linear operator.

Given the meromorphic matrix $\alpha = \alpha(z)$, define by recursion the meromorphic matrices

$$\mathfrak{A}_0 = I, \quad \mathfrak{A}_{n+1} = (D - \alpha)\mathfrak{A}_n\,,$$

or $\mathfrak{A}_n = (D - \alpha)^n I$. These are all matrices in $M_n(K)$, and make up a modification of what is sometimes called the **Jurkat sequence** of α. For example

$$\mathfrak{A}_1 = \alpha, \quad \mathfrak{A}_2 = D\alpha - \alpha^2\,.$$

The significance of these is simple:

2.14. Lemma. *For all k*

$$\nabla_\alpha^m y = \sum_0^m \binom{m}{k} (-1)^k\, \mathfrak{A}_{m-k} D^k y\,.$$

Proof. By induction.

Remark. The sequence defined by Jurkat differs from mine. He definbes

$$A_n = (d/dz - \alpha(z)/z)^n I\,.$$

It is not difficult to relate this to ours. One relation between the two is that

$$\mathrm{ord}(A_n) = \mathrm{ord}(\mathfrak{A}_n) + n\,.$$

I recall that a **lattice** in K^n is a free $\mathfrak{o}$-module of rank n,a nd that the **order** of a matrix $A(z)$ is the maximum order of poles among its entries.

Part of the following result is classical, but one part is due to Jurkat, and another to Katz.

2.15. Theorem. *Suppose $\alpha(z)$ to be in $M_n(K)$. The following are equivalent:*

(a) *the differential equation*

$$Dy = \alpha(z)y$$

has a regular singularity at 0;

(b) *there exists a lattice L in K^n taken into itself by ∇_α;*

(c) *there exists a lattice L in K^n and an integer m such that*

$$\nabla_\alpha^k L \subseteq z^m L$$

for all $k \ge 0$;

(d) *for every lattice L in K^n there exists an integer m such that*

$$\nabla_\alpha^k L \subseteq z^m L$$

for all $k \ge 0$;

(e) *there exists an integer m such that $\mathrm{ord}(\mathfrak{A}_k) \le m$ for all k.*

Proof. Condition (a) is equivalent to (b) since all lattices are equivalent under $\mathrm{GL}_n(K)$. Conditions (d) and (e) are equivalent because of Lemma 2.14. That (b) implies (c) is trivial. The hardest implication goes from (c) to (a). All proofs that I know of pass through a well known and subtle result due to Hugh Turrittin that I won't deal with, and I shall not say anything about it.

What I shall do is prove that (c) implies (d). Given a basis $e = (e_i)$ of K^n, there exists for each k a matrix $A_k(z)$ such that

$$\nabla^k e = eA_k(z)\,.$$

Condition (c) means we can choose M such that $\mathrm{ord}(A_k) \le M$ for all k.

Suppose $f = (f_i)$ to be another basis. Let $f = e\Phi$, and suppose

$$\nabla^k f = fB_k(z)$$

That (c) implies (d) means that the requirement is valid for an arbitrary basis. But we can find a formula for the B_k in terms of the A_k, since

$$\begin{aligned}
\nabla^k f &= \nabla^k e\,\Phi \\
&= \sum_0^k \binom{k}{i} \nabla^{k-i} e \cdot D^i\Phi \\
&= \sum_0^k \binom{k}{i} eA_{k-i} \cdot D^i\Phi \\
&= \sum_0^k \binom{k}{i} f \cdot \Phi \cdot A_{k-i} \cdot D^i\Phi\,,
\end{aligned}$$

so

$$B_k = \sum_0^k \binom{k}{i} \Phi \cdot A_{k-i} \cdot D^i \Phi \,.$$

This implies what we want, since the order of $D\Phi$ is no more than the order of Φ.

The following more precise result, found in [Jurkat-Lutz:1971], makes this criterion practical.

2.16. Theorem. *The differential equation*

$$Dy = z^{-p} \alpha(z)$$

has a regular singularity if and only if

$$\operatorname{ord}(\mathfrak{A}_k) \le (n-1)p$$

for all $k = n, n+1, \dots, (n-1)(2np-1)$.

Checking this can involve a fair amount of work. Here is a small improvement found in [Dietrich:1978(b)]:

2.17. Theorem. *The system has a regular singularity if for some* M

$$\operatorname{ord}(\mathfrak{A}_k) \le M$$

for all $k \le nM + 1$.

One consequence of the Theorem is a simple necessary condition for regularity that's illustrated in a previous example:

2.18. Corollary. *If* $p > 0$ *and the system*

$$Dy = \frac{1}{z^p} \cdot \alpha(z) y$$

has a regular singularity, then the constant term α_0 *of* $\alpha(z)$ *is a nilpotent matrix.*

Proof. Since the order of Dy is the same as that of y, but that of αy is generally more than that of y if α is not holomorphic, this implies immediately that α_0 must be nilpotent.

The original version of this can be found in [Horn:1892], in which it is proved that the characteristic polynomial $\det(\lambda I - \alpha_0)$ is a power of λ.

The results of [Jurkat-Lutz:1971] and [Dietrich:1978(b)] are reasonably practical, but not terribly efficient. There are other algorithms that deduce not only regularity but produce an explicit $T(z)$ in $GL_n(K)$ such that

$$A \longmapsto T^{-1}AT - T \cdot DT$$

effects the meromorphic equivalence. We'll see one of these later on.

3. Second order equations

As we have seen, a differential equation

$$y'' + A(z)y' + B(z)y = 0$$

has a regular singularity at $z = 0$ if $A(z)$ is a meromorphic function of the form $a(z)/z$ with $a(z)$ analytic at $x = 0$, and similarly $B(z) = b(z)/z^2$, so the equation can be rewritten as

$$z^2 y'' + za(z)y' + b(z)y = 0 \,.$$

This can also be rewritten more intelligibly. Set $\partial = d/dz$ and recall that $D = xd/dx = x\partial$ is the multiplicatively invariant derivative. Then

$$\begin{aligned} Dy &= z\,\partial y \\ D^2y &= z(\partial y + z\,\partial^2 y) \\ &= z\,\partial y + z^2\,\partial^2 y \\ zy' &= Dy \\ z^2y'' &= D^2y - Dy\,, \end{aligned}$$

so the equation becomes

$$D^2y + (a(z) - 1)Dy + b(z)y = 0\,.$$

Thus differential equations with regular singularities at 0 are those that can be written as

$$a(z)D^2y + b(z)Dy + c(z) = 0$$

in which a, b, c are analytic at $x = 0$ and $a(0) \neq 0$. Special cases are those of the form

$$aD^2y + bDy + c = 0$$

with $a \neq 0$, b, c constants. These are **Euler equations**. They can be solved in terms more familiar (at least to engineering instructors) by a change of independent variable $z = e^x$ to arrive at the equivalent equation

$$ay'' + by' + c = 0\,.$$

This is solved by exponential functions, leading to solutions of the form $y = z^r$ for Euler's equation itself. The exponent r must be a root of the **indicial equation**

$$ar^2 + br + c = 0\,.$$

The basic fact about Euler's equations is this:

3.1. Lemma. *Suppose given an Euler's equation*

$$aD^2y + bDy + c = 0$$

in which $a \neq 0$, b, c are constants. Then

(a) *if the indicial equation has two roots r_i, z^{r_1} and z^{r_2} are a basis of solutions;*
(b) *if there is just one root r (with multiplicity two), z^r and $z^r \log|z|$ are a basis.*

Now suppose given the linear differential operator

$$L\colon\; y \longmapsto a(z)D^2y + b(z)Dy + c(z)$$

with $a(z)$, $b(z)$, $c(z)$ all analytic at 0 and $a(0) \neq 0$. Define the **associated Euler operator**

$$L_0\colon\; y \longmapsto a(0)D^2 + b(0)Dy + c(0)y\,.$$

The solutions of the original equation are closely related to those of the associated Euler's equation $L_0y = 0$.

Let $\mathcal{O}$ be the ring of convergent series at 0.

3.2. Proposition. *In the circumstances above, suppose the roots of the associated indicial equation to be r_1, r_2. Then:*

(a) *if* $r_1 - r_2 \notin \mathbb{Z}$ *there exists a basis of solutions of the form* $y_i = z^{r_i} f_i$;
(b) *if* $r_1 = r_2 =$ *(say)* r *there exists a basis of solutions of the forms* $y_1 = z^r f_1$, $y_2 = z^r f_2(z) + y_1 \log|z|$;
(c) *if* $r_1 - r_2$ *is a positive integer* n, *there exists a basis of solutions of the forms* $y_1 = z^{r_1} f_1$, $y_2 = z^{r_2} f_2(z) + \mu y_1 \log|z|$ *for some constant* μ.

In this, all f_i are in $\mathcal{O}$.

To find the solutions explicitly, one looks for the candidate series through recursion relations among the coefficients. I recall from Lemma 2.3 that any formal power series that solves the equation will actually converge. In most cases this will be evident. We'll see an example shortly.

THE EUCLIDEAN LAPLACIAN. In a moment I am going to look at Bessel's differential equation, but I'll motivate what is to come by looking first at a somewhat familiar topic that motivates well what we are going to see later.

The Laplacian in the Euclidean plane is

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} .$$

Suppose we want to find solutions of the eigenvalue equation

$$\Delta f = \lambda f$$

in a region with circular symmetry—for example, want to solve the wave equation describing a vibrating drum. The first step is separation of variables in polar coordinates. To do this one first expresses Δ in those coordinates:

$$\Delta f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} .$$

Solutions we are looking for can be expressed in terms of Fourier series

$$f(r,\theta) = \sum_n f_n(r) e^{in\theta}$$

so we are led to look for solutions of the form

$$f(r,\theta) = \varphi(r) e^{in\theta} .$$

This leads to the ordinary differential equation

(3.3)
$$\varphi''(r) + \frac{1}{r}\cdot\varphi'(r) - \frac{n^2}{r^2}\cdot\varphi(r) = \lambda\varphi$$

For $\lambda \ne 0$ solutions of this equation are **Bessel functions**. The equation is evidently singular at $r = 0$. If one changes variables $t = 1/r$ to see what happens at $r = \infty$ one also sees that it has a singularity at infinity. In fact, it has a regular singularity at the origin and an **irregular** singularity at infinity. It is the first that I am most interested in now.

BESSEL FUNCTIONS. Let's look at an example based on (3.3), written now as

$$D^2 y - n^2 y = \lambda z^2 y .$$

To avoid trivial cases, assume $\lambda \ne 0$, and just for simplicity take $n = 0$. The equation to be solved is

$$D^2 y = \lambda z^2 y .$$

The indicial root is $r = 0$ with multiplicity two. The calculation of the solutions will be simpler if I note that the solution is even, hence odd terms in the series vanish. So we proceed:

$$\begin{aligned} y_1 &= 1 + c_2 z^2 + \quad c_4 z^4 + \cdots + \quad c_{2n} z^{2n} + \cdots \\ D^2 y_1 &= \quad 4c_2 z^2 + 16c_4 z^4 + \cdots + 4n^2 c_{2n} z^{2n} + \cdots \\ z^2 \lambda y_1 &= \qquad \lambda z^2 + \lambda c_2 z^4 + \cdots + \lambda c_{2n-2} z^{2n} + \cdots \end{aligned}$$

getting the recursion relations

$$c_0 = 1, \quad c_{2n} = \frac{\lambda}{4n^2} \cdot c_{2n-2} \,.$$

This defines an entire function analytically varying with λ. For the second solution we proceed:

$$\begin{aligned} y_2 &= w_1 + y_1 \log x \\ Dy_2 &= Dw_1 + Dy_1 \log x + y_1 \\ D^2 y_2 &= D^2 w_1 + D^2 y_1 \log x + 2Dy_1 \\ \lambda z^2 y_2 &= \lambda z^2 w_1 + \lambda z^2 y_1 \log z \\ (D^2 - \lambda z^2) y_2 &= (D^2 - \lambda z^2) w_1 + 2Dy_1 \,, \end{aligned}$$

and then solve for w_1.

In all cases—i.e. all n—there is just a one-dimensional family of solutions that are not singular at 0, defining the usual Bessel functions.

Near infinity the equation looks more or less like the equation with constant coefficients

(3.4) $$\varphi''(r) = \lambda \varphi(r) \,.$$

The solutions of the original equation possess solutions whose *asymptotic behaviour* is suggested by the approximating equation (3.4).

Justification of previous remarks comes down to some relatively simple algebra. Let $\mathcal{A}_r$ be the space spanned by germs $z^r f(z) \log^n z$. How does D act on $\mathcal{A}^r/\mathcal{A}^{r+1}$? Let A be a differential operator, $\mathcal{L} = \log z$, the indicial polynomial $\alpha = \alpha(r)$ of the associated Euler operator

$$A_0 = a_k(0) D^k + a_{k-1}(0) D^{k-1} + \cdots + a_0(0) \,.$$

Let $\overline{A}$ be the graded operator. Then

$$D z^r \mathcal{L}^n = r z^r \mathcal{L}^n + n z^r \mathcal{L}^{n-1}$$

leading to

$$\overline{A}\, z^r \mathcal{L}^n = z^r \Big(\alpha(r) \mathcal{L}^n + n\alpha'(r) \mathcal{L}^{n-1} + n(n-1) \frac{\alpha''(r)}{2} \mathcal{L}^{n-2} + \cdots \Big) \,,$$

which explains the traditional formulas.

4. Dietrich's algorithm

There are a number of algorithms in the literature that will decide, at least in principle, whether or not an equation

$$Dy = \frac{1}{z^p} \cdot \alpha(z)y \quad (\alpha(z) \text{ holomorphic at } 0,\, p \ge 1)$$

has a regular singularity at the origin. One of the early papers in the subject, that of Horn, proposes such an algorithm although his explanation is somewhat obscure and I am not aware that anyone has made it into something practical. The next important step seems to have been in [Moser:1960], who made a major contribution to the problem by proving that there exists a series of relatively elementary steps by which the reduction to simple form could be made. It does not seem impossible to me that one could deduce a practical algorithm from Moser's argument, but here too there are difficulties in deciphering part of what he wrote. In addition, Moser's method would seem to require finding some version of the Jordan forms of matrices. There are algorithms known that will do that, but this is not an elementary problem. (Of course it is impractical to work with matrices over $\mathbb{R}$ or $\mathbb{C}$, since in this business exact arithmetic is required. So what we are looking at here are rational Jordan forms.)

As we have seen, Jurkat's criterion for regularity (Theorem 2.15) was made reasonably practical in [Jurkat-Lutz:1971], and this was improved slightly by [Dietrich:1978(b)]. However, what is apparently the most efficient algorithm of all was laid out in [Dietrich:1978(a)]. Curiously, it remains neglected, and has had no subsequent history that I know about. It is Dietrich's algorithm that I wish to explain here. One problem that Dietrich does not address is motivation, and—at least in this current version—I shall not address it either. I have merely expanded Dietrich's own explanation of what is going on, which is accurate if rather succinct.

More precisely, I follow §§5–6 of [Dietrich:1968(a)]. But this is based on an idea originally due to Moser, which I recall.

MOSER'S REDUCTIONS. Suppose we are considering the $n \times n$ differential equation

$$Dy = \alpha(z)y = \frac{1}{z^p}\,(\alpha_0 + \alpha_1 z + \alpha_2 z + \cdots)\alpha(z)y\,,$$

with

$$\alpha_0 \ne 0\,.$$

Moser says that the equation is **reducible** if there exists a meromorphic matrix $T(z)$ such that

$$\beta = T^{-1}\alpha T + T^{-1}\cdot DT = \frac{1}{z^p}\,(\beta_0 + \beta_1 z + \cdots)$$

with

$$0 \le \operatorname{rank} \beta_0 < \operatorname{rank} \alpha_0\,.$$

Equivalent is the condition that for Moser's invariant

$$m(\beta) < m(\alpha)\,.$$

The condition will be true in particular if the order of the pole of β is less than that of α, since then $\beta_0 = 0$.

Moser associates also to α the minimum possible value $\mu(\alpha)$ of the values of $m(\beta)$ as β ranges over meromorphically equivalent matrices. He calls A reducible if $\mu(\alpha) < m(\alpha)$. He discovered an elegant condition for reducibility. Suppose $p > 0$, and let r be the rank of α_0. Define the polynomial

$$\mathfrak{P}_\alpha(\lambda) = z^r \det(\lambda + \alpha_0/z + \alpha_1)_{z=0}$$

The following is due to Moser.

4.1. Proposition. *Suppose $p > 0$. The following are equivalent:*

(*a*) *the system is reducible;*
(*b*) *the polynomial* $\mathfrak{P}_\alpha(\lambda)$ *vanishes identically;*
(*c*) *there exist a holomorphic matrix* $T(z) = T_0 + T_1 z$ *and a diagonal matrix* $\Lambda = \mathrm{diag}(0, \ldots, 0, 1, \ldots, 1)$ *such that* $T = (T_0 + T_1 z) z^\Lambda$ *effects a reduction of* α.

We know that if A is reducible, then according to Proposition 2.12 we can effect the reduction by a matrix of the form $P(z)\, z^\Lambda$. Moser made this more precise.

4.2. Corollary. *If* α_0 *is non-singular or semi-simple, the system is not reducible.*

Proof. By conjugation of α_0 to Jordan form.

In principle, this tells us how to decide whether the system has a regular singularity, First of all, if $p = 0$ it is simple. Otherwise, compute $\mathfrak{P}_\alpha(\lambda)$. If it does not vanish, the system is not reducible and does not have a regular singularity. If it does vanish, look for one of Moser's simple reductions. The original system is reducible if and only if this new one is, and we are faced with a problem onee degree simpler. There is a definite bound on the number of steps we shall require, since each step reduces the rank of α_0 by at least one.

As I have remarked, Moser's argument suggests vaguely an algorithm to find explicitly a series of simple eductions, but details are obscure. Dietrich's contribution was to offer a very elementary and explicit process to find Moser's reductions. It involves only the usual matrix elimination. In addition, Dietrich improves Moser's result by finding a reducing matrix of the form

$$T(z) = T_0 \cdot \mathrm{diag}(1, \ldots, z)\,.$$

Furthermore, the matrix T Dietrich finds is shown by him to be in some sense optimal.

REMARKS ON COLUMN AND ROW OPERATIONS. Dietrich's computations repeat a certain simple basic patter, They apply column or row operations as in Gaussian elmination, and then apply an opposite operation to make it into a conjugation. For example, the simplest row operation will *subtract* from a row some multiple of a *previous, i.e. higher* row. This will amount to multiplying on the left by a lower triangular matrix L. Then one multiplies on the right by L^{-1}, which *adds* a multiple of a *later* column to a column:

$$\begin{bmatrix} 1 & \circ \\ -x & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} r_1 \\ -x \cdot r_1 + r_2 \end{bmatrix}, \quad \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} 1 & \circ \\ x & 1 \end{bmatrix} = \begin{bmatrix} c_1 + x c_2 & c_2 \end{bmatrix}$$

These conjugations must preserve a certain structure in the matrix, and the whole subtlety of Dietrich's paper is (I believe) to manage to do this.

NORMALIZING A_0. The first steps in Dietrich's algorithm conjugate $A(z)$ so that its leading terms α_0 and α_1 possess a special form. First it conjugates α_0 into a matrix that looks like

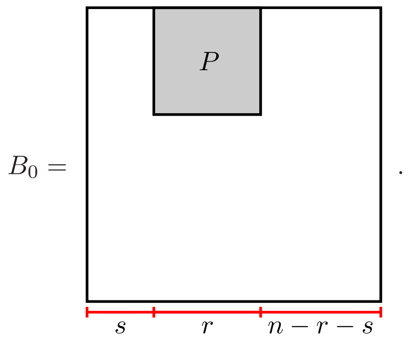
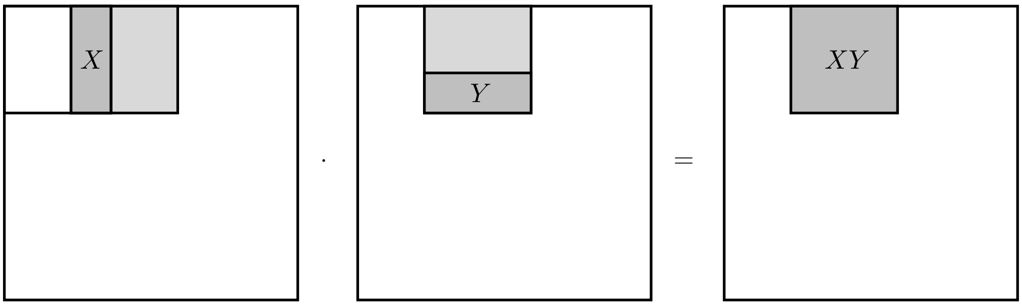


Figure 1

In this and subsequent figures, blank space represents zeroes. Here P is a non-singular $r \times r$ matrix, and r is hence the rank of α_0. Let the width of the first column be s. It is specified also that $s \le r$.

4.3. Proposition. *The excess $r-s$ is the rank of α_0^2.*

Proof. The following figure shows how R_0^2 is calculated.



Here, X and Y are both of maximal rank $r-s$, and the (i,j) entries of XY are the dot products of rows of X and columns of Y. Row reduction applied to X and and column reduction applied to Y will give equations

$$X = L \begin{bmatrix} I \\ 0 \end{bmatrix}, \quad Y = [\,I \quad 0\,]\,R, \quad XY = L \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} R\,,$$

which implies that their product is also of rank $r-s$.

It is instructive to see how this goes when α_0 is in Jordan canonical form, but I'll leave that as an exercise. According to Proposition 4.3 it is at least easy to see that s is the number of nilpotent blocks in it of size 2×2 and larger, while $n-r-s$ is the number of 1×1 blocks with a single 0. Although this structure is related to Jordan form, it is not necessary to find the Jordan form in order to perform the conjugation.

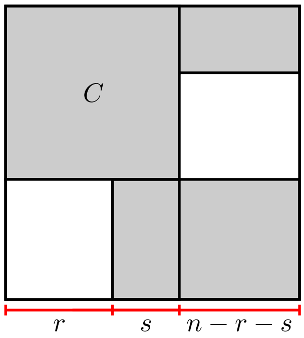
• Apply ordinary row reduction to get r linearly independent rows at the top and all 0-rows at the top. Say we do this through multiplication by L. Then multiply on the right by L^{-1} to calculate the conjugate. This effects column operations and does not change the bottom rows of zeroes. (In practice, we don't even have to look at them.)

• Apply the simplest column operations to find the crudest reduced form. These amount to subtracting multiples of earlier columns from columns. It gives us r non-zero columns that will be linearly independent. There may also be several 0-columns. Apply the corresponding inverse operations on the left, which amount to subtracting subsequent rows from rows. They don't modify the 0-columns or the bottom 0-rows.

• If a 0-column appears among the first r columns, swap it all the way to left. Swap the remainder of the 0-columns all the way to the right. There is a simple way to do this sort of thing—suppose we are given an ordered list of the 0-columns in a matrix, that is to say of the column index. Run through this list in order. If the i-th column in this list is not the i-th column of the matrix (which will not be a 0-column), swap the two columns. Next, apply the inverse operations on the left. Again, these operations will not change the bottom 0-rows.

In a moment, we shall find useful:

4.4. Lemma. *A matrix commutes with the matrix in Figure 1 if and only if it is of the form*



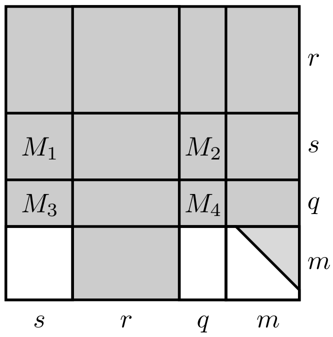
with

$$C\begin{bmatrix}0 & P\\ 0 & 0\end{bmatrix} = \begin{bmatrix}0 & P\\ 0 & 0\end{bmatrix} C\,.$$

I remind you that blank space denotes zeroes.

Proof. Straighforward computation.

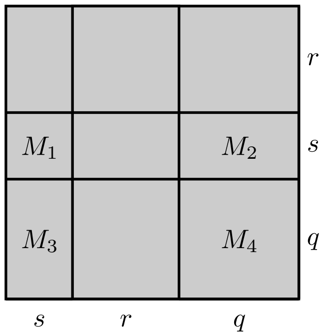
NORMALIZING A_1. This is more complicated. The basic result is that we can find a matrix that commutes with α_0 (now assumed to be in the form displayed in Figure 1) and conjugates α_1 to a matrix of the form



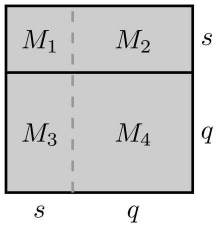
In the figure, m is just $n - r - s - q$. We further impose the condition:

> **(D)** *If ρ is the rank of*
>
> $$[\,M_1 \quad M_2\,]$$
>
> *then the rank of*
>
> **(4.5)** $$\begin{bmatrix}M_1 & M_2\\ M_3 & M_4\end{bmatrix}$$
>
> *is equal to $q + \rho$.*

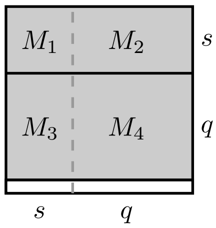
I.e. there is no condition on the rank of $[\,M_1 \quad M_2\,]$, but given this matrix the matrix (4.5) has the maximum possible rank. This time, too, we shall find this form by applying row and column operations. We start with the original α_1, about which we assume we know nothing, and for which we take $m = 0$:



As long as condition (D) is not satisfied, we do some row and column operations, increment m and decrement q. In doing this, we keep applying the same routine to the matrix made up of blocks of the M_i alone. The basic routine thus looks at a matrix



and, if the corresponding condition (D) does not hold, makes it



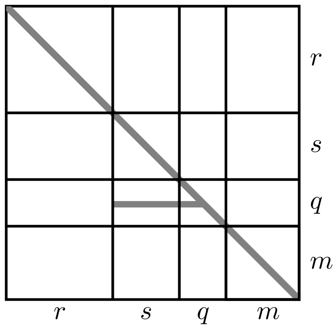
We then apply the same operations to the embedded matrix. Repeat.

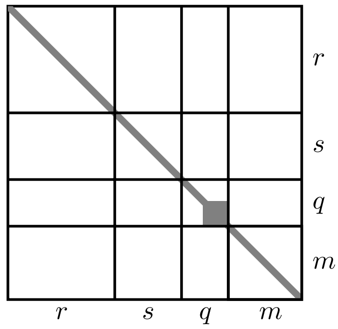
We need to be sure that the conjugation will commute with α_0, or in other words that the row-reducing matrix is of the form in Lemma 4.4. Just for simplicity, we first apply what I call simple row operations, which amount to subtracting multiples of previous rows from the current row. Thus at every point we have (in principle) a lower triangular matrix L with

$$L \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} A' \\ B' \end{bmatrix}$$

(so to speak). If a 0-row appears in B', the corresponding row of L records it as a linear combination of previous rows of $[A : B]$.

We then apply the corresponding row operation to the full matrix. After this, apply the corresponding column operations to make a conjugate. These column operations do not affect M_1 or M_3, and operate on columns of M_2 and M_4 by subtracting subsequent columns of themselves. Then swap that row to the bottom, and again the corresponding column operation swaps columns of M_2 and M_4. (It is this compatibility that I find remarkable about Dietrich's paper. The whole process is much like putting together a very intricate jig-saw puzzle. I do not understand how he arrived at such a complicated thing.)





REDUCTION IN STEPS. The main result of Dietrich is that once we have found α satisfying the conditions on α_0 and α_1, we can tell whether Moser's condition for reducibility holds.

4.6. Lemma. (Dietrich) *Suppose α_0 and α_1 to be in the form described previously. Set Λ to be the diagonal matrix $(0 \times (r+q), 1 \times (n-r-q))$,*

$$M = \begin{array}{cccc|c} I & & & & r \\ & & I & & s \\ & I & & & q \\ & & & I & m \\ \hline r & q & s & m & \end{array}$$

and $T = Mz^{\Lambda}$. The following are equivalent:

(a) $\mathfrak{P}_\alpha(\lambda) \equiv 0$;
(b) the rank of $[\, M_1 \;\; M_2 \,]$ is less than s;
(c) the matrix T implements a reduction;
(d) α is reducible.

Also following an idea of Moser, Dietrich describes an algorithm that attempts successively to decrease $m(A(z))$ by meromorphic equivalences until it cannot be made any smaller. At that point, either $m(\alpha) \leq 1$ and the equation is simple, or $m(\alpha) > 1$ and the equation does not have a regular singularity.

In each step, if the system is not yet simple, we carry out the conjugations outlined above. We are then given the matrix $\alpha(z)$ with α_0 and α_1 in Dietrich's form. If the rank of $[\, M_1 \;\; M_2 \,]$ is less than s, we define T as in the Lemma and replace $\alpha(z)$ by

$$\alpha_T(z) = z^{-\Lambda} M^{-1} \alpha M z^{\Lambda} - z^{-\Lambda} M^{-1} \cdot M \Lambda z^{\Lambda} = z^{-\Lambda} M^{-1} \alpha M z^{\Lambda} - \Lambda \,.$$

for which Moser's invariant will be smaller.

Remark. One great virtue of Dietrich's algorithm is that it preserves rationality. So that if one starts with a matrix α whose entries are rational numbers, the whole process will be carried out in rational arithmetic.

5. Examples

In this section I'll look at some examples, mostly cribbed from other sources.

2×2. Consider the equation

$$Dy = \alpha(z)y = \frac{1}{z^p} \cdot \big(\alpha_0 + \alpha_1 z + \alpha_2 z^2 + O(z^3)\big)$$

in which α is 2×2, $\alpha_0 \neq 0$, and $p > 0$. The first step is to find which of the three possible forms of Dietrich we are dealing with:

$$\begin{gathered} [\,P\,] \\ \begin{bmatrix} P & \circ \\ \circ & \circ \end{bmatrix} \\ \begin{bmatrix} \circ & P \\ \circ & \circ \end{bmatrix}. \end{gathered}$$

According to Corollary 4.2, it is only the last in which α is reducible. We may therefore assume that

$$\alpha_0 = \begin{bmatrix} \circ & b_0 \\ \circ & \circ \end{bmatrix} \quad (b_0 \neq 0)\,.$$

Set for each i

$$\alpha_i = \begin{bmatrix} a_i & b_i \\ c_i & d_i \end{bmatrix}.$$

Then

$$\mathfrak{P}_\alpha(\lambda) = z \cdot \det \begin{bmatrix} \lambda + a_1 & b_0/z + b_1 \\ c_1 & \lambda + d_1 \end{bmatrix}_{z=0} = -c_1 b_0\,,$$

so that according to Moser's criterion the system is reducible if and only if

$$\alpha_1 = \begin{bmatrix} a_1 & b_1 \\ \circ & d_1 \end{bmatrix}.$$

This agrees with Dietrich's criterion. To apply Dietrich's procedure, set

$$T_0 = I, \quad \Lambda = \begin{bmatrix} \circ & \circ \\ \circ & 1 \end{bmatrix},$$

and then we get

$$\begin{aligned} \alpha_T &= \frac{1}{z^p} \cdot \begin{bmatrix} 1 & \circ \\ \circ & 1/z \end{bmatrix} \left(\begin{bmatrix} \circ & b_0 \\ \circ & \circ \end{bmatrix} + \begin{bmatrix} a_1 & b_1 \\ \circ & d_1 \end{bmatrix} z + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} z^2 + \cdots \right) \begin{bmatrix} 1 & \circ \\ \circ & z \end{bmatrix} - \begin{bmatrix} \circ & \circ \\ \circ & 1 \end{bmatrix} \\ &= \frac{1}{z^p} \cdot \left(\begin{bmatrix} a_1 & b_0 \\ c_2 & d_1 \end{bmatrix} z + \begin{bmatrix} a_2 & b_1 \\ c_3 & d_2 \end{bmatrix} z^2 + \cdots \right) - \begin{bmatrix} \circ & \circ \\ \circ & 1 \end{bmatrix} \\ &= \frac{1}{z^{p-1}} \cdot \left(\begin{bmatrix} a_1 & b_0 \\ c_2 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_1 \\ c_3 & d_2 \end{bmatrix} z + \cdots \right) - \begin{bmatrix} \circ & \circ \\ \circ & 1 \end{bmatrix} \end{aligned}$$

If $p = 1$ this will be simple. Otherwise we loop.

To be cont'd.

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