Introduction to admissible representations of p-adic groups

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Chapter I. Analysis on profinite groups

All $\mathfrak{p}$-adic spaces have a topology in which every point has a countable basis of compact open neighbourhoods. Analysis on such spaces is essentially algebra, and in particular the theory of Haar measures on $\mathfrak{p}$-adic groups is elementary. In order to emphasize this, I shall introduce such spaces in purely combinatorial terms.

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SUMMARY. I define an LP (for **L**ocally **P**rofinite) space to be a locally compact Hausdorff topological space in which every point has a neighbourhood basis of compact open subsets. The principal result of the first part is that any locally closed subset of the set of $\mathfrak{k}$-rational points on an algebraic variety defined over a $\mathfrak{p}$-adic field $\mathfrak{k}$ is an LP space. This discussion is self-contained if mildly unorthodox in its avoidance of the strong form of the Axiom of Choice.

The second part is about basic analysis on such spaces. To emphasize the elementary nature of things, I use a somewhat arbitrary coefficient field $\boldsymbol{C}$ of characteristic 0, instead of $\mathbb{C}$, where possible. This will be true throughout all of these noteds, except for places where some property of $\mathbb{C}$ is necessary—for example, when having to deal with convergence or when it is important that something is positive.

I begin by discussing the Schwartz spaces $C_c^\infty(X, V)$ of an LP space X, when V is a vector space over $\boldsymbol{C}$. One important if easy result is the excision theorem (Proposition I.5.3): *If Y is a closed subset of the* LP *space X and V is a vector space over $\boldsymbol{C}$, then the natural maps make an exact sequence*

$$0 \to C_c^\infty(X - Y, V) \to C_c^\infty(X, V) \to C_c^\infty(Y, V) \to 0 \ .$$

Integration of Schwartz functions on an LP group against a Haar measure reduces directly to summation. On homogeneous spaces the situation is a bit more complicated. If G is an arbitrary locally compact group and H a closed subgroup, there is in general no G-invariant measure on $H\backslash G$, which is to say there is no linear functional from $C_c^\infty(H\backslash G)$ to $\boldsymbol{C}$ invariant under right translation by elements of G.

Instead of functions one integrates **one-densities** of compact support, which are linear functions on the space of smooth functions on $H\backslash G$. What happens here ought to be familiar, since on oriented real manifolds one integrates differential forms, not functions. If δ_H, δ_G are the modulus characters of H, G and

$$\delta_{H\backslash G}(h) = \delta_H(h)/\delta_G(h)$$

then continuous one-densities on $H\backslash G$ may be identified, but not canonically, with continuous functions f on G such that $f(hg) = \delta_{H\backslash G}(h)f(g)$. The standard reference for this is [Weil:1965], but in the case of locally profinite groups this material is more elementary, and I shall explain it here. The most useful consequence for representation theory (Theorem I.9.3) is that *if G is an algebraic group defined over $\mathfrak{k}$ and H a closed subgroup, then the space of smooth one-densities on $H\backslash G$ may be identified with the space of smooth $\mathbb{C}$-valued functions on G such that*

$$f(hg) = \left|\det \mathrm{Ad}_{\mathfrak{h}\backslash\mathfrak{g}}(h)\right|^{-1} f(g)$$

for all h in H, g in G. We also have in these circumstances, upon choosing right-invariant measures on H and G, the integral formula:

$$\int_G f(g)\,d_r g = \int_{H\backslash G} \left(\int_H \left|\det \mathrm{Ad}_{\mathfrak{h}\backslash\mathfrak{g}}(h)\right| f(hx)\,d_r h \right) d_r\overline{x}\ .$$

Part I. Topology

1. The p-adic numbers

For an integer $m \neq 0$, let $\mathrm{ord}_p m$ be the exponent of p in its prime factorization and $|m|_p$ be the inverse of the p-factor itself. Thus $|p^k|_p = p^{-k}$ and has limit 0 as k goes to ∞. Set $|0|_p = 0$. For rational numbers define the p-norm

$$\left|\frac{m}{n}\right|_p = |m|_p/|n|_p\ .$$

This norm is multiplicative and satisfies the additive inequality

$$|x+y|_p \le \sup |x|_p, |y|_p\ .$$

The field $\mathbb{Q}_p$ of p-adic numbers is the completion of $\mathbb{Q}$ with respect to this norm. That is to say, the p-adic numbers are defined to be the set of Cauchy sequences (x_n) of rational numbers modulo a certain equivalence condition. A Cauchy sequence (x_n) is one satisfying the condition that

for any $\varepsilon > 0$ there exists N such that $|x_n - x_m|_p < \varepsilon$ for all m, $n > N$.

and two sequences (x_n), (y_n) are equivalent if

for any $\varepsilon > 0$ there exists N such that $|x_n - y_m|_p < \varepsilon$ for all m, $n > N$.

The ring of p-adic integers $\mathbb{Z}_p$ is defined to be the closure of $\mathbb{Z}$. Every non-trivial ideal in it is (p^r) for some r, so $\mathbb{Z}_p$ is a principal ideal domain.

I.1.1. Proposition. *The image in $\mathbb{Q}_p$ of the rational number m/n (m, n relatively prime) lies in $\mathbb{Z}_p$ if and only if* $\gcd(n,p) = 1$.

Proof. If $\gcd(n,p) = 1$, we can find an integer n_* such that $nn_* \equiv 1$ modulo p. Then $m/n = mn_*/nn_*$ and it suffices to prove the lemma when $n \equiv 1$ modulo p. But if $n = 1 + \ell p$ then in the ring of p-adic integers

$$\frac{1}{1+\ell p} = 1 - \ell p + \ell^2 p^2 - \cdots$$

This proves one half the Lemma.

On the other hand, suppose m/n to lie in $\mathbb{Z}_p$. Suppose that p divides n. Since $\gcd(m,n) = 1$, m is then relatively prime to p. The assumption implies that we can find some integer q such that $|m/n - q|_p > 0$. But

$$\frac{m}{n} - q = \frac{m - nq}{n} \,.$$

But $m - nq$ must be relatively prime to p, a contradiction.

The map from $\mathbb{Z}$ to $\mathbb{Z}/p^r$ induces a homomorphism from $\mathbb{Z}_p$ to $\mathbb{Z}/p^r$ as well. If $r \ge s$ there is a canonical projection from $\mathbb{Z}/p^r$ to $\mathbb{Z}/p^s$. This leads to:

I.1.2. Proposition. *The ring $\mathbb{Z}_p$ may be identified with the projective limit of the finite rings $\mathbb{Z}/p^r$.*

Proof. Any sequence of integers (x_n) such that $x_{n+1} \equiv x_n \bmod p^n$ is a Cauchy sequence in the p-adic norm, and the equivalence class of the sequence depends only on the x_n modulo p^n.

There is a very concrete way to represent p-adic numbers—every p-adic rational can be expressed uniquely as an infinite reduced sum

$$\sum c_i \, p^i$$

where only a finite number of the c_i with $i < 0$ are non-zero, and $0 \le c_i < p$ for all i. Indeed, this is perhaps the most straightforward way to define p-adic numbers. A key step in this specification, at least in algorithms, is the reduction of any series $\sum c_i p^i$ with c_i in $\mathbb{Z}$ to one with $0 \le c_i < p$ for all i.

There is one more way to characterize the p-adic numbers—in terms of **Witt vectors**. This is a very interesting matter, but it won't be relevant to what will be needed in these essays, and I won't say anything about it here.

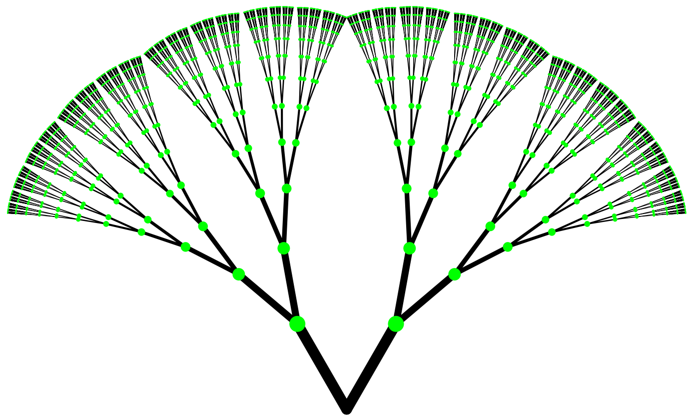
Another example of a topological group with a similar structure is the ring $\mathbb{A}_f$ of finite adèles. It is the **restricted product** of the fields $\mathbb{Q}_p$—i.e. the subset of all (x_p) in $\prod_p \mathbb{Q}_p$ for which all but a finite number of the x_p lie in $\mathbb{Z}_p$. This has as basis of neighbourhoods of 0 the products $\prod K_p$ where each K_p is open in $\mathbb{Q}_p$ and $K_p = \mathbb{Z}_p$ for all but a finite number of p. The additive subgroup $\prod \mathbb{Z}_p$ is the projective limit of finite quotients by open subgroups.

2. Trees and topologies

A p-adic integer can be identified with a sequence (x_n) of compatible integers in the finite rings $\mathbb{Z}/p^n$. We can make a graph with directed edges out of these data: there is one node in the graph for each pair (n, x) with $n > 0$ and x in $\mathbb{Z}/p^n$, and an edge from (n, x) to $(n-1, y)$ if $n \ge 2$ and $y \equiv x \mod p^{n-1}$. A p-adic integer then amounts to a one-way path of nodes in this graph, coming from infinity and terminating at one of the initial nodes in $\mathbb{Z}/p$. These observations should motivate the following discussion.

A **rooted tree** consists of (1) a set of **nodes**, (2) a designated **root node**, and (3) for every node other than the root an assignment of **immediate predecessor** node $\backslash Pred/(x)$, satisfying the condition that from any node x there exists a unique sequence $x_n = x$, $x_{n-1}, \ldots, x_0$ with each x_{i-1} the predecessor of x_i and x_0 the root node. A **predecessor** is defined inductively by the condition that it be either an immediate predecessor or a predecessor of an immediate predecessor.

A node x is an **immediate successor** of another node y if $y = \backslash Pred/(x)$, and a successor if linked by a chain of immediate successors.



A rooted tree is said to be **locally finite** if the number of immediate successors of every node is finite. A **chain** in the tree rooted at the node x is a finite sequence of nodes $x_0 = x, x_1, \ldots, x_n$ where each x_{i+1} is an immediate successor of x_i. It is said to have length n. A **branch** of a rooted tree is a sequence (finite or infinite) of nodes (x_i) where x_0 is the root and each x_{i+1} is an immediate successor of x_i, satisfying the condition that the sequence stops only at a node with no successors.

Locally finite rooted trees possess a recursive structure, since if T is any locally finite rooted tree and x is a node of T then the set of all successors of x in T, together with x itself, make up a locally finite tree with root x. Any subset of T that contains along with a node its predecessor will also be a locally finite tree with the same root as T.

I.2.1. Lemma. *In any locally finite tree, the number of chains of length n rooted at a given node is finite.*

Proof. That this is true for chains of length 1 is the definition of locally finite. The proof proceeds by induction on chain length.

I.2.2. Lemma. (König's Lemma) *In every locally finite rooted tree with an infinite number of nodes there exists an infinite branch.*

Proof. Let ρ be the root of the given tree, assumed to possess an infinite number of nodes. It follows from the previous result that there exist chains rooted at ρ of arbitrary length, and in particular that the set C_0 of all chains rooted at $x_0 = \rho$ is infinite. Since the number of immediate successors of x_0 is finite, the subset C_1 of chains in C_0 passing through some one of them, say x_1, is infinite. Similarly there must exist an infinite number among the chains in C_1 whose third nodes agree. By induction, we obtain for each n a sequence of sets of chains

$$C_0 \supseteq C_1 \supseteq \ldots \supseteq C_n \supseteq C_{n+1} \supseteq \ldots$$

where all the chains in C_n agree with each other in their first $n+1$ nodes, and agree in their first n nodes with the chains in C_{n-1}. By choosing x_n to be the common n-th node of the chains in C_n we assemble an infinite branch.

The reasoning here, although plausible, is specious or at least highly subtle, since disguised in it is a weak form of the Axiom of Choice. König's Lemma is one of those results that hovers on the edge of obviousness, like a dim star one cannot quite focus on directly. The degree to which it is not obvious becomes more apparent when one sees (in, for example, the enlightening discussion to be found in §2.3.4.3 of [Knuth:1973]) some of its immediate consequences.

Suppose T to be a locally finite rooted tree with root ρ. For every node x in T define Ω_x to be the set of all branches passing through x. König's Lemma guarantees that this is never empty. Let Ω_T be the set of all branches of T, which is the same as Ω_ρ. We can make a topological space out of Ω_T by defining as basis of open sets the sets Ω_x—if ω is any branch in the tree then the sets Ω_x for each of its nodes x define a basis of neighbourhoods of ω. Two distinct branches must eventually diverge, and therefore it is easy to see that this topology is Hausdorff. The next result shows that this topology is otherwise somewhat special.

I.2.3. Proposition. *If T is a locally finite rooted tree and x a node in T then Ω_x is closed as well as open in Ω_T.*

In other words, any point of Ω_T has a basis of neighbourhoods that are both open and closed.

Proof. Let $x_0 = \rho$, x_1, … , $x_n = x$ be the chain from ρ to x, and let $X = \{x_i\}$. Let Y be the union of all of the immediate successors of the x_i for $i < n$, except for x_{i+1}. In other words, y is in Y if it is an immediate successor of some x_i and is not in X. Since T is locally finite, Y is finite. Every branch in T that does not pass through x has to branch off at one of the x_i with $i < n$, and therefore the complement of Ω_x is the union of the Ω_y for y in Y.

I.2.4. Proposition. *If T is a locally finite rooted tree then the topological space Ω_T is compact.*

Proof. This amounts to the following assertion:

> *Suppose X to be a set of nodes of T such that the sets Ω_x (x in X) cover Ω_T. Then there exists a finite set of Ω_x (x in X) covering Ω_T.*

The assumption means that every branch in Ω_T lies in some Ω_x with x in X, or equivalently that every branch in Ω_T has a node in X. If X contains the root node, we are immediately through.

Otherwise, let $X_{\min}$ be the set of nodes in X that are minimal—that is to say, x lies in $X_{\min}$ if it lies in X and in the path from the root to x there are no elements of X other than x. It is clear that that the Ω_x for x in $X_{\min}$ cover Ω_T, since every branch has to have a first element in X.

It suffices to show that $X_{\min}$ is finite, since a path from x in X to the root must pass through a node in $X_{\min}$. Let Y be the set of nodes in T that are not in X and none of whose predecessors are in X. In particular, Y contains the root node of T. Any predecessor of a node in Y will also be in Y, so that Y itself is a rooted tree and we can apply König's Lemma to it. It is not possible for Y to contain an infinite branch, since any infinite branch in Y would also be a branch in T, and by assumption every branch in T must contain a node in X. König's Lemma tells us that Y must be finite. But since X does not contain the root node, every element of $X_{\min}$ is the successor of some node in Y, and since the tree is locally finite $X_{\min}$ must be finite.

Applying this to each of the rooted trees Ω_x:

I.2.5. Proposition. *Every point in the topological space Ω_T possesses a countable basis of compact open neighbourhoods.*

3. Locally profinite spaces

Suppose we are given a sequence of finite sets $X_0, X_1, \ldots$ and for each $n > 0$ a surjection $\pi_n\colon X_n \to X_{n-1}$. We can make a finite union of trees from these data by taking the nodes of our graph to be the points of the X_n and defining the predecessor of x in X_n to be $\pi_n(x)$. The branches of this tree are the infinite sequences (x_n) where $\pi_n(x_n) = x_{n-1}$, and for a given point x in X_n the set Ω_x consists of all sequences with $x_n = x$. Let X be the set of such sequences, and let Π_n be the canonical surjection $(x_n) \mapsto x_n$ from X to X_n. The topological space X is the **projective limit** of the given sequence of finite sets. According to Proposition I.2.4 it possesses a natural Hausdorff topology with respect to which it is compact. The point $x = (x_n)$ has as a basis of neighbourhoods the sets $\Omega_{x_n} = \Pi_n^{-1}(\Pi_n(x))$.

More generally, suppose Σ to be a directed set: Σ is ordered, and given any two α, β in Σ there exists $\gamma \ge \alpha, \beta$. Suppose given (a) for each α in Σ a finite set X_α and (b) for each $\alpha > \beta$ a map

$$p_{\beta,\alpha}\colon X_\alpha \to X_\beta \,.$$

The maps are required to be (a) surjective and (b) consistent, in the sense that

$$p_{\gamma,\beta} p_{\beta,\alpha} = p_{\gamma,\alpha}$$

whenever $\alpha > \beta > \gamma$. The projective limit of this system is defined to be the subset of all (x_α) in $\prod X_\alpha$ such that $p_{\beta,\alpha} x_\alpha = x_\beta$. The product has a natural topology with respect to which it is compact, and the projective limit is closed in it, hence also compact. Proving compactness of the product (Tychonov's Theorem) requires a stronger the Axiom of Choice, so that the sequentially profinite systems I have examined are definitely simpler.

A topological space X is said to be profinite if it is a projective limit of finite sets. It is said to be totally disconnected if every point has a basis of neighbourhoods that are both open and closed.

I.3.1. Theorem. *Suppose X to be a topological space. The following are equivalent:*

(a) it is profinite;
(b) it is Hausdorff, compact, and totally disconnected.

I'll not prove this result, because it will not be used subsequently, except to justify terminology.

I.3.2. Corollary. *If X is a Hausdorff topological space, the following are equivalent:*

(a) it is locally profinite;
(b) every point has a basis of compact, open sets.

The equivalence justifies applying the convenient term 'locally profinite' to spaces satisfying these conditions. I shall call them more succinctly LP spaces. A profinite space is the same as a compact LP space, and is also just a compact Hausdorff space in which every point has a basis of compact open sets. The spaces dealt with in earlier sections are those in which every point possesses a countable basis of compact open neighbourhoods. I'll call these **König spaces**.

I.3.3. Lemma. *Every covering of a compact* LP *space by compact open subsets possesses a disjoint refinement by compact open sets.*

Proof. Suppose X to be a profinite space and suppose given a covering by compact open subsets U_i, which we may assume to be a finite covering. The union of any of the U_i is compact, hence closed in X, so that the sets

$$U_{*,i} = U_i - \bigcup_{j<i} U_j$$

are also open, and also cover X.

I.3.4. Proposition. *If (X_n) is a sequence of König profinite spaces then the product $X = \prod X_n$ is also a König space.*

Proof. I recall that open sets in the product topology are of the form $\prod U_n$ with each U_n open in X_n and all but a finite number of $U_n = X_n$.

Let $\Pi_{n,m}$ be the canonical surjection from X_n to the finite set $X_{n,m}$. As a basis of open sets in the topology of X we therefore have the finite products $\prod_{n \le N} \Pi^{-1}_{n,m}(x_{n,m})$ with $x_{n,m}$ in $X_{n,m}$. In order to prove the Proposition we need to find a cofinal sequence of finite sets onto which X surjects. This can be the sequence

$$X_{1,1},\ X_{1,2} \times X_{2,1},\ X_{1,3} \times X_{2,2} \times X_{3,1}, \ldots$$

4. $\mathfrak{p}$-adic spaces

The ring of p-adic integers $\mathbb{Z}_p$ is the projective limit of the finite rings $\mathbb{Z}/p^n$. The ordinary integers $\mathbb{Z}$ may be embedded in $\mathbb{Z}_p$, since m may be identified with the sequence $(m \bmod p^n)$. If q is an integer relatively prime to p then for every $n > 0$ there exists a multiplicative inverse of q modulo p^n, so that all rational numbers m/q with q prime to p may also be identified with elements of $\mathbb{Z}_p$. More generally, the p-adic integers with multiplicative inverses in $\mathbb{Z}_p$ are precisely those whose image in $\mathbb{Z}/p$ does not vanish. A p-adic rational number other than 0 can be identified with a unique expression m/p^k where m is a unit in $\mathbb{Z}_p$. These make up the field $\mathbb{Q}_p$.

Generalizing this construction, I define **$\mathfrak{p}$-adic field** to be a field $\mathfrak{k}$ containing a ring $\mathfrak{o}$ and an ideal $\mathfrak{p}$ of $\mathfrak{o}$ satisfying these conditions:

(a) $\mathfrak{o}/\mathfrak{p}$ is a finite field, say of q elements;
(b) the canonical projections from $\mathfrak{o}$ to $\mathfrak{o}/\mathfrak{p}^n$ and from $\mathfrak{o}/\mathfrak{p}^{n+1}$ to $\mathfrak{o}/\mathfrak{p}^n$ identify the ring $\mathfrak{o}$ with the projective limit of the quotients $\mathfrak{o}/\mathfrak{p}^n$;
(c) if ϖ lies in $\mathfrak{p} - \mathfrak{p}^2$, then every non-zero element of $\mathfrak{k}$ may be expressed uniquely as $u\varpi^n$ where u lies in $\mathfrak{o} - \mathfrak{p}$.

If x is an element of $\mathfrak{o} - \mathfrak{p}$ there exists an element y of $\mathfrak{o} - \mathfrak{p}$ such that $xy \equiv 1$ modulo $\mathfrak{p}$. If then $m = xy - 1$ the series

$$u = 1 - m + m^2 - m^3 + \cdots$$

converges to an element of $\mathfrak{o}$ because of condition (b), since modulo any power of $\mathfrak{p}$ the series terminates. The limit will be a multiplicative inverse of xy, so that yu will be a multiplicative inverse of x. Hence every element of $\mathfrak{o} - \mathfrak{p}$ is a unit of $\mathfrak{o}$. The ideal $\mathfrak{p}$ is the only prime ideal of $\mathfrak{o}$ other than (0), and every non-zero ideal of $\mathfrak{o}$ is a power of $\mathfrak{p}$. If ϖ lies in $\mathfrak{p} - \mathfrak{p}^2$ then multiplication by ϖ^n induces a bijection of $\mathfrak{o}/\mathfrak{p}$ with $\mathfrak{p}^n/\mathfrak{p}^{n+1}$, so that $\mathfrak{o}/\mathfrak{p}^n$ is a finite ring of cardinality q^n.

The field $\mathbb{Q}_p$ of p-adic integers is a $\mathfrak{p}$-adic field with $\mathfrak{o} = \mathbb{Z}_p$ and $\mathfrak{p} = (p)$. Every finite algebraic extension of a $\mathfrak{p}$-adic field is a $\mathfrak{p}$-adic field. The completion of any algebraic number field of finite degree with respect to any non-zero prime ideal of its ring of algebraic integers is a $\mathfrak{p}$-adic field, and is a finite extension of $\mathbb{Q}_p$. The field of power series in x with coefficients in $\mathbb{F}_q$ and a finite number of negative powers of x

$$c_{-n}x^{-n} + c_{-(n-1)}x^{-(n-1)} + \cdots$$

make up the quotient field $\mathbb{F}_q((x))$ of the ring of formal power series $\mathbb{F}_q[[x]]$. It is a $\mathfrak{p}$-adic field with $\mathfrak{p} = (x)$. The completion of any $\mathbb{F}_q$-rational local ring on a non-singular algebraic curve over $\mathbb{F}_q$ is isomorphic to it. Conversely, any $\mathfrak{p}$-adic field is either some $\mathbb{F}_q((x))$ or a finite algebraic extension of $\mathbb{Q}_p$.

Throughout this book we will work with a fixed $\mathfrak{p}$-adic field $(\mathfrak{k}, \mathfrak{o}, \mathfrak{p})$, where $\mathfrak{o}/\mathfrak{p} \cong \mathbb{F}_q$. If $x = u\varpi^n$ with u a unit in $\mathfrak{o}$ then its norm $|x| = |x|_\mathfrak{p}$ is defined to be

$$|x| = q^{-n} \ .$$

Thus when $n \geq 0$ the index of the ideal (x) in $\mathfrak{o}$ is $|\ x\ |^{-1}$.

The ring $\mathfrak{o}$ is a profinite space. The ideals $\mathfrak{p}^n$ form a basis of neighbourhoods of 0, and any locally closed subspace of a finite dimensional vector space over $\mathfrak{k}$ will be a König space. Hence:

I.4.1. Theorem. *Any locally closed subset of the set of $\mathfrak{k}$-rational points on an algebraic variety defined over $\mathfrak{k}$ is a König space.*

Now let F be a global field. The ring $\mathbb{A} = \mathbb{A}_{F,f}$ of finite adèles is the **restricted product** of the non-Archimedian completions F_v—the subset of (x_v) in $\prod F_v$ for which all but a finite number of the x_v lie in the integer ring $\mathfrak{o}_v$. The product $\prod \mathfrak{o}_v$ is an open neighbourhood of 0 in $\mathbb{A}$, and according to Proposition I.3.4:

I.4.2. Theorem. *The ring of finite adèles of F is a König space.*

Part II. Analysis

5. Smooth functions on an LP space

For the moment, fix a coefficient field $\boldsymbol{C}$ of characteristic 0. Much later it will be assumed to be algebraically closed, and even more specifically $\mathbb{C}$, but for the moment it might even be just $\mathbb{Q}$.

Suppose X to be a locally profinite space and V a vector space over $\boldsymbol{C}$. Define

$$\begin{aligned}
C(X, V) &= \text{space of continuous functions on } X \text{ with values in } V \\
C_c(X, V) &= \text{subspace of those with compact support} \\
C^\infty(X, V) &= \text{space of locally constant functions with values in } V \\
C_c^\infty(X, V) &= \text{subspace of those with compact support} \\
\mathcal{D}(X) &= \text{linear dual of } C_c^\infty(X) = C_c^\infty(X, \boldsymbol{C}) \ .
\end{aligned}$$

The functions in C^∞ are called **smooth**, and they are continuous. The elements of $\mathcal{D}$ are **distributions**.

For any open set U of X let $\mathfrak{char}_U$ be its characteristic function.

I.5.1. Lemma. *If Y is a closed subset of the locally profinite space X and f in $C_c^\infty(Y,V)$, then there exists a finite sum $\sum \mathfrak{char}_U \cdot v_U$ whose restriction to Y is f.*

Proof. Suppose f to be in $C_c^\infty(Y)$. Because of Proposition I.2.5, we can find a covering $\{U_i\}$ of the support of f by compact open subsets U_i of X with the property that f is constant on each $U_i \cap Y$. Apply Lemma I.3.3 to get a refinement by disjoint sets $U_{*,i}$.

On each $U_{*,i} \cap Y$ the function f takes a constant value v_i. The linear combination $\sum \mathfrak{char}_{U_{*,i}} \cdot v_i$ then lies in $C_c^\infty(X,V)$ and has image f in $C_c^\infty(Y,V)$.

If $Y = X$:

I.5.2. Corollary. *Every function in $C_c^\infty(X,V)$ is a finite sum of functions $\mathfrak{char}_U \cdot v$ with v in V.*

I.5.3. Proposition. *(Excision Lemma) If Y is a closed subset of the locally profinite space X, then the natural maps make an exact sequence*

$$0 \to C_c^\infty(X - Y, V) \to C_c^\infty(X,V) \to C_c^\infty(Y,V) \to 0 \,.$$

Proof. The only interesting point is the final surjectivity, which follows from Lemma I.5.1.

6. Locally profinite groups

I define an LP **group** to be a Hausdorff topological group possessing a basis of neighbourhoods of 1 that are compact open subgroups.

In $\mathrm{GL}_n(\mathfrak{k})$ the subgroup $\mathrm{GL}_n(\mathfrak{o})$ of invertible matrices with coefficients in $\mathfrak{o}$ is a compact open subgroup, with the congruence subgroups

$$\mathrm{GL}_n(\mathfrak{p}^m) = \left\{ g \in \mathrm{GL}_n(\mathfrak{o}) \,\middle|\, g \equiv I \ (\mathfrak{p}^m) \right\}$$

forming a basis of neighbourhoods of the identity. The group $\mathrm{GL}_n(\mathfrak{o})$ may be identified with the projective limit of the groups $\mathrm{GL}_n(\mathfrak{o}/\mathfrak{p}^n)$. Hence the group of $\mathfrak{k}$-rational points on any closed subgroup of $\mathrm{GL}_n(\mathfrak{k})$, and in particular the group of $\mathfrak{k}$-rational points on any affine algebraic group defined over $\mathfrak{k}$, is a locally profinite group.

SMOOTH FUNCTIONS ON AN LP GROUP.

I.6.1. Proposition. *Suppose X to be any locally profinite space on which the locally profinite group G acts continuously. Then for every compact open set Ω in X there exists a compact open subgroup K of G such that $K\Omega = \Omega$.*

Proof. Because the action of G is continuous, left multiplication in G is continuous. For any point x of Ω there exists a compact subgroup K and a neighbourhood U of x such that $KU \subseteq \Omega$. Since Ω is compact, Ω will be covered by a finite number of these, say by the $K_i U_i$. Then Ω will be stable with respect to the intersection of the K_i.

If G is a locally profinite group, it acts by the right- and left-regular representations on $C^\infty(G)$, $C_c^\infty(G)$, and $\mathcal{D}(G)$ according to the recipes

$$\begin{aligned} L_g f(x) &= f(g^{-1}x) \\ R_g f(x) &= f(xg) \\ \langle L_g \Phi, f \rangle &= \langle \Phi, L_{g^{-1}} f \rangle \\ \langle R_g \Phi, f \rangle &= \langle \Phi, R_{g^{-1}} f \rangle \,. \end{aligned}$$

We have

$$L_{g_1 g_2} = L_{g_1} L_{g_2}, \quad R_{g_1 g_2} = R_{g_1} R_{g_2} \,.$$

Define the subspace $C_u^\infty(G)$ to be that of smooth functions on G with values in $\mathbb{Q}$ that are left-invariant under multiplication by elements of some compact open subgroup. Because of Proposition I.6.1, it contains $C_c^\infty(G)$. A **smooth distribution** is one that is locally right-invariant under some open subgroup of G. **Uniform** smoothness for distribution means global right-invariance under some fixed open subgroup.

I.6.2. Lemma. *If G is a locally profinite group and f a function on G with values in V with compact support, the following are equivalent:*

(a) the function f lies in $C_c^\infty(G, V)$;
(b) there exists a compact open subgroup K of G such that $L_k f = f$ for all k in K;
(c) there exists a compact open subgroup K of G such that $R_k f = f$ for all k in K.

This is straightforward.

HAAR MEASURES. The theory of Haar measures on locally profinite groups is very simple. In practice, as will be explained in a moment, integrals are nearly always sums—even though occasionally infinite.

I.6.3. Proposition. *Let G be an* LP *group. Given a compact open subgroup K and a constant c_K in $\mathbb{Q}^\times$, there exists a unique right G-invariant measure μ on G such that*

$$\langle \mu, \mathfrak{char}_K \rangle = c_K \ .$$

Proof. First I assign measures to compact open subsets of G.

Suppose that the measure μ is known to exist. If K_* is a compact open subgroup contained in K then

$$\langle \mu, \mathfrak{char}_{K_*} \rangle = [K \colon K_*]^{-1} c_K$$

since K is the disjoint union of the $K_* x$ as x runs over representatives of $K_* \backslash K$, and $\langle \mu, \mathfrak{char}_{K_* x} \rangle = \langle \mu, \mathfrak{char}_{K_*} \rangle$. If K_* is an arbitrary compact open subgroup, then

$$\langle \mu, \mathfrak{char}_{K_*} \rangle = \frac{[K_* \colon K \cap K_*]}{[K \colon K \cap K_*]} \, c_K \ .$$

But knowing $\langle \mu, \mathfrak{char}_{K_*} \rangle$ for all compact open subgroups K_*, together with right G-invariance, determines $\langle \mu, f \rangle$ for any smooth function f of compact support, since f is a linear combination of $\mathfrak{char}_{K_* x}$ for some one K_* and a finite set of x in G. This argument when run backwards gives the recipe for constructing μ as a distribution.

To define it as a measure, one must evaluate $\langle \mu, f \rangle$ for any continuous function of compact support. But if f is a continuous function of compact support, there exist arbitrarily close functions in $C_c^\infty(G)$, with the same support, and one can define $\langle \mu, f \rangle$ as a limit, exactly as one defines Riemann sums.

I shall call such any $\mathbb{Q}$-distribution on G with positive c_K that is right G-invariant a (rational) **right Haar measure** on G, and if $d_r x$ is one write

$$\mathrm{meas}(U, d_r x) = \langle d_r x, \mathfrak{char}_U \rangle$$
$$\int_G f(x) \, d_r x = \langle d_r x, f \rangle \ .$$

I have said that, in practice, integration on G amounts to summation. Let's make this explicit.

I.6.4. Lemma. *Suppose (a) V is a vector space over $\mathbb{Q}$, (b) f lies in $C_c^\infty(G, V)$, and (c) K is a compact open subgroup of G. If f is left-invariant with respect to K, then*

$$\int_G f(x) \, d_r x = \mathrm{meas}(K) \sum_{K \backslash G} f(x) \, ,$$

and if it is right-invariant then

$$\int_G f(x)\,d_r x = \mathrm{meas}(K) \sum_{G/K} \delta_G(g) f(x)\,,$$

Proof. Straightforward. For example:

$$\begin{aligned} \int_G f(x)\,d_r x &= \sum_{K\backslash G} \int_{Kg} f(x)\,d_r x \\ &= \mathrm{meas}(K) \sum_{K\backslash G} f(Kg)\,. \end{aligned}$$

Here is a frequently useful characterization:

I.6.5. Proposition. *Suppose G to be an* LP *group, assigned a right-invariant Haar measure $d_r g$. For f in $C_c^\infty(G)$,*

$$\int_G f(g)\,d_r g = 0$$

if and only f is a linear combination of functions of the form $R_g\varphi - \varphi$, with $\varphi \in C_c^\infty(G)$

Proof. One way is trivial. As for the other, suppose that

$$\int_G f(g)\,d_r g = 0\,.$$

If f is left-invariant under K, this means that

$$\sum_{K\backslash G} f(g_i) = 0\,,$$

if the support of f is the disjoint union of the Kg_i. Let $\mathfrak{char}_g$ be the characteristic function of Kg. Then $\mathfrak{char}_g = R_{g^{-1}}\mathfrak{char}_1$ and so

$$\begin{aligned} f &= \sum_i f(g_i)\mathfrak{char}_{g_i} \\ &= \Big(\sum_i f(g_i)\mathfrak{char}_{g_i}\Big) - \Big(\sum f(g_i)\Big)\mathfrak{char}_1 \\ &= \sum f(g_i)\big(R_{g_i^{-1}}\mathfrak{char}_1 - \mathfrak{char}_1\big)\,. \end{aligned}$$

THE MODULUS CHARACTER. If $d_r x$ is any right Haar measure on G then any left translation $L_g\,d_r x$ is also a right Haar measure, and must be therefore a scalar multiple of $d_r x$. In other words for each g in G there exists a scalar $\delta_G(g)$ such that

$$\int_{gU} d_r x = \delta_G(g) \int_U d_r x$$

for all compact open subsets U of G, or in brief

$$d_r gx = \delta_G(g)\,d_r x\,.$$

The constant $\delta_G(g)$ is independent of the right Haar measure chosen, since all others are just scalar multiples of it. It depends multiplicatively on g:

$$\delta_G(g_1 g_2) = \delta_G(g_1)\delta_G(g_2) \ .$$

This multiplicative character of G with values in the positive rational numbers is called the **modulus character** of G. If $d_r x$ is a right Haar measure on G then $\delta_G(x)^{-1} d_r x$ is a left Haar measure.

The modulus character also clearly characterizes how conjugation affects measures, since

$$\mathrm{meas}(g\,U g^{-1}) = \delta_G(g)\,\mathrm{meas}(U) \ .$$

The group is called **unimodular** if δ_G is trivial. If K is a compact subgroup of G then the image of any character of K with values in the positive rational numbers has to be a torsion group, hence trivial. Therefore every compact group is unimodular.

SMOOTH DISTRIBUTIONS. Assume for the moment a right-invariant Haar measure $d_r g$ chosen on G. If φ is a smooth function on G with values in V then the formula

$$\langle D_\varphi, f\rangle = \int_G \varphi(x) f(x)\, d_r x \quad \big(f \in C_c^\infty(G, \mathbb{Q})\big)$$

defines a distribution D_φ on G with values in V, and

$$R_g D_\varphi = D_{R_g \varphi}, \quad L_g D_\varphi = \delta_G(g)^{-1} D_{L_g \varphi} \ .$$

If φ is right-invariant under an open group K, then D_φ will also be right-invariant under K.

Conversely, suppose D to be a smooth distribution, which will be locally right-invariant by some compact open subgroup K. We can associate to it a function value at g in G by the formula

$$\varphi(g) = \frac{\langle D, \mathfrak{char}_{gK}\rangle}{\mathrm{meas}(gK)} \ .$$

Local right-invariance of D implies immediately that

$$\langle D, \mathfrak{char}_{gK}\rangle = \langle D, \mathfrak{char}_{gK_*}\rangle \left(\frac{\mathrm{meas}(gK)}{\mathrm{meas}(gK_*)}\right) \ .$$

for any compact open subgroup K_* of K, which means that the definition of $\varphi(g)$ is independent of the choice of K with respect to which D is right-invariant. It is also straightforward to see that D is then the same as D_φ. We have proved:

I.6.6. Proposition. *Suppose V to be a vector space over $\mathbb{Q}$. Given a right-invariant Haar measure on G, the correspondence $\varphi \mapsto D_\varphi$ is a right-G-equivariant isomorphism between the space of smooth functions with values in V and that of V-valued smooth distributions on G.*

There is one large class of distributions we shall often use. Suppose H to be any compact subgroup of G (not necessarily open). Then associated to H is the distribution μ_H (not necessarily smooth) defined by the formula

$$\langle \mu_H, f\rangle = \frac{1}{[H\colon H\cap K]} \sum_{H/H\cap K} f(h)$$

if f in $C_c^\infty(G)$ is right-invariant under K. In effect it evaluates the average value of f on H.

7. Smooth vector bundles

In this section, I'll explain a version of equivariant vector bundles on a quotient $X = H\backslash G$, with G an LP group and H a closed subgroup. This quotient is also an LP space, which can be covered by translates of the profinite quotients $(H \cap K)\backslash K$, for K a compact open subgroup of G. The principal application I have in mind will be covered in the next section, and in fact my real purpose in this section is only to motivate definitions in the next.

This section is optional as well as formal, and I should first explain the point. There is generally no G-invariant integral of functions on X. Instead, one integrates **one-densities**. There is in practice a very simple way to deal with this, but it is non-canonical. In this section I'll indicate how to specify a canonical identification of the space of one-densities.

BUNDLES. Let $\mathbb{D}$ be an arbitrary field of characteristic 0 and V a vector space over $\mathbb{D}$. A **smooth representation** of H on V is a homomorphism σ from H to $\mathrm{GL}_{\mathbb{D}}(V)$ with the property that the stabilizer of each v in V is an open subgroup. For example, if V has finite dimension and $\mathbb{D} = \mathbb{R}$ or $\mathbb{C}$, this will be true if and only if σ is continuous, since in this case $\mathrm{GL}_{\mathbb{D}}(V)$ does not possess arbitrarily small subgroups. A smooth representation may also be characterized as one that is continuous if V is assigned the discrete topology—that is to say, one for which the associated map $H \times V \to V$ is continuous with respect to the discrete topology.

• *From now on, I shall assume V to be given this topology.*

The group H acts on the product $V \times G$—the element h takes

$$(v, g) \longmapsto (\sigma(h)v, hg)\,.$$

The quotient $H\backslash(V \times G)$ is the **fibre product** $\mathcal{V} = V \times_H G$. It possesses the natural quotient topology, according to which a set U in $\mathcal{V}$ is open if and only if its inverse image in $H \times V$ is open.

There is a canonical projection Π from $\mathcal{V}$ to X, making this diagram commutative:

$$\begin{array}{ccc} V \times G & \longrightarrow & V \times_H G \\ \downarrow & & \downarrow \Pi \\ G & \longrightarrow & X \end{array}$$

The **fibre** over any point x is the inverse image $\Pi^{-1}(x)$. Let $[g]$ be the fibre containing g The embedding

$$v \longmapsto \text{ the image in } \mathcal{V} \text{ of } (v, 1)$$

is an isomorphism of V with the fibre over $[1]$. Let η be its inverse, a map from the fibre over $[1]$ to V. Any two fibres are isomorphic to each other and to V, but there is in general no canonical choice of isomorphism.

The group G acts on both G and $V \times G$ on the right. It commutes with the left action of H, hence also acts compatibly on both X and $\mathcal{V}$. I shall sometimes find it convenient to describe this as a left action:

$$\lambda_g\colon\ x \longmapsto g{\circ}x = x \cdot g^{-1}\,.$$

The map λ_g maps the fibre over $y = \Pi(x)$ to that over $y \cdot g^{-1}$. In particular, it is an isomorphism of the fibre over $[g]$ with that over $[1]$. The map $\Lambda = \eta{\circ}\lambda_g$ is an isomorphism of the fibre over $[g]$ with V. The group H takes the fibre over $[1]$ to itself, and the map η is H-equivariant.

A **section** of the bundle is a map s from X back to $\mathcal{V}$—i.e. it assigns to each element of X an element of its fibre. I'll call it **smooth** if it is continuous. Equivalently, for every x in X there exists a compact open subgroup K such that $s(x \cdot k) = s(x) \cdot k$ for each k in K. For every open U in X, let $\Gamma(U, \mathcal{V})$ be the space of smooth sections of $\mathcal{V}$ over U.

For example, if σ is the trivial bundle, then $\Gamma(X, \mathcal{V})$ is just the space of locally constant functions on $H\backslash G$.

If s is a section of $\mathcal{V}$, define $F = F_s$ from G to V:

$$F(g) = \Lambda(s([g]))\,.$$

This make sense—$v = s([g])$ is in the fibre over $[g]$, $\lambda_g(v)$ is in the fibre over $[1]$, and η takes this to V. Since η is H-equivariant, this map satisfies the condition

(I.7.1) $$F(hg) = \sigma(h)F(g) \ \text{ for all } h \text{ in } H,\, g \text{ in } G.$$

I.7.2. Lemma. *The map defined in this way is a bijection between smooth sections of $\mathcal{V}$ with locally constant functions F from G to V satisfying (I.7.1) .*

Proof. There are two things to be verified. The first is that F is locally constant. The second is that every suitable F gives rise to a section. I leave these as exercises.

SHEAVES. The vector bundle $\mathcal{V}$ determines a sheaf $\mathcal{S} = \mathcal{S}_V$ whose sections are sections of $\mathcal{V}$. For each open U in X, let

(I.7.3) $$\mathcal{S}(U) = \Gamma(U, \mathcal{V})\,.$$

I.7.4. Proposition. *In the notation of (I.7.3) :*

(a) the sheaf $\mathcal{S}$ is G-equivariant;
(b) sections are locally constant;
(c) each $\mathcal{S}(U)$ is a module over $C^\infty(U, \mathbb{D})$ such that

$$R_g(\varphi\cdot s) = R_g(\varphi)\cdot R_g(s)\,.$$

What (a) means is that for every g in G and U open in X the map $y \mapsto y\cdot g$ induces a map

$$R_g\colon\ \mathcal{S}(U) \longrightarrow \mathcal{S}(U\cdot g^{-1})$$

through the formula

$$[R_g s](x) = s(x\cdot g)\cdot g^{-1}\,.$$

In terms of Lemma I.7.2 this becomes

$$R_g F(x) = F(xg)\,.$$

What (b) means is that if U is any open subset of X, s is in $\mathcal{S}(U)$, and x is in U, then there exists a compact open K with $U\cdot K = U$ such that $R_k(s) = s$ for all k in K. This implies that if the image of s in the fibre at x vanishes, then there exists a compact open subgroup K such that $s|x\cdot K = 0$.

As for (c), it is interpreted according to the formula

$$[\varphi\cdot s](x) = \varphi(x)\cdot s(x)\,,$$

which makes sense because each fibre is a vector space.

One important consequence is that fibres of the bundle are determined canonically in terms of the sheaf. Let $\mathfrak{m}_x$ be the ideal of $C^\infty(X, \mathbb{D})$ vanishing at x, and let $M = \mathcal{S}(X)$. In these terms, the **fibre** at x is the quotient $\mathcal{S}_x = M/\mathfrak{m}_x M$. In particular, V is isomorphic to $M/\mathfrak{m}_{[1]}M$. The group H takes $\mathfrak{m}_{[1]}$ into itself, and hence the action of H on V may also be defined in terms of $\mathcal{S}$.

The stalk of the sheaf $\mathcal{S}$ at x is the direct limit of the $\mathcal{S}(U)$ as U shrinks to $\{x\}$. There is a canonical map from the stalk at x to the fibre at x. Because sections are locally constant, this is an isomorphism.

The real point of the present discussion is a converse to the conclusion of this discussion:

I.7.5. Theorem. *If $\mathcal{S}$ is any sheaf on $H\backslash G$ satisfying the conditions in the previous proposition then it is that associated to the vector bundle determined by the representation of H on $\mathcal{S}_{[1]}$.*

In particular, I'll eventually define a line bundle in terms of the sheaf whose sections are analogues of smooth differential volume forms.

Proof. Pretty much straightforward.

PROJECTIONS. Given f in $C_c^\infty(G, V)$, define

$$[\mathcal{P}f](g) = \int_H \sigma^{-1}(h) f(hg)\, d_r h\,.$$

If $F = \mathcal{P}f$ then

$$F(hg) = \sigma(h)F(h)\,.$$

for all h in H, g in G.

I.7.6. Proposition. *The map $\mathcal{P}$ from $C_c^\infty(G, V)$ to $\Gamma_c(H\backslash G)$ is a surjection.*

This is an immediate consequence of the following.

I.7.7. Lemma. *There exists a continuous section of the canonical projection $G \longrightarrow H\backslash G$ over all of $H\backslash G$.*

Consequently, such vector bundles are always topologically trivial.

Proof. Given a compact open subgroup K, the quotient is covered by subsets HgK as g ranges over some subset of G. But $HgK = H\cdot gKg^{-1}\cdot g$, and gKg^{-1} is again compact and open. It therefore suffices to prove the Lemma when $G = K$ itself is compact.

I'll give the proof, in fact, only when K is sequentially profinite. For the general case, I refer to [Serre:1964].

Let K_n be a shrinking sequence of compact open normal subgroups of K, and let $H_n = K_n \cap H$. The first claim is that for each n there exist continuous sections of the canonical projections

$$H_n\backslash K \longrightarrow H\backslash K\ .$$

The quotient $H\backslash K$ is the union of some finite number of cosets HkK_n. But K_n is normal in K, so $HkK_n = HK_nk$. Hence it suffices to prove that there exists a section over $(K_n \cap H)\backslash K_n$. However, $(K_n \cap H)\backslash K_n$ embeds into both $H_n\backslash K$ and $H\backslash K$, so of course there is a section over it.

We now consider a tree whose nodes are continuous sections

$$s_n\colon H\backslash G \longrightarrow H_n\backslash G$$

and where the successor of a section is its composition with the canonical projection from $H_{n+1}\backslash G$ to $H_n\backslash G$. Since H_{n+1} has finite index in H_n, such systems form a locally finite tree. By König's Lemma (Lemma I.2.2) there exists an infinite branch in this tree, hence a sequence of compatible continuous sections $s_n\colon H\backslash G \to H_n\backslash G$, hence a map from $H\backslash G$ to the projective limit of the $H_n\backslash G$, which can be canonically identified with G.

8. Integration on quotients

One-densities are for arbitrary locally compact spaces the analogue of differential forms on oriented manifolds. As one might expect from this analogy, one-densities are sections of a line bundle. The main result of this section is an explicit form of this assertion.

THE SHEAF OF ONE-DENSITIES. A distribution on an open set U of an LP space is an element of the dual of $C_c^\infty(U, \mathbb{D})$. Define a sheaf Ω^∞ by the requirement that $\Omega^\infty(U)$ be the vector space of locally constant distributions on U. It follows immediately from the somewhat abstract Theorem I.7.5 that the sheaf Ω^∞ is the sheaf associated to a vector bundle over $H\backslash G$. The proof will exhibit a practical formula for the evaluation of integrals of one-densities.

Defining smooth volume forms on $H\backslash G$ when G is a Lie group requires choosing local coordinates x_i, specifying a form as $\omega(x)\,dx_1\ldots dx_n$. This is usually done by defining the form as a ratio of forms on H and G. Integration then reduces to integration over a subset of $\mathbb{R}^n$. The analogous procedure here will be to choose right-invariant measures on H and G. So I begin by fixing such measures.

Let $X = H\backslash G$. For f in $C_c^\infty(G)$ define

(I.8.1)
$$\overline{f}(g) = \int_H f(hg)\,d_rh\,.$$

I.8.2. Lemma. *The map $f \mapsto \overline{f}$ is a surjection from $C_c^\infty(G)$ to $C_c^\infty(X)$.*

Proof. This is a corollary of Proposition I.7.6, but I'll give a direct proof that does not require sections. Suppose f to be in $C_c^\infty(X)$, say fixed by elements of K. Then it is a finite linear combination of functions constant on sets HgK, so we may assume that f is equal to the characteristic function of HgK. But then it is, up to scalar, equal to $\overline{f}$ with f equal to the characteristic function of gK. ▯

If ω is a smooth linear functional on $C_c^\infty(H\backslash G)$, then we can define a smooth distribution D_ω on G according to the formula
$$\langle D_\omega, f\rangle = \langle \omega, \overline{f}\rangle\,.$$

The map taking ω to D_ω is injective, according to the previous Lemma. Since

$$\overline{L_hf}(g) = \int_H f(h^{-1}xg)\,d_rx = \delta_H(h)\overline{f}(g)$$

since
$$\langle L_hD_\omega, f\rangle = \langle D_\omega, L_{h^{-1}}f\rangle = \langle \omega, \overline{L_{h^{-1}}f}\rangle$$

we must have

(I.8.3)
$$L_hD_\omega = \delta_H^{-1}(h)D_\omega$$

for all h in H. Since D_ω is smooth, Proposition I.6.6 tells us that for some smooth function $\varphi = \varphi_\omega$ on G

$$\langle D, f\rangle = \int_G \varphi(g)f(g)\,d_rg\,.$$

The condition (I.8.3) on D translates to the condition $\delta_G(h)^{-1}L_h\varphi = \delta_H(h)^{-1}\varphi$ or, equivalently,

(I.8.4)
$$\varphi(hg) = \delta_H(h)\delta_G(h)^{-1}\varphi(g)$$

for all h in H, g in G.

Conversely, suppose φ to be a smooth function on G satisfying (I.8.4) , f in $C_c^\infty(G)$. Suppose K to be a compact open subgroup of G such that f is right-K-invariant and that φ is right-K-invariant on the support of f. Then

$$\int_G \varphi(x)f(x)\,d_rx = \sum_{H\backslash G/K}\int_{HgK}\varphi(x)f(x)\,d_rx\,,$$

and since $h \mapsto hgK$ is a bijection between $H/H\cap gKg^{-1}$ and HgK/K

$$\begin{aligned}\int_{HgK}\varphi(x)f(x)\,d_rx &= \sum_{H/H\cap gKg^{-1}}\varphi(hg)f(hg)\,\mathrm{meas}(hgK)\\ &= \sum_{H/H\cap gKg^{-1}}\delta_H(h)\delta_G^{-1}(h)\varphi(g)\cdot f(hg)\cdot\delta_G(h)\,\mathrm{meas}(gK)\\ &= \mathrm{meas}(gK)\varphi(g)\left(\sum_{H/H\cap gKg^{-1}}\delta_H(h)f(hg)\right)\\ &= \varphi(g)\left[\frac{\mathrm{meas}_G(gK)}{\mathrm{meas}_H(H\cap gKg^{-1})}\right]\int_H f(hg)d_rh\\ &= \varphi(g)\left[\frac{\mathrm{meas}_G(gK)}{\mathrm{meas}_H(H\cap gKg^{-1})}\right]\cdot\overline{f}(g)\,.\end{aligned} \tag{I.8.5}$$

I.8.6. Theorem. *Suppose given right-invariant measures on H and G. The map $\omega \mapsto \varphi_\omega$ is an isomorphism of the space of smooth distributions on $H\backslash G$ with the space of smooth functions φ on G such that*

$$\varphi(hg) = \delta_H(h)\delta_G(h)^{-1}\varphi(g)\;. \tag{I.8.7}$$

Explicitly, if f and φ are right K-invariant then

$$\int_{H\backslash G}\varphi(\overline{x})f(\overline{x})\,d\overline{x} = \sum\nolimits_{H\backslash G/K}\varphi(g)f(g)\left[\frac{\mathrm{meas}_G(gK)}{\mathrm{meas}_H(H\cap gKg^{-1})}\right]\,.$$

You can check quickly that the terms in this sum do not depend on the choice of representatives g in $H\backslash G/K$.

I.8.8. Corollary. *There exists a unique right G-invariant linear map $d_r\overline{x}$ from $\Omega_c^\infty(H\backslash G)$ to* ***C****, which will be denoted simply by integration, such that*

$$\int_G f(g)\,d_rx = \int_{H\backslash G}\left(\int_H \delta_H^{-1}(h)\delta_G(h)f(hx)\,d_rh\right)\,d_r\overline{x}\;.$$

for any f in $C_c^\infty(G)$.

This can be written

$$\int_G f(g)\,d_rx = \int_{H\backslash G}\overline{f}_\Omega(\overline{x})\,d_r\overline{x}\;.$$

if

$$\overline{f}_\Omega(x) = \int_H \delta_H^{-1}(h)\delta_G(h)f(hx)\,d_rh\,.$$

I.8.9. Corollary. *For φ in $\Omega_c^\infty(H\backslash G)$ fixed on the right by elements of the compact open subgroup K*

$$\int_{H\backslash G}\varphi(\overline{x})\,d\overline{x} = \sum_{H\backslash G/K}\varphi(g)\left[\frac{\mathrm{meas}_G(gK)}{\mathrm{meas}_H(H\cap gKg^{-1})}\right]\,.$$

I add a supplement. Suppose

$$\int_{H\backslash G} F(\overline{x})\, d\overline{x} = 0\,.$$

Then we can find f in $C_c^\infty(G)$ such that $\overline{f}_\Omega = F$, and then

$$\int_G f(g)\, d_r g = 0$$

as well. According to Proposition I.6.5, we may express f as a linear combination of $R_g\varphi - \varphi$. We deduce the following generalization of Proposition I.6.5:

I.8.10. Corollary. *If F lies in $\Omega_c^\infty(H\backslash G)$ then*

$$\int_{H\backslash G} F(\overline{x})\, d\overline{x} = 0$$

if and only if F is a linear combination of smooth densities of the form $\Phi - R_g\Phi$.

9. Measures on $\mathfrak{p}$-adic manifolds

In the special case that G is an algebraic group defined over the $\mathfrak{p}$-adic field $\mathfrak{k}$, we can derive measures on G in a way that one must often use when one wants to treat coherently all local fields associated to a global one.

First let V be the additive group $\mathfrak{k}^n$. A translation-invariant measure dx on V is a constant multiple of the measure $dx = dx_1\ \ldots\ dx_n$ that assigns measure 1 to $\mathfrak{o}^n$. This assigns measure q^{-nk} to $(\mathfrak{p}^n)^k$. This determines the measure on any compact open subgroup, since it will be the disjoint union of translates of one of these neighbourhoods of 0.

If T is any matrix in $\mathrm{GL}_n(\mathfrak{k})$ and dx is translation-invariant then the transform by T is also translation-invariant, hence a multiple of dx. The constant is determined by what T does to $\mathfrak{o}^n$. If D is a diagonal matrix with diagonal entries d_i the volume of $T\mathfrak{o}^n$ is $\mu\prod|d_i|$. If γ is in $\mathrm{GL}_n(\mathfrak{o})$ then $\gamma\mathfrak{o}^n = \mathfrak{o}^n$, hence it preserves volumes. The principal divisor theorem tells us that any matrix T in $\mathrm{GL}_n(\mathfrak{k})$ is a product $\gamma_1 D\gamma_2$ with the γ_i in $\mathrm{GL}_n(\mathfrak{o})$ and D diagonal, so T multiplies volumes by $|\det(T)|$. Consequently:

I.9.1. Proposition. *If φ is an analytic isomorphism of subsets X and Y of $\mathfrak{k}^n$, then setting $y = \varphi(x)$, $g(x) = f(\varphi(x))$ we have the change of variables formula*

$$\int_Y f(y)\, dy = \int_X g(x)\, \big|\det(\partial y/\partial x)\big|\, dx\,.$$

Now suppose X to be an arbitrary $\mathfrak{k}$-analytic manifold, given with a countable, locally finite atlas $\{X_i\}$. For each X_i we are given an open embedding into $\mathfrak{k}^n$, and on overlaps two coordinate systems differ by an invertible analytic function. Details about what this means can be found in [Serre:1965]. Because of the implicit function theorem for $\mathfrak{k}$-analytic maps, every non-singular algebraic variety over $\mathfrak{k}$ has such a structure.

Following what happens for real manifolds, I define a **smooth one-density** on X to be a family of compatible smooth measures on the sets X_i. Compatibility means only that the measures agree on overlaps. In particular, if one is given a non-vanishing differential form ω on an algebraic variety M, then $|\omega|$ defines a smooth one-density. If one applies this to an algebraic group, one obtains:

I.9.2. Proposition. *If G is an algebraic group defined over $\mathfrak{k}$, then the modulus character is $\big|\det(\mathrm{Ad}_{\mathfrak{g}})\big|$.*

I.9.3. Theorem. *If G is an algebraic group defined over $\mathfrak{k}$ and H a closed subgroup, then the space of smooth one-densities on $H\backslash G$ may be identified with the space of smooth $\mathbb{C}$-valued functions on G such that*

$$f(hg) = \left|\det^{-1}\mathrm{Ad}_{\mathfrak{h}\backslash\mathfrak{g}}(h)\right| f(g)$$

for all h in H, g in G.

If G is unimodular

$$\left|\det^{-1}\mathrm{Ad}_{\mathfrak{h}\backslash\mathfrak{g}}(h)\right| = \left|\det\mathrm{Ad}_{\mathfrak{h}}(h)\right| .$$

10. References

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