Essays on topological vector spaces

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Quasi-complete TVS

Suppose G to be a locally compact group. In the theory of representations of G, an indispensable role is played by an action of the convolution algebra $C_c(G)$ on the space V of a continuous representation of G. In order to define this, one must know how to make sense of integrals

$$\int_G f(x)\,dx$$

where f is a continuous function of compact support on G with values in V. This is not quite a trivial matter. If V is assumed to be complete, a definition by Riemann sums works quite well, but this is an unnecessarily strong requirement. The natural condition on V seems to be that it be **quasi-complete**, a somewhat technical but nonetheless valuable condition. Continuous representations of a locally compact group are almost always assumed to be on a quasi-complete topological vector space.

I can illustrate here quite briefly what the problem is. Suppose M to be a manifold assigned a smooth measure dm, and suppose f a continuous, compactly supported function on M with values in the TVS (i.e. locally convex, Hausdorff, topological vector space) V. *How is one to define the integral*

$$I = \int_M f(m)\,dm\,?$$

If V is complete, as I say, this is not difficult. In general, what one can do, according to the Hahn-Banach Theorem, is characterize the integral uniquely by the condition that

$$\langle F, I\rangle = \Big\langle F, \int_M f(m)\,dm\Big\rangle = \int_M \langle F, f(m)\rangle\,dm$$

for any F in the continuous linear dual of V. But although the vector I is certainly unique, why does it exist? Why is there a vector I in v actually satisfying this equation? In other words, we have defined the intetgral in the double dual of V, but why is it actually in V? That's the first question I answer in this essay.

The second question is rather similar, but is about differentiation. Suppose $f(x)$ to be a continuous function on an open subset Ω of $\mathbb{R}^n$ with values in the TVS V, and suppose that for every F in $\widehat{V}$ the function $F(f(x))$ is smooth. Is $f(x)$ itself then smooth? The answer is again affirmative, if V is quasi-complete.

In this essay, a TVS will always be a locally convex, Hausdorff, topological vector space. Such spaces possess a topology defined by a collection of semi-norms $\|v\|_\rho$, or equivalently by a basis of neighbourhoods of 0 that are convex, balanced, and absorbing. I assume some familiarity with these spaces, but the most important thing required will be the Hahn-Banach theorem, which guarantees the existence of sufficiently many linear functions on V—enough, for example, to separate distinct points.

I start out by recalling the definition of quasi-complete spaces, and then I go on to prove the main theorems about integration and differentiation.

Contents

1. Quasi-complete spaces

Exposition in this section were modified after I had read [Taylor:1995].

The point of the following definition is that in defining V-valued integrals one does not need the whole space to be complete.

I recall that a subset of a TVS is **bounded** if every (continuous) semi-norm is bounded on it. A subset is **complete** if every Cauchy net in it converges.

Definition. *A vector space is said to be* **quasi-complete** *if every closed bounded subset is complete.*

An equivalent condition is that every bounded Cauchy net converge. If the vector space is complete, then of course it is also quasi-complete, but the converse is not necessarily true. There are useful spaces which are quasi-complete although not complete and almost all topological vector spaces that one encounters in practice are quasi-complete. They make up the natural class of TVS to use in representation theory, but the published literature is sparse.

Quasi-completeness is related to compactness. A subset C of V is **totally bounded** if for every neighbourhood X of 0 in V the set C is covered by a finite number of translates of X. It is elementary to verify that every totally bounded subset of a TVS is bounded.

1.1. Lemma. *The following conditions on the subset C of a TVS are equivalent:*

(a) every net in C possesses a Cauchy subnet;
(b) the set C is totally bounded.

A set satisfying these conditions is often called **precompact**, for reasons that will apppear in a moment. The equivalence of the two criteria is extremely useful.

Proof. This is a standard result in the theory of uniform spaces—see, for example, Theorem 6.32 of [Kelley:1955]—and the proof in the general case involves a few notions I don't want to introduce. But I'll prove it in a special case, when V possesses a countable basis of semi-norms, and is hence metrizable. In this case, we can restrict consideration to sequences instead of nets. Let $|u - v|$ be the metric.

Suppose C to satisfy (b), and that v_i is a sequence in C. For every $n \geq 0$ the set Ω can be covered by a finite number U_n of open sets of the form $|c - c_k| < 1/2^n$. Since U_0 is finite, there exist an infinite number of the v_i in some single set in it, hence a subsequence v_{i_k} at most a unit distance apart from each other. But then this in turn must contain a subsequence with all of its elements at distance at most $1/2$ from each other. And so on. Therefore (c_i) contains a Cauchy subsequence.

Conversely, suppose that C does not satisfy (b). Then for some $\varepsilon > 0$ it cannot be covered by a finite number of disks of radius ε. Choose an arbitrary element c_1 of C. The disk $|x - c| < \varepsilon$ doesn't cover C, so we can choose c_2 not in it. But then we can choose c_3 with $|c - c_i| \geq \varepsilon$ for $i = 1, 2$. And so on. The sequence you get has no Cauchy subsequence. ∎

1.2. Proposition. *A subset C of a TVS is compact if and only if it is complete and totally bounded.*

Proof. I again restrict myself to a metrizable space. A subset C is compact if and only if every sequence contains a convergent subsequence, or (by Heine-Borel) if every covering of C by open sets contains a finite sub-covering. That every compact subset is totally bounded is thus a matter of definition.

Suppose, conversely, that C is totally bounded. Since C is bounded, its closure is complete and every Cauchy sequence in it converges. Combining this with (a) of Lemma 1.1, we see that C is compact. ∎

1.3. Corollary. *A subset C of a quasi-complete TVS is compact if and only if it is closed and totally bounded.*

1.4. Lemma. *The convex hull of any totally bounded subset of an arbitrary TVS is totally bounded.*

Proof. Let U be a neighbourhood of 0 in V, and U_* a convex neighbourhood such that $U_* + U_* \subseteq U$. Suppose X totally bounded. We can find a finite set of x_i in X with X contained in the union of the $x_i + U_*$. Let C be the convex hull of the x_i, which is contained in the finite-dimensional space they span. It is closed and bounded in that space, hence compact, hence we can find a finite set of of y_j such that C is contained in the union of the $y_j + U_*$. But then since U_* is convex, the convex hull of X is contained in the union of the $y_j + U$. ∎

It can happen in an arbitrary TVS that the closure of the convex hull of a compact set is not compact. For example, let V be the space of all sequences (e_i) of finite support, given the L^2 norm. For each n let e_n be the sequence with $e_{n,i} = 0$ unless $i = n$, and $e_{n,n} = 1/n$. Then e_n converges to 0 in V. The set

$$\Omega = \{e_i\} \cup \{0\}$$

is contained in this space, and the closure of its convex hull is not compact. But:

1.5. Proposition. *In a quasi-complete TVS, the closure of the convex hull of a compact subset is compact.*

Proof. Combine Lemma 1.4 and Lemma 1.1. ∎

Remark. It can happen even in a complete space that the convex hull of a compact set is not compact. For example, let V be the Hilbert space of all square-summable sequences $(x_n$. The set Ω is compact in V, but its convex hull is not even closed.

None of these phenomena occurs in a TVS of finite dimension. In a TVS of infinite dimension, the convex hull of a set X may involve sums of an arbitrarily large number of elements of X. But the required size is bounded in spaces of finite dimension.

1.6. Proposition. (Carathéodory's Theorem) *Suppose X to be a nonempty subset of $\mathbb{R}^n$. Then:*

(a) if $y \neq 0$ is a non-negative linear combination of points in X, it may be expressed as a positive linear combination of linearly independent points of X;
(b) any point in the convex hull of X lies in the convex hull of some subset of points of X of size $\leq n+1$.

In other words, in (a) every point in the convex cone generated by X is contained in a simplicial cone generated by elements of X, and in (b) every element in the convex hull of X is contained in a simplex whose vertices are in X.

Proof. The proof is an algorithm.

(•) Suppose $v = \sum c_i x_i$ with $c_i > 0$, x_i in X. Let N be the number of x_i. If the x_i are linearly independent, exit.

Otherwise, there exists an equation

$$\sum a_i x_i = 0\,.$$

Negating if necessary, we may assume one $a_i > 0$. Then for any b

$$v = \sum (c_i - b\,a_i) x_i\,.$$

The line $y = c_i - x a_i$ in $\mathbb{R}^2$ starts off at $(0, c_i)$, and has slope $-a_i$. At least one of them crosses the positive x-axis; suppose the first one crosses at $(b, 0)$. Then all $c_i - b a_i$ are non-negative, and at least one is 0. This gives an expression for y involving a smaller number of x_i. Loop to (•).

The procedure cannot go on forever, since $y \neq 0$ and a single element of X is a linearly independent set.

For (b), let Ω be the convex hull of X. The claim follows from (a) applied to the cone over $(\Omega, 1)$ in $\mathbb{R}^{n+1}$. ∎

1.7. Corollary. *In a finite-dimensional real vector space*

(a) the convex hull Ω of a compact set X is compact;
(b) the convex cone generated by a finite set is closed.

Proof. For (a), let Σ be the simplex of all (c_i) in $\mathbb{R}^{n+1}$ with $\sum c_i = 1$. According to the proposition, the set Ω is the image of $\Sigma \times X^{n+1}$ with respect to the map

$$(c, x) \longmapsto \sum c_i x_i \,.$$

For (b), the claim is certainly true when the points are linearly independent, and the cone is simplicial. But Carathéodory's Theorem reduces the claim to that case.

As a final remark about quasi-complete spaces, here is what [Reed-Simon:1972] call the BLT (for **B**ounded **L**inear **T**ransformation) theorem in a narrower context:

1.8. Proposition. *Suppose V and W to be TVS, U a subspace of V with the property that every point of V is in the closure of a bounded subset of U. If $F\colon U \to W$ is a linear map from U to W, continuous in the induced topology, and W is quasi-complete, then F extends uniquely to a continuous linear map from all of V to W.*

Proof. The space U is dense in V, so that by the standard extension theorem for complete spaces the map f may be extended to a continuous map $\overline{f}$ into the completion $\overline{W}$ of W. But since every point of V is in the closure of a bounded subset B of U, $\overline{f}(v)$ is in the closure of $f(B)$. But $f(B)$ is closed in W, so its closure is complete, and $\overline{f}(v)$ lies in W.

As I have already mentioned, quasi-complete TVS are ubiquitous. I refer to §34.3 of [Treves:1967] for proofs of the following claims:

1.9. Proposition. *A TVS is quasi-complete if any of the following is true:*

(a) it is a Fréchet space;
(b) it is the weak dual of a Fréchet space;
(c) it is the inductive limit of Fréchet spaces.

2. Integration of vector-valued functions

From now on, assume V always to be quasi-somplete/

One of the principal reasons for interest in quasi-complete spaces is that they furnish the right context for integration, as I want to explain in this section.

The problem is how to make sense of integrals

$$I(f) = \int_X f(x)\, dx$$

where X is a locally compact space, dx a good measure on X, and f a function with compact support on X and values in a TVS. To tell the truth, in just about all cases we'll deal with the integral will have a direct interpretation by-passing the machinery I'll introduce here, but it is worth something not to have to deal with each case separately.

Suppose X to be a locally compact space that's the union of a countable number of compact subsets. There are two different ways to define the notion of a positive measure on X compatible with its topology. The space $C(X)$ is that of all continuous functions on X, with semi-norms

$$\|f\|_{\infty,\Omega} = \sup_{x\in\Omega} \big|f(x)\big|$$

where Ω is a compact subset of X. (This is standard notation. The ∞ is there because in a finite-dimensional space the norm

$$\|x\|_p = \Big(\sum x_i^p\Big)^{1/p}$$

has limit $\sup|x_i|$ as $p \to \infty$.) For each compact subset Ω of X let $C_c(\overline{\Omega})$ be the space of functions f in $C(X)$ with support in $\overline{\Omega}$ assigned the semi-norm $\|f\|_{\infty,\overline{\Omega}}$. The TVS $C_c(X)$ is the inductive limit of the $C_c(\overline{\Omega})$. The simplest definition of a positive measure μ on X compatible with its topology is that it is a continuous linear functional on $C_c(X)$ such that $\langle \mu, f\rangle \geq 0$ if $f \geq 0$. By the **Riesz-Markov Theorem** (see page 111 of [Reed & Simon:1972]), this gives rise also to a measure in the usual sense—called a **Baire measure**—one that assigns a measure to various subsets of X, including the open ones. A compactly supported measure is the same as a continuous linear functional on $C(X)$ itself.

2.1. Theorem. *Suppose X to be a locally compact space with a positive Baire measure dx. Suppose $f(x)$ to be a continuous function of compact support on X with values in a quasi-complete TVS V. There is a unique vector I_f in V such that*

$$\big\langle \widehat{v}, I_f\big\rangle = \int_X \big\langle \widehat{v}, f(x)\big\rangle\, dx$$

for all $\widehat{v}$ in $\widehat{V}$. The integral I_f lies in the closed convex hull of $f(X)$ scaled by $\operatorname{meas}(X)$.

I'll take up the proof in just a moment. This has as immediate consequence:

2.2. Corollary. *If ρ is a (continuous) semi-norm of V then*

$$\big\|I_f\big\|_\rho \leq \int_X \big\|f(x)\big\|_\rho\, dx\,.$$

Proof of the Theorem. Replacing X if necessary by the support of f, and then perhaps scaling, I may assume that $\operatorname{meas}(X) = 1$. I may also assume that V is a vector space over $\mathbb{R}$.

Defining the integral when V has finite dimension is instructive. Choose a basis e_i of V, express

$$f(x) = \sum f_i(x) e_i\,,$$

and then calculate

$$\int_X f(x)\, dx = \sum_i \left(\int_X f_i(x)\, dx\right) e_i\,.$$

In general, the trick is the same—to reduce the definition of the integral to an integral of scalar functions. In this example, the function $f_i(x)$ is the same $\langle \widehat{e}_i, f(x)\rangle$ where $\widehat{e}_i$ is the basis dual to e_i. The vector space V is locally convex, so by the Hahn-Banach theorem we know that vectors v in V are distinguished by the values of $\langle \widehat{v}, v\rangle$ where $\widehat{v}$ is a continuous linear function on V. This tells us that the integral we are looking for is determined uniquely by the integrals

$$\int_X \big\langle \widehat{v}, f(x)\big\rangle\, dx$$

as $\widehat{v}$ ranges over all elements of the continuous dual $\widehat{V}$, *once we know how to interpret the integral as an element of V*. It's the last fact, the existence of the integral, that causes trouble.

There seem to be two approaches to a proof in the literature, rather different from each other. One is that of [Bourbaki:1959–65]. The same technique occurs frequently in French expositions. The integral exists always, without further hypothesis, in a very weak sense. Let $\widetilde{\widehat{V}}$ be the *simple linear dual* of $\widehat{V}$. Each v in V determines an element of $\widetilde{\widehat{V}}$, but more generally an element of $\widetilde{\widehat{V}}$ is specified by any linear function on

$\widehat{V}$, with no topological restriction imposed. The formula above tells us that we can interpret the integral without trouble as an element of $\widetilde{\widehat{V}}$. The question is whether it lies in the image of the canonical embedding $\iota\colon V \hookrightarrow \widetilde{\widehat{V}}$. That it does takes several steps of rather delicate reasoning about double duals. I prefer to follow here the note [Garrett:2011], which reduces the matter to a few simple facts about finite-dimensional spaces. The first step is to prove the theorem when V has finite dimension. We have already seen the definition of the integral in this case, and it remains only to prove that the integral in this case lies in the closure of the convex hull of $f(X)$. But the Hahn-Banach theorem asserts that the closure of the convex hull of any set is the intersection of all the closed half-spaces containing the set, so what must be shown is that if λ is any linear function on V such that $\langle \lambda, f(X)\rangle \le C$ then

$$\langle \lambda, \int_X f(x)\,dx\rangle = \int_X \langle \lambda, f(x)\rangle\,dx \le \operatorname{meas}(X)\,C\,.$$

But this is a basic fact about integration of Radon measures.

I recall that a **polyhedron** is the convex hull of a finite number of points in some $\mathbb{R}^n$.

2.3. Lemma. *A polyhdron in $\mathbb{R}^n$ may be partitioned into a finite number of simplices of dimension n.*

The simplices may be degenerate.

Proof. The result is trivially true in dimension one, since a polyhedron is just an interval. One may assume the given points to be extremal for the convex hull. Fix one of these vertices, say P, and look at the maximal faces of the polyhedron not containing it. By induction, each of these may be partitioned into simplices of dimension $n-1$, and then the original polyhedron is the union of cones with these as bases and P as vertex. ∎

In an arbitrary quasi-complete TVS it is only the *closure* of the convex hull of a compact set that is necessarily compact, but in a finite-dimensional space things are simpler:

2.4. Lemma. *In a finite-dimensional vector space the convex hull of a compact set is compact.*

Proof. Say V has dimension n. Any point in the convex hull of the compact set Ω will be in the convex hull of a finite number of points of Ω. By the Lemma just proved, it will be in the convex hull of $n+1$ of them. If

$$\Sigma_n = \left\{(x_i)\,\middle|\,\sum x_i = 1, x_i \ge 0\right\}$$

is the standard n-dimensional simplex then the map

$$\Sigma_n \times \Omega^{n+1} \to V$$

therefore has image Ω, but it also has compact image. ∎

I have proved the Theorem for finite-dimensional V, and so we know that the integral in that case lies in the closure of the convex hull of $f(X)$ scaled by $\operatorname{meas}(X)$. But now we see that this closure is just the convex hull itself. Hence:

2.5. Lemma. *If V is finite-dimensional, the integral*

$$\int_X f(X)\,dx$$

lies in the convex hull of $f(X)$ scaled by $\operatorname{meas}(X)$.

Now I return to the proof of Theorem 2.1. We are looking for a vector I_f in V such that

$$\langle F, I_f\rangle = I_{F,f} = \int_X \langle F, f(x)\rangle\,dx$$

for all F in $\widehat{V}$. We are going to reduce this to the case when V has finite-dimension. If $\mathcal{F}$ is a finite set of continuous linear functions on V, we can define the map

$$F_{\mathcal{F}}\colon\ V \to \mathbb{R}^{\mathcal{F}}, \quad v \longmapsto (\langle F, v\rangle)\,.$$

We obtain a map from X to $\mathbb{R}^{\mathcal{F}}$ by composing $F_{\mathcal{F}}$ with f. We know that the integral of this composite over X lies in the convex hull of its image, so that the set $\mathcal{C}_{\mathcal{F}}$ of all v in the convex hull of $f(X)$ such that

$$\langle F, v\rangle = \int_X \langle F, f(x)\rangle\, dx$$

for all F in $\mathcal{F}$ is not empty. Since f has compact support, $f(X)$ is compact. The closure of its convex hull is therefore compact, by Proposition 1.5. Hence the closures of the $C_{\mathcal{F}}$ are also compact.

If Φ is a finite set of sets $\mathcal{F}$, then

$$\textstyle\bigcap_{\mathcal{F}\in\Phi} V_{\mathcal{F}} = V_{\cup\Phi}$$

so we may apply the following elementary result, which I leave as an exercise, to see that there exists v in the intersection of all $\mathcal{C}_F$, which is exactly what we are looking for.

2.6. Lemma. *Suppose A to be a directed set and $A \to K_A$ a directed map from A to compact subsets of V such that no K_a is empty. Then the intersection of all K_a is non-empty.*

The assumption means that if $a \prec b$ then $K_a \supseteq K_b$.

3. Differentiation of vector-valued functions

In this section, continue to assume V to be quasi-complete.

Suppose Ω to be an open subset of $\mathbb{R}^n$, V a TVS over $\mathbb{R}$. Define $C(\Omega, V) = C^0(\Omega, V)$ to be the space of all continuous functions from Ω to V. If $f\colon \Omega \to V$ is in $C^0(\Omega, V)$, it is said to be differentiable at u if for each vector v in $\mathbb{R}^n$ the directional derivative

$$\lim_{t\to 0} \frac{f(x+tv) - f(x)}{t} = [\partial_v f](x)$$

exists. If it is differentiable at every point of Ω this defines a map from $\Omega \times \mathbb{R}^n$ to V, the derivative f' of f. The function f is said to be continuously differentiable if this is turn is continuous. Define $C^1(\Omega, V)$ to be the space of all continuously differentiable functions from Ω to V.

Here is a very useful observation:

3.1. Lemma. *If f is in $C^1(\Omega, V)$ then for every $\widehat{v}$ in $\widehat{V}$ the function $\langle \widehat{v}, f(x)\rangle$ is in $C^1(\Omega, \mathbb{C})$.*

This is elementary. We shall see later that the converse is also true, but not elementary. Given this, many results about calculus for V-valued functions reduce to the standard arguments of ordinary calculus, found for example in §§3–4 of Chapter II in [Courant:1936].

Proof. Let

$$F(x) = \langle \widehat{v}, f(x)\rangle\,.$$

Then

$$\frac{F(x+tv) - F(x)}{t} = \frac{\langle \widehat{v}, f(x+tv) - f(x)\rangle}{t}\,.$$

If the limit $[\partial_v f](x)$ exists, so does the limit $[\partial_v F](x)$, and

$$[\partial_v F](x) = \langle \widehat{v}, [\partial_v f](x)\rangle\,.$$

If $[\partial_v f](x)$ is continuous on $\Omega \times \mathbb{R}^n$, so is $[\partial_v F](x)$.

3.2. Lemma. *If f lies in $C^1(\Omega, V)$ then the derivative $\partial_v f$ is a linear function of v.*

Proof. Linear behaviour with respect to scalar multiplication is immediate from the definition. So now we want to show that if v, w are in $\mathbb{R}^n$ then $[\partial_{u+v} f](x) = [\partial_u f](x) + [\partial_v f](x)$. Here we apply the previous Lemma. Elementary calculus assures us that differentiation of C^1 scalar functions is linear. But then the Hahn-Banach Theorem tells us that continuous linear functionals separate points of V, so f' itself is linear.

The function $\partial_v f$ hence determines its **differential** dF, a map from Ω to the TVS $\mathrm{Hom}(\mathbb{R}^n, V)$. It is easy to verify that if f lies in $C^1(\Omega, V)$ then $\partial_v f$ depends only on the tangent vector v, not the explicit coordinate at hand. At any rate, we can therefore continue. The m-th derivative is a function from $\Omega \times T^m(\mathbb{R}^n)$ to V, where

$$T^m(\mathbb{R}^n) = \bigotimes^m \mathbb{R}^n \,.$$

3.3. Lemma. *Suppose Ω to be a convex set and f to lie in $C(\Omega, V)$, φ in $C(\Omega \times \mathbb{R}^n)$. Then $\varphi(x, v) = \partial_v f(x)$ for every x in Ω if and only if*

$$f(x + tv) - f(x) = \int_0^t \varphi(x + sv, v)\, ds \,.$$

whenever x, $x + tv$ lie in Ω.

Proof. Apply a function in $\widehat{V}$ and Hahn-Banach, reducing to the Fundamental Theorem of calculus.

Again applying Hahn-Banach, the following result again reduces to the case $V = \mathbb{C}$:

3.4. Lemma. *If f lies in $C^2(\Omega, V)$, mixed partial derivatives commute.*

Thus the map from T^m factors through S^m, the space of symmetric tensors. A form in which we shall often apply this Lemma is:

3.5. Proposition. *If X, Y are two smooth vector fields on Ω and f lies in $C^2(\Omega, V)$ then*

$$X \cdot Y \cdot f - Y \cdot X \cdot f = [X, Y] \cdot f \,.$$

We have a deceasing filtration

$$C^0(\Omega, V) \supset C^1(\Omega, V) \supset \ldots \supset C^m(\Omega, V) \supset \ldots \,.$$

Let $C^\infty(\Omega, V)$ be the intersection of all of these.

Fix a basis $(\alpha_i$ of $\mathbb{R}^n$. The space $C^m(\Omega, V)$ may be made into a topological vector space by means of the semi-norms

$$\|f\|_{U,\rho,m} = \sup_{k \le m, u \in U} \|(\alpha \cdot f)(x)\|_\rho$$

where U is a compact subset of Ω, $0 \le k \le m$, ρ a semi-norm of V, and α a product of the α_i in $S^k(\mathbb{R}^n)$.

3.6. Proposition. *The space $C^m(\Omega, V)$ is quasi-complete. If V is complete, so is it.*

Recall that I am assuming V to be quasi-complete.

Proof. Immediate from Lemma 3.3.

We have used the fact, easy to prove, that if F is a function in $\widehat{V}$ and f is in C^m then $\langle F, f(x)\rangle$ is a scalar function in $C^m(\Omega, \mathbb{C})$. One of the pleasant if curious things about vector-valued derivatives is that although the converse it not known to be true for all TVS, it is known to be almost true, under our assumption on V:

3.7. Theorem. *If V is quasi-complete and the function*

$$u \longmapsto \langle F, f(x)\rangle$$

lies in $C^m(\Omega, \mathbb{C})$ for all F in $\widehat{V}$, then f itself is in $C^{m-1}(\Omega, V)$.

This seems to have been folklore in the early nineteen-fifties, at least for functions with values in Banach spaces. The result for quasi-complete spaces is due originally to Grothendieck, apparently first appearing in a sketch in [Grothendieck:1953], with more details in [Schwartz:1954] and [Grothendieck:1958]. (Incidentally, Schwartz' paper has a very readable introduction to quasi-complete spaces.) As you might guess from the nature of both hypothesis and conclusion, as well as the identities of these authors, this result has subtle aspects. At any rate, it has as immediate consequence:

3.8. Corollary. *Suppose V to be a quasi-complete TVS. If*

$$u \longmapsto \langle F, f(x)\rangle$$

lies in $C^\infty(\Omega, \mathbb{C})$ for all F in $\widehat{V}$, then f itself is in $C^\infty(\Omega, V)$.

This has useful applications. One I have in mind implies that certain vectors in a unitary representation of a Lie group are smooth vectors, because the matrix coefficients they determine are solutions of elliptic differential operators, hence smooth.

Proof of Theorem 3.7. The proof that Grothendieck and Schwartz give, like their proof of the existence of integrals that I mentioned in the last section, relies on delicate facts about double duals. Instead, I follow [Garrett:2008].

The case $m = 1$ is quite different from the rest. Assume

$$u \longmapsto \langle F, f(x)\rangle$$

to be in $C^1(\Omega, V)$ for all x in Ω, F in $\widehat{V}$. Fix x_0 in Ω. By assumption, the set of values

$$\frac{f(x) - f(x_0)}{|x - x_0|}$$

is weakly bounded as x varies over some disk in Ω, hence by by a well known theorem (see, for example, V.23 of [Reed-Simon:1972]) it is bounded. Thus

$$f(x) - f(x_0) \in |x - x_0|\, B$$

for some bounded set B in V, hence f is continuous at x_0.

From now on we argue by induction, and easily reduce to the case $m = 2$. So we assume that

$$x \longmapsto \langle F, f(x)\rangle$$

is in $C^2(\Omega, \mathbb{C})$ for every F in $\widehat{V}$, and we want to show (1) f itself is differentiable everywhere and (2) its derivative is continuous. Once (1) is proven, (2) will follow from the case $m = 1$ we have already seen. I want to show that

$$\lim_{t \to x} \frac{f(x+t) - f(x)}{t}$$

exists for every x in Ω. For this, because V is quasi-complete, it suffices to show that this set makes up a bounded Cauchy net. But since $\varphi_F(x) = \langle F, f(x)\rangle$ is C^2, we can write

$$\varphi_F(x+t) - \varphi_F(y) = t\varphi_F'(y) + \frac{t^2}{2!}\,\varphi_F''(y) + o(t2)\,,$$

uniformly for t near 0. Then

$$\varphi_F(x+t) - \varphi_F(x+s) = O(s-t)^2$$

uniformly for s, t near 0. Therefore

$$\frac{\varphi_F(x+t) - \varphi_F(x+s)}{t-s}$$

is bounded for all F, which means that

$$\frac{\varphi(x+t) - \varphi(x+s)}{t-s}$$

is weakly bounded, hence strongly bounded. Here

$$\varphi(y) = \frac{f(y) - f(x)}{y-x}\,.$$

But then $\varphi(t+x)$ is a bounded Cauchy net in V.

4. References

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