Essays on the structure of reductive groups

Root systems of rank one and two

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Understanding root systems of higher rank depends critically on understanding systems of ranks one and two. Many arguments reduce quickly to one of these cases.

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I recall one method by which root systems are sepcified. If V is a Euclidean vector space and v in V then define

$$v^\diamond = \frac{2v}{\|v\|^2},$$

and define $v^\vee$ to be the element of the dual space $V^\vee$ corresponding to $v^\diamond$, so that by definition

$$\langle v^\vee, u\rangle = v^\diamond \bullet u$$

for all u in V. A finite subset Σ together with the map

$$\Sigma \longrightarrow \Sigma,\ \lambda \longmapsto \lambda^\vee$$

will form of a root system if

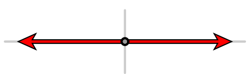
(a) $\lambda \bullet \mu^\diamond \in \mathbb{Z}$ for all λ, μ in Σ
(b) for each λ in Σ the orthogonal reflection

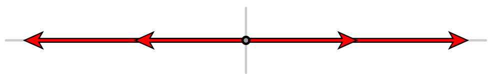
$$s_\lambda\colon\ v \longmapsto v - (v \bullet \lambda^\diamond)\lambda$$

takes Σ to itself.

1. Root systems of rank one

The simplest system is that containing just a vector and its negative. There is one other system of rank one, however, which we have already seen as that of SU_{ω_3}:





Throughout this section and the next I shall exhibit root systems by Euclidean diagrams, implicitly leaving it as an exercise to verify the conditions laid out above.

That these are the only rank one systems follows from this:

[non-reduced] **1.1. Lemma.** *If λ and $c\lambda$ are both roots, then $|c| = 1/2, 1,$ or 2.*

Proof. On the one hand $(c\lambda)^\vee = c^{-1}\lambda^\vee$, and on the other $\langle\lambda, (c\lambda)^\vee\rangle$ must be an integer. Therefore $2c^{-1}$ must be an integer, and similarly $2c$ must be an integer.

A root λ is called **indivisible** if $\lambda/2$ is not a root. It is easy to see that:

[indivisible] **1.2. Proposition.** *If Σ is a set of roots, so is the system obtained made up of its indivisible roots.*

2. Root systems of rank two

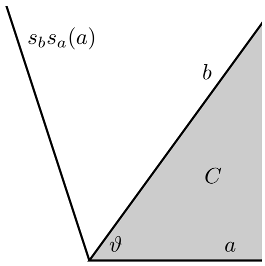
REFLECTION-INVARIANT LINE CONFIGURATIONS. I start with a situation generalizing that occurring for root systems of rank two. Assume for a while that we are given

- a Euclidean plane U;
- a finite set $\mathcal{L}$ of at least two lines;

satisfying the condition:

- the set $\mathcal{L}$ is stable under orthogonal reflections in its lines.

I'll call this a **geometric root configuration**. Each connected component of the complement of the lines in $\mathcal{L}$ is a wedge. Two of them must be acute and two obtuse. Let C be one of the acute ones, and let a and b be rays bounding it. Let θ be the angle between them. We may assume U to be given an orientation and θ positive, a to b.



roots-images/rays.eps

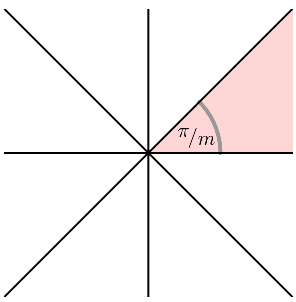
We may choose our coordinate system so that a is the positive x-axis. The product $\tau = s_b s_a$ is a rotation through angle 2θ. The line $\tau^k b$ will lie in $\mathcal{L}$ and lie at angle $2k\theta$. Since $\mathcal{L}$ is finite, $2m\theta$ must be $2\pi p$ for some positive integers m, p and

$$\theta = \frac{\pi p}{m}$$

where we may assume $p < m$ relatively prime.

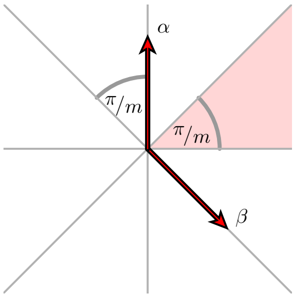
I claim that $p = 1$. Suppose k to be inverse to p modulo m, say $kp = 1 + Nm$. The ray $\tau^k a$ will then lie at angle $\pi/m + N\pi$. Since the angle of a line is only determined up to π, if $p \neq 1$ this gives us a line through the interior of C, a contradiction. Therefore $\theta = \pi/m$ for some integer $m > 1$.

There are m lines in the whole collection. In the following figure, $m = 4$.



roots-images/cox2.eps

Suppose that α and β are vectors perpendicular to a and b, respectively, and on the sides indicated in the diagram:



roots-images/rank2.eps

Then the angle between α and β is $\pi - \pi/m$, and hence:

[rank-two] **2.1. Proposition.** *Suppose C to be a connected component of the complement of the lines in $\mathcal{L}$. If*

$$C = \{\alpha > 0\} \cap \{\beta > 0\}$$

then

$$\alpha \bullet \beta \le 0\,.$$

It is 0 if and only if the rays a and b are perpendicular.

Since $\langle \alpha, \beta^\vee \rangle = 2(\alpha \bullet \beta / \beta \bullet \beta)$:

[abvee] **2.2. Corollary.** *Under the same hypothesis*

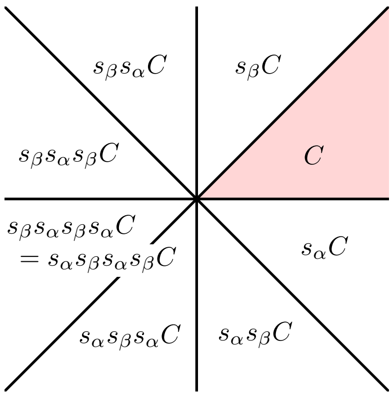
$$\langle \alpha, \beta^\vee \rangle, \langle \beta, \alpha^\vee \rangle \le 0\,.$$

In all cases, the region C is a fundamental domain for W.

As the following figure shows, the generators s_α and s_β satisfy the **braid relation**

$$s_\alpha s_\beta \ldots = s_\beta s_\alpha \ldots \qquad (m \text{ terms on each side})\,.$$

This also follows from the **rotation relation** $(s_\beta s_\alpha)^m = 1$, since the s_α, s_β are involutions. Let W_* be the abstract group with generators σ_α, σ_β and relations $\sigma_*^2 = 1$ as well as the rotation relation. The map $\sigma_\alpha \mapsto s_\alpha$, $\sigma_\beta \mapsto s_\beta$ is a homomorphism, even a surjection.



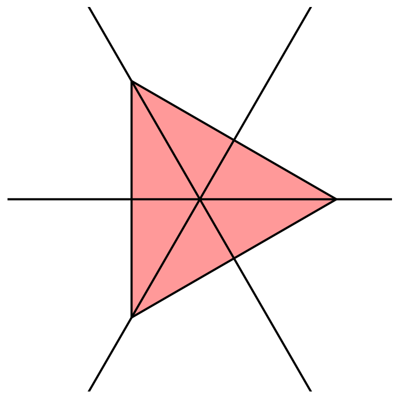
roots-images/braid2.eps

[braid2] **2.3. Proposition.** *This map from W_* to W is an isomorphism.*

Proof. Any word w in s_α and s_β may be turned into one of the form $s_\alpha s_\beta \ldots$ or $s_\beta s_\alpha \ldots$ by deletions of redundant reflections. But $m_{\alpha,\beta}$ is the order of $s_\alpha s_\beta$. $\square$

Conversely, to each $m > 1$ there exists an essentially unique configuration $\mathcal{L}$ satisfying the conditions under consideration, namely the lines at angles $k\pi/m$. The group generated by the reflections in lines of $\mathcal{L}$ has order $2m$. It may also be described as the symmetry group of a regular polygon of order m.

The lines of symmetry of any regular polygon form a planar root configuration. These are exactly the systems explored earlier. The lines at right, for the equilateral triangle, are the lines of root reflection for the case $m = 3$.



INTEGRAL ROOT CONFIGURATIONS IN THE PLANE. For the rest of this section, suppose $(V, \Sigma, V^\vee, \Sigma^\vee)$ to be a root system in the plane that actually spans the plane. The subspaces $\langle v, \lambda^\vee \rangle = 0$ are lines, and the set of all of them is a finite set of lines stable with respect to the reflections s_λ. The initial working assumption of this section holds, but that the collection of lines arises from a root system imposes severe restrictions on the integer m.

With the wedge C chosen as earlier, again let α and β be roots such that C is where $\alpha \bullet v > 0$ and $\beta \bullet v > 0$. The matrices of the corresponding reflections with respect to the basis (α, β) are

$$s_\alpha = \begin{bmatrix} -1 & -\langle \beta, \alpha^\vee \rangle \\ 0 & 1 \end{bmatrix}, \quad s_\beta = \begin{bmatrix} 1 & 0 \\ -\langle \alpha, \beta^\vee \rangle & -1 \end{bmatrix}$$

and that of their product is

$$s_\alpha s_\beta = \begin{bmatrix} -1 & -\langle \beta, \alpha^\vee \rangle \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\langle \alpha, \beta^\vee \rangle & -1 \end{bmatrix} = \begin{bmatrix} -1 + \langle \alpha, \beta^\vee \rangle \langle \beta, \alpha^\vee \rangle & \langle \beta, \alpha^\vee \rangle \\ -\langle \alpha, \beta^\vee \rangle & -1 \end{bmatrix}.$$

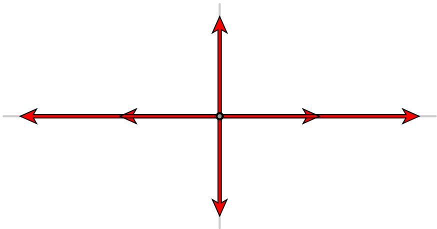
This product must be a non-trivial Euclidean rotation. Because it must have eigenvalues of absolute value 1, its trace $\tau = -2 + \langle \alpha, \beta^\vee \rangle \langle \beta, \alpha^\vee \rangle$ must satisfy the inequality

$$-2 \le \tau < 2\,,$$

which imposes the condition

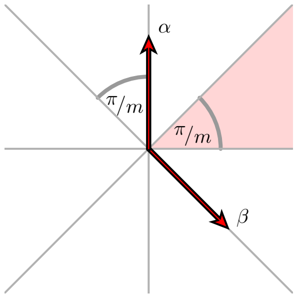
$$0 \le n_{\alpha,\beta} = \langle \alpha, \beta^\vee \rangle \langle \beta, \alpha^\vee \rangle < 4\,.$$

But $n_{\alpha,\beta}$ must also be an integer. Therefore it can only be 0, 1, 2, or 3. It will be 0 if and only if s_α and s_β commute, which means that Σ is the orthogonal union of two rank one systems:



roots-images/a1xsu2.eps

So now suppose the root system to be irreducible. Recall the picture:



roots-images/rank2.eps

Here, $\alpha \bullet \beta$ will actually be negative. By switching α and β if necessary, we may assume that one of these cases is at hand:

- $\langle \alpha, \beta^\vee \rangle = -1$, $\langle \beta, \alpha^\vee \rangle = -1$;
- $\langle \alpha, \beta^\vee \rangle = -1$, $\langle \beta, \alpha^\vee \rangle = -2$;
- $\langle \alpha, \beta^\vee \rangle = -1$, $\langle \beta, \alpha^\vee \rangle = -3$.

Since

$$\langle \alpha, \beta^\vee \rangle = 2 \left(\frac{\alpha \bullet \beta}{\beta \bullet \beta} \right), \quad \langle \beta, \alpha^\vee \rangle = 2 \left(\frac{\beta \bullet \alpha}{\alpha \bullet \alpha} \right)$$
$$\|\alpha\|^2 \langle \alpha, \beta^\vee \rangle = \|\beta\|^2 \langle \beta, \alpha^\vee \rangle$$

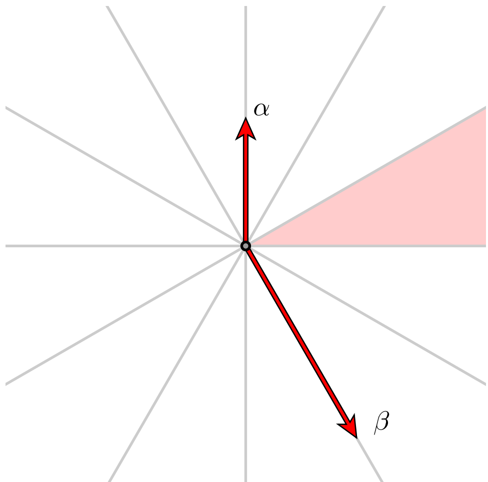
we also have

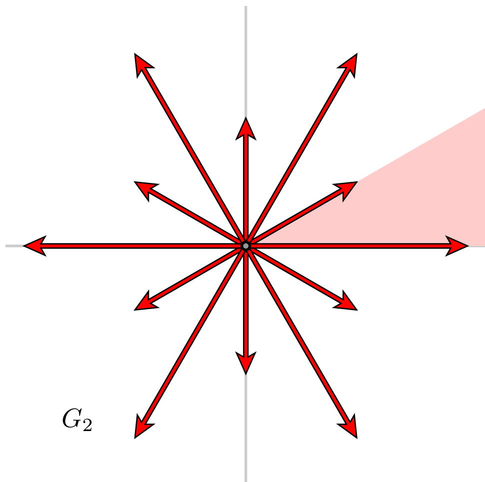
$$\frac{\|\beta\|^2}{\|\alpha\|^2} = \frac{\langle \beta, \alpha^\vee \rangle}{\langle \alpha, \beta^\vee \rangle} .$$

Let's now look at one of the three cases, the last one. We have $\langle \beta, \alpha^\vee \rangle = -3$ and $\langle \alpha, \beta^\vee \rangle = -1$. Therefore $\|\beta\|^2 / \|\alpha\|^2 = 3$. If φ is the angle between α and β,

$$\cos \varphi = \frac{\alpha \bullet \beta}{\|\alpha\| \, \|\beta\|} = \frac{1}{2} \, \frac{\langle \alpha, \beta^\vee \rangle \|\beta\|}{\|\alpha\|} = -\sqrt{3}/2 \, .$$

Thus $\varphi = \pi - \pi/6$ and $m = 6$. Here is the figure and its saturation:





roots-images/g2basis.eps
roots-images/g2roots.eps

This system is called G_2.

Taking the possibility of non-reduced roots into account, we get all together three more possible irreducible systems:

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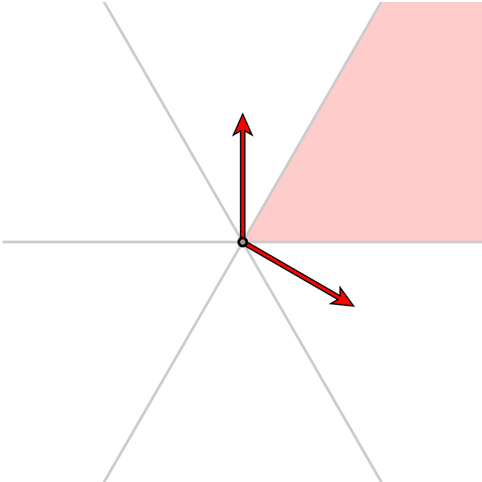

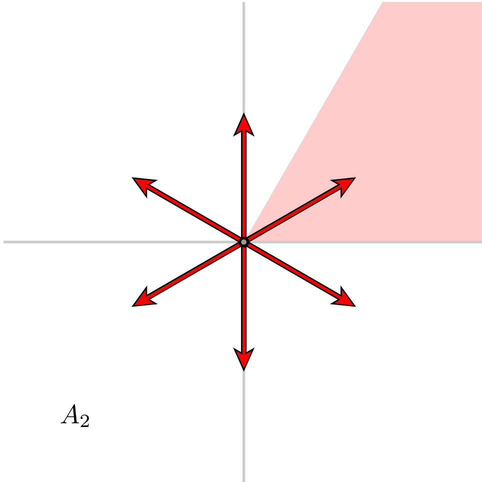

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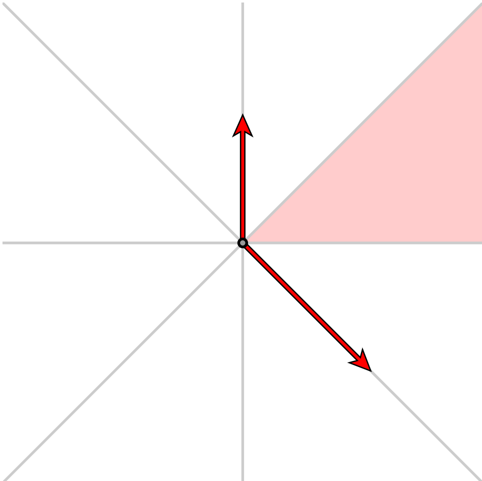

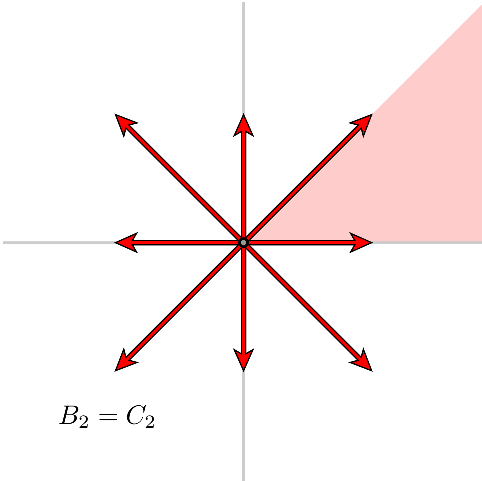

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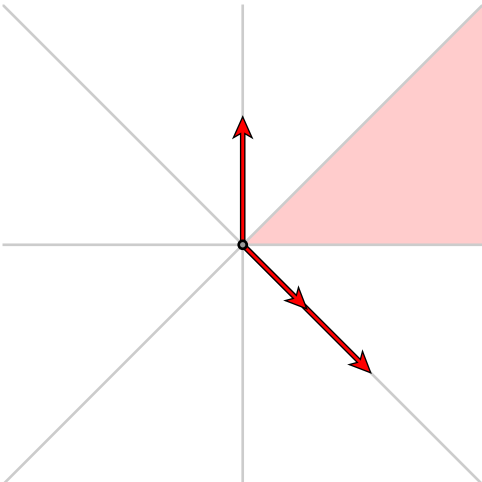

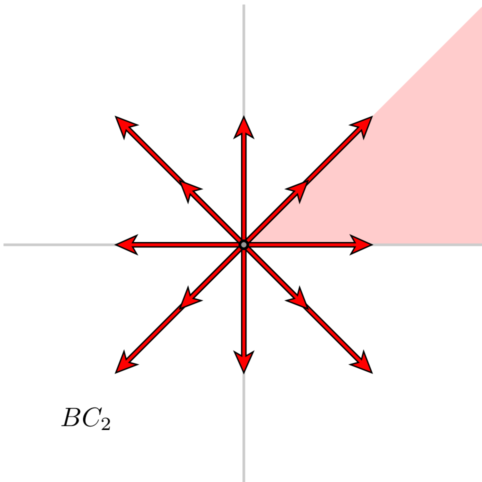

The first three are reduced.

In summary:

[W-roots] **2.4. Proposition.** *Suppose that* $(V, \Sigma, V^\vee, \Sigma^\vee)$ *is a root system of semi-simple rank two. Let* C *be one of the complements of the root reflection lines, equal to the region* $\alpha > 0$ $\beta > 0$ *for roots* α, β. *Swapping* α *and* β *if necessary, we have one of these cases:*

$$\begin{array}{llll}
\langle\alpha,\beta^\vee\rangle = 0, & \langle\beta,\alpha^\vee\rangle = 0, & \|\beta\|/\|\alpha\| \textit{ indeterminate}, & m_{\alpha,\beta} = 2\,; \\
\langle\alpha,\beta^\vee\rangle = -1, & \langle\beta,\alpha^\vee\rangle = -1, & \|\beta\|/\|\alpha\| = \phantom{\sqrt{}}1, & m_{\alpha,\beta} = 3\,; \\
\langle\alpha,\beta^\vee\rangle = -1, & \langle\beta,\alpha^\vee\rangle = -2, & \|\beta\|/\|\alpha\| = \sqrt{2}, & m_{\alpha,\beta} = 4\,; \\
\langle\alpha,\beta^\vee\rangle = -1, & \langle\beta,\alpha^\vee\rangle = -3, & \|\beta\|/\|\alpha\| = \sqrt{3}, & m_{\alpha,\beta} = 6\,.
\end{array}$$

There is another way to see the restriction on the possible values of m for planar root systems. The rotations $s_\alpha s_\beta$ are of order m and act on the lattice L spanned by the roots, thus making that $L_\mathbb{Q} = L \otimes \mathbb{Q}$ into a vector space over the field of m-roots of unity, necessarily of dimension one. This field must therefore be a quadratic extension of $\mathbb{Q}$.