Continuous characters of real tori

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In this essay I hope to explain the classification of characters of algebraic tori defined over $\mathbb{R}$, particularly Langlands' parametrization in terms of homomorphisms from the Weil group into the associated L-group. I'll follow more or less the treatment in §§2 and 3 of [Langlands:1974/1989], avoiding subsequent improvements. The only slightly new thing is how I relate Langlands' formula for characters more directly to the structure of a real torus.

Langlands' classification for tori is a special case of his results for more general reductive groups. The basic idea can be stated very briefly for an important subset of cases. Suppose F to be an arbitrary local field, $\overline{F}$ a separable algebraic closure. Let W_F be the associated Weil group. For $\mathfrak{p}$-adic fields this is a dense subgroup of the Galois group $\mathcal{G}(\overline{F}/F)$, and I'll recall the definition for $\mathbb{R}$ and $\mathbb{C}$ later on.

Suppose G to be a quasi-split reductive group defined over F. To it is associated a connected complex group $G^\vee$ whose root datum is dual to that of G, and the structure of G determines an algebraic action of $\mathcal{G}(\overline{F}/F)$ on $G^\vee$. Define ${}^LG$ to be the semidirect product $G^\vee \rtimes W_F$. Langlands has conjectured a partition of a large part of the set of irreducible representations of $G(F)$ into finite packets (called L packets, because each constituent in a packet is associated to the same local L-function). The conjecture is that these packets are parametrized by certain continuous homomorphisms from W_F to ${}^LG$ compatible with the projection from ${}^LG$ to W_F. The bijection should relate in a natural way to the local factors of L-functions Langlands associates to automorphic forms. Indeed, one main point of Langlands' conjectures is to relate representations to local L-functions. If F is $\mathbb{C}$ or $\mathbb{R}$ this partitions the entire set of irreducible representations, but if F is $\mathfrak{p}$-adic one must introduce a group slightly larger than W_F in order to account for certain ℓ-adic representations of the Galois group.

In this essay I'll be concerned only with tori, but in some version, just not the current one, discuss some applications to certain features of the general conjecture.

Another, much briefer, survey of the same material can be found in Chapter III of [Borel:1979]. A very different approach can be found in [Adams-Vogan:1992].

Contents

Check: $a = e^{\pi i \alpha}$; discrete series? Gamma functions?

1. Complex tori

In this section I'll recall some general facts about algebraic tori, show what happens in particular for complex tori, and then prove Langlands' theorem for them.

Langlands' parametrization of characters of complex tori is essentially just a matter of definition. The Weil group $W_\mathbb{C}$ is just $\mathbb{C}^\times$, and all tori defined over $\mathbb{C}$ are just products of copies of the multiplicative group $\mathbb{G}_m$. But, before I explain Langlands' parametrization in this simple case, I'll recall relevant definitions.

ALGEBRAIC CHARACTER AND COCHARACTER GROUPS. For the moment, let F be an arbitrary field. A **split algebraic torus** defined over F is an algebraic group isomorphic to a product of copies of the multiplicative group $\mathbb{G}_m$. Its group of points rational over F is isomorphic to a product of copies of $F^\times$.

If T is a split algebraic torus of dimension n defined over F, the lattice $X^*(T)$ is the group of algebraic characters—i.e. algebraic homomorphisms from T to $\mathbb{G}_m$. Given coordinates on T, this becomes isomorphic to $\mathbb{Z}^n$, with (n_i) corresponding to

$$(x_i) \longmapsto \prod x_i^{n_i} .$$

Let $X_*(T)$ be the free $\mathbb{Z}$-module dual to $X^*(T)$. This may be identified canonically with the cocharacter group of T, the group of algebraic homomorphisms from $\mathbb{G}_m$ to T, according to the rule

$$\lambda(\mu^\vee(x)) = x^{\langle \lambda, \mu^\vee \rangle} \quad (x \in \mathbb{C}^\times) .$$

The torus T is completely specified by $X^*(T)$. Its affine ring $A_F[T]$ is the group ring $F[X^*(T)]$, which is also $F[x_i^{\pm 1}]$. If S is another split torus defined over F, then the group of algebraic homomorphisms from S to T may be identified with the group of homomorphisms from $X^*(T)$ to $X^*(S)$, or equivalently from $X_*(S)$ to $X_*(T)$. Explicitly, if f is an algebraic homomorphism from S to T, it determines maps f_*, f^* from $X_*(S)$ to $X_*(T)$ and $X^*(T)$ to $X^*(S)$:

$$\begin{aligned} [f_*\lambda^\vee](x) &= f(\lambda^\vee(x)) \ \text{ for any } \lambda^\vee \in X_*(S), x \in F^\times \\ [f^*\lambda](x) &= \lambda(f(x)) \ \text{ for any } \lambda \in X^*(T), x \in S(F) \end{aligned}$$

.

Either of the maps f_* or f^* determines f completely, and algebraic homomorphisms of tori are in bijection with maps of character and cocharacter groups.

If F is algebraically closed, then any torus defined over F is split. If F is not algebraically closed, algebraic tori over F are classified easily by Galois descent. I'll recall in a later section how this works for $\mathbb{R}$.

CONTINUOUS CHARACTERS OF COMPLEX TORI. In this essay, a continuous character of a torus over $\mathbb{R}$ or $\mathbb{C}$ will be a continuous homomorphism into $\mathbb{C}^\times$. I'll explain in this section how to parametrize all of these for complex tori.

First I'll show what happens for $T = \mathbb{G}_m$. The group $\mathbb{C}^\times$ can be factored as $\mathbb{S} \times \mathbb{R}^\times_{>0}$, say $z = (z/\|z\|) \times \|z\|$, where $\|x + iy\| = \sqrt{x^2 + y^2}$. The continuous characters of $\mathbb{S}$ are all of the form $z \mapsto z^n$ for n in $\mathbb{Z}$, while those of the positive real numbers are of the form $x \mapsto x^s$ for s in $\mathbb{C}$. So the continuous characters of $\mathbb{C}^\times$ are of the form

$$z \longmapsto (z/\|z\|)^n \cdot \|z\|^s$$

for unique s in $\mathbb{C}$, n in $\mathbb{Z}$. But

$$(z/\|z\|)^n \|z\| = z^{s+n/2} \overline{z}^{s-n/2} .$$

so we can also parametrize such characters by pairs λ, μ in $\mathbb{C}$ such that $\lambda - \mu$ is an integer. One can also see this in terms of the exponential map $x \mapsto e^x$ from $\mathbb{C}$ to $\mathbb{C}^\times$. The $\mathbb{R}$-linear maps from $\mathbb{C}$ to $\mathbb{C}$ are of the form

$$x \longmapsto \lambda x + \mu \overline{x}$$

for λ, μ in $\mathbb{C}$. Composing this with the exponential gives a homomorphism from $\mathbb{C}$ to $\mathbb{C}^\times$:

$$x \longmapsto e^{\lambda x} e^{\mu \overline{x}} .$$

This will factor through the exponential map taking x to e^x if and only if $\lambda y + \mu \overline{y} = (\lambda - \mu) y$ lies in $2\pi i \mathbb{Z}$ for every y in $2\pi i \mathbb{Z}$. But this happens if and only if $\lambda - \mu$ lies in $\mathbb{Z}$. The final character will be written

$$\chi^{\lambda,\mu}(z) = z^\lambda \overline{z}^\mu .$$

For a complex algebraic torus T? The vector space $\mathbb{C} \otimes X_*(T)$ may be identified with $\mathrm{Hom}(X^*(T), \mathbb{C})$. There is an exponential map from $\mathbb{C} \otimes X_*(T)$ to T, characterized by the formula

$$\lambda(e^x) = e^{\langle \lambda, x \rangle}$$

for all λ in $X^*(T)$. The kernel of this exponential is $2\pi i X_*(T)$.

For any λ, μ in $\mathbb{C}\otimes X^*(T)$

$$x \longmapsto \langle\lambda, x\rangle + \langle\mu, \overline{x}\rangle$$

is an $\mathbb{R}$-linear map from $\mathbb{C}\otimes X_*(T)$ to $\mathbb{C}$. It descends to define the character of T

(1.1) $$e^x \longmapsto \chi^{\lambda,\mu}(e^x) = e^{\langle\lambda,x\rangle}e^{\langle\mu,\overline{x}\rangle}$$

if and only if $\lambda-\mu$ lies in $X^*(T)$. If λ, μ lie in $\mathbb{C}\otimes X^*(T)$, I call them a **coherent pair** if $\lambda-\mu$ lies in $X^*(T)$.

1.2. Proposition. *The map taking (λ,μ) to $\chi^{\lambda,\mu}$ is a bijection of the group of coherent pairs with the group of continuous characters of T.*

Another way to characterize $\chi^{\lambda,\mu}$ is to note that

$$\langle\chi^{\lambda,\mu}, \alpha^\vee(z)\rangle = z^{\langle\lambda,\alpha^\vee\rangle}\overline{z}^{\langle\mu,\alpha^\vee\rangle}$$

for all $\alpha^\vee$ in $X_*(T)$.

We shall also want to know the continuous homomorphisms from $\mathbb{C}^\times$ to T. These are of the dual form

$$z \longmapsto z^\lambda\overline{z}^\mu \quad \big(\lambda,\mu\in\mathbb{C}\otimes X_*(T), \lambda-\mu\in X_*(T)\big)\,.$$

Langlands' classification is now, as I said before, a matter of definition. Suppose T to be an algebraic torus defined over $\mathbb{C}$. Its dual $T^\vee$ is the complex torus such that $X^*(T^\vee)=X_*(T)$. If a homomorphism

$$\varphi\colon W_{\mathbb{C}} = \mathbb{C}^\times \longrightarrow T^\vee(\mathbb{C})$$

is specified by the pair λ, μ then the character of T is characterized by the same pair. This makes sense precisely because $X^*(T)=X_*(T^\vee)$.

As I have already said, Langlands' parametrization of characters of complex tori is essentially just a matter of definition.

2. The classification of real tori

Let $T=(\mathbb{G}_m)^n$. According to the theory of Galois descent, together with the observation that all automorphisms of a split torus are rational, isomorphism classes of tori of dimension n defined over $\mathbb{R}$ are in bijection with conjugacy classes of involutions in $\mathrm{GL}(X^*(T))$. If θ^* is such an involution, let θ be the corresponding involution of $T(\mathbb{C})$, and θ_* that of $X_*(T)$. The corresponding conjugation of $T(\mathbb{C})$ is

$$\sigma=\sigma_\theta\colon z\longmapsto \overline{z}^\theta\,.$$

The $\mathbb{R}$-rational points of the real torus defined by it are those z in $T(\mathbb{C})$ such that $z=z^\sigma$.

The simplest tori over $\mathbb{R}$ are the split ones, for which $\theta=I$. Other tori are classified, in effect, by comparison with this one.

What do one-dimensional real tori look like? There are two involutions in $\mathrm{GL}_1(\mathbb{Z})$, hence two real tori of dimension one. One is the split torus $\mathbb{G}_m$, the other is $\mathbb{S}$, whose group of real points is isomorphic to the group of z in $\mathbb{C}$ with $\|z\|=1$. In the second case, the associated involution is scalar multiplication by -1 in $\mathbb{Z}=X^*(T)$, and the points in $T(\mathbb{R})$ are the z in $\mathbb{C}$ with $z=\overline{z}^{-1}$. As a real variety, this is the set of (x,y) in $\mathbb{R}^2$ such that $x^2+y^2=1$. The involution swaps the two complex characters $x\pm iy$, which are inverses of each other.

Another real torus is that whose group of real points is isomorphic to $\mathbb{C}^\times$. It is the group $R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m)$ obtained from $\mathbb{G}_m$ by restriction of scalars from $\mathbb{C}$ to $\mathbb{R}$. More explicitly, it may be identified with the group of real (x, y, z) such that

$$(x^2 + y^2) \cdot z = 1$$

Its real dimension is two. The lattice of characters is spanned by the maps $(x, y, z) \mapsto x \pm iy$, and the corresponding involution in $X^*(T)$ is a swap of factors in $\mathbb{Z}^2$.

I can now give a complete classification of two-dimensional tori.

2.1. Proposition. *Any involution in* $\mathrm{GL}_2(\mathbb{Z})$ *is conjugate to exactly one of the following:*

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

2.2. Corollary. *Every two-dimensional real torus is a product of one-dimensional tori or isomorphic to the restriction of* $\mathbb{G}_m$ *from* $\mathbb{C}$ *to* $\mathbb{R}$*.*

Proof of the Proposition. Let $L = \mathbb{Z}^2$, let $L_\pm$ be the intersection of L with the eigenspace on which the involution ι acts by ± 1, and let $M = L_+ + L_-$ with basis $(e_\pm)$. It is a direct sum, and since

$$2v = (v + \theta(v)) + (v - \theta(v))$$

we see that $2L \subseteq M$. It cannot happen that $M = 2L$, since the $e_\pm$ are primitive vectors in L. If L is not equal to M, then according to the principal divisor theorem there exists a basis λ, μ for L for which 2λ, μ is a basis for M. We then have

$$\begin{aligned} 2\lambda &= ae_- + be_+ \\ \mu &= ce_- + de_+ \end{aligned} \qquad \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(\mathbb{Z}) \right).$$

In particular, a and b, a and c are relatively prime. This implies that e_+ is in the lattice spanned by 2λ and aL. If a were even, then e_+ would lie in $2L$, which would mean that e_+ is not a basis element of L_+. Hence a is odd, and a is relatively prime to $2c$.

We have

$$\begin{aligned} (\theta - I)\lambda &= a\, e_- \\ (\theta - I)\mu &= 2c\, e_- \,. \end{aligned}$$

Therefore

$$(\theta - I)\alpha = 0 \quad (\alpha = -2c\lambda + a\mu)\,.$$

Since a and $2c$ are relatively prime, α is part of a basis (α, β) of L. Since ι has order two, ι swaps β and $\theta(\beta)$. If these formed a basis of L, we'd be through. The matrix of ι in the basis (α, β) must be of the form

$$\begin{bmatrix} 1 & m \\ 0 & -1 \end{bmatrix}.$$

Say $m = 2n + \varepsilon$, with $\varepsilon = 0$ or 1. Then

$$\theta(\beta) = -\beta + (2n + \varepsilon)\alpha, \quad \theta(\beta - n\alpha) = -(\beta - n\alpha) + \varepsilon\alpha\,.$$

If m were even (and $\varepsilon = 0$) this is similar to a diagonal matrix, so $\varepsilon = 1$. If $\gamma = \beta - n\alpha$ then γ and $\iota(\gamma) = \gamma + \alpha$ form a basis of L and are swapped by ι. ▯

A slightly more complicated argument (which I learned from Fokko du Cloux) proves this generalization:

2.3. Proposition. *If* ι *is an involution on* $\mathbb{Z}^n$*, then* $\mathbb{Z}^n$ *decomposes into a direct sum of lattices on which* ι *acts by one of (a) the identity; (b) multiplication by* -1*; (c) swaps of two copies of* $\mathbb{Z}$*.*

A proof of this can be found in [Casselman:2008]) I'll include a slightly expanded version of it in an appenidx.

2.4. Corollary. *Every real torus is a direct product of copies of one of the three basic tori* $\mathbb{G}_m$, $\mathbb{S}$ *or* $R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m)$.

The factors $T_\mathbb{R}$, $T_\mathbb{S}$, and $T_\mathbb{C}$ themselves in this result are not unique. This can be shown already by the torus $\mathbb{C}^\times \times \mathbb{R}^\times$, in which any image of the map

$$t \longmapsto (t^n, t)$$

may be taken as $T_\mathbb{R}$. However, their dimensions are an invariant of the torus. The following is easily verified by examining the three basic cases.

2.5. Lemma. *Suppose* T *to be an algebraic torus defined over* $\mathbb{R}$. *Then*

(a) the dimension of $T_\mathbb{R}$ *is the dimension over* $\mathbb{F}_2$ *of*

$$\frac{\mathrm{Ker}(I-\theta^*)}{\mathrm{Im}(I+\theta^*)};$$

(b) the dimension of $T_\mathbb{S}$ *is the dimension over* $\mathbb{F}_2$ *of*

$$\frac{\mathrm{Ker}(I+\theta^*)}{\mathrm{Im}(I-\theta^*)}.$$

The classification of du Cloux is not essential in this business, but it makes some proofs much simpler. I'll use it in a very straightforward way—to verify case by case formulas that are stated in general terms.

Here is another useful consequence of du Cloux' result. If χ lies in $X_*(T)$ then $\chi(-1)$ is an element of order two in $T(\mathbb{C})$. If $\chi = \chi^\theta$ then it lies in $T(\mathbb{R})$. Let T° be the connected component of $T(\mathbb{R})$. It is the image of the map $t \mapsto t^{1+\theta}$ from $T(\mathbb{C})$ to itself.

2.6. Proposition. *The map taking* χ *to* $\chi(-1)$ *induces an isomorphism of* $\mathrm{Ker}(I-\theta_*)$ *modulo* $\mathrm{Im}(I+\theta_*)$ *with* $T(\mathbb{R})/T^\circ$.

Proof. Apply du Cloux' theorem.

Remark. I can now say something about continuous characters of real tori. If χ is a character in $X^*(T)$ such that $\chi^{\theta^*} = \chi^{-1}$ then its restriction to the 2-torsion τ_2 of $T(\mathbb{R})$ is a surjection onto the characters of τ_2. This induces an isomorphism of the dual of the component group of $T(\mathbb{R})$ with $\mathrm{Ker}(\theta^*+I)/\mathrm{Im}(\theta^*-I)$. This is the simplest case of a much more general result found in §3.3 of [Kottwitz:1984], that is valid for all local fields, about the characters of local tori of finite order.

3. Characters

Suppose T to be a real torus defined by the involution θ on $T(\mathbb{C})$, so that

$$T(\mathbb{R}) = \{t \in T(\mathbb{C}) \,|\, t = \overline{t}^\theta\}.$$

Let $X_* = X_*(T)$, $X^* = X^*(T)$.

THE EXPONENTIAL MAP. The points of $T(\mathbb{R})$ are conveniently parametrized by the exponential map. Let

$$\mathfrak{t}_\mathbb{C} = \mathbb{C} \otimes X_*.$$

For x in $\mathfrak{t}_\mathbb{C}$ define its exponential in $T(\mathbb{C})$ to be e^x. It is characterized, as I have already mentioned, by the equation

$$\lambda(e^x) = e^{\langle\lambda, x\rangle}$$

for λ in X^*. The kernel of this map is $2\pi i X_*$. The inverse image of $T(\mathbb{R})$ is the set

$$\mathfrak{t}_\mathbb{R} = \left\{x \in \mathfrak{t}_\mathbb{C} \,\middle|\, x - \overline{x}^{\theta_*} \in 2\pi i X_*\right\},$$

and

$$T(\mathbb{R}) \cong \frac{\mathfrak{t}_{\mathbb{R}}}{\mathfrak{t}_{\mathbb{R}} \cap 2\pi i X_*} .$$

Examples. (a) Let T be $\mathbb{R}^{\times}$. Then $X_*(T) = \mathbb{Z}$, $\theta = I$, $\mathbb{C} \otimes X_* = \mathbb{C}$, and

$$\mathfrak{t}_{\mathbb{R}} = \mathbb{R} \oplus (\pi i + \mathbb{R}) .$$

(b) Let T be $\mathbb{S}$. Then $X_*(T) = \mathbb{Z}$, $\theta = -I$, $\mathbb{C} \otimes X_* = \mathbb{C}$, and

$$\mathfrak{t}_{\mathbb{R}} = 2\pi i \mathbb{R} .$$

(c) Let $T = \mathbb{C}^X$. Then $X_*(T) = \mathbb{Z}^2$, θ swaps coordinates, $\mathbb{C} \otimes X_* = \mathbb{C}^2$, and

$$\mathfrak{t}_{\mathbb{R}} = \{(z, w) \mid z = \overline{w}\} .$$

The group $T(\mathbb{R})$ may not be connected. We have an exact sequence

$$1 \longrightarrow T^\circ \longrightarrow T(\mathbb{R}) \longrightarrow T(\mathbb{R})/T^\circ \longrightarrow 1 .$$

Each of the outer terms has a relatively simple description. The connected component T° is the image with respect to the exponential map of the connected component

$$\mathfrak{t}^\circ_{\mathbb{R}} = \{x \in \mathfrak{t}_{\mathbb{C}} \mid x = \overline{x}^\theta\}$$

of $\mathfrak{t}_{\mathbb{R}}$. The exponential map induces an isomorphism

$$T^\circ \cong \frac{\mathfrak{t}^\circ_{\mathbb{R}}}{\mathfrak{t}^\circ_{\mathbb{R}} \cap 2\pi i X_*(T)} .$$

The component group $T(\mathbb{R})/T^\circ$ has several interesting descriptions. One describes it in terms of du Cloux' factorization:

3.1. Proposition. *Suppose T to be a real torus, T° the connected component of $T(\mathbb{R})$. The embedding of $T_{\mathbb{R}}(\mathbb{R})$ into $T(\mathbb{R})$ induces an isomorphism of $T_{\mathbb{R}}(\mathbb{R})/T^\circ_{\mathbb{R}}$ with $T(\mathbb{R})/T^\circ$.*

Proof. The groups $T_{\mathbb{S}}$ and $T_{\mathbb{C}}$ are connected.

Hence $T(\mathbb{R})/T^\circ$ is a product of copies of $\{\pm 1\}$.

It also follows from du Cloux' theorem that the canonical projection from T to T/T° splits. No splitting is canonical, since the factorization is not canonical.

If λ lies in X_* and $\lambda^\theta = \lambda$, then $\lambda(-1)$ lies in $T(\mathbb{R})$. If $\lambda = \mu^{I+\theta}$ then $\lambda(-1)$ lies in T°. The following may be verified by applying du Cloux' factorization:

3.2. Lemma. *The map taking λ to $\lambda(-1)$ induces an isomorphism*

$$\frac{\mathrm{Ker}(I - \theta_*)_{X_*}}{\mathrm{Im}(I + \theta_*)_{X_*}} \cong T(\mathbb{R})/T^\circ .$$

If $\lambda = \lambda^{\theta_*}$ and $x = \pi i \lambda$ in $\mathfrak{t}_{\mathbb{C}}$ then $e^x = \lambda(-1)$. Furthermore, $\overline{x}^\theta = -x$.

Here is another way to see what $T(\mathbb{R})/T^\circ$ looks like. Given x in $\mathfrak{t}_{\mathbb{R}}$, the difference $x - \overline{x}^{\theta_*} = x - \sigma_*(x)$ lies in $2\pi i X_*$. We get thus a map

(3.3)
$$\mathfrak{t}_{\mathbb{R}} \longrightarrow \mathrm{Ker}(I + \sigma_*)_{2\pi i X_*} .$$

Du Cloux' classification shows immediately:

3.4. Corollary. *The map in (3.3) induces an isomorphism*

$$T(\mathbb{R})/T^\circ \cong \frac{\mathrm{Ker}(I+\sigma_*)_{2\pi i X_*}}{\mathrm{Im}(I-\sigma_*)_{2\pi i X_*}}\,.$$

Proof. It might be worthwhile pointing out a direct proof of this. The following sequence is exact:

$$0 \longrightarrow 2\pi i\mathbb{Z} \longrightarrow \mathbb{C} \xrightarrow{\exp} \mathbb{C}^\times \longrightarrow 1$$

Tensoring with X_* we get

$$0 \longrightarrow 2\pi i X_* \longrightarrow \mathbb{C}\otimes X_* \longrightarrow T(\mathbb{C}) \longrightarrow 1\,.$$

If $\mathcal{G}$ is the Galois group of $\mathbb{C}/\mathbb{R}$, the associated long exact cohomology sequence gives us

$$0 \longrightarrow (2\pi i X_*)^{\mathcal{G}} \longrightarrow (\mathbb{C}\otimes X_*)^{\mathcal{G}} \longrightarrow T(\mathbb{R}) \longrightarrow H^1(\mathcal{G}, 2\pi i X_*) \longrightarrow 0\,.$$

This gives the isomorphism

$$T(\mathbb{R})/T^\circ \cong H^1(\mathcal{G}, 2\pi i X_*)\,,$$

which is just a slightly different formulation of the corollary. ▯

DUALITY. What do these results imply about the continuous characters of $T(\mathbb{R})$? The basic fact we start from is that the map from continuous characters of $T(\mathbb{C})$ to those of $T(\mathbb{R})$ is surjective. A character of $T(\mathbb{C})$ is of the form

$$\chi^{\lambda,\mu}\colon\ t \longmapsto t^\lambda \overline{t}^\mu$$

for λ, μ in $\mathbb{C}\otimes X^*$ such that $\lambda-\mu\in X^*$. In terms of the covering space $\mathfrak{t}_\mathbb{C}$, we can express this as

$$x \longmapsto \langle\lambda, x\rangle + \langle\mu, \overline{x}\rangle\,.$$

But the map $x\mapsto \overline{x}^{\theta_*}$ is a real automorphism of order two, and we can express the eigenvector decomposition as

$$x = \frac{x+\overline{x}^{\theta_*}}{2} + \frac{x-\overline{x}^{\theta_*}}{2} = x_+ + x_- \quad (x_+ = \overline{x}_+^{\theta_*},\ x_- = -\overline{x}_-^{\theta_*})$$

and then

$$\begin{aligned}
\langle\lambda, x\rangle + \langle\mu, \overline{x}\rangle &= \langle\lambda, x_+ + x_-\rangle + \langle\mu, \overline{x}_+ + \overline{x}_-\rangle \\
&= \langle\lambda, x_+ + x_-\rangle + \langle\mu, x_+^{\theta_*} - x_-^{\theta_*}\rangle \\
&= \langle\lambda + \mu^{\theta^*}, x_+\rangle + \langle\lambda - \mu^{\theta^*}, x_-\rangle \\
&= \langle\alpha, x_+\rangle + \langle\beta, x_-\rangle \qquad (\alpha = \lambda+\mu^\theta,\ \beta = \lambda-\mu^\theta)\,.
\end{aligned}$$

3.5. Lemma. *The map taking λ, μ such that $\lambda-\mu\in X^*$ to $\alpha=\lambda+\mu^\theta$, $\beta=\lambda-\mu^\theta$ is a bijection of the set of coherent pairs with the set of all (α,β) such that*

$$(\alpha-\alpha^\theta) - (\beta+\beta^\theta) \in 2X^*(T)\,.$$

In other words, the presence of θ gives us a new way to parametrize continuous characters of $T(\mathbb{C})$.

Proof. We have

$$\lambda = \frac{\alpha+\beta^\theta}{2}, \mu^\theta = \frac{\alpha-\beta^\theta}{2}\,,$$

and then

$$\lambda - \mu = \frac{(\alpha - \alpha^{\theta}) - (\beta + \beta^{\theta})}{2} \,.$$

In these circumstances, define

$$\chi_{\alpha,\beta}\colon\ e^x \longmapsto e^y \quad (y = \langle \alpha, (x + \overline{x}^{\theta_*})/2\rangle + \langle \beta, (x - \overline{x}^{\theta_*})/2\rangle)\,.$$

It is a character of $T(\mathbb{C})$, and we are interested in its restriction to $T(\mathbb{R})$. If x lies in $\mathfrak{t}_{\mathbb{R}}$ then $x - \overline{x}^{\theta_*} = 2\pi i\kappa$ with $\kappa = -\kappa^{\theta_*}$, so

(3.6) $$\langle \alpha, (x + \overline{x}^{\theta_*})/2\rangle + \langle \beta, (x - \overline{x}^{\theta_*})/2\rangle) = \langle \alpha, x + \pi\kappa\rangle + \langle \beta, \pi i\kappa\rangle\,.$$

If $x = \overline{x}^{\theta}$ then $\kappa = 0$ and $\chi_{\alpha,\beta}(e^x)$ depends only on α. The parameter α, in fact, has a simple interpretation.

3.7. Lemma. *Suppose (λ, μ) to be a coherent pair and $\alpha = \lambda + \mu^{\theta^*}$. Then*

(a) $\alpha - \alpha^{\theta^*} \in X^*(T)$ and
(b) $\langle \alpha, \kappa\rangle \in \mathbb{Z}$ for every κ in $X_*(T)$ such that $\kappa = -\kappa^{\theta_*}$

Proof. The first is a straightforward calculation:

$$\alpha - \alpha^{\theta^*} = (\lambda + \mu^{\theta^*}) - (\lambda + \mu^{\theta^*})^{\theta^*} = (\lambda - \mu) - (\lambda - \mu)^{\theta^*}\,.$$

As for the second, suppose $\kappa = -\kappa_*$ in X_* so that if $x = 2\pi i\kappa$ then $2\pi i x$ lies in $\mathfrak{t}^\circ \cap X_*$. Then

$$\langle \alpha, \kappa\rangle = \langle \lambda + \mu^{\theta}, \kappa\rangle = \langle \lambda - \mu, \kappa\rangle \in \mathbb{Z}\,.$$

Combining these, we see that the map

$$\chi_\alpha\colon\ x \longmapsto e^{\langle \alpha, x\rangle}$$

for x in $\mathfrak{t}^\circ_{\mathbb{R}}$ defines a continuous character of T°.

3.8. Lemma. *The map $\alpha \mapsto \chi_\alpha$ is an isomorphism of the set of α in $\mathbb{C} \otimes X^*$ satisfying the conditions of the previous Lemma with the set of continuous characters of T°.*

Proof. Case by case, applying du Cloux' factorization.

3.9. Lemma. *The map taking a coherent pair (λ, μ) to $\lambda + \mu^{\theta^*}$ is a surjection onto the set of α satisfying the conditions of Lemma 3.8.*

Proof. Apply du Cloux' factorization.

A character $\chi_{\alpha,\beta}$ determines by restriction a continuous character of $T(\mathbb{R})$. When do two pairs (α, β_1) and (α, β_2) determine the same one?

Suppose (α, β_1) to satisfy the conditions of Lemma 3.5, and let $\beta_2 = \beta_1 + (I - \theta^*)\gamma$. Then

$$(\alpha + \alpha^{\theta^*}) + (\beta_2 - \beta_2^{\theta^*}) = (\alpha + \alpha^{\theta^*}) + (\beta_1 - \beta_1^{\theta^*})\,,$$

so that (α, β_2) still satisfy the conditions of Lemma 3.5. If $x \in \mathfrak{t}_{\mathbb{R}}$ with $x - \overline{x}^{\theta_*(} = 2\pi i\kappa$ and as (3.6) tells us

$$\begin{aligned}\langle \alpha, x_+\rangle + \langle \beta_2, x_-\rangle &= \langle \alpha, x + \pi i\kappa\rangle + \langle \beta_2, \pi i\kappa\rangle \\ &= \langle \alpha, x + \pi i\kappa\rangle + \langle \beta_1 + \gamma + \gamma^{\theta^*}, \pi i\kappa\rangle\,.\end{aligned}$$

But $\kappa = -\kappa^{\theta^*}$, so

$$\langle \gamma + \gamma^{\theta}, \pi i\kappa\rangle = 0\,.$$

Therefore the restriction of χ_{α,β_2} to $\mathfrak{t}_{\mathbb{R}}$ is the same as that of χ_{α,β_1}. In fact:

3.10. Theorem. *The map taking* (α, β) *to the restriction of* $\chi_{\alpha,\beta}$ *to* $\mathfrak{t}_{\mathbb{R}}$ *is a surjection onto the group of continuous characters of* $T(\mathbb{R})$*. Pairs* (α_1, β_1) *and* (α_2, β_2) *determine the same character if and only if* $\alpha_1 = \alpha_2$ *and* $\beta_2 - \beta_1$ *is in the image of* $I + \theta^*$*.*

Proof. Apply du Cloux' factorization. ∎

A character of $T(\mathbb{R})$ is called **unramified** if its restriction to aits maximal compact subgroup is trivial. These are easily described. Suppose λ to be an element of $(\mathbb{C} \otimes X^*(T))$ fixed by θ^*. Then $\lambda - \lambda^{\theta^*} = 0$ and we may choose $c = 1$ to define a character of $T(\mathbb{R})$. If s is in the maximal compact subgroup of $T(\mathbb{R}$ then $s^\theta = s^{-1}$, and $\chi_{\lambda,c}(s) = 1$.

3.11. Proposition. *The map from* $X_{\mathrm{nr}}(T(\mathbb{R})) = (\mathbb{C} \otimes X^*(T))^{\mathcal{G}}$ *to the group of continuous characters of* $T(\mathbb{R})$ *identifies* $X_{\mathrm{nr}}(T(\mathbb{R}))$ *with the group of unramified characters of* $T(\mathbb{R})$*.*

Proof. The simplest proof checks case by case, applying du Cloux' classification. ∎

4. Admissible homomorphisms

Suppose T to be an algebraic torus defined over $\mathbb{R}$, corresponding to the involution θ. Its **Langlands dual** is the complex torus $T^\vee$ with character group $X^*(T^\vee) = X_*(T)$. Its **L-group** ${}^LT$ in this essay is the semi-direct product $T^\vee \rtimes \mathcal{G}$, generated by $1 \times \sigma \neq 1$ and a copy of $T^\vee$ such that

$$(1 \times \sigma)^2 = 1$$
$$(1 \times \sigma)(z \times 1) = (z^{\theta^\vee} \times 1)(1 \times \sigma)\,.$$

I write $z \mapsto z^{\theta^\vee}$ for the involution of $T^\vee$, but it is important to keep in mind that although T and its Langlands dual are both complex tori they play very different roles. *There is no real structure on* $T^\vee$. It is just a complex torus with an algebraic involution θ. Whereas there are two involutions of $T(\mathbb{C})$ in play, conjugation $x \mapsto \overline{x}$ as well as θ. They commute.

THE WEIL GROUP. I'll now explain Langlands' parametrization of continuous characters of $T(\mathbb{R})$. An **admissible homomorphism** from the Weil group $W_\mathbb{R}$ to ${}^LT$ that is continuous on $\mathbb{C}^\times$ and making the diagram

$$\begin{array}{ccc} W_\mathbb{R} & \longrightarrow & {}^LT \\ & \searrow \quad \swarrow & \\ & \mathcal{G} & \end{array}$$

commutative. The basic claim is that the characters of $T(\mathbb{R})$ are in bijection with conjugacy classes of admissible homomorphisms from $W_\mathbb{R}$ to ${}^LT$. We have already seen a proof of the analogous claim for complex tori.

The group $W_\mathbb{R}$ is an extension of $\mathbb{C}^\times$ by $\mathcal{G} = \mathcal{G}(\mathbb{C}/\mathbb{R}) = \{1, \sigma\}$, fitting into an exact sequence

$$1 \longrightarrow \mathbb{C}^\times \longrightarrow W_\mathbb{R} \longrightarrow \mathcal{G} \longrightarrow 1\,.$$

It is generated by the copy of $\mathbb{C}^\times$ and an element σ mapping onto the non-trivial element of $\mathcal{G}$, with relations

$$z \cdot \sigma = \sigma \cdot \overline{z}, \quad \sigma^2 = -1$$

Since -1 is not the norm of a complex number, the extension does not split. It is not a coincidence that it is isomorphic to the normalizer of a copy of $\mathbb{C}^\times$ in the unit group of the Hamilton quaternions $\mathbb{H}$. Very generally, if E/F is any finite Galois extension of local fields, then $W_{E/F}$ may be embedded into the normalizer of a copy of $E^\times$ in the multiplicative group of the division algebra over F defined by the fundamental class in the Brauer group $H^2(\mathcal{G}(E/F), E^\times)$.

The **norm map** NM from $W_\mathbb{R}$ to $\mathbb{R}^\times$ extends the norm on its subgroup $\mathbb{C}^\times$, mapping

$$z \longmapsto z\overline{z}, \quad \sigma \longmapsto -1\,.$$

It is a surjective homomorphism with kernel $\mathbb{S}$. Since $z\sigma z\sigma^{-1} = z/\overline{z}$, the subgroup $\mathbb{S}$ is the commutator subgroup of $W_\mathbb{R}$, and hence the norm map identifies $\mathbb{R}^\times$ as the maximal abelian quotient.

ADMISSIBLE HOMOMORPHISMS. Suppose given an admissible homorphism φ from $W_\mathbb{R}$ to ${}^LT$. Restricted to $\mathbb{C}^\times$ it gives a continuous homomorphism into $T^\vee$, hence is defined, as we have seen earlier, by a pair λ, μ such that $\lambda - \mu$ lies in $X_*(T^\vee)$. I recall that I call this a coherent pair. The image of σ will be of the form $c \times \sigma$, with c in $T^\vee$.

4.1. Lemma. *Suppose λ, μ to be an coherent pair, and let φ be the map from $\mathbb{C}^\times$ to $T^\vee$ that they define. Then c in $T^\vee$ defines an extension to $W_\mathbb{R}$ if and only if*

(a) $$\lambda = \mu^{\theta^*}$$

(b) $$a^{1+\theta} = (-1)^{\lambda-\mu}\,.$$

Two elements in $T^\vee$ define conjugate admissible homomorphisms if and only if their quotient is of the form $c^{1-\theta}$.

I leave the proof as exercise. The possible c may therefore be considered as cosets of the group

(4.2)
$$\frac{\mathrm{Ker}(I+\theta^\vee)}{\mathrm{Im}(I-\theta^\vee)} .$$

Of course this expression looks very similar to some in the last section.

Condition (b) imposes a restriction on λ. For example, suppose $T = \mathbb{S}$, for which θ takes z to z^{-1} and $\theta^* = -I$. The basic condition on λ is that $\lambda - \lambda^\theta = 2\lambda$ lie in $\mathbb{Z}$. But then condition (b) requires that $1 = (-1)^{2\lambda}$, which requires that 2λ be an even integer, so that λ itself must be an integer.

The simplest case is when $T = \mathbb{G}_m$. In this case ${}^LT$ is just the direct product of $\mathbb{C}^\times \times \{1, \sigma\}$ and an admissible homomorphism amounts to a character of $W_\mathbb{R}$. It also amounts to λ in $\mathbb{C}$ and a complex c such that $c^2 = 1$. However, the norm map identifies $W_\mathbb{R}/\mathbb{S}$ with $\mathbb{R}^\times$. Hence Langlands' parametrization of characters of real tori is a generalization of the well known result that the characters of $W_\mathbb{R}$ are the lifts of those of $\mathbb{R}^\times$, via the norm map.

THE PARAMETRIZATION. We now know (1) that continuous characters of $T(\mathbb{R})$ are parametrized by pairs (α, β) in $\mathbb{C} \otimes X^*(T)$ such that

(4.3)
$$(\beta + \beta^{\theta^*}) - (\alpha - \alpha^{\theta^*}) \in 2X^*(T)$$

and (2) that admissible homomorphisms are parametrized by a pair λ in $\mathbb{C} \otimes X^T$ such that $\lambda - \lambda^{\theta^*} \in X^*(T) = X_*(T^\vee)$ and c in $T^\vee$ such that

(4.4)
$$c^{1+\theta^\vee} = (-1)^{\lambda - \lambda^{\theta^*}} .$$

How are these related? Very simply—set $\lambda = \alpha$ and $c = e^{\pi i \beta}$. Then (4.3) is equivalent to (4.4). Further, we know that both parametrizations are subject to equivalences, but these turn out to be equivalent, too.

4.5. Theorem. *The map taking (α, β) to $(\lambda, e^{\pi i \beta})$ induces a bijection of continuous characters of $T(\mathbb{R})$ with admissible homomorphisms from $W_\mathbb{R}$ to ${}^LT$.*

I leave details of a proof as an exercise.

Remark. How do we know that this bijection between admissible homomorphisms and characters is in any sense the 'correct' one? For one thing, as Langlands pointed out, the bijection of this theorem and its analogue for $W_\mathbb{C}$ are compatible with restriction of scalars. The bijection is also functorial with respect to homomorphisms from one torus to another. It is certainly compatible with du Cloux's factorization, so we must only decide if it is what we desire for the basic tori. Two of these basic examples are directly related to characters of the multiplicative groups of local fields, for which all is certainly correct. There are also several natural requirements of compatibility. For example, $\mathbb{C}^\times$ is the quotient of $\mathbb{S} \times \mathbb{R}^\times$.

Finally, the assignment is consistent with that of Γ-factors to representations of the Weil group, which might be considered the principal motivation for Langlands' parametrization.

5. *Appendix.* Computing the factorization of real tori

In this appendix I am going to explain how to compute the factorization of real tori into simple components.

We are given an involution τ in $\mathrm{GL}_n(\mathbb{Z})$, and I shall decompose $\mathbb{Z}^n$ into pieces of dimension one on which τ acts as $\pm I$ and pieces of dimension two on which τ acts as a swap.

I'll begin with a preliminary result:

5.1. Lemma. *The canonical projection from* $\mathrm{SL}_n(\mathbb{Z})$ *to* $\mathrm{GL}_n(\mathbb{Z}/2) = \mathrm{SL}_n(\mathbb{Z}/2)$ *is surjective.*

This is a very simple case of the strong approximation theorem for simply connected semi-simple algebraic groups defined over Dedekind domains. In the particular case at hand it is extremely easy to prove and to implement constructively.

Proof. If F is any field, then applying Gauss-Jordan elimination, we may write any matrix in $\mathrm{GL}_n(F)$ as $n_1 w a n_2$ where the n_i are upper triangular, a is diagonal, and w is a permutation matrix. We may multiply w by a diagonal matrix with entries ± 1 if necessary to make it a monomial matrix with entries ± 1 and $\det(w) = 1$. The matrices n_i are certainly in the image of the projection, as is w. And in our case, since all units in $\mathbb{Z}/2$ are 1, a happens to be I. ▯

Let $L = \mathbb{Z}^n$, and suppose that we are given an integral matrix τ of order 2.

Let $M_\pm$ be the intersection of the ± 1 eigenspaces with L. Each of these is a summand of L. Let $M = M_+ \oplus M_-$. If $M = L$, there is no problem—we set $L_\pm = M_\pm$, $L_{\mathrm{swap}} = \{0\}$. Otherwise, things are more interesting.

Some motivation for the proof will come from understanding how τ acts on the quotient $L/2L$. This is a finite-dimensional vector space over $\mathbb{F}_2$. According to the Jordan decomposition theorem it must break up into a direct sum of spaces on which τ acts by

$$\begin{bmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & \dots & 0 \\ 0 & 0 & \lambda & \dots & 0 \\ & \dots & & & \\ 0 & 0 & 0 & \dots & \lambda \end{bmatrix}$$

but since $\tau^2 = 1$ it must in fact be a sum of one-dimensional spaces on which τ is the identity, plus a direct sum of two-dimensional space on each of which τ acts as the matrix

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad (\text{over } \mathbb{F}_2)\,.$$

This matrix is equivalent to a swap (in characteristic 2), as we can deduce from the following observation:

5.2. Lemma. (Swap Lemma) *The integral matrix*

$$\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}.$$

is that of a swap.

Proof. If

$$\begin{aligned} \tau(e) &= \quad e \\ \tau(f) &= -f + e \end{aligned}$$

then

$$\tau(e - f) = e - (-f + e) = f$$

so τ swaps $e - f$ and f. ▯

The lattice decomposition we are looking for is closely related to this decomposition of $L/2L$, except that the two lattices $L_\pm$ of course both collapse to the single eigenspace modulo 2.

I now start the proof in detail. The first step is to find the Smith normal form of $\tau + I$. This gives us a basis $e_1, \ldots, e_j, e_{j+1}, \ldots e_k, e_{k+1}, \ldots, e_n$ of L where the $2e_i$ for $i \le j$ and the e_i for $j < i \le k$ make up a basis of $(\tau + I)L$. This implies that the e_i for $i \le k$ are basis of M_+. This implies also that the e_i with $i \le k$ make up a basis of M_+. The calculation also gives us equations

$$(\tau + I)e_\ell = \sum_{i \le j} 2c_i e_i + \sum_{j < i \le k} c_i e_i$$

for $\ell > k$. But then

$$(\tau + I)\Big(e_\ell - \sum_{i \le j} c_i e_i\Big) = \sum_{i \le j} 2c_i e_i + \sum_{j<i\le k} c_i e_i - \sum_{i\le j} 2c_i e_i = \sum_{j<i\le k} c_i e_i \,.$$

If we replace the e_ℓ for $\ell > k$ by $e_\ell - \sum_{i \le j} c_i e_i$, we see that the lattice L_* spanned by the e_i with $i > j$ is stable under τ, as is its complement, the lattice L_+ spanned by the e_i with $i \le j$. We can therefore split off L_+, on which $\tau = I$, and work only with L_*. We may as well take L to be L_*. We may also assume we have a basis $e_1, e_2, \ldots, e_m$ of L with the e_i for $i \le k$ a basis of $(\tau + I)L$ which is now the same as M_+. With the new basis the matrix of τ is of the form

$$\begin{bmatrix} I & C \\ 0 & -I \end{bmatrix} .$$

Our goal now is to show that L may be decomposed into $L_- \oplus L_{\mathrm{swap}}$.

We next find the Smith normal form of

$$\tau - I = \begin{bmatrix} 0 & C \\ 0 & -2I \end{bmatrix}$$

which means, in effect, finding that of

$$\begin{bmatrix} C \\ -2I \end{bmatrix} .$$

Because the e_i for $i \le k$ are a basis of M_+, the reduction modulo 2 of the columns of C are sufficiently linearly independent. This is already in a particularly good Hermite normal form. Since the pivot entries are all 2, the calculation of its Smith normal form is relatively simple. At any rate, we get a basis f_i of L such that the f_i for $1 \le i \le j$ and $2f_i$ for $j < i \le m - k$ are a basis of $(\tau - I)L$. Let L_- be the lattice spanned by the f_i with $i > j$. As before, we can find a τ-stable complement L_* of L_-, and split off L_-.

As a result of what we have done so far, we have a decomposition $\mathbb{Z}^n = L_+ \oplus L_* \oplus L_-$, where $\tau = \pm I$ on $L_\pm$. Furthermore, in L_*, which I may as well now take to be L, *the intersection $M_\pm$ of the ± 1-eigenspace of τ coincides with $(\tau \pm I)L$.* We also have (a) a basis (e_i) of L such that the e_i with $i \le k$ form a basis of M_+ and (b) a basis (f_i) $(i \le m - k)$ of M_-, expressed in terms of the e_i.

5.3. Lemma. *Suppose τ to be an involution of the lattice L, $M_\pm$ the intersection of the ± 1-eigenspaces of τ with L. If $(\tau \pm I)L = M_\pm$, then τ is a direct sum of swaps.*

This is really the crux of the matter. The proof will find an explicit decomposition. In practice, however, this explicit decomposition is rarely necessary. What one most often needs to know are the dimensions of the spaces $L_\pm$ and L_{swap} and that, given the truth of this lemma, we know now.

Proof. I claim first that

- *the kernel of the map $(\tau \pm I)$ from L to $M_\pm/2M_\pm$ is $M = M_- \oplus M_+$.*

Because if $(\tau + I)e = 2m = (\tau + I)m$ for m in M_+, then $(\tau + I)(e - m) = 0$ so that $e - m$ lies in M_- and $e = (e - m) + m$ lies in M. Therefore $\tau \pm I$ is an embedding of the $\mathbb{F}_2$-vector space L/M into $M_\pm/2M_\pm$.

As a consequence of this, we must have $m - k = k$, $m = 2k$, and $L/M \cong (\mathbb{Z}/2)^k$.

The e_i with $i \le k$ and the f_i with $i \le k$ make up a basis of $M_+ \oplus M_-$. We next find the Smith normal form of the matrix whose columns are these e_i and the f_i, assuming the e_i to be the basis of L. This matrix looks like

$$\begin{bmatrix} I & F_1 \\ 0 & F_2 \end{bmatrix} .$$

Applying elementary column operations, we may reduce this to

$$\begin{bmatrix} I & 0 \\ 0 & G \end{bmatrix} .$$

But since the index of $L/M \cong (\mathbb{Z}/2)^k$, the Smith normal form of G is the diagonal matrix with all diagonal entries equal to 2. Therefore G is itself divisible by 2. So the columns g_i of $G/2$, together with the e_i for $i \le k$, are a basis of L such that these e_i together with the $2g_i$ are a basis of M.

Now $(\tau + I)$ induces an isomorphism of L/M with $M_+/2M_+ \subseteq L/2L$. The images $\overline{g}_i$ in L/M make up a basis of that vector space over $\mathbb{F}_2$, as are the images $\overline{e}_i$ of the e_i in $M_+/2M_+$. We therefore have

$$(\tau + I)\overline{g}_i = \sum \overline{c}_i^j \overline{e} j$$

where $\overline{C} = (\overline{c}_i^j)$ is an invertible $k \times k$ matrix over $\mathbb{F}_2$. Since according to strong approximation the canonical projection from $\mathrm{SL}_n(\mathbb{Z})$ to $\mathrm{SL}_n(\mathbb{Z}/2)$ is surjective, we may replace the basis e_i for M_+, and assume that $(\tau + I)g_i = e_i$ modulo $2M_+$.

But if $(\tau + I)g_i = e_i + 2m_i$ with m_i in M_+, then

$$(\tau + I)(g_i - m_i) = e_i + 2m_i - 2m_i = e_i \, .$$

We may replace the basis g_i by the $g_i - m_i$ so that

$$(\tau + I)g_i = e_i, \quad \tau(g_i) = -g_i + e_i$$

The lattice L is the direct sum of the τ-stable two-dimensional lattices spanned by g_i and e_i, and on this the matrix of τ is

$$\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} .$$

The Swap Lemma tells us that we are through.

6. References

1. Jeffrey Adams and David Vogan, 'L-groups, projective representations, and the Langlands classification', *American Journal of Mathematics* **114** (1992), 45–138.

2. Armand Borel, 'Automorphic L-functions', pp. 27–61 in part 2 of [Borel and Casselman:1979].

3. —— and William Casselman (editors), **Automorphic forms, representations, and L-functions**, proceedings of a conference in Corvallis, *Proceedings of Symposia in Pure Mathematics* **33**, A.M.S., 1979.

4. Bill Casselman, 'Computations in real tori', pp. 137–151 in **Representation theory of real reductive Lie groups**, *Contemporary Mathematics* **472**, American Mathematical Society, 2008.

5. R. E. Kottwitz, 'Stable trace formula: cuspidal tempered terms', *Duke Mathematicas Journal* **51** (1984), 611–650.

6. Robert Langlands, 'On the classification of irreducible representations of real groups', pp. 101–179 in **Representation theory and harmonic analysis on semisimple Lie groups**, *Mathematical Surveys Monographs* **31**, American Mathematical Society, 1989. Originally a preprint from the IAS, dated 1974.

7. ——, 'Representations of abelian algebraic groups', *Pacific Journal of Mathematics* (special issue dedicated to Olga Taussky Todd) (1997), 231–250. Originally a Yale University preprint.

8. John Tate, 'Number theoretic background', pp. 3–26 in part 2 of [Borel-Casselman:1979].