Essays on Coxeter groups

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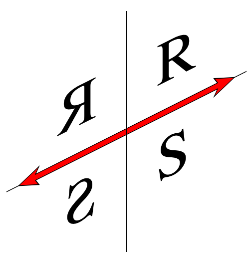
Reflections



A **linear reflection** of a real vector space V is a linear transformation fixing points on a hyperplane passing through the origin and acting as multiplication by -1 on a transverse line through the origin. More explicitly, a reflection in the hyperplane where $\alpha = 0$ is a linear map of the form

$$s_{\alpha,\alpha^\vee}\colon\ v \longmapsto v - \langle \alpha, v\rangle \alpha^\vee$$

taking $\alpha^\vee$ to $-\alpha^\vee$. Necessarily, $\langle \alpha, \alpha^\vee\rangle = 2$. The pair α, $\alpha^\vee$ are determined uniquely up to multiplication by c, c^{-1}.

All reflections are essentially the same, in the sense that they are all conjugate in $\mathrm{GL}(V)$. This essay is concerned with classifying **pairs** of reflections up to conjugacy, which is a more interesting problem, apparently first dealt with in §2 of [Vinberg:1971]. This essay will be a somewhat leisurely exploration of matters discussed by Vinberg. This should provide motivation for certain ways of dealing with more general groups generated by reflections—i.e. realizations of Coxeter groups. Much of this theory reduces to an analysis of pairs.

The main results are mildly technical, but some idea of what they are all about can be stated without too much trouble. Given s and t, let

$$c_{s,t} = \langle \alpha, \beta^\vee\rangle, \quad c_{t,s} = \langle \beta, \alpha^\vee\rangle\,.$$

These depend on the choice of α etc. but the product

$$n_{s,t} = c_{s,t}c_{t,s}$$

is a conjugation invariant of the pair. Whether either factor is 0 is also invariant under conjugation.

Theorem 1. (Classification of pairs of reflections) *For $n \ne 0, 4$ there exists exactly one conjugacy class of reflection pairs s, t with $n_{s,t} = n$. In the exceptional cases:*

- $n = 0$. *There exist two conjugacy classes of pairs: (a) $c_{s,t} = c_{t,s} = 0$, in which case s and t commute, generating a group of order 4; (b) exactly one of $c_{s,t}$ and $c_{t,s} = 0$.*
- $n = 4$. *There exist three classes in dimension 2 and four in dimensions 3 or more: (a) $s = t$; (b) α and β are linearly dependent, but $\alpha^\vee$ and $\beta^\vee$ are not; (c) $\alpha^\vee$ and $\beta^\vee$ are linearly dependent, but α and β are not; (d) (only in dimensions 3 or more) both α, β and $\alpha^\vee$, $\beta^\vee$ are linearly independent.*

Details will be filled in. In each case, we shall especially want to understand the way the group W generated by the reflections acts. *Can one find a useful partition into sets stable under W?* In every case, the group W acts discretely on the complement of a finite union of sets of lower dimension. *Can one describe a fundamental domain for this action? Does V possess a convex open set on which W acts discretely? Can we relate the geometry of the action of W to its combinatorics?*

All these questions will be answered.

Contents

1. Single reflections

I begin by recalling what happens for a single reflection.

Suppose s to be a reflection, $\alpha = \alpha_s = 0$ the equation of the hyperplane H_s in which it reflects. There exists a unique eigenvector $\alpha^\vee = \alpha_s^\vee$ of s with $\langle \alpha, \alpha^\vee \rangle = 2$, and then for all v in $V^\vee$

$$s(v) = v - \langle \alpha, v \rangle \alpha^\vee .$$

In these circumstances, if more precision is required I shall refer to s as $s_{\alpha,\alpha^\vee}$, but sometimes also as just s_α. If α is replaced by $c\alpha$ ($c \neq 0$), $\alpha^\vee$ is replaced by $c^{-1}\alpha^\vee$.

1.1. Proposition. *Reflections make up a single conjugacy class in* $\mathrm{GL}(V)$.

Proof. They are all diagonalizable with the same eigenvalue multiplicities.

Classically, a reflection is defined to be orthogonal with respect to some Euclidean metric. This is a restrictive assumption, but there is a useful generalization. When are reflections orthogonal transformations with respect to an arbitrary quadratic form?

1.2. Lemma. *Suppose given on* $\mathbb{R}^n$ *a non-degenerate quadratic form with associated inner product* $u \bullet v$. *A reflection* $s_{\alpha,\alpha^\vee}$ *is orthogonal if and only if* $\alpha^\vee$ *is non-isotropic and perpendicular to the hyperplane* $\alpha = 0$. *In these circumstances*

$$\langle \alpha, v \rangle = 2 \left(\frac{\alpha^\vee \bullet v}{\alpha^\vee \bullet \alpha^\vee} \right) .$$

In other words, the inner product induces a linear map from $V^\vee$ to its linear dual space V, and α is the image of $2\alpha^\vee/(\alpha^\vee \bullet \alpha^\vee)$. As a consequence, the reflection takes

(1.3) $$v \longmapsto v - 2 \left(\frac{\alpha^\vee \bullet v}{\alpha^\vee \bullet \alpha^\vee} \right) \alpha^\vee .$$

Proof. Suppose that $\alpha^\vee \bullet \alpha^\vee \neq 0$ and $\alpha^\vee$ is perpendicular to $\alpha = 0$. We may decompose any vector v into components

$$v = u + k\,\alpha^\vee$$

with $\langle \alpha, u \rangle = 0$. Then

$$sv = u - k\alpha^\vee, \quad \langle \alpha, v \rangle = 2k$$

and

$$v \bullet \alpha^\vee = k(\alpha^\vee \bullet \alpha^\vee), \quad k = \frac{\alpha^\vee \bullet v}{\alpha^\vee \bullet \alpha^\vee} .$$

Conversely, suppose $s = s_{\alpha,\alpha^\vee}$ to be orthogonal. If $\langle \alpha, u \rangle = 0$ then for every v in V

$$u \bullet v = s(u) \bullet s(v) = u \bullet v - \langle \alpha, v \rangle (u \bullet \alpha^\vee) .$$

This implies that

$$\langle \alpha, v \rangle \, (u \bullet \alpha^\vee) = 0$$

for all v in V. Since $\alpha \neq 0$ and the form is non-degenerate, this implies that $u \bullet \alpha^\vee = 0$. Since the dot-product is assumed to be non-degenerate, there exists $c \neq 0$ such that $u \bullet \alpha^\vee = c \langle \alpha, u \rangle$ for all u.

Since $\langle \alpha, \alpha^\vee \rangle = 2$, $\alpha^\vee$ does not lie in this hyperplane and $\alpha^\vee$ cannot be isotropic. But then the reflection (1.3) has the same effect as s.

2. Introduction to pairs

In contrast to a single reflection, pairs of reflections turn out to make up what is generically a one-parameter family of conjugacy classes.

Suppose s and t to be a pair of reflections, corresponding to α, $\alpha^\vee$ and $\beta, \beta^\vee$. Define associated real constants

$$c_{s,t} = \langle \alpha, \beta^\vee \rangle, \quad c_{t,s} = \langle \beta, \alpha^\vee \rangle, \quad n_{s,t} = c_{s,t} c_{t,s} \, .$$

If (s_*, t_*) is another pair of reflections conjugate to s, t under the linear transformation g, then

$$\begin{aligned} g\alpha = c_\alpha \alpha_*, \quad g\alpha^\vee = c_\alpha^{-1} \alpha_*^\vee, \\ g\beta = c_\beta \beta_*, \quad g\beta^\vee = c_\beta^{-1} \beta_*^\vee \end{aligned}$$

for some non-zero constants c_α and c_β, and consequently

$$c_{s_*,t_*} = (c_\beta / c_\alpha) c_{s,t}, \quad c_{t_*,s_*} = (c_\alpha / c_\beta) c_{t,s} \, .$$

The constants $c_{s,t}$ and $c_{t,s}$ are not conjugation-invariant, but the product $n_{s,t}$ is. Also invariant is whether $c_{s,t}$ or $c_{t,s}$ is 0. For generic pairs, $n_{s,t}$ will turn out to possess a familiar interpretation. For all but two values, the conjugacy class of a pair is determined by $n_{s,t}$

The **signature matrix** of a pair is

$$\begin{bmatrix} \langle \alpha, \alpha^\vee \rangle & \langle \alpha, \beta^\vee \rangle \\ \langle \beta, \alpha^\vee \rangle & \langle \beta, \beta^\vee \rangle \end{bmatrix} = \begin{bmatrix} c_{s,s} & c_{s,t} \\ c_{t,s} & c_{t,t} \end{bmatrix} = \begin{bmatrix} 2 & c_{s,t} \\ c_{t,s} & 2 \end{bmatrix} .$$

Its determinant is $4 - n_{s,t}$. It is determined up to conjugation by a diagonal matrix

$$\begin{bmatrix} a & 0 \\ 0 & 1/a \end{bmatrix} .$$

3. When $n = 4$

One way to understand conjugation-invariant phenomena is to consider geometrical configurations—for example, how the various hyperplanes and lines associated to reflections relate qualitatively to each other. As we shall see, this works to distinguish $n = 0$ or 4 very simply from the other cases.

The case $n = 4$ is particularly distinctive. If either α, β or $\alpha^\vee$, $\beta^\vee$ are linearly dependent, the signature matrix

$$\begin{bmatrix} \langle \alpha, \alpha^\vee \rangle & \langle \alpha, \beta^\vee \rangle \\ \langle \beta, \alpha^\vee \rangle & \langle \beta, \beta^\vee \rangle \end{bmatrix} = \begin{bmatrix} 2 & c_{s,t} \\ c_{t,s} & 2 \end{bmatrix}$$

must be singular, hence:

3.1. Lemma. *If either α, β or $\alpha^\vee$, $\beta^\vee$ are linearly dependent then $n_{s,t} = 4$.*

In analyzing the pairs of reflections s, t with $n = 4$ there are *a priori* four possibilities to be considered:

(a) *Both α, β and $\alpha^\vee$, $\beta^\vee$ are dependent.*

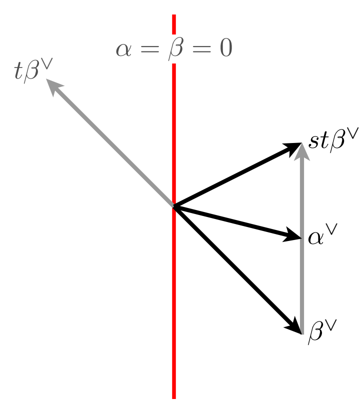
Here $s = t$.

(b) *α, β are dependent but $\alpha^\vee$, $\beta^\vee$ are not.*

The reflection hyperplanes of s and t are the same. We may as well assume also that $\alpha = \beta$.

3.2. Lemma. *If the reflections s and t are reflections in the same hyperplane then st is a shear parallel to that hyperplane. In this case,*

$$st(v) = v + \langle \beta, v \rangle (\alpha^\vee - \beta^\vee) .$$



When interpreting the figures involving reflections, keep in mind that locating $\alpha^\vee$ tells you on which side of the line $\alpha = 0$ the linear function α is positive.

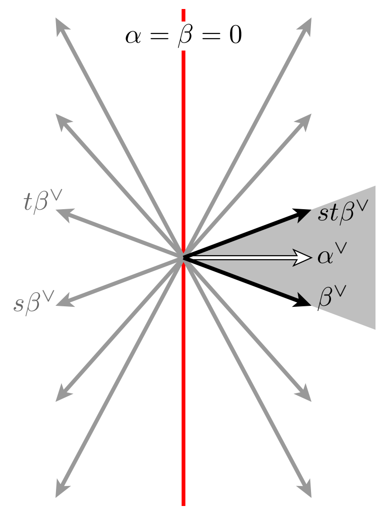
Proof. Suppose s, t to be reflections in the same hyperplane. We may assume $\alpha = \beta$, so $\langle \alpha, \beta^\vee \rangle = \langle \beta, \beta^\vee \rangle = 2$ and $\alpha^\vee - \beta^\vee$ is parallel to the hyperplane of reflection. Now

$$\begin{aligned} s\beta^\vee &= \beta^\vee - \langle \alpha, \beta^\vee \rangle \alpha^\vee \\ t\beta^\vee &= -\beta^\vee \\ st(\beta^\vee) &= -\beta^\vee + 2\alpha^\vee \\ &= \beta^\vee + 2(\alpha^\vee - \beta^\vee) . \end{aligned}$$

If $v = u + k\beta^\vee$ with $\langle \beta, u \rangle = 0$ then $k = \langle \beta, v \rangle / 2$, so

$$st(v) = v + \langle \beta, v \rangle (\alpha^\vee - \beta^\vee) .$$

In a little more detail:

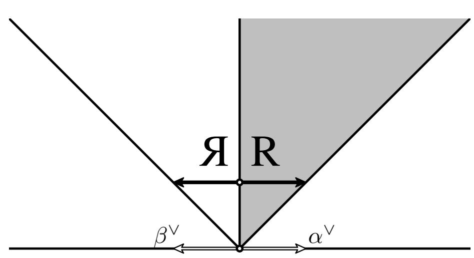


When $\alpha = \beta$

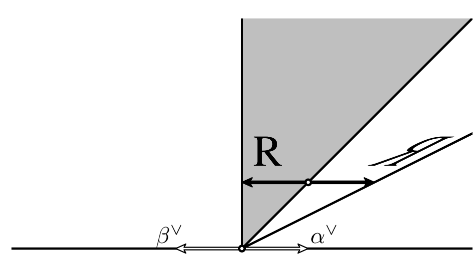
In this case, we can choose our coordinate system so that $\alpha = \beta = x$. By swapping and scaling, if necessary, we may assume that $\alpha^\vee = (2, 0)$ and $\beta^\vee = (2, -1)$. As we have just seen in Lemma 3.2, st is a shear parallel to $\alpha = 0$, and the group of all powers of st is a normal subgroup in W of index 2. It takes each side of the reflection line into itself, while the reflections swap them. The vector space may be partitioned into two subsets on which W acts in a well understood manner—the line $\alpha = 0$ and its complement. All of W acts trivially on that line. The region between the lines through $\beta^\vee$ and $st(\beta^\vee)$ where $\alpha \ge 0$ is a fundamental domain for the action of W on its complement.

In the remaining cases **(c)** and **(d)**, α and β are independent. Let L be the linear space of codimension two on which they both vanish. The reflections s, t induce reflections on the plane V/L.

With this in mind, suppose at first that the dimension of V is 2. Since $c_{s,t}c_{t,s} = 4$, and we can scale α so that in fact $c_{s,t} = c_{t,s} = -2$. The vectors α and β are a basis of the dual of V, so a vector in V is distinguished by the pair $\langle \alpha, v\rangle$ and $\langle \beta, v\rangle$. Therefore $\alpha^\vee = -\beta^\vee$, and they span a single line. Both s and t transform a vector in a direction parallel to that line.

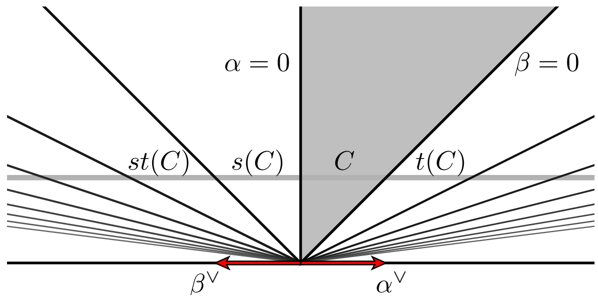


The reflection s



The reflection t

The space V is the union of three pieces on which W acts nicely—each of the open halves of the complement of the x-axis, and the x-axis itself. On the x-axis, each reflection acts as scalar multiplication by -1. The region C on either side of the x-axis and in between the two lines $\alpha = 0$ and $\beta = 0$ will be a fundamental domain for the group acting on the corresponding open half-plane. On one side of the line, this region is that where $\alpha \ge 0$ and $\beta \ge 0$. In this case the group W is infinite, consisting of all products $1, s, t, st, ts, \ldots$



Now abandon the assumption that V have dimension two. Then s and t induce reflections on the two-dimensional quotient V/L. In this plane the previous argument implies that modulo L we may assume that $\alpha = -\beta$, so that in V we have $\beta^\vee = -\alpha^\vee + \lambda^\vee$ with $\lambda^\vee$ in L. There are two possibilities—either $\lambda^\vee = 0$ or not.

The space L is of codimension one in the hyperplane $\alpha = 0$, so there exists $\gamma^\vee$ such that $\langle \alpha, \gamma^\vee \rangle = 0$, $\langle \beta, \gamma^\vee \rangle = 2$. If $\lambda^\vee = 0$, then $\alpha^\vee$ and $\gamma^\vee$ span a plane transverse to L, and V is the direct sum of this plane an L. This takes care of case **(c)**.

But now suppose $\lambda^\vee \neq 0$, which means we are looking at **(d)**. The space U spanned by $\alpha^\vee$, $\beta^\vee$, $\gamma^\vee$ has dimension three, and is stable under both s and t. The intersection of L with U is the line spanned by $\lambda^\vee$, so U is transversal to any complement in L to that line, and the action of W is the direct sum of U and that complement.

In U, the plane spanned by $\alpha^\vee$ and $\beta^\vee$ is stable under s and t, and falls in case **(b)**, which I have already handled. The representation of W on U is hence reducible but indecomposable.

In both cases, the region $\alpha \geq 0$, $\beta \geq 0$ is a fundamental domain for W.

This second case occurs as a feature of Kac-Moody algebras determined by the loop group associated to SL_2, as is discussed in Chapters 1 and 6 of [Kac:1985].

From now on, I assume that $n_{s,t}$ is not equal to 4.

4. When $n = 0$

The next result is straightforward:

4.1. Proposition. *When $n_{s,t}$ is not equal to 4, then α and β are linearly independent, as are $\alpha^\vee$ and $\beta^\vee$. The plane spanned by $\alpha^\vee$ and $\beta^\vee$ is transverse to the linear subspace of codimension 2 where α and β are both equal to 0.* In effect:

We may now restrict ourselves to dimension two.

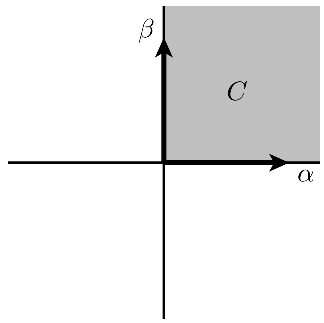
In these circumstances, reflections fix lines. And we may make explicit matrix computations.

4.2. Lemma. *Take $\alpha^\vee$ and $\beta^\vee$ as a basis of V. The relevant matrices are now*

$$s = \begin{bmatrix} -1 & -c_{s,t} \\ 0 & 1 \end{bmatrix}, \quad t = \begin{bmatrix} 1 & 0 \\ -c_{t,s} & -1 \end{bmatrix}, \quad st = \begin{bmatrix} -1 + n_{s,t} & c_{s,t} \\ -c_{t,s} & -1 \end{bmatrix}.$$

As one consequence, $-2 + n_{s,t}$ is the trace of st.

In this section, I'll look at the second exceptional case, in which $n = 0$. This happens only when either $c_{s,t} = \langle \alpha, \beta^\vee \rangle = 0$ or $c_{t,s} = \langle \beta, \alpha^\vee \rangle = 0$, or both. If both are equal to 0, then s and t are a commuting pair of reflections whose product amounts to multiplication by -1. The group generated by s and t has four elements, and any one of the four quadrants is a fundamental domain.



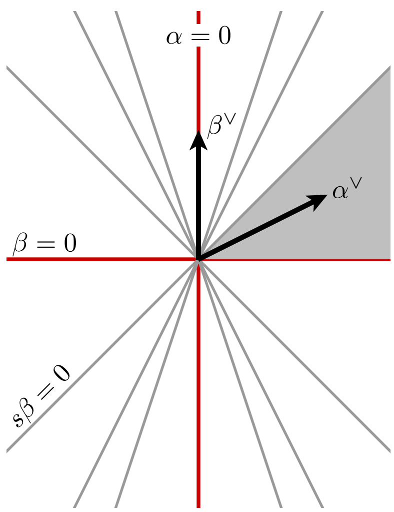
Otherwise, by swapping s and t if necessary (and swapping amounts to conjugation in $\mathrm{GL}(V)$) we may suppose that $\langle \alpha, \beta^\vee \rangle = 0$, but $\langle \beta, \alpha^\vee \rangle \neq 0$. The matrix of st is

$$\begin{bmatrix} -1 & 0 \\ -c_{t,s} & -1 \end{bmatrix}.$$

All choices are conjugate, since all matrices

$$\begin{bmatrix} -1 & 0 \\ c & -1 \end{bmatrix}$$

are conjugate.



The line $\alpha = 0$ is fixed by both s and t. The reflection s acts trivially on it, while t swaps its two halves. The space V may be decomposed into two pieces on which W acts nicely—the y-axis and its complement. The reflection s acts trivially on the y-axis, while t acts on it as scalar multiplication by -1. The reflection s interchanges the two sides of the y-axis, but t preserves them. The pair of reflections t and sts now both stabilize each side of the line $\alpha = 0$, and the n-invariant of this pair is 4, since they share the eigenvector $\beta^\vee$. From what we have already learned in this case, the group generated by these two reflections takes each side of the y-axis to itself. This group is normal, of index two in W.

From now on, I assume that $n_{s,t}$ is neither 0 nor 4.

5. When $n \neq 0, 4$

Of course I continue to assume that the dimension is two.

5.1. Proposition. *If n is neither 0 nor 4, there is a unique conjugacy class of pairs (s, t) of reflections with $n_{s,t} = n$.*

Proof. Let s be any reflection, say corresponding to α, $\alpha^\vee$. All choices of s are certainly conjugate. We may as well take α to be the function x and the vector $\alpha^\vee$ to be $(2, 0)$. The matrices commuting with s are then the diagonal matrices, and it acts transitively on the complement of the axes.

We now look for β and $\beta^\vee$ with $\langle\beta, \beta^\vee\rangle = 2$, $\langle\alpha, \beta^\vee\rangle\langle\beta, \alpha^\vee\rangle = n$, and we may scale $(\beta, \beta^\vee)$ by (c, c^{-1}) without changing t. Since $n \neq 0$, we may therefore fix $\langle\alpha, \beta^\vee\rangle = -1$ and $\langle\beta, \alpha^\vee\rangle = -n$. Thus

$$\begin{aligned} \beta &= (-n/2, a) \\ \beta^\vee &= (-1, b) \end{aligned}$$

for some unknowns a, b subject to the condition

$$n/2 + ab = 2\,.$$

Since $n \neq 4$, neither a nor b is 0. For every choice of b there is a unique choice of a. All choices of β and $\beta^\vee$ therefore lie in the open orbit of the diagonal matrices, and all possible t are therefore conjugate to each other by a diagonal matrix.

How does the Weyl group act in these cases?

5.2. Proposition. *If $n_{s,t}$ is neither 0 nor 4, there exists a non-degenerate quadratic form invariant under both s and t, unique up to scalar multiple. It will be definite if and only if $0 < n_{s,t} < 4$.*

Proof. Suppose $u \bullet v$ to be a non-degenerate inner product on U. It will be invariant under s and t if and only if $\alpha^\vee$ is perpendicular to the line $\alpha = 0$, and similarly for $\beta^\vee$ and β. The space of vectors fixed by s is spanned by $c_{s,t}\alpha^\vee - 2\beta^\vee$, and that fixed by t is spanned by $2\alpha^\vee - c_{t,s}\beta^\vee$. For invariance, therefore, we must have

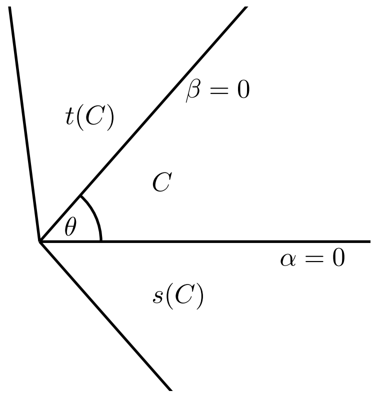
$$\begin{aligned} (c_{s,t}\alpha^\vee - 2\beta^\vee)\bullet\alpha^\vee &= 0 \\ (2\alpha^\vee - c_{t,s}\beta^\vee)\bullet\beta^\vee &= 0 \\ 2(\beta^\vee\bullet\alpha^\vee) &= c_{s,t}(\alpha^\vee\bullet\alpha^\vee) \\ c_{t,s}(\beta^\vee\bullet\beta^\vee) &= 2(\alpha^\vee\bullet\beta^\vee) \end{aligned}$$

Since $n_{s,t}$ is not 0, neither $c_{s,t}$ nor $c_{t,s}$ vanish. Therefore the single number $\alpha^\vee\bullet\beta^\vee$ determines the inner product on all of V. The matrix of the quadratic form is thus

$$\begin{bmatrix} \alpha^\vee\bullet\alpha^\vee & \alpha^\vee\bullet\beta^\vee \\ \alpha^\vee\bullet\beta^\vee & \beta^\vee\bullet\beta^\vee \end{bmatrix} = (\alpha^\vee\bullet\beta^\vee)\begin{bmatrix} 2/c_{s,t} & 1 \\ 1 & 2/c_{t,s} \end{bmatrix}.$$

In order for the form to be definite, it is necessary and sufficient that $c_{s,t}$ and $c_{t,s}$ have the same sign, and that the determinant be positive. The first condition occurs if and only if $n_{s,t} > 0$, and the second if and only if $n_{s,t} < 4$.

DEFINITE FORMS. Suppose $0 < n < 4$. We may assume the invariant form to be the Euclidean norm, and s to be orthogonal reflection in the x-axis, t to be orthogonal reflection in the line at angle θ to the x-axis.



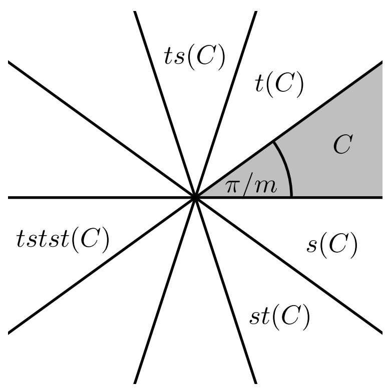
The matrices of s, t are

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix},$$

and $n_{s,t} = 4\cos^2\theta$.

The product ts amounts to rotation by 2θ. The group W therefore acts discretely if and only if $\theta = p/m$ for some relatively prime p, m. If $kp + \ell m = 1$, then $(ts)^k$ is a rotation though $2\pi/m$. We can write it as $ts \ldots t \cdot s$. The transformation $ts \ldots t$ is a reflection in the line at angle π/m to the x-axis. We can choose

$$\alpha^\vee = (0, 1), \quad \beta^\vee = (\sin\pi/m, -\cos\pi/m)\,.$$



The group W therefore has $2m$ elements, and its fundamental domain is the region in the wedge of angle θ. This is also the region where $\alpha \ge 0$ and $\beta \ge 0$. The numbers $c_{s,t} = c_{t,s}$ are negative.

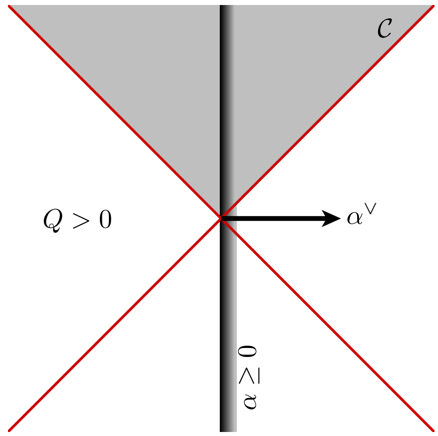
INDEFINITE FORMS. Suppose now that the invariant quadratic form is indefinite, and hence that either $n < 0$ or $n > 4$.

By choosing coordinates suitably, we may assume the form to be $Q(x, y) = x^2 - y^2$, and we may also assume $\alpha^\vee = (1, 0)$, so that $Q(\alpha^\vee) = 1$. If

$$Q(v) = v \bullet v\,,$$

then

$$\langle \alpha, v \rangle = 2(\alpha^\vee \bullet v)\,.$$



The region where $Q < 0$ has two components. Let $\mathcal{C}$ be that where $y > 0$. The reflection s swaps the two components of the region $Q > 0$, but takes each of the components $\pm\mathcal{C}$ into itself.

What about t? There are two possibilities—either $Q(\beta^\vee) > 0$ or $Q(\beta^\vee) < 0$. In the first case, t takes each of the two components $\pm\mathcal{C}$ into itself, while in the second it swaps them.

The special orthogonal group in these circumstances is that of hyperbolic rotations

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix}$$

with $a^2 - b^2 = 1$. There are two components. In the connected component of the identity $a > 0$, while in the other one $a < 0$. Rotations in the first take each of the components $\pm\mathcal{C}$ to itself, while those in the second swap them.

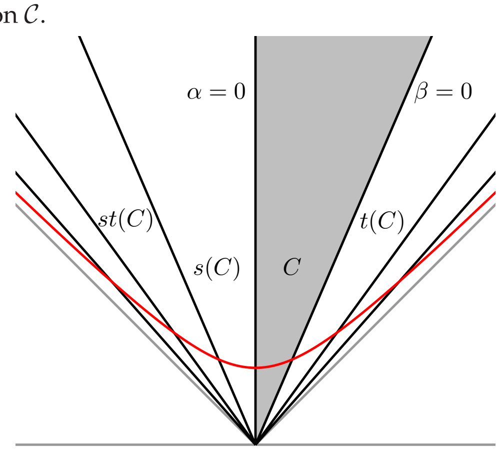
In the first case

$$ts = \begin{bmatrix} \cosh 2\theta & \sinh 2\theta \\ \sinh 2\theta & \cosh 2\theta \end{bmatrix}, \quad n_{s,t} = 4\cosh^2\theta > 4$$

for some θ, while in the second

$$ts = \begin{bmatrix} -\cosh 2\theta & -\sinh 2\theta \\ -\sinh 2\theta & -\cosh 2\theta \end{bmatrix}, \quad n_{s,t} = 4(1 - \cosh^2\theta) < 0\,.$$

In the first case, by choosing $\beta^\vee$ suitably one can see that the region $\alpha \geq 0$, $\beta \geq 0$ is a fundamental domain for the action of W on $\mathcal{C}$.



This concludes the proof of Theorem 1.

6. Stable cones

What is important in applications are the cases in which W acts discretely on an open convex cone. An examination of the discussion so far will show:

Theorem 2. *There exists a convex open subset $\mathcal{C}$ of V on which W acts discretely if and only if one of the following holds:*

(a) $n_{s,t} = c_{s,t} = c_{t,s} = 0$;
(b) $n_{s,t} = 4\cos^2(p\pi/m)$ *for some integers* $0 < p < m$ *with* $m \geq 3$, $p/m \neq 1/2$;
(c) $n_{s,t} = 4$ *and the reflection hyperplanes* $\alpha = 0$ *and* $\beta = 0$ *are distinct;*
(d) $n_{s,t} > 4$.

Choices of α and β can be made so as to have both $c_{s,t}$ and $c_{t,s} \leq 0$, in which case the region $\alpha \geq 0$, $\beta \geq 0$ is a fundamental domain for W.

This list corresponds nicely to this classification:

6.1. Lemma. *There are three possibilities for a convex open cone in a two-dimensional real vector space V:*

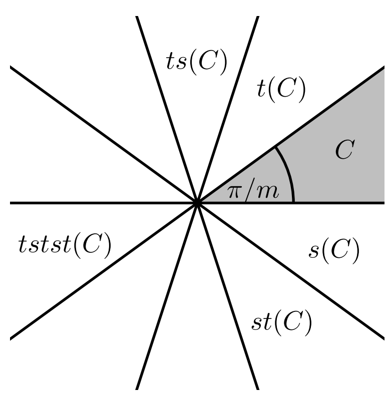
(a) *it is all of V;*
(b) *it is an open half-plane;*
(c) *it is an acute wedge.*

Proof. Let $\mathcal{C}$ be the cone. Place a circle around the origin. If all of the circle is contained in $\mathcal{C}$, then $\mathcal{C}$ is all of U. Otherwise, there exists a point on the circle not in $\mathcal{C}$. The intersection of $\mathcal{C}$ with the circle may then be defined as a proper subinterval of $(0, 2\pi)$. If the arc has length more than π, then $\mathcal{C}$ must contain the origin, a contradiction. So it must be of length at most π. If it is π, then $\mathcal{C}$ is a half-plane. If not, it is an acute wedge.

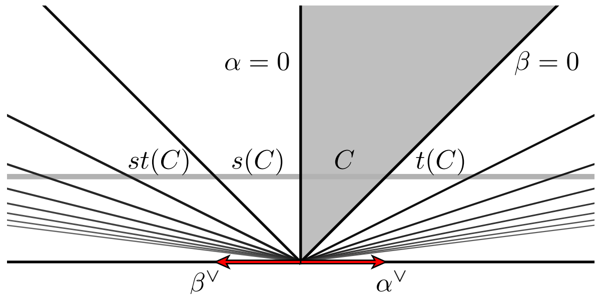
These three cases correspond to where W is finite, equivalent to a group of affine reflections, or is a group of hyperbolic orthogonal transformations

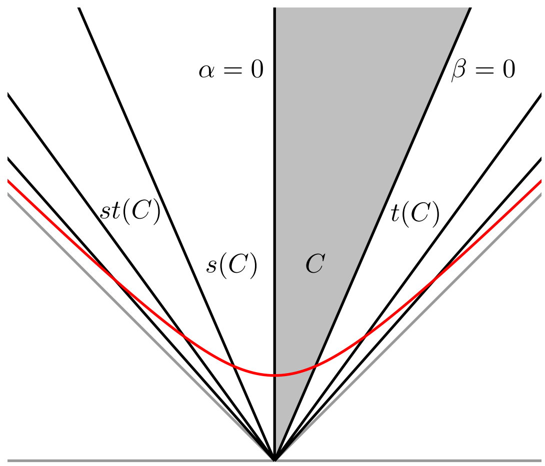
Other important things can be deduced from an inspection of the diagrams. There are three possibilities. In one, the group is finite, and in fact if it has order $2m$ it is the group with generators s and t and relations

$$s^2 = t^2 = 1, \quad (st)^m = 1\,.$$



In a second and third the group has generators s and t but as relations just $s^2 = t^2 = 1$.





In each case, the group has a fundamental domain $C \subset \mathcal{C}$. As generators of the group W one may take reflections s, t in the rays bounding it. Choose linear functions α, β defining the lines through those rays that are positive on C, and corresponding $\alpha^\vee$, $\beta^\vee$ defining the reflections.

Of course every element of W can be represented as a product of s and t. For the infinite groups, these are arbitrary strings

$$stst\dots, \quad tsts\dots$$

Here, every w in W has a unique such representation, and vice-versa.

For the finite case, every element except one can be represented as a unique minimal word

$$(st)^n,\ (st)^n s,\ (ts)^n,\ (ts)^n t\,,$$

with $n < m$. The exception is

$$(st)^m = (ts)^m\,.$$

In all cases, there is an evident link between geometry and word expressions:

6.2. Proposition. *A minimal word for w starts with s if and only if $w(C)$ lies on the opposite side of $\alpha = 0$ from C, and similarly for t.*

7. Affine reflections

I have mentioned that there are reflections of interest that are not linear, but also that they are in fact derived from linear ones. In this section I'll deal with affine reflections.

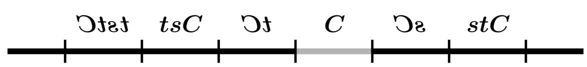
An **affine reflection** reflects points in an affine hyperplane $f(x) = 0$ for some affine function $f(x)$. The function f can be expressed as

$$f(x) = \langle \alpha, x \rangle + c$$

for some linear function α and constant c. The linear function α is the **gradient** ∇f of f, and for any point x in V and vector v we have $f(x+v) = f(x) + \langle \alpha, v \rangle$. The reflection then takes a point x to $x - f(x)\alpha^\vee$ for some vector $\alpha^\vee$ in V with $\langle \alpha, \alpha^\vee \rangle = 2$.

An affine reflection is a special case of a linear one, through the familiar trick of embedding an affine space of dimension n into a vector space U of dimension $n+1$. Thus v in V maps to $(v, 1)$ in U. Let δ be the function on U taking (v, x) to x. The affine reflection on V corresponding to α, $\alpha^\vee$, and c is the restriction to V of the reflection defined by the linear function $\alpha + c\delta$ on U. In the discussion of pairs of reflections with $n = 4$ we have already seen an implicit example of this.

Here is what happens for the group generated by affine reflections in the points at the end of a line segment in one dimension.



It is an infinite Coxeter group with two generators, say s and t, and with $m_{s,t}$ infinite. As we have already seen, the realization by affine reflections is the restriction to a line of a linear Coxeter group in dimension 2.

8. Non-Euclidean reflections

A **non-Euclidean reflection** reflects points of a non-Euclidean space in a non-Euclidean hyperplane. This also can be explained in terms of linear reflections.

The non-Euclidean space H_n of n dimension is the component $Q(x) = x_{n+1}^2 - x_1^2 - ... - x_n^2 = 1, x_{n+1} > 0$ of the hyperbolic sphere, which can be parametrized by Euclidean space E_n according to the formula

$$x_{n+1} = \sqrt{x_1^2 + \cdots + x_n^2} \, .$$

It is taken into itself by the connected component of the orthogonal group of Q and there is a unique Riemannian metric on it invariant under this group and restricting to the Euclidean metric at $(0, 0, ..., 0, 1)$. There are two models of this in Euclidean space of n dimensions, the familiar Poincaré model and the slightly less familiar Klein model. (There is also, for $n = 2$, the upper half-plane, but that is another story.) Both models identify H_n+ with the interior of the unit ball in E_n centred at the origin.

For the Klein model, this ball is identified with the intersection of the slice $x_{n+1} = 1$ of the region $Q(x) > 0$ (the interior of a homogeneous cone). A point x on H maps to $x_* = x/x_{n+1}$. It happens that a non-Euclidean geodesic line between two points x and y is the intersection with H of the plane through the origin containing x and y, and this intersects the slice in a line between x_* and y_*. Thus in the Klein model. points of H are identified with points of the interior of a unit ball, and geodesics between such points are line segments in the ball.

The Poincaré model is a transformation of the Klein model. A point x in the interior of the unit ball in E_n maps to the point above it on the upper hemisphere in E_{n+1}, then by South pole stereographic projection back onto the equator. In this transformation, geodesics become arcs of circles perpendicular to the boundary of the unit ball.

The point for us is that *non-Euclidean reflections are those induced on the hyperbolic sphere, or either of its models, by linear reflections in* E_{n+1}. For both models, we start with a hyperplane $\alpha = 0$ which intersects the region $Q(x) > 0$ and a vector $\alpha^\vee$ transverse to it. In the Klein model, we take a point x in the slice $Q(x) > 0$, $x_{n+1} = 1$ to $x - \langle\alpha, x\rangle\alpha^\vee$ and then project it back onto the slice. We have seen examples of this also in the discussion above of pairs of linear reflections with $n > 4$. In the Poincaré model we unravel by stereographic projection, reflect, and ravel again.

9. References

1. Victor Kac, **Infinite dimensional Lie algenras**, Cambridge University Press, (second edition) 1985.

2. Ernest Borisovich Vinberg, 'Discrete linear groups generated by reflections', *Math. USSR Izvestia* **5** (1971), 1083–1119.