The principal series of SL(2)

Bill Casselman
University of British Columbia
cass@math.ubc.ca

Contents
1. The Fourier transform on finite fields
2. The projective line
3. The principal series representations of SL(2)

1. The Fourier transform on finite fields

Let $F$ be the finite field $\mathbb{F}_q$. Suppose $\psi$ to be a non-trivial character of the additive group of $F$. For example, suppose $\tau$ be the trace from $F$ to $\mathbb{F}_p = \mathbb{Z}/p$. As a consequence of the linear independence of Galois automorphisms, $\tau$ is surjective. Set

$$\psi(x) = e^{2\pi i \tau(x)/p}.$$ 

For every $x$ in $F$ let

$$\psi_x(y) = \psi(xy).$$ 

The character $\psi_x$ is the trivial character of $F$ if and only if $x = 0$. Therefore

$$\sum_{x \in F} \psi(xy) = \begin{cases} q & \text{if } y = 0 \\ 0 & \text{otherwise.} \end{cases}$$

I define the Fourier transform from $\mathbb{C}(F)$ to itself:

$$\hat{f}(y) = \frac{1}{\sqrt{q}} \sum_{x \in F} f(x) \psi(-xy).$$

[ft-finite-fields] Proposition 1.1. The Fourier transform is an isomorphism of $\mathbb{C}(F)$ with itself. Its inverse is

$$f(x) = \frac{1}{\sqrt{q}} \sum_{y \in F} \hat{f}(y) \psi(xy).$$

Proof. We have

$$\frac{1}{\sqrt{q}} \sum_{y \in F} \hat{f}(y) \psi(xy) = \frac{1}{q} \sum_{y,z} f(z) \psi(-xy) \psi(yz)$$

$$= \frac{1}{q} \sum_{z} f(z) \left( \sum_{y} \psi(y(z - x)) \right)$$

$$= \frac{1}{q} \sum_{z} f(z) \cdot \begin{cases} 0 & \text{if } z \neq x \\ q & \text{if } z = x \end{cases}$$

$$= f(x) \cdot 0.$$
Proposition 1.2. For \( f \) in \( C(F) \)
\[
\sum_{x \in F} |f(x)|^2 = \sum_{y \in F} |\hat{f}(y)|^2.
\]

It is for this that the \( 1/\sqrt{q} \) factor is in the definition of the Fourier transform.

Proof. A straightforward calculation.

Suppose \( \chi \) to be a character of the multiplicative group \( F^\times \) of \( F \). Extend it to be 0 at 0. Its Fourier transform is
\[
\hat{\chi}(y) = \frac{1}{q} \sum_{x} \chi(x)\psi(-xy).
\]

If \( \chi \) is the trivial character then
\[
\hat{\chi}(y) = \begin{cases} 
q - 1 & \text{if } y = 0 \\
-1 & \text{otherwise}.
\end{cases}
\]

If \( \chi \) is not trivial:

Lemma 1.3. If \( \chi \) is a non-trivial character of \( F^\times \) then
\[
\hat{\chi}(y) = \chi^{-1}(-y)G_{\chi,\psi}.
\]

where
\[
G_{\chi,\psi} = \frac{1}{\sqrt{q}} \sum \chi(x)\psi(x), \quad |G_{\chi,\psi}| = 1.
\]

The Fourier inversion theorem implies that
\[
\hat{\hat{f}}(x) = f(-x).
\]

Applied to \( \chi \) this tells us
\[
G_{\chi,\psi}G_{\chi^{-1},\psi} = \chi(-1).
\]

2. The projective line

In the next section, I shall investigate certain representations of the group \( SL_2(\mathbb{Z}/p) \). It will help if we know something about elementary projective geometry. Let \( F \) be an arbitrary field, and further let
\[
G = SL_2(F) \quad A = \text{the subgroup of diagonal matrices} \\
N = \text{the subgroup of upper triangular unipotent matrices} \\
B = \text{the group of upper triangular matrices} = AN.
\]

Thus \( A \cong B/N \), and is isomorphic to \( F^\times \), while \( N \) is isomorphic to the additive group of \( F \).

In this section, suppose \( F \) to be an arbitrary field. The group \( SL_2(F) \) acts by matrix multiplication on \( F^2 \), taking lines through the origin to other lines through the origin. It therefore acts on the set \( \mathbb{P} = \mathbb{P}(F) \) of all such lines in \( F^2 \). How can we describe these? Through every point \((x, y)\) of \( F^2 \) other than \((0, 0)\) there goes exactly one line, which I’ll call \([x, y]\). Two points determine the same line if and only if one is a non-zero multiple of the other. If \( y \neq 0 \) the line through \((x, y)\) is the same as that through \((x/y, 1)\); these lines are therefore parametrized by \( F \) itself: \( x \mapsto [x, 1] \). If \( y = 0 \), then we can divide by \( x \), so there
is exactly one line remaining, the line \([1,0]\). By convention we say that \(1/0 = \infty\), so \(\mathbb{P}\) may be expressed as \(\mathbb{F} \cup \{\infty\}\). If \(\mathbb{F} = \mathbb{R}\), the thing we have constructed is obtained by adding to \(\mathbb{R}\) a single point at infinity, and is called the real projective line.

The subgroup of \(G\) that fixes \(\infty = [1,0]\) turns out be \(B\), since
\[
\begin{bmatrix}
1 & \beta \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
1
\end{bmatrix} = \begin{bmatrix} x + \beta \\ 1 \end{bmatrix},
\]
so the subgroup \(N\) of \(B\) acts by translation in \(\mathbb{F}\). Next, \(A\):
\[
\begin{bmatrix} a & 0 \\
0 & 1/a \end{bmatrix}
\begin{bmatrix} x \\
1
\end{bmatrix} = \begin{bmatrix} ax \\ 1/a \end{bmatrix} \sim \begin{bmatrix} ax^2 \\ 1 \end{bmatrix},
\]
so \(A\) acts by multiplication.

Thus \(\mathbb{P}\) is the union of two orbits of \(B\), \(\infty\) and \(\mathbb{F}\). If
\[
w = \begin{bmatrix} 0 & -1 \\
1 & 0 \end{bmatrix}
\]
then \(w(\infty) = 0\) and the matrix
\[
\begin{bmatrix} 1 & x \\
0 & 1 \end{bmatrix}
\]
translates \(0\) to \(x\), I have in fact shown that if \(g\) is not in \(B\) then it may be factored uniquely as \(nwb\) with \(n\) in \(N\), \(b\) in \(B\).

We’ll need an explicit factorization. If \(g\) is not in \(B\) then
\[
g = \begin{bmatrix} \alpha & \beta \\
\gamma & \delta \end{bmatrix}
\]
with \(\gamma \neq 0\). Then, as we have seen, if we identify \(\mathbb{P}\) with \(\{\infty\} \cup \mathbb{F}\) we have \(g(\infty) = \alpha/\gamma\) so
\[
g = \begin{bmatrix} 1 & \alpha/\gamma \\
0 & 1 \end{bmatrix}
\begin{bmatrix} 0 & -1 \\
1 & 0 \end{bmatrix}
\begin{bmatrix} a & x \\
0 & 1/a \end{bmatrix}
\]
for some \(a, x\). A simple calculation tells us
\[
a = \gamma, \quad x = \delta.
\]
so we can write
\[
g = \begin{bmatrix} 1 & \alpha/\gamma \\
0 & 1 \end{bmatrix}
\begin{bmatrix} 0 & -1 \\
1 & 0 \end{bmatrix}
\begin{bmatrix} \gamma & \delta \\
0 & 1/\gamma \end{bmatrix}
\]
\[
= \begin{bmatrix} 1 & \alpha/\gamma \\
0 & 1 \end{bmatrix}
\begin{bmatrix} 0 & -1 \\
1 & 0 \end{bmatrix}
\begin{bmatrix} \gamma & 0 \\
0 & 1/\gamma \end{bmatrix}
\begin{bmatrix} 1 & \delta/\gamma \\
0 & 1 \end{bmatrix}
\]
\[
= \begin{bmatrix} 1 & \alpha/\gamma \\
0 & 1 \end{bmatrix}
\begin{bmatrix} 1/\gamma & 0 \\
0 & 1 \end{bmatrix}
\begin{bmatrix} 0 & -1 \\
\gamma & 1 \end{bmatrix}
\begin{bmatrix} 1 & \delta/\gamma \\
0 & 1 \end{bmatrix}.
\]

To summarize, \(G\) is the disjoint union
\[
G = NwB \cup B.
\]

Since \(B = AN\) and \(w\) normalizes \(A\), we have \(Aw = wA\), and we can also say:

**Lemma 2.1.** Any \(g\) in \(\text{SL}_2(\mathbb{F})\) is either in \(B\) or factors uniquely as \(bwn\) with \(b\) in \(B\), \(n\) in \(N\).
3. The principal series representations of SL(2)

Even the brief discussion about the representations of $\mathfrak{S}_n$ should suggest that there is no universal method for finding representations of complicated groups. Every interesting group offers its own path to salvation. We shall see this assertion reinforced in this section.

In this section let $F$ be the field $\mathbb{Z}/p$. I'll assume $p$ to be an odd prime, since if $p = 2$ the group $\text{SL}_2(\mathbb{Z}/2)$ is isomorphic to $\mathfrak{S}_3$, which we have already investigated. I recall (Fermat's 'little' theorem) that the multiplicative group $F^\times$ of $F$ is cyclic (of order $p-1$). The group $\text{SL}_2(F)$ acts transitively on $F^2 - \{0\}$, and the subgroup fixing $(1,0)$ is $N$, so $|G| = p(p^2 - 1)$.

The representations of $G$ are of two types, the principal series and the cuspidal representations. We'll look here only at the first, which are by far the simpler to construct, and still interesting.

Suppose $\chi$ to be a character of $F^\times$. Through the projection
\[
\begin{bmatrix} a & x \\ 1 & 1/a \end{bmatrix} \mapsto \chi(a)
\]
it determines a character of $B$, trivial on its normal subgroup $N$. (In fact, any character of $B$ is of this form—i.e. any character of $B$ is trivial on $N$.) Define

\[ I(\chi) = \text{Ind}(\chi | B, G) . \]

Since $|G| = p(p^2 - 1)$ and $|B| = p(p - 1)$ it has dimension $p + 1$.

Here is the main result, which it will take a while to prove:

[ps-reps] Theorem 3.1. If $\chi$ is a character of $F^\times$, then:

(a) the representation $I(\chi)$ is irreducible if and only if $\chi^2 \neq 1$;
(b) the representation $I(\chi)$ is isomorphic to $I(\rho)$ if and only if $\chi = \rho^{\pm 1}$;
(c) if $\chi = 1$ then $I(\chi)$ contains the trivial representation of $G$, and its complement is an irreducible representation of dimension $p$;
(d) if $\chi$ is the unique character of order two, then $I(\chi)$ is the direct sum of two distinct irreducible representations, each of dimension $(p + 1)/2$.

Proof. Proofs of all these assertions depends on the same idea used in applying Frobenius reciprocity—an answer to the question: What is $\text{Hom}(I(\chi), I(\rho))$? How do operators in this space act on $I(\chi)$?

Frobenius reciprocity tells us this space may be identified with
\[ \text{Hom}_B(\text{Ind}(\chi), \rho) , \]
so we must figure out what the restriction of $I(\chi)$ to $B$ looks like. The principal conclusion of the previous section is that
\[ G = BwN \cup B = BwB \cup B \]

where
\[
\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} .
\]

Therefore the restriction of $I(\chi)$ to $B$ is the direct sum of two pieces—the functions in $I(\chi)$ with support on $B$ and those with support on $BwN$. Let $I_1(\chi)$ be the subspace of $I(\chi)$ of functions with support in $B$, and let $I_w(\chi)$ be the space of those with support on $BwN$.

If $V$ is any representation of $B$ and $\rho$ is a character of $B$ trivial on $N$, then any $B$-equivariant map from $V$ to $\rho$ is by restriction an $N$-equivariant map from $V$ to the trivial representation. I shall now expin how to find all of these, and then see that answering this question answers the original one.
(1) For any function in $I_1(\chi)$, $f(b) = \chi(b)f(1)$, so the dimension of $I_1(\chi)$ is one. The map

$$\Omega_1: f \mapsto f(1)$$

is $N$-invariant, and

$$\langle \Omega_1, R_b f \rangle = f(b) = \chi(b)f(1),$$

so $\Omega_1$ lies in $\text{Hom}_B(I(\chi), \chi)$. By Frobenius reciprocity, this corresponds to the identity map from $I(\chi)$ to itself.

(2) A function in $I_w(\chi)$ is determined by its restriction to $wN$, since its support lies in $BwN$. As a representation of $N$, it is isomorphic to $\mathbb{C}(N)$. There is up to scalar only one $N$-invariant function from $\mathbb{C}(N)$ to $\mathbb{C}$, the average over $N$. On $I_w$ I’ll take this to be

$$\Omega_w: f \mapsto \gamma \sum_{x \in N} f(wx).$$

for some constant $\gamma \neq 0$ that I’ll specify later.

**Lemma 3.2.** The map $f \mapsto \langle \Omega_w, f \rangle$ is a $B$-equivariant map from $I(\chi)$ to $\chi^{-1}$.

**Proof.** If $b = an$ with $a$ in $A$, $n$ in $N$ we have

$$\langle \Omega_w, R_b f \rangle = \gamma \sum_{x \in N} f(wxan)$$

$$= \gamma \sum_{x \in N} f(wa \cdot a^{-1}xa \cdot n)$$

$$= \gamma \sum_{y \in N} f(wa \cdot y)$$

$$= \gamma \sum_{y \in N} f(waw^{-1} \cdot wy)$$

$$= \gamma \sum_{y \in N} f(a^{-1} \cdot wy)$$

$$= \chi^{-1}(a)\langle \Omega_w, f \rangle$$

$$= \chi^{-1}(b)\langle \Omega_w, f \rangle \cdot 0$$

Therefore

$$\dim \text{Hom}_B(I(\chi), \rho) = \dim \text{Hom}_G(I(\chi), I(\rho)) = \begin{cases} 0 & \text{if } \rho \neq \chi^\pm 1 \\ 1 & \text{if } \rho = \chi^{-1} \text{ but } \chi^2 \neq 1 \\ 2 & \text{if } \rho = \chi \text{ and } \chi^2 = 1. \end{cases}$$

If $\chi = 1$, then $I(\chi)$ contains the constant functions, making up the trivial representation of $G$. Since the dimension of $\text{End}(I(\chi))$ in this case is two, the complement, which is of dimension $p$, is irreducible.

This implies all the claims to be verified except when $\chi$ is the quadratic character $\text{sgn}$, in which case this argument proves only that $I(\chi)$ is the sum of two irreducible components. It remains to show the components have the same dimension.

To prove this, we must examine more closely the effect of the operator $T$ which corresponds by Frobenius reciprocity to $\Omega_w$. For this, we shall want to have at hand a good basis of $I_w$. 
The basis I construct is that of eigenfunctions of $N$. First of all, there is

$$
\varphi_1(g) = \begin{cases} 
\chi(b) & \text{if } g = b \text{ lies in } B \\
0 & \text{otherwise.}
\end{cases}
$$

Then there the functions with support on $BwN$:

$$
\varphi_w(g) = \begin{cases} 
0 & \text{for } g \in B \\
\chi(b) & \text{if } g = bwn.
\end{cases}
$$

Then for each nontrivial additive character $\psi$ of $N$ or, effectively, $F$:

$$
\varphi_{\psi}(g) = \begin{cases} 
0 & \text{if } g \text{ lies in } B \\
\chi(b)\psi(n) & \text{if } g = bwn.
\end{cases}
$$

*How does the operator $T$ act on these?* Frobenius reciprocity gives us

$$
[Tf](g) = (\Omega_w, R_g f) = \int_N f(wn g) \ dn.
$$

Since $T$ commutes with $G$, it commutes with $N$, and therefore takes the subspace $I(\chi)^N$, which is spanned by $\varphi_1$ and $\varphi_w$, into itself; and also takes each $\varphi_{\psi}$ into a multiple of itself. A function in $I(\chi)^N$ is determined by its values at 1 and $w$, and one of the $\varphi_1$ is determined by its value at $w$. From here on, I go by cases.

We shall need the following factorization (with $x \neq 0$), which is a special case of the one leading to [Lemma 2.1.](#)

$$
a_x = \begin{bmatrix} x & 0 \\
0 & 1/x
\end{bmatrix}, \quad n_x = \begin{bmatrix} 1 & x \\
0 & 1
\end{bmatrix}.
$$

Then

$$
wn_xw = \begin{bmatrix} -1 & 0 \\
x & -1
\end{bmatrix} = n_{-1/x}a_{1/x}wn_{-1/x}.
$$

Since $w^2 = -I$, for $f$ in $I(\chi)$ this gives us

$$
\sum_{n \in N} f(wnw) = f(-I) + \sum_{x \neq 0} \chi^{-1}(x)f(wn_{-1/x}).
$$

* What is $T\varphi_1$?

$$
[T\varphi_1](1) = \gamma \sum_{n \in N} \varphi_1(wn) = 0 \quad \text{since } f_1 = 0 \text{ on } BwN
$$

$$
[T\varphi_1](w) = \gamma \sum_{n \in N} \varphi_1(wnw)
$$

$$
= \gamma \varphi_1(w^2) \quad \text{since } \varphi_1 = 0 \text{ on } BwB
$$

$$
= \gamma \chi(-1).
$$

Thus

$$
T\varphi_1 = \gamma \chi(-1) \varphi_w.
$$
• What is $T\varphi_w$?

$$[T\varphi_w](1) = \gamma \sum_{n \in N} \varphi_w(wn)$$

$$= \gamma q$$

$$[T\varphi_w](w) = \gamma \left( \varphi_w(-I) + \sum_{x \neq 0} \chi(1/x) \right)$$

$$= \begin{cases} 
\gamma(q-1) & \text{if } \chi = 1 \\
0 & \text{otherwise.} 
\end{cases}$$

Thus

$$T\varphi_w = \begin{cases} 
\gamma(q\varphi_1 + (q-1)\varphi_w) & \text{if } \chi = 1 \\
\gamma q\varphi_1 & \text{otherwise.} 
\end{cases}$$

• What is $T\varphi_\psi$? If $\chi = 1$ we get

$$[Tf_\psi](w) = \gamma \sum_{x \neq 0} \chi(1/x)\psi(1/x))$$

$$= -\gamma.$$ 

while if $\chi \neq 1$

$$[Tf_\psi](w) = \gamma \sum_{x \neq 0} \chi(1/x)\psi(1/x))$$

$$= \gamma \sqrt{q} G_{\chi,\psi}.$$ 

so

$$T\varphi_\psi = \begin{cases} 
-\gamma\varphi_\psi & \text{if } \chi = 1 \\
\gamma \sqrt{q} G_{\chi,\psi} \varphi_w & \text{otherwise.} 
\end{cases}$$

I now choose $\gamma = 1/\sqrt{q}$. In summary, if $\chi = 1$ then

$$T\varphi_1 = \frac{1}{\sqrt{q}} \varphi_w$$

$$T\varphi_w = \frac{1}{\sqrt{q}} (q\varphi_1 + (q-1)\varphi_w)$$

$$T\varphi_\psi = -\gamma \varphi_\psi$$

and if $\chi \neq 1$

$$T\varphi_1 = \frac{\chi(-1)}{\sqrt{q}} \varphi_w$$

$$T\varphi_w = \sqrt{q} \varphi_1$$

$$T\varphi_\psi = G_{\chi,\psi} \varphi_\psi$$

One nice feature is that if $\chi \neq 1$ then $T^2 = \chi(= 1) \cdot I$, since $G_{\chi,\psi} G_{\chi,-1,\psi} = \chi(-1)$. I leave it as an exercise to check that if $\chi = \text{sgn}$ then $I(\chi)$ decomposes into the eigenspaces of $\chi$, with eigenvalues the two square roots of $\text{sgn}(-1)$. They are of equal dimension.