The computation of structure constants according to Jacques Tits

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Suppose $\mathfrak{g}$ to be a complex semi-simple Lie algebra. In 1955, Chevalley showed that one can assign to it an essentially unique $\mathbb{Z}$-structure, with structure constants of a simple form, themselves unique up to some choices of sign. An algorithm to compute these is also due implicitly to Chevalley, explained explicitly in [Carter:1972] and more recently in [Cohen-Murray-Taylor:2004]. It is reasonably efficient, and has been implemented in the computer algebra program MAGMA. The principle behind the algorithm is simple—it is based on a straightforward transcription of the Jacobi identity. [Cohen-Murray-Taylor:2004] goes on to explain in detail how to compute in semi-simple algebraic groups associated to the Lie algebra in terms of the Bruhat decomposition.

Another account of structure constants was presented in [Frenkel-Kac:1980] for simply laced systems, and this has been extended to all cases by [Rylands:2000], using the technique of defining the other systems by 'folding' ones that are simply laced.

These algorithms are all very efficient, and work ell in practice. But there are reasons for trying another approach to the computation of structure constants. The most obvious is that, as far as I can tell, they do not extend to arbitrary Kac-Moody algebras. Although there are obvious problems due to the infinite structure of the root system, the really essential reason is the interference caused by imaginary roots. I shan't elaborate on this, but instead simply offer a somewhat different approach that, at least in principle, will work for all Kac-Moody algebras, finite- or infinite-dimensional. For finite systems it does not seem to be a lot less efficient than the one currently in use. Conjecturally, even the problems caused by a possibly infinite root system can be overcome.

This method is one due implicitly to [Tits:1966a]. In this scheme, all computations take place in the extended Weyl group. Tits himself has made remarks that indicate how his method may be applied to arbitrary Kac-Moody algebras, but I do not believe he ever published details.

Much of this paper just explains results of Tits in a slightly different perspective. What is new here are details that translate these results into a practical algorithm.

I might remark that this approach seems to me mathematically more interesting than the others.

My principal general reference for finite system has been [Carter:1972], mostly Chapters 3 and 4, although for proofs Carter often refers to [Jacobson:1962]. For Kac-Moody algebras my main references have been [Tits:1987] and [Tits:1990].

I am extremely grateful to the organizers of Nolan Wallach's birthday conference for inviting me to speak in San Diego. I also wish to thank my colleague Julia Gordon for inviting me to speak in a seminar on an early version of this material. Without that stimulus, I might never have written this.

Contents

1. Introduction

Suppose

$$\begin{aligned} \mathfrak{g} &= \text{a semi-simple complex Lie algebra} \\ \mathfrak{h} &= \text{a Cartan subalgebra} \\ \Sigma &= \text{roots of } \mathfrak{g} \text{ with respect to } \mathfrak{h} \\ \Delta &= \text{a basis of simple roots in } \Sigma. \end{aligned}$$

For each λ in Σ, let $\mathfrak{g}_\lambda$ be the one-dimensional root subspace of $\mathfrak{g}$ on which $\mathfrak{h}$ acts by λ. If for each root λ we choose $e_\lambda \neq 0$ in $\mathfrak{g}_\lambda$, a basis of $\mathfrak{g}$ will be made up of a basis (h_i) of $\mathfrak{h}$ together with all the e_λ. Then

$$[e_\lambda, e_\mu] = \begin{cases} \text{some sum } \sum c_i h_i & \text{if } \mu = -\lambda \\ N_{\lambda,\mu} e_{\lambda+\mu} & \text{if } \lambda+\mu \text{ is a root (with } N_{\lambda,\mu} \neq 0) \\ \quad 0 & \text{otherwise.} \end{cases}$$

The $N_{\lambda,\mu}$ are called the **structure constants** of the system. The problem at hand is to find a way to specify a good set of e_λ and a good algorithm for computing the corresponding structure constants.

For each α in Δ choose $e_\alpha \neq 0$ in $\mathfrak{g}_\alpha$ and let $\mathcal{E}$ be the set $\{e_\alpha\}$, or rather the map

$$\Delta \longrightarrow \mathfrak{g}, \quad \alpha \longmapsto e_\alpha\,.$$

I call the pair $\mathfrak{h}$, $\mathcal{E}$ a **frame** for $\mathfrak{g}$ (called an **épinglage** by the French). The importance of frames is that they rigidify the algebra—the *isomorphism theorem* (3.5.2 of [Carter:1972]) asserts that if we are given two semi-simple Lie algebras together with frames giving rise to the same Cartan matrix, there exists a unique isomorphism of the two taking one frame into the other. In particular, all frames for $\mathfrak{g}$ are conjugate under the automorphism group of $\mathfrak{g}$, and may be taken as equivalent.

If α lies in Δ and $e_{\pm\alpha}$ lies in $\mathfrak{g}_{\pm\alpha}$ then $[e_\alpha, e_{-\alpha}]$ lies in $\mathfrak{h}$. Given e_α there exists a unique $e_{-\alpha}$ in $\mathfrak{g}_{-\alpha}$ such that $\langle \alpha, [e_\alpha, e_{-\alpha}]\rangle = -2$. In these circumstances, let $h_\alpha = -[e_\alpha, e_{-\alpha}]$ (so $\langle \alpha, h_\alpha \rangle = 2$). For example, if $\mathfrak{g} = \mathfrak{sl}_2$, we have the standard triplet

$$\begin{aligned} h &= \begin{bmatrix} 1 & \circ \\ \circ & -1 \end{bmatrix} \\ e_+ &= \begin{bmatrix} \circ & 1 \\ \circ & \circ \end{bmatrix} \\ e_- &= \begin{bmatrix} \circ & \circ \\ -1 & \circ \end{bmatrix}\,. \end{aligned}$$

The more usual convention is to require $[e_\alpha, e_{-\alpha}] = h_\alpha$, but the choice made here, as [Tits:1966a] emphasizes, has virtues that ought not to be ignored.

The triple h_α, $e_{\pm\alpha}$ span in $\mathfrak{g}$ a copy $\mathfrak{sl}_{2,\alpha}$ of $\mathfrak{sl}_2$, which therefore acts on $\mathfrak{g}$ through the adjoint action. Since $\mathrm{SL}(2,\mathbb{C})$ is connected and simply connected, this comes from a unique representation of $\mathrm{SL}(2,\mathbb{C})$ as well. From this observation, it is not difficult to deduce that $C = (c_{\alpha,\beta}) = (\langle \alpha, h_\beta \rangle)$ is a Cartan matrix, which means

(a) all $c_{\alpha,\beta}$ are integral;
(b) $c_{\alpha,\alpha} = 2$;
(c) $c_{\alpha,\beta} \le 0$ for $\alpha \ne \beta$;
(d) there exists a diagonal matrix D with positive entries such that CD is positive definite.

In particular, the roots Σ are contained in the lattice L_Δ spanned by Δ, and the h_α may be identified with elements $\alpha^\vee$ of its dual lattice $L_\Delta^\vee$. The positive definite matrix CD gives rise to Euclidean norms on L and $\mathfrak{h}$:

$$\alpha \bullet \beta = \langle \alpha, \beta^\vee \rangle d_{\beta,\beta} \,.$$

This implies that

$$\langle \alpha, \beta^\vee \rangle = 2 \cdot \frac{\alpha \bullet \beta}{\beta \bullet \beta} \,,$$

hence that each linear transformation

$$s_\alpha \colon v \longmapsto v - \langle v, \alpha^\vee \rangle \alpha$$

is an orthogonal reflection in L_Δ taking Σ to itself. The group generated by these reflections may be identified with the Weyl group W of $(\mathfrak{g}, \mathfrak{h})$, and the set Σ is the W-orbit of Δ. The adjoint group $G_{\mathrm{adj}}(\mathbb{C})$ acts by the adjoint action as automorphisms of $\mathfrak{g}$. Let H_{adj} be the torus of G_{adj} whose Lie algebra is $\mathfrak{h}$. If $\lambda = w\alpha$ and g in the normalizer of H_{adj} in G_{adj} has image w in W then the subalgebra

$$\mathfrak{sl}_{2,\lambda} = \mathrm{Ad}(g)\, \mathfrak{sl}_{2,\alpha} \,.$$

depends only on w, although its identification with $\mathfrak{sl}_2$ depends on g.

To each root λ is associated a coroot according to the formula $(w\alpha)^\vee = w\alpha^\vee$. The definition of the inner product and the root reflections may be generalized to arbitrary roots. This leads to the following observation:

Lemma 1.1. *If λ and μ are roots, then $\langle \lambda, \mu^\vee \rangle \langle \mu, \lambda^\vee \rangle \ge 0$. This product vanishes only if both factors vanish.*

Proof. From the calculation

$$\langle \lambda, \mu^\vee \rangle \langle \mu, \lambda^\vee \rangle = 4\, \frac{(\lambda \bullet \mu)^2}{\|\lambda\|^2 \|\mu\|^2} \,. \;\square$$

There is a *canonical* W-invariant Euclidean norm on the lattice L_Δ. Define the linear map

$$\rho\colon\; L_\Delta \longmapsto L_{\Delta^\vee}, \quad \lambda \longmapsto \sum_{\mu \in \Sigma^\vee} \langle \lambda, \mu^\vee \rangle \mu^\vee \,,$$

and then define the positive definite norm

$$\|\lambda\|^2 = \lambda \bullet \lambda = \langle \lambda, \rho(\lambda) \rangle = \sum_{\Sigma^\vee} \langle \lambda, \mu^\vee \rangle^2 \,.$$

Lemma 1.2. *For every root λ*

$$\|\lambda\|^2 \lambda^\vee = 2\rho(\lambda) \,.$$

Thus although $\lambda \mapsto \lambda^\vee$ is not linear it deviates from linearity in a manageable way.

Proof. For every μ in Σ

$$\begin{aligned} s_{\lambda^\vee}\mu^\vee &= \mu^\vee - \langle\lambda,\mu^\vee\rangle\lambda^\vee \\ \langle\lambda,\mu^\vee\rangle\lambda^\vee &= \mu^\vee - s_{\lambda^\vee}\mu^\vee \\ \langle\lambda,\mu^\vee\rangle^2\,\lambda^\vee &= \langle\lambda,\mu^\vee\rangle\mu^\vee - \langle\lambda,\mu^\vee\rangle s_{\lambda^\vee}\mu^\vee \\ &= \langle\lambda,\mu^\vee\rangle\mu^\vee + \langle s_\lambda\lambda,\mu^\vee\rangle s_{\lambda^\vee}\mu^\vee \\ &= \langle\lambda,\mu^\vee\rangle\mu^\vee + \langle\lambda, s_{\lambda^\vee}\mu^\vee\rangle s_{\lambda^\vee}\mu^\vee \end{aligned}$$

But since $s_{\lambda^\vee}$ is a bijection of $\Sigma^\vee$ with itself, we can conclude by summing over μ in Σ. ▯

As a consequence of the isomorphism theorem, there exists a unique automorphism θ of $\mathfrak{g}$ acting as -1 on $\mathfrak{h}$ and taking e_λ to $e_{-\lambda}$. It is called the *canonical involution* determined by the frame. For SL_n as well as classical groups embedded in it suitably, it is the map taking X to ${}^tX^{-1}$.

By construction $\theta(e_\alpha) = e_{-\alpha}$ for α in Δ. More generally, if λ is any root and $e_\lambda \neq 0$ lies in $\mathfrak{g}_\lambda$ then there exists a unique $e_{-\lambda}$ in $\mathfrak{g}_{-\lambda}$ such that

$$\langle\lambda, h_\lambda\rangle = 2 \quad \big([e_\lambda, e_{-\lambda}] = -h_\lambda\big)\,.$$

We must have $\theta(e_\lambda) = ae_{-\lambda}$ for some $a \neq 0$. If we replace e_λ by be_λ, we replace $e_{-\lambda}$ by $(1/b)e_\lambda$. We shall have $\theta(be_\lambda) = (1/b)e_{-\lambda}$ if and only if $b^2a = 1$, so there exist exactly two choices of e_λ such that $\theta(e_\lambda) = e_{-\lambda}$. We get in this way a family of bases of $\mathfrak{g}$ well behaved under θ. I shall call these **Chevalley bases**, although there is a difference in sign from the standard meaning of that term (since the standard convention is to require $[e_\lambda, e_{-\lambda}] = h_\lambda$).

For Chevalley bases, the structure constants are particularly simple. For roots λ, μ, the set of roots in $\mathbb{Z}\cdot\lambda + \mu$ is called the **λ-chain** through μ. The sum of the spaces $\mathfrak{g}_\nu$ as ν ranges over this chain is stable under $\mathrm{SL}_\lambda(2)$, hence under the image in it of the reflection

$$\begin{bmatrix} \circ & 1 \\ -1 & \circ \end{bmatrix}.$$

The λ-chain consists of all roots of the form $\mu + n\lambda$ where n lies in an interval $[-p, q]$. (What is important is that there are no gaps.) Define $p_{\lambda,\mu}$ to be the least integer p such that $\mu - p\lambda$ is a root, and $q_{\lambda,\mu}$ to be the maximum q such that $\mu + q\lambda$ is one. Since every pair of roots is contained in the W-transform of the root system spanned by some α, β in Δ, the possibilities are limited by what happens for systems of rank two. We'll look at these more closely later.

The fundamental fact about Chevalley's $\mathbb{Z}$-structure on $\mathfrak{g}$ is that with a Chevalley basis

$$|N_{\lambda,\mu}| = p_{\lambda,\mu} + 1\,.$$

(See §4.2 of [Carter:1972]. I'll recall later Tits' argument.) The $N_{\lambda,\mu}$ are thus determined up to a sign. Some of these are arbitrary, since we can always change the sign of the e_λ. After such a basic choice, one is faced with the problem of computing all the other signs. The computational problems that the rest of this paper is concerned with are these:

> *Given the Cartan matrix C: (1) construct the root system Σ; (2) specify a Chevalley basis for the corresponding Lie algebra; (3) calculate the set of structure constants for the system.*

I'll pose this last problem in terms close to those of Tits:

> *Given Chevalley basis elements e_λ, find the functions ε such that*
>
> $$[e_\lambda, e_\mu] = \varepsilon(\lambda,\mu,\nu)(p_{\lambda,\mu} + 1)\,e_{-\nu}$$
>
> *whenever λ, μ, ν are roots with $\lambda + \mu + \nu = 0$.*

Chevalley's theorem tells us that $\varepsilon = \pm 1$, but I'll not assume that to start with, and in fact deduce it along the way.

2. Digression on root chains

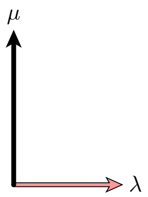
I recall that $p_{\lambda,\mu}$ is the largest integer p such that $\mu - p\lambda$ is a root, $q_{\lambda,\mu}$ the largest q such that $\mu + q\lambda$ is one. The reflection

$$s_\lambda\colon\ \mu \longmapsto \mu - \langle \mu, \lambda^\vee \rangle \lambda$$

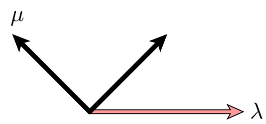
inverts this chain, and swaps p and q. A quick sketch will convince you that

$$p_{\lambda,\mu} - q_{\lambda,\mu} = \langle \mu, \lambda^\vee \rangle \,.$$

In particular at the right-hand end where $q_{\lambda,\mu} = 0$ and $p_{\lambda,\mu}$ takes its maximum value, we have $p_{\lambda,\mu} = \langle \mu, \lambda^\vee \rangle$. Because the paiur λ, μ lie in some subsystem of rank two, this implies that $p_{\lambda,\mu}$ must be one of the integers 0, 1, 2, 3, giving a definite bound to the lengths of λ-chains. Since every pair λ, μ lies in a root system of rank two, and these can be easily classified, one can easily classify them:

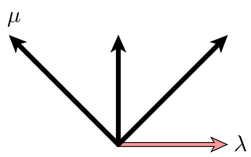


Length 1

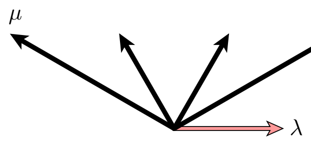


Length 2

For the first, the only condition imposed is that the roots be perpendicular, and in the second that $\|\lambda\| \ge \|\mu\|$.

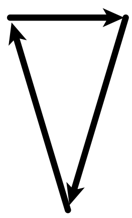


Length 3



Length 4

In this group, the first may be embedded into a system of type C_2, the second into one of type G_2, so angles and the ratios of lengths are uniquely determined. What is especially important is that we cannot have this configuration in a root system:



An easy geometric argument will show this directly. What this means is:

Lemma 2.1. *Given roots λ, μ, ν with $\lambda + \mu + \nu = 0$, we can so name them that*

$$\|\lambda\| \ge \|\mu\| = \|\nu\| \,.$$

This makes the following result interesting.

Lemma 2.2. *Suppose λ, μ, ν to be roots with $\lambda + \mu + \nu = 0$. The following are equivalent:*

(a) $s_\lambda \mu = -\nu$;
(b) $\langle \mu, \lambda^\vee \rangle = -1$;
(c) $\|\lambda\| \ge \|\mu\|$ *and* $\|\lambda\| \ge \|\nu\|$.

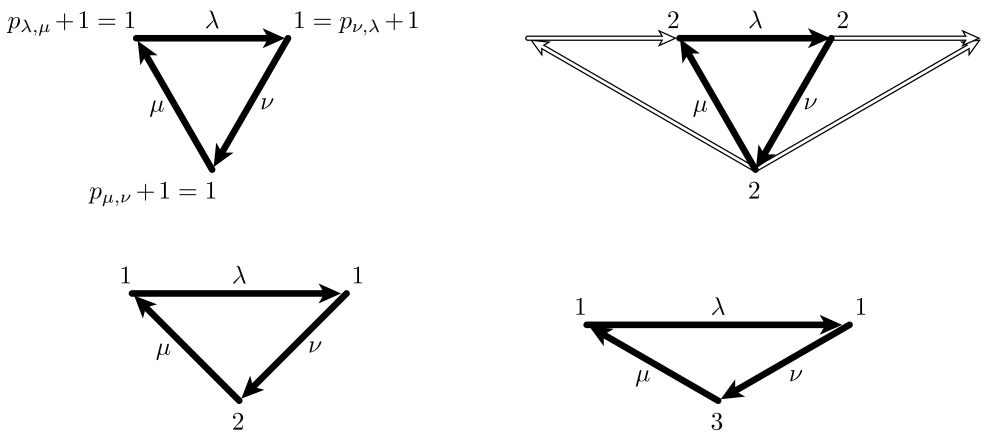
Proof. This is a consequence of the inadmissibility of the configuration in the figure. ∎

I define a triple (λ, μ, ν) to be **admissible** if $\lambda > 0$ and one (hence all) of these hold. Properties of admissible triples are crucial in Tits' calculation of structure constants.

Lemma 2.3. *Given roots* λ, μ, ν *with* $\lambda + \mu + \nu = 0$

$$\frac{p_{\lambda,\mu} + 1}{\|\nu\|^2} = \frac{p_{\mu,\nu} + 1}{\|\lambda\|^2} = \frac{p_{\nu,\lambda} + 1}{\|\mu\|^2} .$$

Proof. This can be seen by perusing rank two systems. It can also be proved slightly more directly, using the fact that λ-chains are of length at most four.



The case where all three lengths are the same can be seen directly, since there exists a symmetry of the root system cycling the three roots. ∎

Lemma 2.4. *Given roots* λ, μ, ν *with* $\lambda + \mu + \nu = 0$ *we have*

$$(p_{\lambda,\mu} + 1)\nu^\vee + (p_{\mu,\nu} + 1)\lambda^\vee + (p_{\nu,\lambda} + 1)\mu^\vee = 0 .$$

Proof. According to Lemma 1.2 the linear map 2ρ takes the root λ to the vector

$$\|\lambda\|^2 \lambda^\vee ,$$

so if $\lambda + \mu + \nu = 0$ we also have

$$\|\lambda\|^2 \lambda^\vee + \|\mu\|^2 \mu^\vee + \|\nu\|^2 \nu^\vee = 0 .$$

Apply the previous Lemma. ∎

3. Digression on representations of SL(2)

Representations of $\mathrm{SL}(2)$ play an important role in both Carter's and Tits' approach to structure constants. In this section I recall briefly what is needed.

Let

$$u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

be the standard basis of $\mathbb{C}^2$, on which $\mathrm{SL}(2, \mathbb{C})$ and $\mathfrak{sl}_2$ act. They also act on the symmetric space $S^n(\mathbb{C})$, with basis $u^k v^{n-k}$ for $0 \le k \le n$. Let π_n be this representation, which is of dimension $n+1$.

$$\begin{aligned} \pi_n\left(\begin{bmatrix} e^x & \circ \\ \circ & e^{-x} \end{bmatrix}\right)&\colon\ u^k v^{n-k} \longmapsto e^{(2k-n)x} u^k v^{n-k} \\ \pi_n\left(\begin{bmatrix} 1 & x \\ \circ & 1 \end{bmatrix}\right)&\colon\ u^k v^{n-k} \longmapsto u^k (v + xu)^{n-k} \\ \pi_n\left(\begin{bmatrix} 1 & \circ \\ -x & 1 \end{bmatrix}\right)&\colon\ u^k v^{n-k} \longmapsto (u - xv)^k v^{n-k} . \end{aligned}$$

Now to translate these formulas into those for the action of $\mathfrak{g}$. Let $w_k = u^k v^{n-k}$.

$$\begin{aligned} \pi_n(h)&\colon\ w_k \longmapsto (2k-n)\, w_k \\ \pi_n(e_+)&\colon\ w_k \longmapsto (n-k) w_{k+1} \\ \pi_n(e_-)&\colon\ w_k \longmapsto (-1)^k k w_{k-1} . \end{aligned}$$

For many reasons, it is a good idea to use a different basis of the representation. Define **divided powers**

$$u^{[k]} = \frac{u^k}{k!} ,$$

and set $w_{[k]} = u^{[k]} v^{[n-k]}$. Then

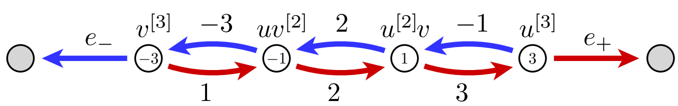
$$\begin{aligned} \pi_n(h)&\colon\ w_{[k]} \longmapsto (2k-n)\, w_{[k]} \\ \pi_n(e_+)&\colon\ w_{[k]} \longmapsto (k+1) w_{[k+1]} \\ \pi_n(e_-)&\colon\ w_{[k]} \longmapsto (-1)^k (n-k+1) w_{[k-1]} . \end{aligned}$$

If

$$\sigma = \begin{bmatrix} \circ & 1 \\ -1 & \circ \end{bmatrix}$$

then we also have

$$\pi_n(\sigma)\colon\ w_{[k]} \longmapsto (-1)^k w_{[n-k]} .$$



The representation π_3

The π_n exhaust the irreducible finite dimensional representations of both $\mathrm{SL}(2, \mathbb{C})$ and $\mathfrak{sl}_2$, and every finite-dimensional representation of either is a direct sum of them.

Let $d = n+1$, the dimension of π_n, assumed even. The weights of this with respect to h are

$$-d, \ldots, -1, 1, \ldots, d ,$$

The formulas above imply immediately:

Lemma 3.1. *Suppose v to be a vector of weight -1 in this representation. Then $\pi(e_+)v$ and $\pi(\sigma)v$ are both of weight 1, and*

$$\pi(e_+)v = -(-1)^{d/2}\binom{d}{2}\pi(\sigma)v\,.$$

I recall that the sum of weight spaces $\mathfrak{g}_\mu$ for μ in a given λ-chain is a representation of $\mathfrak{sl}_2$. The formulas laid out in this section relate closely to the numbers $p_{\lambda,\mu}$ and $q_{\lambda,\mu}$. For one thing, as the picture above suggests and is easy to verify, they tell us that we can choose basis elements e_μ for each μ in the chain so that

$$\begin{aligned}[e_\lambda, e_\mu] &= (p_{\lambda,\mu}+1)e_{\mu+\lambda} \\ [e_{-\lambda}, e_\mu] &= (-1)^{p_{\lambda,\mu}}(q_{\lambda,\mu}+1)e_{\mu-\lambda}\end{aligned}$$

The choice of basis for one chain unfortunately affects other chains as well, so this observation doesn't make the problem of structure constants trivial. But it is our first hint of a connection between structure constants and chains.

Although the explicit formulas for π_n depend on the choice of the basis, the product $\pi(e_+)\pi(e_-)$ acts as scalars, independent. So we can conclude that with respect to any basis of $\mathfrak{g}$.

$$N_{\lambda,\mu}N_{-\lambda,\lambda+\mu} = -(-1)^{p_{\lambda,\mu}}(p_{\lambda,\mu}+1)q_{\lambda,\mu}\,.$$

4. Carter's algorithm

It will be useful for purposes of comparison to know something about the standard approach to the problem. Suppose given a root system Σ and a basis Δ in Σ of positive roots. Fix a vector space with basis h_α for α in Δ, and e_λ for λ in Σ. Assume

$$[e_\lambda, e_{-\lambda}] = -h_\lambda, \quad \langle\lambda, h_\lambda\rangle = 2$$

for all roots λ. Automatically, we have

$$\begin{aligned}[h_\alpha, h_\beta] &= 0 \\ [h_\alpha, e_\lambda] &= \langle\lambda, \alpha^\vee\rangle e_\lambda\,.\end{aligned}$$

Lemma 4.1. *Together with the formulas above, the formulas*

$$[e_\lambda, e_\mu] = \begin{cases} 0 & \text{if } \lambda+\mu \text{ is not a root} \\ N_{\lambda,\mu}e_{\lambda+\mu} & \text{if } \lambda+\mu \text{ is a root}\end{cases}$$

define a Lie algebra if and only if

(a) $N_{\mu,\lambda} = -N_{\lambda,\mu}$;

(b) $N_{\lambda,\mu}N_{-\lambda,\lambda+\mu} = (-1)^{p_{\lambda,\mu}+1}(p_{\lambda,\mu}+1)q_{\lambda,\mu}$;

(c) *whenever* $\lambda+\mu+\nu=0$

$$\frac{N_{\lambda,\mu}}{\|\nu\|^2} = \frac{N_{\mu,\nu}}{\|\lambda\|^2} = \frac{N_{\nu,\lambda}}{\|\mu\|^2}\,;$$

(d) *whenever* $\lambda+\mu+\nu+\rho=0$ *but no pair is opposite*

$$\frac{N_{\lambda,\mu}N_{\nu,\rho}}{\|\lambda+\mu\|^2} + \frac{N_{\mu,\nu}N_{\lambda,\rho}}{\|\mu+\nu\|^2} + \frac{N_{\nu,\lambda}N_{\mu,\rho}}{\|\lambda+\mu\|^2} = 0\,.$$

For discussion and proof, see §4.1 of [Carter:1972].

These properties are directly related to the two defining properties of a Lie bracket. Property (a) is equivalent to $[x, y] = -[y, x]$. The rest are deductions in various cases from the Jacobi identity

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0\,.$$

Property (b), as we have seen in the previous section, is related to the adjoint representation of the copy of $\mathfrak{sl}_2$ associated to the λ-chain. It can also be deduced directly from the Jacobi identity for e_λ, $e_{-\lambda}$, e_μ as is done in [Carter:1972]. The norms in the formulas appear as a consequence of the identification of $\mu^\vee$ with $2\mu/\|\mu\|^2$ induced by the Euclidean metric.

The first step in Carter's algorithm is to specify certain e_λ for $\lambda > 0$. The e_α for $\alpha \in \Delta$ are arbitrary. After that, one way to proceed is to first order Δ, then for $\mu > 0$ but $\mu \notin \Delta$ set $e_\mu = [e_\alpha, e_\lambda]/(p_{\alpha,\lambda} + 1)$ if α is least with $\mu - \alpha \in \Sigma$. The calculation uses the equations of Lemma 4.1 in a straightforward manner together with recursion on root height to compute the structure constants. Details can be found in [Cohen-Murray-Taylor:2004].

5. Structure constants and the extended Weyl group

In this section, I start describing the method I have in mind for computing structure constants based on ideas of [Tits:1966a].

Choosing a frame and Chevalley basis elements e_λ determines on $\mathfrak{g}$ the structure of a Lie algebra defined over $\mathbb{Z}$. Let G be the split, simply connected group scheme defined over $\mathbb{Z}$ with Lie algebra $\mathfrak{g}_\mathbb{Z}$, and let H be the torus in G whose Lie algebra is $\mathfrak{h}_\mathbb{Z}$. Let L be the **weight lattice** of H, which may be identified with the dual of the lattice $L_{\Delta^\vee}$ spanned by $\Delta^\vee$. The group $H(\mathbb{Z})$ of integral points on H is isomorphic to $\mathrm{Hom}(L, \pm 1) \cong \{\pm 1\}^\Delta$. More explicitly, since

$$0 \longrightarrow L_{\Delta^\vee} \longrightarrow L_{\Delta^\vee} \otimes \mathbb{C} \xrightarrow{\exp} H(\mathbb{C}) \longrightarrow 1$$

is exact, it may be identified with

$$(1/2)L_{\Delta^\vee}/L_{\Delta^\vee} \cong L_{\Delta^\vee} \otimes \{\pm 1\}\,,$$

which may be in turn identified for computational purposes with $L_{\Delta^\vee}/2L_{\Delta^\vee}$. A basis of this is made up of the images of $\alpha^\vee$ for α in Δ.

To each root λ and choice of e_λ corresponds an embedding

$$\lambda_*\colon\ \mathfrak{sl}_2 \hookrightarrow \mathfrak{g}, \qquad \begin{bmatrix} \circ & 1 \\ \circ & \circ \end{bmatrix} \longmapsto e_\lambda, \qquad \begin{bmatrix} 1 & \circ \\ \circ & -1 \end{bmatrix} \longmapsto h_\lambda\,,$$

and also one of $\mathrm{SL}(2)$ into G. The elements $\lambda_*(-I)$ may be identified with $\lambda^\vee$ in $L_{\Delta^\vee}/2L_{\Delta^\vee}$.

The normalizer of $H(\mathbb{Z})$ in $G(\mathbb{Z})$ fits into an exact sequence

$$1 \longrightarrow H(\mathbb{Z}) \longrightarrow N_H(\mathbb{Z}) \longrightarrow W \longrightarrow 1\,.$$

This is a non-trivial extension but, as we shall see shortly, we can describe it very conveniently purely in terms of the root system. In particular, computation in this group reduces essentially to computation in the Weyl group itself.

As an example, the group $H(\mathbb{Z})$ in $\mathrm{SL}_n(\mathbb{Z})$ consists of all diagonal matrices in it with entries ± 1, and the normalizer is the subgroup of monomial matrices with all entries ± 1. Equivalently, it consists of products tw with $\det(tw) = 1$ in which t is a diagonal matrix with $t_i = \pm 1$ and w a permutation matrix.

The basic result upon which this paper depends is this:

Theorem 5.1. *(Jacques Tits) The computation of bracket operations in $\mathfrak{g}_{\mathbb{Z}}$ can be carried out entirely within $N_H(\mathbb{Z})$.*

This rather vague if surprising statement will eventually be expanded into an algorithm.

The connection between N_H and nilpotent elements of the Lie algebra can be seen already in $\mathrm{SL}(2)$. In this group, the normalizer breaks up into two parts, the diagonal matrices and the inverse image M of the non-trivial element of the Weyl group, made up of matrices

$$\sigma(x) = \begin{bmatrix} \circ & x \\ -1/x & \circ \end{bmatrix}$$

with x invertible. The matrix $\sigma = \sigma(x)$ satisfies

$$\sigma^2 = \begin{bmatrix} \circ & x \\ -1/x & \circ \end{bmatrix}^2 = \begin{bmatrix} -1 & \circ \\ \circ & -1 \end{bmatrix}, \quad \sigma^{-1} = -\sigma\,.$$

According to the Bruhat decomposition, every element of G is either upper triangular or can be factored as $n_1\sigma(x)n_2$ with the n_i unipotent and upper triangular. Making this explicit:

$$\begin{bmatrix} 1 & \circ \\ x & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1/x \\ \circ & 1 \end{bmatrix} \begin{bmatrix} \circ & -1/x \\ x & \circ \end{bmatrix} \begin{bmatrix} 1 & 1/x \\ \circ & 1 \end{bmatrix}$$

$$\text{which implies that} \quad \begin{bmatrix} \circ & x \\ -1/x & \circ \end{bmatrix} = \begin{bmatrix} 1 & x \\ \circ & 1 \end{bmatrix} \begin{bmatrix} 1 & \circ \\ -1/x & 1 \end{bmatrix} \begin{bmatrix} 1 & x \\ \circ & 1 \end{bmatrix}.$$

Some easy calculating will verify further the following observation of Tits:

Lemma 5.2. *For any σ in the non-trivial coset of N_H in $\mathrm{SL}(2)$ there exist unique e_+ in $\mathfrak{g}_+$ and e_- such that*

$$\sigma = \exp(e_+)\exp(e_-)\exp(e_+)\,.$$

In this equation, any one of the three determines the other two.

Applying the involution $x \mapsto {}^t x^{-1}$ we see that the roles of e_+ and e_- may be reversed. In other words, specifying an upper triangular nilpotent e_+ or a lower triangular nilpotent e_- is equivalent to specifying an element of M.

We shall apply these considerations in the case $x = \pm 1$, that is to say in $\mathrm{SL}(2, \mathbb{Z})$. Let $H_\lambda(\mathbb{Z})$ be the image of the diagonal matrices in $\mathrm{SL}(2, \mathbb{Z})$, and let $M_\lambda(\mathbb{Z})$ be the image of the matrices

$$\begin{bmatrix} \circ & \pm 1 \\ \mp 1 & \circ \end{bmatrix}$$

in $N_H(\mathbb{Z})$. According to Tits' observation, for each root λ we have bijections

$$\begin{tikzcd} \mathfrak{g}_\lambda(\mathbb{Z}) \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] & & \mathfrak{g}_{-\lambda}(\mathbb{Z}) \arrow[dl, leftrightarrow] \\ & M_\lambda(\mathbb{Z}) & \end{tikzcd}$$

Map $\sigma \in M_\lambda(\mathbb{Z})$ to $e_{\lambda,\sigma}$. *The ambiguity in choice of e_λ is the same as the ambiguity in choosing σ_λ.*

For g in $N_H(\mathbb{Z})$ with image w in W we have

$$gM_\lambda(\mathbb{Z})g^{-1} = M_{w\lambda}(\mathbb{Z})\,.$$

and furthermore for σ in M_λ

$$\mathrm{Ad}(g)e_{\lambda,\sigma} = e_{w\lambda, g\sigma g^{-1}}\,.$$

So now we can at least see some connection between structure constants and the extended Weyl group. At the very beginning of the computation, we have to choose basis elements e_λ (in a Chevalley basis) for $\lambda > 0$. We now see that this is equivalent to choosing for each $\lambda > 0$ an element σ_λ in $M_\lambda(\mathbb{Z})$. In the classical algorithm, we can choose $e_{\lambda+\alpha}$ to be $[e_\alpha, e_\lambda]/(p_{\alpha,\lambda}+1)$ according to some suitable rule. In the new scheme we'll choose some σ_λ in $M_\lambda(\mathbb{Z})$ in a similar fashion. I'll explain precisely how this goes in the next section.

In Carter's algorithm, once we have chosen the e_λ we then find $N_{\lambda,\mu,\nu}$ so that

$$[e_\lambda, e_\mu] = N_{\lambda,\mu,\nu} e_{-\nu}$$

whenever $\lambda + \mu + \nu = 0$. What is the equivalent in the new scheme?

Suppose $\lambda + \mu + \nu = 0$. Choose elements σ_λ etc. in $M_\lambda(\mathbb{Z})$ etc. These determine nilpotent elements $e_{\pm\lambda,\sigma_\lambda}$ in $\mathfrak{sl}_{2,\lambda}(\mathbb{Z})$. According to our definitions,

$$[e_{\lambda,\sigma_\lambda}, e_{\mu,\sigma_\mu}] = \varepsilon(\lambda,\mu,\nu,\sigma_\lambda,\sigma_\mu,\sigma_\nu)(p_{\lambda,\mu}+1)e_{-\nu,\sigma_\nu}\,.$$

The dependence on the roots λ, μ, ν is manifest but weak, in that σ_λ determines both e_λ and $e_{-\lambda}$. In fact we don't need to take it into account at all. If $\lambda+\mu+\nu = 0$ then the only other linear combination that vanishes is $-\lambda-\mu-\nu$. Since $\theta(e_+) = e_-$ we know that $\varepsilon(-\lambda,-\mu,-\nu,\dots) = \varepsilon(\lambda,\mu,\nu,\dots)$, so we can just write $\varepsilon(\sigma_\lambda,\sigma_\mu,\sigma_\nu)$:

$$[e_{\lambda,\sigma_\lambda}, e_{\mu,\sigma_\mu}] = \varepsilon(\sigma_\lambda,\sigma_\mu,\sigma_\nu)(p_{\lambda,\mu}+1)e_{-\nu,\sigma_\nu}\,.$$

How do we compute $\varepsilon(\sigma_\lambda,\sigma_\mu,\sigma_\nu)$? From §2.9 of [Tits:1966a], we get four relevant facts. Assume in the hypotheses of all four that λ, μ, ν are roots with $\lambda+\mu+\nu = 0$.

Lemma 5.3. *We have*

$$\varepsilon(\sigma_\lambda,\sigma_\mu,\sigma_\nu) = -\varepsilon(\sigma_\mu,\sigma_\lambda,\sigma_\nu)\,.$$

Proof. This is elementary, since $[e_{\lambda,\sigma_\lambda}, e_{\mu,\sigma_\mu}] = -[e_{\mu,\sigma_\mu}, e_{\lambda,\sigma_\lambda}]$. It can also be proven directly that $\sigma_\lambda\sigma_\mu\sigma_\lambda^{-1} = \sigma_\nu$ if and only if $\sigma_\lambda\sigma_\nu\sigma_\lambda^{-1} = \mu^\vee(-1)\cdot\sigma_\mu$, and as the next result shows this is an equivalent claim. ∎

Lemma 5.4. *We have*

$$\varepsilon(\sigma_\lambda,\sigma_\mu,\sigma_\nu^{-1}) = -\varepsilon(\sigma_\lambda,\sigma_\mu,\sigma_\nu)\,.$$

Proof. This is also elementary, since $e_{\nu,\sigma_\nu^{-1}} = -e_{\nu,\sigma_\nu}$. ∎

It is the next two results that are significant. The first tells us that there is one case in which we can calculate the constant explicitly, and the second tells us that we can manipulate any case so as to fall in this one.

Lemma 5.5. *Suppose* $\|\lambda\| \ge \|\mu\|, \|\nu\|$. *In this case* $s_\lambda\mu = \nu$ *and* $s_\nu = s_\lambda s_\mu s_\lambda$, *so* $\sigma_\lambda\sigma_\mu\sigma_\lambda^{-1}$ *lies in* $M_\nu(\mathbb{Z})$ *and satisfies*

$$\varepsilon(\sigma_\lambda,\sigma_\mu,\sigma_\lambda\sigma_\mu\sigma_\lambda^{-1}) = (-1)^{p_{\lambda,\mu}}\,.$$

Proof. Lemma 3.1 tells that in this case

$$[e_{\lambda,\sigma_\lambda}, e_{\mu,\sigma_\mu}] = (-1)^{p_{\lambda,\mu}}(p_{\lambda,\mu}+1)\mathrm{Ad}(\sigma_\lambda)e_{\mu,\sigma_\mu} = (-1)^{p_{\lambda,\mu}}(p_{\lambda,\mu}+1)e_{s_\lambda\mu,\sigma_\lambda\sigma_\mu\sigma_\lambda^{-1}}\,.$$ ∎

The representations of $\mathfrak{sl}_2$ associated to the λ-chain that occur are those with dimensions 2 and 4, and in those cases $p_{\lambda,\mu} = 0$ and 1. (The second occurs only for system G_2.)

Lemma 5.6. *The function ε is invariant under cyclic permutations:*

$$\varepsilon(\sigma_\lambda, \sigma_\mu, \sigma_\nu) = \varepsilon(\sigma_\mu, \sigma_\nu, \sigma_\lambda)\,.$$

Proof. This reduces to the fact that the Jacobi relation has cyclic symmetry. That relation for e_λ, e_μ, and e_ν tells us that

$$\begin{aligned}
&[[e_\lambda, e_\mu], e_\nu] + [[e_\mu, e_\nu], e_\lambda] + [[e_\nu, e_\lambda], e_\mu] \\
&\quad = \varepsilon(\sigma_\lambda, \sigma_\mu, \sigma_\nu)(p_{\lambda,\mu} + 1)[e_{-\nu}, e_\nu] \\
&\qquad\qquad + \varepsilon(\sigma_\mu, \sigma_\nu, \sigma_\lambda)(p_{\mu,\nu} + 1)[e_{-\lambda}, e_\lambda] \\
&\qquad\qquad\qquad + \varepsilon(\sigma_\nu, \sigma_\lambda, \sigma_{mu})(p_{\nu,\lambda} + 1)[e_{-\mu}, e_\mu] \\
&\quad = \varepsilon(\sigma_\lambda, \sigma_\mu, \sigma_\nu)(p_{\lambda,\mu} + 1)h_\nu + \varepsilon(\sigma_\mu, \sigma_\nu, \sigma_\lambda)(p_{\mu,\nu} + 1)h_\lambda + \varepsilon(\sigma_\nu, \sigma_\lambda, \sigma_\mu)(p_{\nu,\lambda} + 1)h_\mu \\
&\quad = 0\,.
\end{aligned}$$

But according to Lemma 2.4,

$$(p_{\lambda,\mu} + 1)h_\nu + (p_{\mu,\nu} + 1)h_\lambda + (p_{\nu,\lambda} + 1)h_\mu = 0\,.$$

The vectors h_λ and h_μ are linearly independent, so the conclusion may be deduced from the following trivial observation: if u, v, w are vectors, two of them linearly independent, and

$$\begin{aligned}
au + bv + cw &= 0 \\
u + \ \ v + \ \ w &= 0
\end{aligned}$$

then $a = b = c$. ∎

The relevance to computation should be evident. If $\lambda + \mu + \nu = 0$, then one of the three roots will at least weakly dominate in length, and by Lemma 5.6 we can cycle to get the condition assumed in Lemma 5.5.

From this result follows incidentally:

Proposition 5.7. (Chevalley) *The function ε always takes values ± 1.*

There is a hint of circular reasoning involved here, which I shall ignore. The problem is that I have used the simply connected group scheme G defined over $\mathbb{Z}$ to get to this point, but Chevalley uses his theorem to construct it. Since we don't really need all of $G(\mathbb{Z})$ but only its subgroup $N_H(\mathbb{Z})$, this shouldn't be a difficult obstacle to get around.

At any rate, we are now faced with two algorithmic problems:

(1) *How to assign to each $\lambda > 0$ an element σ_λ in $M_\lambda(\mathbb{Z})$?*
(2) *Suppose $\lambda + \mu + \nu = 0$ with $\|\lambda\| \ge \|\mu\| = \|\nu\|$. Then $\nu = s_\lambda \mu$ and $\sigma_\lambda \sigma_\mu \sigma_\lambda^{-1}$ lies in M_ν. Is it equal to σ_ν or σ_ν^{-1}?*

If $\sigma_\lambda \sigma_\mu \sigma_\lambda^{-1} = \nu^\vee(\pm 1)\sigma_\nu$ we can deduce now:

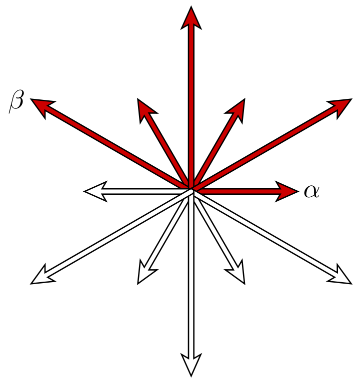
$$[e_{\lambda,\sigma_\lambda}, e_{\mu,\sigma_\mu}] = \pm(-1)^{p_{\lambda,\mu}}(p_{\lambda,\mu} + 1)e_{\nu,\sigma_\nu}\,.$$

In the next section, I begin detailed discussion of the calculations involved in answering these questions.

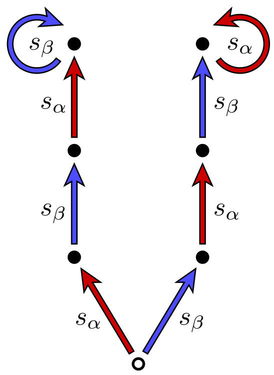
6. The root graph

There are now several computations to be explained. The starting point (input datum) will be a Cartan matrix C. From it we must (a) construct the full set of roots and associated data; (b) specify a Chevalley basis in terms of a choice of σ_λ in $M_\lambda(\mathbb{Z})$; (c) compute conjugates $\sigma_\lambda \sigma_\mu \sigma_\lambda^{-1}$ as $\sigma_\nu^{\pm 1}$ when $\lambda + \mu + \nu = 0$ and $\|\lambda\| \geq \|\mu\| = \|\nu\|$.

Constructing the full set of roots is equivalent to constructing the full set of positive roots. The basic idea is quite simple—the set Σ is the smallest subset of L_Δ containing Δ that's stable under the reflections s_α. To find it, I build what I call the **positive root graph**. Recall that the height of $\sum \lambda_\alpha \alpha$ is $\sum \lambda_\alpha$. The nodes of the root graph are, with one exception, the positive roots λ. The edges are oriented, with an edge $\lambda \prec \mu$ from λ to $\mu = s_\alpha \lambda$ if the height of μ is greater than or equal to that of λ. This happens precisely when $\langle \lambda, \alpha^\vee \rangle \leq 0$. This edge will be labeled by s_α. I add also a 'dummy' node at the bottom, with an edge from it to α labeled by s_α.



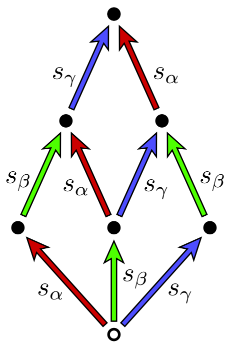
The roots of G_2



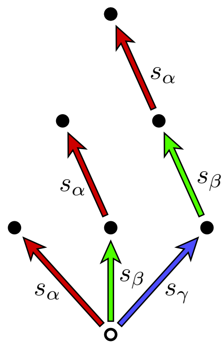
The positive root graph of G_2

In drawing the root graph, the loops are redundant, since the total number of edges in and out of a node must be equal to the rank.

There are several ways to label the positive roots. One is by the array of λ_α with $\lambda = \sum_{\alpha \in \Delta} \lambda_\alpha \alpha$. Another is by the array of $\langle \lambda, \alpha^\vee \rangle$. The first identifies it as an element in the free module $\mathbb{Z}^\Delta$ spanned by Δ, the second as an element of the dual of the free $\mathbb{Z}$-module spanned by the coroots in $\Delta^\vee$. There is a third, which will be important later on. Assign an order (arbitrary) to Δ. Inside the root graph is what I call the ISL tree—same nodes, but an edge from λ to $\mu = s_\alpha \lambda$ only if α is the least in Δ with $s_\alpha \mu \prec \mu$.



The root graph of A_3



The ISL tree of A_3

To every root is a unique path in the ISL tree from the base, and hence the roots may be labeled by these paths. (This is the point of adding the 'dummy' root.)

How to construct the positive roots? We want to make a list of them, and for each λ also a list of its reflections $s_\alpha \lambda$ ($\alpha \in \Delta$), along with some other data such as its norm $\|\lambda\|^2$ scaled suitably.

(1) We maintain a list of roots we have found, and also a second dynamic list of roots λ we have found for which not all $s_\alpha\lambda$ have been calculated. This last list will be a **queue**, for reasons that will become apparent later on. This is a dynamic array from which items are removed in the order in which they are put on. This requires that they are placed on one end and removed from the other. (This is as opposed to a **stack**, an array in which items are placed and removed from the same end.) I call these two lists the **root list** and the **process queue**. I also maintain a look-up table of roots, using one of the three labeling schemes I mentioned above (although for other reasons, it is probably best to index by (λ_α)). Of course I also have to decide, what exactly is a root? For me the answer is a largish collection of data including the arrays (λ_α) and $(\langle\lambda,\alpha^\vee\rangle)$ mentioned above, similar data for the coroots $\lambda^\vee$, the reflection table, and the norms $\|\lambda\|^2$.

(2) As a simple preliminary, I run through the roots in Δ, assigning the short ones length 1 and the long ones their lengths accordingly. I also compute the other data for the simple roots. Formally, I set $s_\alpha\alpha$ to be the dummy root. (For all other positive roots λ, $s_\alpha\lambda$ will be another positive root.)

(3) I start the interesting part of the computation by putting the roots of Δ in both the root list and the process queue.

(4) While the process queue is not empty, we remove items from it and calculate their reflections. Say we remove λ. Since the process list is a queue, all of the reflections $s_\alpha\lambda \prec \lambda$ have been calculated (since when we set $s_\alpha\lambda = \mu$ we also set $s_\alpha\mu = \lambda$). So we go through the α for which $s_\alpha\lambda$ has not been calculated. We have the array $(\langle\lambda,\alpha^\vee\rangle$, so we can tell when $s_\alpha\lambda = \lambda$, and in the other cases $\langle\lambda,\alpha^\vee\rangle < 0$ and $s_\alpha\lambda \succ \lambda$. Using the look-up table, we add new roots to the root list and the process queue when necessary.

From the full root graph it is easy to construct the ISL tree. There are many circumstances in which it is useful to traverse the roots by traversing it. There will be two in this paper. The first lays out a procedure for assigning the elements σ_λ for roots λ. First of all, for α in Δ I assign σ_α arbitrarily in $M_\alpha(\mathbb{Z})$. This makes absolutely no difference at all, since all choices are conjugate by an element of $G_{\mathrm{adj}}(\mathbb{Z})$. Then we proceed by induction. Given an edge $\lambda \prec \mu = s_\alpha\lambda$ in the ISL tree, set $\sigma_\mu = \sigma_\alpha\sigma_\lambda\sigma_\alpha^{-1}$. By specifying these σ_λ we also specify, according to Tits trijection, nilpotent elements e_λ according to the rule $e_\mu = \sigma_\alpha e_\lambda \sigma_\alpha^{-1}$.

7. Processing admissible triples

I have now outlined how to deal with (a) and (b) among the tasks mentioned at the beginning of this section. I now assume we are given the ISL tree and have constructed all the σ_λ. I'll now begin to describe (c)—how to compute all the $\sigma_\lambda\sigma_\mu\sigma_\lambda^{-1}$ when $\lambda+\mu+\nu = 0$.

According to Lemma 5.6, we can reduce the computation of the factors ε to the case in which the arguments form an admissible triple, and in this case it reduces more precisely to a comparison of $\sigma_\lambda\sigma_\mu\sigma_\lambda^{-1}$ and σ_ν for admissible triples (λ,μ,ν). We therefore want to make a list of admissible triples, and the ISL tree can be used to do this. We start by dealing directly with all cases in which $\lambda = \alpha$ lies in Δ and $\mu > 0$, following from λ up the ISL tree. The cases where $\mu < 0$ can be dealt with at the same time, since if $\alpha + \mu = \nu$ then $-\nu + \alpha = -\mu$. Then we add the ones where λ is not in Δ. If (λ,μ,ν) is admissible with $s_\alpha\lambda > 0$, then so is

$$s_\alpha(\lambda,\mu,\nu) = (s_\alpha\lambda, s_\alpha\mu, s_\alpha\nu)\,.$$

Thus we can compile a complete list of admissible triples by going up the ISL tree.

We must then compare $\sigma_\lambda\sigma_\mu\sigma_\lambda^{-1}$ to σ_ν for all admissible triples (λ,μ,ν) This computation may also be done inductively in the ISL tree, since

$$\sigma_\alpha\sigma_\lambda\sigma_\alpha^{-1}\cdot\sigma_\alpha\sigma_\mu\sigma_\alpha^{-1}\cdot\sigma_\alpha\sigma_\lambda^{-1}\sigma_\alpha^{-1} = \sigma_\alpha\cdot\sigma_\lambda\sigma_\mu\sigma_\lambda^{-1}\cdot\sigma_\alpha^{-1}\,.$$

To see exactly how this goes, fix for the moment assignments $\lambda \mapsto \sigma_\lambda$. For every triple (λ, μ, ν) with $s_\lambda \mu = \nu$ (not just admissible ones) define the factor $\tau^\vee_{\lambda,\mu,\nu}$ by the formula

$$\sigma_\lambda \sigma_\mu \sigma_\lambda^{-1} = \tau^\vee_{\lambda,\mu,\nu} \cdot \sigma_\nu \,.$$

The factor $\tau^\vee_{\lambda,\mu,\nu}$ will lie in $H_\nu(\mathbb{Z})$, hence will be either 1 or $\nu^\vee(-1)$. I'll show in the next section how to compute these factors when λ lies in Δ. This will depend on something we haven't seen yet. But for the moment assume that the $\tau^\vee_{\alpha,\mu,\nu}$ have been calculated for all α in Δ and μ an arbitrary positive root. We can then proceed to calculate the τ-factors for all admissible triples by induction. Let

$$(\lambda_\bullet, \mu_\bullet, \nu_\bullet) = (s_\alpha \lambda, s_\alpha \mu, s_\alpha \nu) \,,$$

Then

$$\begin{aligned}
\sigma_\lambda \sigma_\mu \sigma_\lambda^{-1} &= \tau^\vee_{\lambda,\mu,\nu} \cdot \sigma_\nu \\
\sigma_\alpha \sigma_\lambda \sigma_\alpha^{-1} \cdot \sigma_\alpha \sigma_\mu \sigma_\alpha^{-1} \cdot \sigma_\alpha \sigma_\lambda^{-1} \sigma_\alpha^{-1} &= \sigma_\alpha \cdot \sigma_\lambda \sigma_\mu \sigma_\lambda^{-1} \cdot \sigma_\alpha^{-1} \\
(\tau^\vee_{\alpha,\lambda,\lambda_\bullet} \cdot \sigma_{\lambda_\bullet})(\tau^\vee_{\alpha,\mu,\mu_\bullet} \cdot \sigma_{\mu_\bullet})(\tau^\vee_{\alpha,\lambda,\lambda_\bullet} \cdot \sigma_{\lambda_\bullet})^{-1} &= \sigma_\alpha \cdot \tau^\vee_{\lambda,\mu,\nu} \sigma_\nu \cdot \sigma_\alpha^{-1} \\
(\tau^\vee_{\alpha,\lambda,\lambda_\bullet} + s_{\lambda_\bullet} \tau^\vee_{\alpha,\mu,\mu_\bullet} + s_{\lambda_\bullet} s_{\mu_\bullet} s_{\lambda_\bullet} \tau^\vee_{\alpha,\lambda,\lambda_\bullet}) \cdot \sigma_{\lambda_\bullet} \sigma_{\mu_\bullet} \sigma_{\lambda_\bullet}^{-1} &= s_\alpha \tau^\vee_{\lambda,\mu,\nu} \cdot \sigma_\alpha \sigma_\nu \sigma_\alpha^{-1} \\
(\tau^\vee_{\alpha,\lambda,\lambda_\bullet} + s_{\lambda_\bullet} \tau^\vee_{\alpha,\mu,\mu_\bullet} + s_{\nu_\bullet} \tau^\vee_{\alpha,\lambda,\lambda_\bullet}) \cdot \sigma_{\lambda_\bullet} \sigma_{\mu_\bullet} \sigma_{\lambda_\bullet}^{-1} &= (s_\alpha \tau^\vee_{\lambda,\mu,\nu} + \tau^\vee_{\alpha,\nu,\nu_\bullet}) \cdot \sigma_{\nu_\bullet}
\end{aligned}$$

leading to:

Lemma 7.1. *If (λ, μ, ν) is an admissible triple and $\lambda \neq \alpha$ then so is*

$$(\lambda_\bullet, \mu_\bullet, \nu_\bullet) = (s_\alpha \lambda, s_\alpha \mu, s_\alpha \nu) \,,$$

and

$$\tau^\vee_{\lambda_\bullet,\mu_\bullet,\nu_\bullet} = s_\alpha \tau^\vee_{\lambda,\mu,\nu} + \tau^\vee_{\alpha,\lambda,\lambda_\bullet} + s_{\lambda_\bullet} \tau^\vee_{\alpha,\mu,\mu_\bullet} + s_{\nu_\bullet} \tau^\vee_{\alpha,\lambda,\lambda_\bullet} + \tau^\vee_{\alpha,\nu,\nu_\bullet} \,.$$

The principal conclusion of these preliminary formulas is that for both the specification of the σ_λ and the calculation of the $\sigma_\lambda \sigma_\mu \sigma_\lambda^{-1}$ we are reduced to the single calculation: *for α in Δ and $\lambda > 0$, given σ_λ how do we calculate $\sigma_\alpha \sigma_\lambda \sigma_\alpha^{-1}$?* I must explain not only how calculations are made, but also how elements of $N_H(\mathbb{Z})$ are interpreted in a computer program—I have yet to explain the basis of calculation in the extended Weyl group.

In understanding how efficient the computation of structure constants will be, we have to know roughly how many admissible triples there are. Following [Carter:1972] and [Cohen-Murray-Taylor:2004], I assume the positive roots to be ordered, and I define a trio of roots λ, μ, ν to be **special** if $0 < \lambda < \mu$ and $\lambda + \mu = -\nu$ is again a root. If λ, μ, ν is any triple of roots with $\lambda + \mu + \nu = 0$, then (as Carter points out) exactly one of the following twelve triples is special:

$$\begin{array}{llll}
(\lambda, \mu, \nu), & (-\lambda, -\mu, -\nu), & (\mu, \lambda, \nu), & (-\mu, -\lambda, -\nu) \\
(\nu, \lambda, \mu), & (-\nu, -\lambda, -\mu), & (\lambda, \nu, \mu), & (-\lambda, -\nu, -\mu) \\
(\mu, \nu, \lambda), & (-\mu, -\nu, -\lambda), & (\nu, \mu, \lambda), & (-\nu, -\mu, -\lambda).
\end{array}$$

Hence there are at most 12 times as many admissible triples as special triples. How many special triples are there? This is independent of the ordering of Σ^+, since it is one-half the number of pairs of positive roots λ, μ with $\lambda + \mu$ also a root. In [Cohen-Murray-Taylor:2004] it is asserted that for all classical systems the number is $O(n^3)$, where n is the rank of the system. Don Taylor has given me the following more precise table:

System	Number of special triples
A_n	$n(n^2-1)/6$
B_n	$n(n-1)(2n-1)/3$
C_n	$n(n-1)(2n-1)/3$
D_n	$2n(n-1)(n-2)/3$
E_6	120
E_7	336
E_8	1120
F_4	68
G_2	5

8. Computation in the extended Weyl group

The principal point involved in computation within the extended Weyl group is that, as explained in [Tits:1966b], there exist good sections of the extension

$$1 \longrightarrow H(\mathbb{Z}) \longrightarrow N_H(\mathbb{Z}) \longrightarrow W \longrightarrow 1\,.$$

First of all, pick for each α in Δ an element $\overset{\diamond}{s}_\alpha$ in $M_\alpha(\mathbb{Z})$. Given an arbitrary w in W, we write it as a reduced expression

$$w = s_1 \ldots s_n\,,$$

and then set

$$\overset{\diamond}{w} = \overset{\diamond}{s}_1 \ldots \overset{\diamond}{s}_n\,.$$

Lemma 8.1. *(Jacques Tits) The element $\overset{\diamond}{w}$ depends only on w, not on the particular reduced expression.*

This is one of the principal results of [Tits:1966b], but it is also discussed briefly in §2.8 of [Tits:1966a]. He shows that the products $\overset{\diamond}{s}_1 \ldots \overset{\diamond}{s}_n$ satisfy the braid relation, and the Lemma follows from the fact that two reduced expressions give rise to the same element of W if and only if they can be linked by a sequence of braid relations (a fundamental if undervalued result to be found in [Tits:1968]).

A calculation in $\mathrm{SL}(2)$ tells us how to multiply in $N_H(\mathbb{Z})$. First of all, I have mentioned that the group $H(\mathbb{Z})$ is isomorphic to $L_{\Delta^\vee}/2L_{\Delta^\vee}$. More precisely, the image of $\lambda^\vee$ corresponds to $\lambda^\vee(-1)$ in $H(\mathbb{Z})$. This identification is W-covariant, which means that

$$(w\lambda)^\vee(-1) = \overset{\diamond}{w}\lambda^\vee(-1)\overset{\diamond}{w}^{-1}\,.$$

Every element of $N_H(\mathbb{Z})$ can be written uniquely as $t^\vee\,\overset{\diamond}{w}$ with $t^\vee$ in $H(\mathbb{Z})$. Finally, if $s_\alpha w < w$ then $w = s_\alpha \cdot s_\alpha w$ with $\ell(w) = \ell(s_\alpha) + \ell(s_\alpha w)$, therefore $\overset{\diamond}{w} = \overset{\diamond}{s}_\alpha (s_\alpha w)^\diamond$. Hence:

$$\overset{\diamond}{s}_\alpha \overset{\diamond}{w} = \begin{cases} (s_\alpha w)^\diamond & \text{if } s_\alpha w > w \\ \alpha^\vee(-1)(s_\alpha w)^\diamond & \text{otherwise} \end{cases}$$

and

$$\overset{\diamond}{w}\,\overset{\diamond}{s}_\alpha = \begin{cases} (w s_\alpha)^\diamond & \text{if } w s_\alpha > w \\ w\alpha^\vee(-1)(w s_\alpha)^\diamond & \text{otherwise.} \end{cases}$$

I am going to simplify notation. Every element of the extended Weyl group may be represented uniquely as $t^\vee(-1)\cdot\overset{\diamond}{w}$, where $t^\vee$ is in $X_*(H) = L_{\Delta^\vee}$ and w is in W. I'll express it as just $t^\vee\cdot\overset{\diamond}{w}$, and of course it is only $t^\vee$ modulo $2X_*(H)$ that counts. Also, I'll refer to the group operation in $L_{\Delta^\vee}$ additively.

Proposition 8.2. *Suppose α to be in Δ, $\lambda > 0$. Then*

$$\overset{\diamond}{s}_\alpha \overset{\diamond}{s}_\lambda \overset{\diamond}{s}_\alpha = \begin{cases} (s_{s_\alpha\lambda})^\diamond & \text{if } \langle\alpha,\lambda^\vee\rangle > 0 \\ \alpha^\vee\cdot\overset{\diamond}{s}_\lambda & \langle\alpha,\lambda^\vee\rangle = 0 \\ (\alpha^\vee + s_\alpha s_\lambda \alpha^\vee)\cdot(s_{s_\alpha\lambda})^\diamond & \langle\alpha,\lambda^\vee\rangle < 0\,. \end{cases}$$

Proof. A preliminary calculation:

$$\begin{aligned} s_\lambda \alpha &= \alpha - \langle \alpha, \lambda^\vee \rangle \lambda \\ s_\alpha s_\lambda \alpha &= -\alpha - \langle \alpha, \lambda^\vee \rangle s_\alpha \lambda \\ &= -\alpha - \langle \alpha, \lambda^\vee \rangle (\lambda - \langle \lambda, \alpha^\vee \rangle \alpha) \\ &= -\alpha - \langle \alpha, \lambda^\vee \rangle \lambda + \langle \alpha, \lambda^\vee \rangle \langle \lambda, \alpha^\vee \rangle \alpha \\ &= -\langle \alpha, \lambda^\vee \rangle \lambda + \big(\langle \alpha, \lambda^\vee \rangle \langle \lambda, \alpha^\vee \rangle - 1 \big) \alpha \end{aligned}$$

Recall that by Lemma 1.1 the product $\langle \alpha, \lambda^\vee \rangle \langle \lambda, \alpha^\vee \rangle \ge 0$. Recall also that $w s_\alpha > w$ if and only if $w\alpha > 0$.

(a) $\langle \alpha, \lambda^\vee \rangle < 0$. Here $s_\lambda \alpha > 0$ and $s_\alpha s_\lambda \alpha > 0$ so $s_\lambda < s_\alpha s_\lambda < s_\alpha s_\lambda s_\alpha$, and $\ell(s_\alpha s_\lambda s_\alpha = \ell(s_\lambda) + 2$, and

$$\mathring{s}_\alpha \mathring{s}_\lambda \mathring{s}_\alpha = (s_\alpha s_\lambda s_\alpha)^\diamond = (s_{s_\alpha \lambda})^\diamond \, .$$

(b) $\langle \alpha, \lambda^\vee \rangle = 0$. Here $s_\alpha \lambda = \lambda$, and $\mathring{s}_\lambda \mathring{s}_\alpha = (s_\lambda s_\alpha)^\diamond$, but $s_\alpha (s_\lambda s_\alpha) = s_\lambda$ so

$$\mathring{s}_\alpha \mathring{s}_\lambda \mathring{s}_\alpha = \mathring{s}_\alpha (s_\lambda s_\alpha)^\diamond = \alpha^\vee \cdot \mathring{s}_\lambda \, .$$

(c) $\langle \alpha, \lambda^\vee \rangle > 0$. Here one sees easily that $s_\lambda \alpha < 0$. But since $\lambda \ne \alpha$ we also have $s_\alpha s_\lambda \alpha < 0$ also. So $\ell(s_{s_\alpha \lambda}) = \ell(s_\lambda) - 2$.

$$\begin{aligned} \mathring{s}_\alpha \mathring{s}_\lambda &= \alpha^\vee \cdot (s_\alpha s_\lambda)^\diamond \\ \mathring{s}_\alpha \mathring{s}_\lambda \mathring{s}_\alpha &= \alpha^\vee \cdot (s_\alpha s_\lambda)^\diamond \, \mathring{s}_\alpha \\ &= \alpha^\vee \cdot (s_{s_\alpha \lambda})^\diamond \cdot \alpha^\vee \cdot 1 \\ &= (\alpha^\vee + s_\alpha s_\lambda \alpha^\vee) \cdot (s_{s_\alpha \lambda})^\diamond \, . \end{aligned}$$ ∎

The explicit cocycle of the extension of W by $H(\mathbb{Z})$ was first exhibited in [Langands-Shelstad:1987].

Corollary 8.3. *Suppose $\sigma_\lambda = t_\lambda^\vee \cdot \mathring{s}_\lambda$. Then for $\alpha \ne \lambda$*

$$\sigma_\alpha \sigma_\lambda \sigma_\alpha^{-1} = \begin{cases} (s_\alpha t_\lambda^\vee + s_\alpha s_\lambda \alpha^\vee) \cdot (s_{s_\alpha \lambda})^\diamond & \langle \lambda, \alpha^\vee \rangle < 0 \\ (s_\alpha t_\lambda^\vee) \cdot (s_{s_\alpha \lambda})^\diamond & \langle \lambda, \alpha^\vee \rangle = 0 \\ (s_\alpha t_\lambda^\vee + \alpha^\vee)(s_{s_\alpha \lambda})^\diamond & \langle \lambda, \alpha^\vee \rangle > 0. \end{cases}$$

Keep in mind that the reflection associated to $s_\alpha \lambda$ is $s_\alpha s_\lambda s_\alpha$.

Proof. We start with

$$\sigma_\alpha \sigma_\lambda \sigma_\alpha^{-1} = \mathring{s}_\alpha t_\lambda^\vee \mathring{s}_\lambda \alpha^\vee \mathring{s}_\alpha = (s_\alpha t_\lambda^\vee + s_\alpha s_\lambda \alpha^\vee) \cdot \mathring{s}_\alpha \mathring{s}_\lambda \mathring{s}_\alpha \, .$$

Apply the Proposition. ∎

There are three cases, according to whether $\langle \alpha, \lambda^\vee \rangle$ is $<$, $=$, or > 0. These correspond to how the length $\ell(s_\alpha s_\lambda s_\alpha)$ relates to $\ell(s_\lambda)$. So now finally we can compute the factors $\tau^\vee_{\alpha, \mu, s_\alpha \mu}$, comparing $\sigma_\alpha \sigma_\mu \sigma_\alpha^{-1}$ to $\sigma_{s_\alpha \mu}$.

Lemma 8.4. *Suppose $s_\alpha \mu = \nu$. If $\sigma_\alpha \sigma_\mu \sigma_\alpha^{-1} = t^\vee \mathring{s}_\nu$ and $\sigma_\nu = t_\nu^\vee \mathring{s}_\nu$ then*

$$\tau^\vee_{\alpha, \mu, \mu} = t^\vee + t_\nu^\vee \, .$$

9. Summary

I summarize here how the computations proceed.

First construct the set of all roots (both positive and negative) along with for each root λ a reflection table, a pair of vectors associated to each of λ and $\lambda^\vee$, and $\|\lambda\|^2$. Also construct a look-up table of roots, indexing λ by the array $\langle\!\langle\lambda\rangle\!\rangle = (\lambda_\alpha)$ with $\lambda = \sum \lambda_\alpha \alpha$. Looking up the array returns the full root structure. The look-up table makes it easy to tell whether $\lambda + \mu$ is a root or not, and to compute $p_{\lambda,\mu}$, $q_{\lambda,\mu}$.

Construct from the root graph its subgraph the ISL tree. Use this to select the σ_λ for $\lambda > 0$, according to the inductive rules (1) $\sigma_\alpha = \mathring{s}_\alpha$; (2) $\sigma_\mu = \sigma_\alpha\sigma_\lambda\sigma_\alpha^{-1}$ for edges $\lambda \prec \mu = \sigma_\alpha\lambda$ in the ISL tree. Write each of these as $t_\lambda\,\mathring{s}_\alpha$. Make a look-up table indexed by $\langle\!\langle\lambda\rangle\!\rangle$, returning the pair $\sigma_\lambda = (t_\lambda, \lambda)$. Define a function calculating $\sigma_\alpha\sigma_\lambda\sigma_\alpha^{-1}$.

Use this in turn to make a table of all admissible triples (λ, μ, ν) with $\lambda > 0$, returning values of $\varepsilon(\sigma_\lambda, \sigma_\mu, \sigma_\nu)$ along with values of $\varepsilon(\sigma_\lambda, \sigma_\mu, \sigma_\nu)$. To find these, we just search for all $\mu > 0$ with $\langle \mu, \alpha^\vee\rangle = -1$, which can be detected among the root data for μ. This is to be indexed by the pair $\langle\!\langle\lambda\rangle\!\rangle$, $\langle\!\langle\mu\rangle\!\rangle$. For this, I start with the case in which λ is a simple root, where it suffices to locate those with $\mu > 0$, since then (λ, ν, μ) will give its twin with $\nu < 0$. From there, follow from λ up the ISL tree. There will be some redundancy if all three roots have the same length, but I ignore this problem. The point is that $\sigma_\lambda\sigma_\mu\sigma_\lambda^{-1}$ will be of the form $t_{\lambda,\mu,\nu}\sigma_\nu$. We record the value 1 if $t = 1$ and -1 if it is $\nu^\vee(-1)$—this tells us the sign of $\varepsilon(\sigma_\lambda, \sigma_\mu, \sigma_\nu)$. We are in effect using the rules

$$
\begin{aligned}
\varepsilon(\sigma, \tau, \upsilon^{-1}) &= -\varepsilon(\sigma, \tau, \upsilon) \\
\varepsilon(\sigma_\lambda, \sigma_\mu, \sigma_\lambda\sigma_\mu\sigma_\lambda^{-1}) &= (-1)^{p_{\lambda,\mu}} \qquad \text{for an admissible triple } \lambda, \mu, \nu = -\lambda - \mu \\
\varepsilon(\sigma, \tau, \upsilon) &= \varepsilon(\tau, \upsilon, \sigma)
\end{aligned}
$$

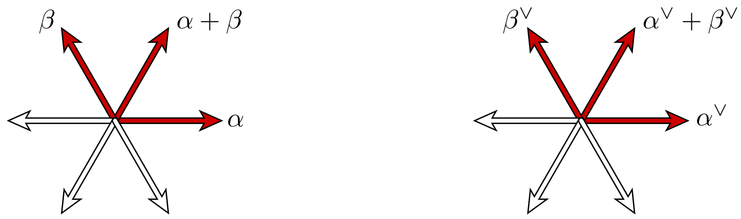
The choices of the σ_λ gives us also nilpotent elements $e_{\pm\lambda} = e_{\pm\lambda,\sigma_\lambda}$. The problem of calculating structure constants is that we are given λ and μ and want to calculate $[e_\lambda, e_\mu]$.

(a) If $\lambda = -\mu$ then this is $-h_\lambda$ which we represent as $\sum \lambda_\alpha h_\alpha$.
(b) Now we may assume $\lambda \neq -\mu$. We can tell from the look-up tables for $p_{\lambda,\mu}$ whether $\lambda + \mu$ is a root or not. If not, the bracket vanishes.
(c) Otherwise, let $\nu = -\lambda - \mu$. Then $N_{\lambda,\mu} = \varepsilon(\sigma_\lambda, \sigma_\mu, \sigma_\nu)(p_{\lambda,\mu} + 1)$. We cycle to make this triple admissible, then use the tables for admissible triples to compute the ε factor.

10. Examples

Let's look at some examples of rank two. In all cases, order $\Delta = \{\alpha, \beta\}$. As for the figures, on the left are the root systems, on the right the dual systems of coroots (suitably scaled).

Example. The Lie algebra $\mathfrak{sl}_3$.



I'll look at the two admissible triples $(\alpha, \beta, -\alpha - \beta)$ and $(\beta, \alpha, -\alpha - \beta)$. Since $\alpha < \beta$ and $s_\alpha\beta = \alpha + \beta$, by definition

$$\sigma_{\alpha+\beta} = \sigma_\alpha\sigma_\beta\sigma_\alpha^{-1} = s_\alpha s_\beta{\cdot}\alpha^\vee{\cdot}\,\mathring{s}_{\alpha+\beta} = \beta^\vee{\cdot}\,\mathring{s}_{\alpha+\beta}\,.$$

and since $p_{\alpha,\beta} + 1 = 1$ we therefore have

$$[e_\alpha, e_\beta] = e_{\alpha+\beta}$$

The other admissible triple is $(\beta, \alpha, -\alpha-\beta)$. Here

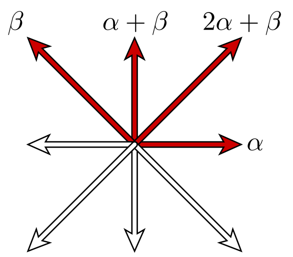
$$\sigma_\beta \sigma_\alpha \sigma_\beta^{-1} = s_\beta s_\alpha \beta^\vee {\cdot} \mathring{s}_{\alpha+\beta} = (\alpha+\beta)^\vee {\cdot} \sigma_{\alpha+\beta}$$

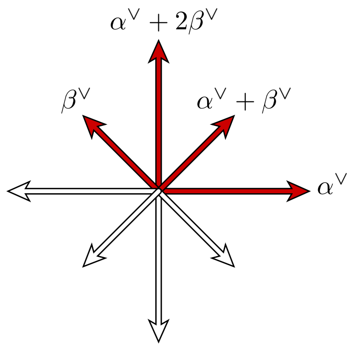
so

$$[e_\beta, e_\alpha] = -e_{\alpha+\beta}\,,$$

which of course we knew anyway.

Example. The Lie algebra $\mathfrak{sp}_4$





Here things are more interesting. Also, a bit more complicated. First of all I define the elements $\sigma_{\alpha+\beta}$ and $\sigma_{2\alpha+\beta}$:

$$\begin{aligned} \sigma_{2\alpha+\beta} &= \sigma_\alpha \sigma_\beta \sigma_\alpha^{-1} \\ &= \mathring{s}_\alpha \mathring{s}_\beta (\alpha^\vee {\cdot} \mathring{s}_\alpha) \\ &= s_\alpha s_\beta \alpha^\vee {\cdot} \mathring{s}_\alpha \mathring{s}_\beta \mathring{s}_\alpha \\ &= (\alpha^\vee + 2\beta^\vee) {\cdot} \mathring{s}_{2\alpha+\beta} \\ &= \alpha^\vee {\cdot} \mathring{s}_{2\alpha+\beta}\,. \end{aligned}$$

$$\begin{aligned} \sigma_{\alpha+\beta} &= \sigma_\beta \sigma_\alpha \sigma_\beta^{-1} \\ &= \mathring{s}_\beta \mathring{s}_\alpha (\beta^\vee {\cdot} \mathring{s}_\beta) \\ &= s_\beta s_\alpha \beta^\vee {\cdot} \mathring{s}_\beta \mathring{s}_\alpha \mathring{s}_\beta) \\ &= (\alpha^\vee + \beta^\vee) {\cdot} \mathring{s}_{\alpha+\beta} \end{aligned}$$

We have also implicitly defined nilpotent elements

$$\begin{aligned} e_{\alpha+\beta} &= \sigma_\beta e_\alpha \sigma_\beta^{-1} \\ e_{2\alpha+\beta} &= \sigma_\alpha e_\beta \sigma_\alpha\,. \end{aligned}$$

Here I'll look at two admissible triples:

$$\begin{aligned} \beta + \alpha + (-\alpha-\beta) &= 0 \\ (2\alpha+\beta) + (-\alpha) + (-\alpha-\beta) &= 0\,. \end{aligned}$$

Proposition 10.1. *We have*

$$\begin{aligned} \varepsilon(\sigma_\beta, \sigma_\alpha, \sigma_{\alpha+\beta}) &= 1 \\ \varepsilon(\sigma_{2\alpha+\beta}, \sigma_\alpha, \sigma_{\alpha+\beta}) &= -1\,. \end{aligned}$$

Proof. We have defined $\sigma_{\alpha+\beta} = \sigma_\beta\sigma_\alpha\sigma_\beta^{-1}$, which implies the first equation.

Since $\sigma_\alpha\sigma_\beta\sigma_\alpha^{-1} = \sigma_{2\alpha+\beta}$, for the other triple we calculate

$$\begin{aligned}
\sigma_{2\alpha+\beta}\sigma_\alpha\sigma_{2\alpha+\beta}^{-1} &= \sigma_\alpha\sigma_\beta\sigma_\alpha^{-1}{\cdot}\sigma_\alpha\sigma_\alpha\sigma_\alpha^{-1}{\cdot}\sigma_\alpha\sigma_\beta^{-1}\sigma_\alpha^{-1} \\
&= \sigma_\alpha{\cdot}\sigma_\beta\sigma_\alpha\sigma_\beta^{-1}{\cdot}\sigma_\alpha^{-1} \\
&= \sigma_\alpha{\cdot}\sigma_{\alpha+\beta}{\cdot}\sigma_\alpha^{-1} \\
&= \mathring{s}_\alpha(\alpha^\vee + \beta^\vee){\cdot}\mathring{s}_{\alpha+\beta}{\cdot}(\alpha^\vee{\cdot}\mathring{s}_\alpha) \\
&= \beta^\vee{\cdot}\mathring{s}_\alpha\mathring{s}_{\alpha+\beta}{\cdot}(\alpha^\vee{\cdot}\mathring{s}_\alpha) \\
&= (\beta^\vee + s_\alpha s_{\alpha+\beta}\alpha^\vee){\cdot}\mathring{s}_\alpha\mathring{s}_{\alpha+\beta}\mathring{s}_\alpha \\
&= (\alpha^\vee + \beta^\vee){\cdot}\mathring{s}_\alpha\mathring{s}_{\alpha+\beta}\mathring{s}_\alpha \\
&= \alpha^\vee{\cdot}\sigma_{\alpha+\beta}\,.
\end{aligned}$$

Hence

$$\varepsilon(\sigma_{2\alpha+\beta}, \sigma_\alpha, \sigma_{\alpha+\beta}) = -1\,. \qquad \square$$

Let's use this to calculate

$$[e_\alpha, e_{\alpha+\beta}] = \varepsilon(\sigma_\alpha, \sigma_{\alpha+\beta}, \sigma_{2\alpha+\beta})p_{\alpha,\alpha+\beta} + 1)e_{2\alpha+\beta}\,.$$

But

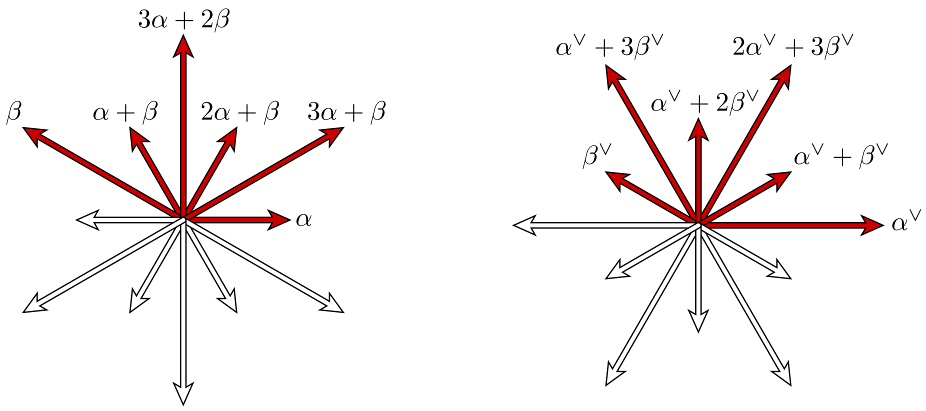
$$\varepsilon(\sigma_\alpha, \sigma_{\alpha+\beta}, \sigma_{2\alpha+\beta}) = \varepsilon(\sigma_{2\alpha+\beta}, \sigma_\alpha, \sigma_{\alpha+\beta}) = -1\,,$$

so

$$[e_\alpha, e_{\alpha+\beta}] = -(p_{\alpha,\alpha+\beta} + 1)e_{2\alpha+\beta} = -2e_{2\alpha+\beta}\,.$$

Of course you can check all these formulas with matrix computations.

Example. The Lie algebra $\mathfrak{g}_2$



Straightforward application of the rule defining the σ_λ gives:

$$\begin{aligned}
\sigma_\alpha &= \mathring{s}_\alpha \\
\sigma_\beta &= \mathring{s}_\beta \\
\sigma_{\alpha+\beta} &= \sigma_\beta \sigma_\alpha \sigma_\beta^{-1} \\
&= \alpha^\vee \cdot \mathring{s}_{\alpha+\beta} \\
\sigma_{2\alpha+\beta} &= \sigma_\alpha \sigma_{\alpha+\beta} \sigma_\alpha^{-1} \\
&= \beta^\vee \cdot \mathring{s}_{2\alpha+\beta} \\
\sigma_{3\alpha+\beta} &= \sigma_\alpha \sigma_\beta \sigma_\alpha^{-1} \\
&= \beta^\vee \cdot \mathring{s}_{3\alpha+\beta} \\
\sigma_{3\alpha+2\beta} &= \sigma_\beta \sigma_{3\alpha+\beta} \sigma_\beta^{-1} \\
&= \alpha^\vee \cdot \mathring{s}_{3\alpha+2\beta} \,.
\end{aligned}$$

These choices of σ_λ give rise to choices of nilpotent e_λ as well. I'll do a few of the interesting examples of calculating Lie brackets. The basic formulas are

$$\begin{array}{ll}
\text{(a)} & [e_\alpha, e_\beta] = \varepsilon(\sigma_\alpha, \sigma_\beta, \sigma_{\alpha+\beta})\, e_{\alpha+\beta} \\
\text{(b)} & [e_\alpha, e_{\alpha+\beta}] = \varepsilon(\sigma_\alpha, \sigma_{\alpha+\beta}, \sigma_{2\alpha+\beta})\, 2e_{2\alpha+\beta} \\
\text{(c)} & [e_\alpha, e_{2\alpha+\beta}] = \varepsilon(\sigma_\alpha, \sigma_{2\alpha+\beta}, \sigma_{3\alpha+\beta})\, 3e_{3\alpha+\beta} \\
\text{(d)} & [e_\beta, e_{3\alpha+\beta}] = \varepsilon(\sigma_\beta, \sigma_{3\alpha+\beta}, \sigma_{3\alpha+2\beta})\, e_{3\alpha+2\beta} \\
\text{(e)} & [e_{\alpha+\beta}, e_{2\alpha+\beta}] = \varepsilon(\sigma_{\alpha+\beta}, \sigma_{2\alpha+\beta}, \sigma_{3\alpha+2\beta})\, 3e_{3\alpha+2\beta}
\end{array}$$

I go through them one by one, but not quite in order.

(a) The triplet $(\beta, \alpha, -\alpha-\beta)$ is admissible. By definition, $\sigma_\beta \sigma_\alpha \sigma_\beta^{-1} = \sigma_{\alpha+\beta}$, so

$$\varepsilon(\sigma_\beta, \sigma_\alpha, \sigma_{\alpha+\beta}) = 1, \quad [e_\alpha, e_\beta] = -[e_\beta, e_\alpha] = -e_{\alpha+\beta} \,.$$

(b) The triplet $(\alpha, \alpha+\beta, -2\alpha-\beta)$ is admissible. By definition, $\sigma_\alpha \sigma_{\alpha+\beta} \sigma_\alpha^{-1} = \sigma_{2\alpha+\beta}$. This is the special case: and

$$\varepsilon(\sigma_\alpha, \sigma_{\alpha+\beta}, \sigma_{2\alpha+\beta}) = (-1)^{p_{\alpha,\alpha+\beta}} = -1, \quad [e_\alpha, e_{\alpha+\beta}] = -2\, e_{2\alpha+\beta} \,.$$

(d) The triplet $(\beta, 3\alpha+\beta, -3\alpha-2\beta)$ is admissible. By definition, $\sigma_\beta \sigma_{3\alpha+\beta} \sigma_\beta^{-1} = \sigma_{3\alpha+2\beta}$, so

$$\varepsilon(\sigma_\beta, \sigma_{3\alpha+\beta}, \sigma_{3\alpha+2\beta}) = 1, \quad [e_\beta, e_{3\alpha+\beta}] = e_{3\alpha+2\beta} \,.$$

(c) The reflection s_α takes the admissible triple $(\beta, \alpha, -\alpha-\beta)$ to $(3\alpha+\beta, -\alpha, -2\alpha-\beta)$ and by definition $\sigma_\alpha \sigma_\beta \sigma_\alpha^{-1} = \sigma_{3\alpha+\beta}$. So we calculate

$$\begin{aligned}
\sigma_\beta \sigma_\alpha \sigma_\beta^{-1} &= \sigma_{\alpha+\beta} \\
\sigma_\alpha \sigma_\beta \sigma_\alpha^{-1} \cdot \sigma_\alpha \sigma_\alpha \sigma_\alpha^{-1} \cdot \sigma_\alpha \sigma_\beta^{-1} \sigma_\alpha^{-1} &= \sigma_\alpha \sigma_{\alpha+\beta} \sigma_\alpha^{-1} \\
\sigma_{3\alpha+\beta} \sigma_\alpha \sigma_{3\alpha+\beta}^{-1} &= \sigma_{2\alpha+\beta}
\end{aligned}$$

so

$$\varepsilon(\sigma_\alpha, \sigma_{2\alpha+\beta}, \sigma_{3\alpha+\beta}) = \varepsilon(\sigma_{3\alpha+\beta}, \sigma_\alpha, \sigma_{2\alpha+\beta}) = 1$$

and

$$[e_\alpha, e_{2\alpha+\beta}] = e_{3\alpha+\beta} \,.$$

(e) Since $\sigma_\beta \sigma_{3\alpha+\beta} \sigma_\beta^{-1} = \sigma_{3\alpha+2\beta}$ we continue on from before

$$\begin{aligned}
\sigma_{3\alpha+\beta}\sigma_\alpha\sigma_{3\alpha+\beta}^{-1} &= \sigma_{2\alpha+\beta} \\
\sigma_\beta\sigma_{3\alpha+\beta}\sigma_\beta^{-1}\cdot\sigma_\beta\sigma_\alpha\sigma_\beta^{-1}\cdot\sigma_\beta\sigma_{3\alpha+\beta}^{-1}\sigma_\beta^{-1} &= \sigma_\beta\sigma_{2\alpha+\beta}\sigma_\beta^{-1} \\
\sigma_{3\alpha+2\beta}\cdot\sigma_{\alpha+\beta}\cdot\sigma_{3\alpha+2\beta}^{-1} &= \mathring{s}_\beta\beta^\vee\cdot\mathring{s}_{2\alpha+\beta}\beta^\vee\cdot\mathring{s}_\beta \\
&= \beta^\vee\cdot\mathring{s}_\beta\,\mathring{s}_{2\alpha+\beta}\beta^\vee\cdot\mathring{s}_\beta \\
&= (\beta^\vee + s_\beta s_{2\alpha+\beta}\beta^\vee)\cdot\mathring{s}_\beta\,\mathring{s}_{2\alpha+\beta}\,\mathring{s}_\beta \\
&= (\beta^\vee + s_\beta s_{2\alpha+\beta}\beta^\vee + \beta^\vee)\cdot\mathring{s}_{2\alpha+\beta} \\
&= \beta^\vee\cdot\mathring{s}_{2\alpha+\beta}\,. \\
&= \sigma_{2\alpha+\beta}\,,
\end{aligned}$$

so

$$\varepsilon(\sigma_{\alpha+\beta}, \sigma_{2\alpha+\beta}, \sigma_{3\alpha+2\beta}) = \varepsilon(\sigma_{3\alpha+2\beta}, \sigma_{\alpha+\beta}, \sigma_{2\alpha+\beta}) = 1, \quad [e_{\alpha+\beta}, e_{2\alpha+\beta}] = 3\, e_{3\alpha+2\beta}\,.$$

Here is what a program produces for all admissible triples:

λ	μ	ν	ε
α	$-2\alpha-\beta$	$\alpha+\beta$	1
α	$\alpha+\beta$	$-2\alpha-\beta$	-1
$\alpha+\beta$	$-2\alpha-\beta$	α	-1
$\alpha+\beta$	α	$-2\alpha-\beta$	1
$2\alpha+\beta$	$-\alpha-\beta$	$-\alpha$	-1
$2\alpha+\beta$	$-\alpha$	$-\alpha-\beta$	1
$3\alpha+\beta$	$-2\alpha-\beta$	$-\alpha$	1
$3\alpha+\beta$	$-\alpha$	$-2\alpha-\beta$	-1
$3\alpha+\beta$	β	$-3\alpha-2\beta$	-1
$3\alpha+\beta$	$-3\alpha-2\beta$	β	1
$3\alpha+2\beta$	$-2\alpha-\beta$	$-\alpha-\beta$	1
$3\alpha+2\beta$	$-\alpha-\beta$	$-2\alpha-\beta$	-1
$3\alpha+2\beta$	$-\beta$	$-3\alpha-\beta$	-1
$3\alpha+2\beta$	$-3\alpha-\beta$	$-\beta$	1
β	$-3\alpha-2\beta$	$3\alpha+\beta$	-1
β	$3\alpha+\beta$	$-3\alpha-2\beta$	1
β	$-\alpha-\beta$	α	-1
β	α	$-\alpha-\beta$	1

The same program takes about two seconds to produce the list of admissible triples for E_8 and the corresponding signs of structure constants.

11. Stop the presses!

I want to prove here a number of formulas about arbitrary root systems. Suppose λ and μ are two vectors in a lattice, $\lambda^\vee$, $\mu^\vee$ in a dual lattice, defining reflections leaving the lattice stable, with $\langle \lambda, \mu^\vee \rangle \langle \mu, \lambda^\vee \rangle \neq 0$. Then we can define a unique inner product $\|v\|^2$ with respect to which the reflections are orthogonal:

$$\begin{aligned} 2\, \lambda \bullet \mu &= -\langle \lambda, \mu^\vee \rangle \langle \mu, \lambda^\vee \rangle \\ \lambda \bullet \lambda &= -\langle \lambda, \mu^\vee \rangle \\ \mu \bullet \mu &= -\langle \mu, \lambda^\vee \rangle\,. \end{aligned}$$

There are at most two lengths of roots, since all are transforms of λ or μ.

Suppose $\langle \mu, \lambda^\vee \rangle \geq 0$. Then $\|\lambda + \mu\| > \|\lambda\|, \|\mu\|$. Since there are only two lengths of roots, all strings intersected with real roots are of the form

$$\mu,\ \lambda + \mu,\ (n-1)\lambda + \mu,\ n\lambda + \mu$$

where $-n = \langle \mu, \lambda^\vee \rangle$.

Thsi imples

$$p_{\lambda,\mu} + 1 \circ over \|\lambda + \mu\|^2 = \frac{q_{\lambda,\mu}}{\|\mu\|^2}\,.$$

We cannot have $s_\lambda \mu = \lambda + \mu$ with $\|\lambda\| \geq \mu$, $n > 1$. Because otherwise we would have a string λ, $\lambda + \mu$, $\lambda + (n-1)\mu$, $\lambda + n\mu$, and $\|\lambda\| > \|\lambda + \mu\|$.

Finally, what about the Geometric Lemma? Easy. Say $\langle \mu, \lambda^\vee \rangle = -1$. (1) If $\|\lambda\| = \|\mu\|$ then reflections in μ and ν move strinsg around, all $p_{*,*}$ are equal. (2) If $\|\lambda\| > \|\mu\|$ then $p_{\lambda,\mu} = 0$, otherwise three lengths. And by the initial reasoning, the other $p_{*,*}$ can be easily calculated, too—if $\|\lambda\|^2 = n$, $\| * \|^2 = 1$, then $\langle \lambda, *^\vee \rangle = -n$ and the other strings have length n.

Chevalley's formula follows from the Geometric Lemma also, plus 2.4 & 4.1(a)–(c).

This also proves Morita's results.

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