Analysis on the tree

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Eigenspaces of the Laplacian Δ on the non-Euclidean plane are important representations of the isometry group $\mathrm{SL}_2(\mathbb{R})$. Is there an analogue for the Bruhat-Tits tree? What is to replace Δ?

If D is a differential operator on either the Eucidean or the non-Euclidean plane that commute with the isometry group, then its symbol, evaluated at the origin, is a polynomial in $\mathbb{C}[x, y]$ invariant under rotations, hence a polynomial in $x^2 + y^2$. Therefore the ring of commuting differential operators is the polynomial ring generated by Δ, whoss symbol this is.

Suppose f to be a smooth function on either and let $\overline{f}_x$ be the function obtained from f by averaging over spheres around the point x. It is a smooth function of $r^2 = \|y - x\|^2$ alone. In $\mathbb{R}^2$ we can see directly that $\Delta r^{2k} = (2k)^2\, r^{2k-2}$, so that

$$\overline{f}_x(y) \sim \sum_{k \geq 0} \frac{r^{2k}}{2^{2k}(k!)^2}\, [\Delta^k f](x)\,.$$

Hence

$$\lim_{r \to 0} \left(\frac{4}{r^2}\right) (\overline{f}_x(r) - f(x)) = [\Delta f](x)\,.$$

There is a similar asymptotic formula for the non-Euclidean plane.

This and classical results of Hecke suggest what to expect for functions on Bruhat-Tits trees. The principal references for this material are [Cartier:1971/2] and [Cartier:1973].

Material in this essay will depend on the geometry of the Bruhat-Tits tree. I'll refer to [Casselman:2018] as [Geometry]. Notation here will be that established in the first section of [Geometry].

Contents

1. Hecke operators

Let $\mathbb{C}(\mathfrak{X})$ be the space of functions on the nodes of $\mathfrak{X}$. It is a Fréchet space with semi-norms

$$\|f\|_\Omega = \sup_{x \in \Omega}$$

for finite sets of nodes Ω. There is a natural embedding of the subspace $\mathbb{C}_c(\mathfrak{X})$ of functions with compact support into the continuous dual of $\mathbb{C}(\mathfrak{X})$:

$$\langle F, f\rangle = \sum\nolimits_\nu F(\nu) f(\nu)\,.$$

A basis of the space of functions with compact support is made up of the Dirac functions:

$$\delta_\nu(x) = \begin{cases} 1 & \text{if } x = \nu \\ 0 & \text{otherwise.} \end{cases}$$

1.1. Lemma. *The embedding of $\mathbb{C}_c(\mathfrak{X})$ into the continuous linear dual of $\mathbb{C}(\mathfrak{X})$ is an isomorphism.*

Proof. By definition of continuity, if F is a continuous linear function on $\mathbb{C}(\mathfrak{X})$ then

$$\langle F, f\rangle \le C\|f\|_\Omega$$

for some $C > 0$ and some finite $\Omega \subset \mathfrak{X}$. But then $\langle F, f\rangle = 0$ if f has support outside Ω, and F is a linear combination of δ_ν with ν in Ω.

The group GL_2 acts continuously on $\mathbb{C}(\mathfrak{X})$ by the left regular representation:

$$[L_g F](x) = F\big(g^{-1}(x)\big)\,.$$

This induces representations of both $\mathrm{PGL}_2(\mathfrak{k})$ and $\mathrm{SL}_2(\mathfrak{k})$. *What are the continuous operators on $\mathbb{C}(\mathfrak{X})$ commuting with these representations?*

The answer should be clearer if I answer a more general question. Suppose for the moment that G is an arbitrary $\mathfrak{p}$-adic group, H an open subgroup. Thus G/H is a discrete set, and one can similarly define $\mathbb{C}(G/H)$ as a topological vector space. If $G = \mathrm{PGL}_2(\mathfrak{k})$ and $H = \mathrm{PGL}_2(\mathfrak{o})$, for example, then G/H may be identified with the set of nodes of $\mathfrak{X}$. There exists a canonical H-equivariant map

$$\omega\colon\ \mathbb{C}(G/H) \longrightarrow \mathbb{C}, \quad f \longmapsto f(1)\,.$$

If (π, V) is any continuous representation of G, then composition with ω determines a map

$$\mathrm{Hom}_{\mathrm{cont},G}(V, \mathbb{C}(G/H)) \longrightarrow \mathrm{Hom}_{\mathrm{cont},H}(V, \mathbb{C})\,. \tag{1.2}$$

1.3. Lemma. (Elementary Frobenius reciprocity) *The map (1.2) is an isomorphism.*

Proof. The inverse is easily defined. Suppose given an H-invariant map F from V to $\mathbb{C}$. Then to v in V associate the function

$$F_v(g) = F(\pi(g^{-1}(v)))\,.$$

Because ω is continuous and π is a continuous representation, F_v lies in $\mathbb{C}(G/H)$. For the same reasons, the map $v \mapsto F_v$ is continuous.

I'll now apply this to the cases we are interested in.

HECKE OPERATORS AS ALGEBRAIC CORRESPONDENCES. The integral **Hecke algebras** of $\mathrm{SL}_2(\mathfrak{k})$ and $\mathrm{PGL}_2(\mathfrak{k})$ are rings of 'algebraic correspondences' on the tree $\mathfrak{X}$. The definitions in these terms mimic Hecke's original definitions of the classical operators T_n. The two cases need to be separated.

PGL(2). Suppose for the moment that $G = \mathrm{PGL}_2(\mathfrak{k})$, $K = \mathrm{PGL}_2(\mathfrak{o})$.

Since the nodes of $\mathfrak{X}$ are in bijection with G/K, Lemma 1.3 tells us that in order to describe the ring of continuous operators on $\mathbb{C}(\mathfrak{X})$ we must describe all the K-invariant functions on $\mathbb{C}(\mathfrak{X})$. These are the finite sums

$$f \longmapsto \sum c_\nu f(\nu)$$

in which $c_{k(\nu)} = c_\nu$ for all k in K. According to Proposition 5.3 of [Geometry] two nodes are in the same K-orbit if and only if they are at the same distance from ν_0. Therefore orbits of K are in bijection with the circles in $\mathfrak{X}$ around ν_0, or equivalently with the nodes ν_m for $m \ge 0$.

Define the Hecke operator

$$[T_m f](x) = \sum_{\substack{y \\ |x:y|=m}} f(y)\,.$$

Let $\mathcal{H}_\mathbb{C}$ be the linear space of operators spanned by the T_m. By Lemma 1.3:

1.4. Proposition. *The embedding of $\mathcal{H}_{\mathbb{C}}$ into the space of continuous linear endomorphisms on $\mathbb{C}(\mathfrak{X})$ that commute with G is an isomorphism.*

Since every node has $q+1$ neighbours, the strict analogue of the Laplacian is $T/(q+1) - I$, so that f is 'harmonic' if and only if at every node it agrees with the average value taken at its neighbours.

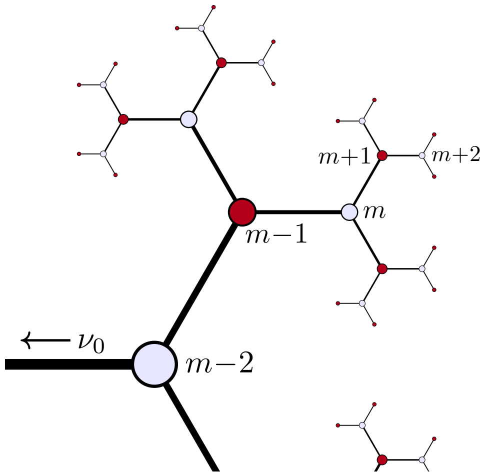
The algebra $\mathcal{H}_{\mathbb{C}}$ is stable under composition, which corresponds to the convolution of integral operators. It is a ring, and in fact its algebraic structure is very simple. Let $T = T_1$.

1.5. Proposition. *The ring $\mathcal{H}_{\mathbb{C}}$ is equal to the polynomial ring $\mathbb{C}[T]$.*

To prove this, we must show that $\mathcal{H}_{\mathbb{C}}$ is generated by T and that the powers of T form a $\mathbb{C}$-basis of it. These are both an immediate consequence of:

1.6. Lemma. *We have*

$$
\begin{aligned}
T \circ T &= T_2 + (q+1)I \\
T \circ T_m &= T_{m+1} + qT_{m-1} \quad (m \geq 2)\,.
\end{aligned}
$$



Proof. As the figure above illustrates, every node has $q+1$ neighbours. If y is at distance $m \geq 1$ from ν_0 it has q neighbours at distance $m+1$ from ν_0 and 1 at distance $m-1$. Thus, for example:

$$
[T \circ T](x) = \sum_{y \sim x, z \sim y} z = \sum_{|z:x|=2} z + \sum_{y \sim x} x = T_2(x) + (q+1)I(x)\,.
$$

Here $x \sim y$ means $|x\!:\!y| = 1$.

SL(2). Now let $G = \mathrm{SL}_2(\mathfrak{k})$, $K = \mathrm{SL}_2(\mathfrak{o})$.

Since there are two orbits of $\mathrm{SL}_2(\mathfrak{k})$ among the nodes of $\mathfrak{X}$, the representation of $\mathrm{SL}_2(\mathfrak{k})$ on $\mathbb{C}[\mathfrak{X}]$ is the direct sum of two components, determined by support.

Let $\mathcal{H}_{\mathbb{C}} = \mathcal{H}_{\mathbb{C}}(G)$ be the algebra generated by $S = T_2$ rather than that generated by T. For this operator we have relations that can be derived from those above:

1.7. Proposition. *If $S = T_2$ then*

$$
\begin{aligned}
S \circ S &= T_4 + (q-1)T_2 + q(q+1)I \\
S \circ T_{2m} &= T_{2m+2} + (q-1)T_{2m} + q^2 T_{2m-2}\,.
\end{aligned}
$$

They can also be derived more directly from this, which the figure also illustrates:

1.8. Lemma. *Suppose $m \geq 2$. Among the nodes at distance 2 from ν_m are q^2 at distance $m+2$ from ν_0, $q-1$ at distance from m from ν_0, and 1 at distance $m-2$ from it.*

1.9. Corollary. *The Hecke algebra of SL_2 is the polynomial ring $\mathbb{C}[S]$.*

HECKE OPERATORS IN TERMS OF CONVOLUTION. There is another way to interpret Hecke operators. For simplicity, I'll look only at $G = \mathrm{PGL}_2(\mathfrak{k})$ (and $K = \mathrm{PGL}_2(\mathfrak{o})$). We may identify $\mathfrak{X}$ with G/K and hence $\mathbb{C}(\mathfrak{X})$ with $\mathbb{C}(G/K)$: f in $\mathfrak{X}$ maps to

$$F_f(g) = f(g \cdot \nu_0)\,.$$

How do the Hecke operators act on functions in $\mathbb{C}(G/K)$?

Let $H_{\mathbb{C}}(G/\!/K)$ be the ring of functions on G with values in $\mathbb{C}$ that are bi-invariant on left and right by elements of K. This space has as basis the characteristic functions $\tau_m = \mathfrak{char}(K\alpha^{-m}K)$.

Functions in $H_{\mathbb{C}}(G/\!/K)$ act by a kind of convolution on $C_c(G/K)$:

$$[\tau(f)](g) = \int_G \tau(x) f(gx)\,dx = \int_G \tau(g^{-1}y) f(y)\,dy\,.$$

I leave the following as an exercise:

1.10. Proposition. *Suppose that $\tau = \mathfrak{char}(KgK)$ and that*

$$KgK = \bigsqcup g_i K\,.$$

Then for f in $C_c(G/K)$

$$[\tau f](g) = \sum f(gg_i)\,.$$

1.11. Corollary. *If f in $\mathbb{C}(G/K)$ is equal to F_φ then*

$$F_{T_m(\varphi)} = \tau_m(f)\,.$$

Proof. According to Proposition 5.3 of [Geometry], the nodes at distance m from ν_0 make up the K-orbit of ν_{-m}, and

$$k \longmapsto k \cdot \nu_{-m}$$

induces a bijection of the orbit with $K/K \cap \alpha^{-m} K \alpha^m$. If k_i traverses this quotient, then the formula for $T_m\varphi$ evaluated at ν_0 is $\sum_i \varphi(k_i\alpha^{-m} \cdot \nu_0)$. Since T_m commutes with G, the evaluation at $\nu = g \cdot \nu_0$ is

$$[T_m f](g\nu_0) = \sum_i \varphi(gk_i\alpha^{-m} \cdot \nu_0) = \sum_i \varphi(gg_i \cdot \nu_0) \quad (g_i = k_i\alpha^{-m})\,. \qquad \blacksquare$$

2. Eigenfunctions

An **admissible representation** (π, V) of a $\mathfrak{p}$-adic group is one with these two properties: (1) every vector in V is fixed by some open subgroup (i.e. it is **smooth**); (2) for any open subgroup, the subspace of vectors fixed by all elements in that subgroup is finite-dimensional.

If G is either $\mathrm{PGL}_2(\mathfrak{k})$ or $\mathrm{SL}_2(\mathfrak{k})$, the action of G on $\mathbb{C}(\mathfrak{X})$ commutes with the Hecke operators, and in consequence it acts on the space of eigenfunctions of a Hecke operator. We can describe these representations rather explicitly.

PGL(2). Let $G = \mathrm{PGL}_2(\mathfrak{k})$, $K = G(\mathfrak{o})$. For λ in $\mathbb{C}$, let V_λ be the space of smooth functions f on the nodes of $\mathfrak{X}$ such that $T(f) = \lambda f$. This is essentially the $\mathfrak{p}$-adic analogue of classical conditions on eigenfunctions of the Laplacian in the Euclidean plane.

2.1. Proposition. *The representation of G on V_λ is admissible, and V_λ^K has dimension* 1.

Proof. The condition on an eigenfunction φ is that

$$\lambda\varphi(x) = \sum_{y \sim x} \varphi(y) .$$

From here, in several steps:

Step 1. For each $n \geq 0$ let K_n be the congruence subgroup $G(\mathfrak{p}^n)$ (those elements congruent to I modulo $\mathfrak{p}^n$). Suppose the eigenfunction φ to be fixed by K_n. Let B_n be the 'ball' of nodes at distance $\leq n$ from ν_0, which are all fixed by K_n. I claim that *the values of φ at all nodes at distance $> n$ from ν_0 are determined by its values on B_n*.

Suppose x to be one of the nodes at distance exactly n from ν_0. Recall that a node y is **exterior** to x if the geodesic to ν_0 passes through x. This includes x itself. Then K_n fixes x and, if $m \geq n$, acts transitively on all nodes at distance m from ν_0 and exterior to x. Hence φ takes the same value, say $\varphi_{x,m}$, at all those nodes. My earlier claim will follow from the new claim that *all these $\varphi_{x,m}$ are determined by the values of φ at the neighbours of x, if any, inside B_n.*

Step 2. First we look at the value of φ at the external neighbours of x at distance 1. There are two cases, according to whether $x = \nu_0$ or not.

Suppose $x = \nu_0$. This happens only when $n = 0$ and φ is fixed by K itself. The function φ takes the same value φ_m at all nodes at distance m from ν_0. There are $q + 1$ neighbours of ν_0 at distance 1, so

$$\lambda\varphi_0 = (q + 1)\,\varphi_1, \quad \varphi_1 = \frac{\lambda}{q + 1} \cdot \varphi_0 . \tag{2.2}$$

If $n \geq 1$, there is one neighbour y inside B_n, q outside, and we must have

$$q\varphi_{x,n+1} = \lambda\varphi_{x,n} - \varphi(y) . \tag{2.3}$$

In either case, the values $\varphi_{x,n+1}$ are determined by the values of φ inside B_n.

Step 3. Suppose now $m \geq n + 1$, y at distance m from ν_0 and external to x. Then all neighbours of y are external to x, and by Lemma 1.8

$$\lambda\varphi_{x,m} = \varphi_{x,m-1} + q\varphi_{x,m+1} .$$

That is to say, the function $\varphi_{x,m}$ for $m \geq n + 1$ satisfies the difference equation

$$q\varphi_{x,m} - \lambda\varphi_{x,m-1} + \varphi_{x,m-2} = 0 . \tag{2.4}$$

This is a difference equation of second order. The function φ also satisfies initial conditions determined by its values $\varphi_{x,n}$ and $\varphi_{x,n+1}$.

Step 4. There is a well known recipe for the solution of a difference equation. Set $\varphi_{x,m} = r^m$. Plugging into the equation, we see that r must be a root of the quadratic equation

$$x^2 - (\lambda/q)x + 1/q = 0\,.$$

If this equation has distinct roots r_1, r_2, then the general solution is of the form $cr_1^m + dr_2^m$. Set $r_1 = z/\sqrt{q}$, $r_2 = z^{-1}/\sqrt{q}$ where now we require that

$$z + z^{-1} = \lambda/\sqrt{q}, \quad \lambda = \sqrt{q}\,(z + z^{-1})\,.$$

This makes the solution

$$\varphi_{x,m} = q^{-m/2}(cz^m + dz^{-m}) \tag{2.5}$$

for constants c, d satisfying the given initial conditions near the boundary of B_n.

Step 5. The exceptional case is when the roots are equal: $z = z^{-1}$ and $\lambda = \pm 2\sqrt{q}$. In this case $r = \pm 1/\sqrt{q}$ is the unique root of the characteristic equation. The solutions of the difference equation are linear combinations of $f_0(m) = q^{-m/2}z^m$ and $f_1(m) = mq^{-m/2}z^m$, with $z = \pm 1$.

In this case, the first function is an eigenfunction of the translation operator $[\tau f](m) = f(m-1)$ since $f_0(m-1) = q^{1/2}z^{-1}\cdot f_0(m)$-. It is unique up to scalar. The second does not possess an intrinsic characterization. Instead it satisfies the equation

$$f_1(m-1) = q^{1/2}z^{-1}\cdot f_1(m) + q^{1/2}z^{-1}\cdot f_0(m)\,,$$

but so does any sum $f_1 + c\,f_0$.

Step 6. To summarize: if $n = 0$ and $x = \nu_0$, there is a unique function φ_m satisfying the difference equation with a given value of φ_0, and proportional to φ_0. Therefore V_λ^K has dimension one. Otherwise ($n > 0$), the function φ is uniquely determined by its values on the ball B_n. This proves the Proposition. ▯

Note that the proof does not give a very explicit description of the subspace of functions in V_λ fixed by K_n, unless $n = 0$. We shall see later a very different way to understand this.

In all cases, φ has a well defined asymptotic behaviour on every branch running out from B_n. (a) If the polynomial

$$x^2 - (\lambda/\sqrt{q})x + 1 = 0$$

has distinct roots $z^{\pm 1}$ then there exist constants c, d such that

$$\varphi(y) = q^{-m/2}(cz^m + dz^{-m})$$

if y lies at distance m from B_n. (b) If it has one root z then

$$\varphi(y) = q^{-m/2}(cz^m + dmz^m)\,.$$

The constants depend on the branch.

When $n = 0$ the argument above does give us a completely explicit formula for φ, at least with a bit more work. Suppose $\varphi_0 = \varphi(\nu_0) = 1$. According to (2.5) and (2.2) we are looking for constants c, d such that

$$\begin{aligned} c + d &= 1 \\ q^{-1/2}(cz + d/z) &= \frac{q^{1/2}(z + 1/z)}{q+1} \\ cz + d/z &= \frac{z + 1/z}{1 + 1/q}\,. \end{aligned}$$

The formulas for c and d are surprisingly complicated. I phrase them in terms I shall justify later.

2.6. Proposition. *The unique solution of the difference equation for φ is*

$$\varphi_m = \frac{q^{-m/2}}{1+1/q}\left(\left(\frac{1-q^{-1}z^{-2}}{1-z^{-2}}\right)z^m + \left(\frac{1-q^{-1}z^{2}}{1-z^{2}}\right)z^{-m}\right)$$

as long as $z \neq \pm 1$.

Once this equation is at hand, one can verify that it is correct by evaluating it for $m = 0, 1$. Finding it in the first place is just a matter of solving a 2×2 system of linear equations.

This formula for φ_m can be put into a curious form.

2.7. Corollary. *For $m \geq 1$*

$$\varphi_m = \frac{1}{1+1/q}\cdot\left(q^{-m/2}(z^m + z^{m-2} + \cdots + z^{-m}) - q^{-(m-2)/2}(z^{m-2} + \cdots + z^{-(m-2)})\right).$$

Proof. Expanding expressions, the formula in the Proposition becomes for $m \geq 1$

$$\begin{aligned}
\varphi_m &= \frac{q^{-m/2}}{1+1/q}\cdot\left(\left(\frac{1-q^{-1}z^{-2}}{1-z^{-2}}\right)z^m + \left(\frac{1-q^{-1}z^{2}}{1-z^{2}}\right)z^{-m}\right)\\
&= \frac{q^{-m/2}}{1+1/q}\cdot\left(\left(\frac{z^{m+1}-z^{-(m+1)}}{z-z^{-1}}\right) - q^{-1}\left(\frac{z^{m-1}-z^{-(m-1)}}{z-z^{-1}}\right)\right)\\
&= \frac{1}{1+1/q}\cdot\left(q^{-m/2}(z^m + z^{m-2} + \cdots + z^{-m}) - q^{-(m-2)/2}(z^{m-2} + \cdots + z^{-(m-2)})\right).
\end{aligned}$$ ∎

This gives immediately a formula for the 'singular' cases $z^2 = 1$.

SL(2). Now I take $G = \mathrm{SL}_2(\mathfrak{k})$. Here we use S instead of T, and consider the set of nodes with even parity, a single $\mathrm{SL}_2(\mathfrak{k})$-orbit. So V_λ is the space of functions on this orbit such that $S\varphi = \lambda\varphi$. The proof of admissibility is essentially the same, but I am interested in an explicit formula when φ is fixed by K. Because of the Cartan decomposition, φ is determined by its values on ν_{2m}. Let $\varphi_{2m} = \varphi(\nu_{2m})$.

According to Lemma 1.8, the difference equation and initial conditions are now

$$\begin{aligned}
\varphi_2 &= \frac{\lambda}{q(q+1)}\cdot\varphi_0\\
0 &= q^2\varphi_{2m} - (\lambda - q + 1)\varphi_{2m-2} + \varphi_{2m-4}\,.
\end{aligned}$$

We get solutions

$$cr_1^m + dr_2^m$$

where the r_i are roots of the indicial equation

$$x^2 - (\mu/q^2)x + 1/q^2 = 0 \quad (\mu = \lambda - q + 1)\,.$$

I set $r_1 = z/q$, $r_2 = z^{-1}/q$, where now

$$z + z^{-1} = \mu/q\,.$$

The final conclusion is:

2.8. Proposition. *When $G = \mathrm{SL}_2(\mathfrak{k})$, the spherical function is*

$$\varphi_{2m} = \frac{q^{-m}}{1+1/q}\left(\left(\frac{1-q^{-1}z^{-1}}{1-z^{-1}}\right)z^m + \left(\frac{1-q^{-1}z}{1-z}\right)z^{-m}\right)$$

as long as $z \neq 1$, and some non-trivial linear combination of $1/q^m$ and m/q^m when $z = 1$.

The case $z = -1$ will turn out to be both simple and especially interesting. Explicitly:

2.9. Corollary. *If $z = -1$ then*

$$\varphi_{2m} = (-q)^m \, .$$

I'll explain elsewhere why there is just a single term. We can also repeat what I did for PGL_2:

2.10. Corollary. *For $m \geq 1$*

$$\varphi_{2m} = \frac{1}{1+1/q} \cdot \left(q^{-m}(z^m + z^{m-1} + \cdots + z^{-m}) - q^{-(m-1)}(z^{m-1} + z^{m-2} + \cdots + z^{-(m-1)})\right)$$

Proof.

$$\begin{aligned} \varphi_{2m} &= \frac{q^{-m}}{1+1/q}\left(\left(\frac{1-q^{-1}z^{-1}}{1-z^{-1}}\right)z^m + \left(\frac{1-q^{-1}z}{1-z}\right)z^{-m}\right) \\ &= \frac{q^{-m}}{1+1/q} \cdot \left(\left(\frac{z^{m+1} - z^{-m}}{z-1}\right) - q^{-1}\left(\frac{z^m - z^{-(m-1)}}{z-1}\right)\right) \\ &= \frac{1}{1+1/q} \cdot \left(q^{-m}(z^m + z^{m-1} + \cdots + z^{-m}) - q^{-(m-1)}(z^{m-1} + z^{m-2} + \cdots + z^{-(m-1)})\right) . \end{aligned}$$

There is a second interpretation of the values of $\varphi_m = \varphi(\nu_{-m})$ in the case when φ is fixed by K, and $\varphi(\nu_0) = 1$. Because the space of all functions fixed by K has dimension one, φ must be an eigenfunction of each T_m.

2.11. Proposition. *In this situation, for $m \geq 0$*

$$T_m(\varphi) = \lambda_m \varphi$$

with

$$\lambda_m = q^{m-1}(q+1)\varphi_m \, .$$

This is true for both PGL_2 amd SL_2.

Proof. Because

$$[T_m(\varphi)](\nu_0) = \lambda_m = \sum\nolimits_{\nu_0:x]=m} \varphi(x) = q^{m-1}(q+1)\varphi_m \, .$$

To summarize: in both cases, PGL_2 and SL_2, not only is the space of eigenfunctions fixed by K of dimension 1, but we know exactly what the functions in the space are. For example, if $z = q$ in the last formula, the representation is on the space of functions φ such that $q(q+1)\varphi(\nu)$ is the sum of the values of φ at the nodes at distance 2 from ν. This contains the constants, and we get $\varphi_m \equiv 1$.

3. Boundary behaviour

Near the identity, the functions φ behave in a rather complicated way, but outside of a finite region they all behave in a much simpler fashion. This suggests considering their asymptotic behaviour in relation to the boundary of the tree.

To see what's going on in the simplest case, let's look at the eigenspace of harmonic functions φ, in which $\lambda = q + 1$ and $z = q^{\pm 1/2}$. Suppose $\mathcal{B}$ to be a branch coming out from ν_0. Then for $m \gg 0$ the value of φ at ν on $\mathcal{B}$ at distance m from ν_0 depends only on m, and

$$\varphi_\nu = q^{-m/2}(cq^{m/2} + dq^{-m/2}) = c + dq^{-m}$$

for some constants c, d depending only on the branch. This formula is consistent with the fact that constant functions form a G-stable subspace. There is a very simple way to interpret this formula. Recall from §4 that the tree may be compactified by adding a copy of $\mathbb{P}^1(\mathfrak{k})$ at infinite distance. *Every harmonic function φ on the tree has a limit along each branch, defining a smooth function on $\mathbb{P}^1(\mathfrak{k})$.*

We have thus defined a G-equivariant map from the space of harmonic functions on $\mathfrak{X}$ to $C^\infty(G/P)$. It turns out this is an isomorphism of representations, but I'll not show that.

This construction can be generalized—the boundary values of functions in all eigenspaces of T will be smooth sections of some representation of G induced from one on P.

I want to demonstrate the basic idea, so I'll assume until the end of this section that $G = \mathrm{PGL}_2(\mathfrak{k})$.

As we have seen, if φ is an eigenfunction of T with eigenvalue λ, then for $m \gg 0$ the restriction of φ to the branch

$$\nu_{-m} — \nu_{-m-1} — \dots$$

satisfies the difference equation

(3.1) $$q\varphi_m + \lambda\varphi_{m-1} + \varphi_{m-2} = 0 \quad (\varphi_m = \varphi(\nu_{-m}))\,.$$

The follwoing is elementary:

3.2. Lemma. *If φ is a solution of (3.1), there is a unique extension of φ to a solution of the difference equation over all of $\mathcal{A}$.*

Let U_λ be the space of all solutions of the difference equation on $\mathcal{A}$. It is a two-dimensional space on which $A/A(\mathfrak{o})$ acts by translation:

$$[L_a f](\nu_{-m}) = f(a^{-1} \cdot \nu_{-m}), \quad [L_\alpha f](\nu_{-m}) = f(\nu_{-m-1})\,.$$

The consequence of the Lemma is that Φ defines a map from V_λ to U_λ.

The group P does not take $\mathcal{A}$ to itself, but nonetheless the map Φ is P-equivariant in some sense.

3.3. Proposition. *Suppose φ to lie in V_λ. Then*

(a) for a in A

$$\Phi(L_a\varphi) = L_a\Phi(\varphi)\,;$$

(b) for n in N

$$\Phi(L_n\varphi) = \Phi(\varphi)\,.$$

Proof. This is because any n in N fixes all nodes ν_{-m} for $m \gg 0$. ▯

This makes U_λ in a natural way a representation of P. Define the 'twisted' representation

$$[\sigma_\lambda(p)]\varphi = \delta^{-1/2}(p)\Phi(L_p\varphi)\,.$$

If (σ, U) is any finite-dimensional representation of P, the representation $\mathrm{Ind}(\sigma \mid P, G)$ induced by σ from P to G is the right regular representation of G on the space of locally constant functions $f: G \to U$ such that

$$f(pg) = \delta^{1/2}(p)\sigma(p)f(g)$$

for all p in P, g in G. Thus $Ind(\delta^{-1/2} \mid P, G)$ may be identified with $C^\infty(P\backslash G)$, and $Ind(\delta^{1/2} \mid P, G)$ with space $\Omega^\infty(P\backslash G)$ of smooth one-densities (the smooth dual of $C^\infty(P\backslash G)$).

There is also here a kind of Frobenius reciprocity—if (π, V) is any smooth representation of G then there is a canonical isomorphism

$$\mathrm{Hom}(V, \mathrm{Ind}(\sigma \mid P, G)) \longrightarrow \mathrm{Hom}_P(V, \delta^{1/2}\sigma)\,.$$

So if $I(\sigma_\lambda)$ is the representation of G induced by $(\sigma_\lambda, U_\lambda)$ from P to G, the asymptotic behaviour of eigenfunctions gives a G-equivariant map

$$V_\lambda \longrightarrow \mathrm{Ind}(U_\lambda \mid P, G)\,.$$

For example, if V_λ is the space of harmonic functions, σ_λ is isomorphic to the direct sum of characters $\delta^{\pm 1/2}$, and we get maps into $C^\infty(G/P)$ and $\Omega^\infty(G/P)$.

4. References

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