Essays on representations of real groups

Bill Casselman
University of British Columbia
cass@math.ubc.ca

Unitary representations

This essay is a brief introduction to unitary representations of a locally compact group. I assume throughout that G is unimodular and locally compact. Although many of the results are valid for arbitrary locally compact groups, all the applications I have in mind concern Lie groups, and occasionally I shall assume that G is one.

Contents

Part I. Introduction

1. Continuity

Suppose V to be a (separable, always separable) Hilbert space. The ring $\mathrm{End}(V)$ of bounded operators on V can be assigned three useful topologies:

(a) The **weak** topology: T_n converges to T if $T_n u \bullet v$ converges to $Tu \bullet v$ for all u, v, in V. Each pair (u, v) defines neighbourhoods of 0

$$\big\{ T \,\big|\, |Tu \bullet v| < \varepsilon \big\} .$$

(b) The **strong** topology: T_n converges to T if $T_n v$ converges to Tv for all v in V. Each v defines neighbourhoods of 0

$$\big\{ T \,\big|\, \|Tv\| < \varepsilon \big\} .$$

(c) The **norm** or **uniform** topology: T_n converges to T if $\|T_n - T\|$ converges to 0. We get uniform neighbourhoods of 0:

$$\big\{ T \,\big|\, \|T\| < \varepsilon \big\} .$$

Since we'll deal mostly with the strong operator topology, I'll make its definition a bit more precise: it is a locally convex topological vector space, with a basis of semi-norms determined by finite sets $\sigma = \{v\}$ of vectors in V

$$\|T\|_\sigma = \sup_{v\in\sigma} \|Tv\| \,.$$

The group $\mathbb{U}(V)$ of unitary operators on V inherits these three topologies. It is closed in $\mathrm{End}(V)$ with respect to the strong operator topology. If π is a homomorphism from G to $\mathbb{U}(V)$, we can require that it be continuous with respect to one of these topologies, thus defining three types of continuous unitary representation: **weakly**, **strongly**, and **uniformly** continuous. Only two of these are interesting:

1.1. Proposition. *A unitary representation of G is weakly continuous if and only if it is strongly continuous. If it is uniformly continuous then its image is finite-dimensional.*

I could have made a definition of continuity even weaker than weak continuity, but it would have been redundant—[Segal-von Neumann:1950] proves that if a unitary representation is weakly measurable, it is already weakly continuous. As for the last claim, it will serve only as negative motivation, so I'll just refer to [Singer:1952].

The theme 'weak is the same as strong' is common enough in this subject that this shouldn't be too much of a surprise.

Proof. That strongly continuous implies weakly continuous is straightforward.

On the other hand, suppose (π, V) to be weakly continuous. Then for any v in V and g in G we have

$$\|\pi(g)v - v\|^2 = 2\|v\|^2 - 2\left(\pi(g)v \bullet v\right).$$

Thus if $g \to 1$ weakly, $\|\pi(g)v - v\| \to 0$.

The point of Singer's result is that uniformly continuous unitary representations are not very interesting. *From now on, when I refer to a unitary representation I'll mean one that's strongly continuous.* This means that for every v in V the map from G to V taking g to $\pi(g)v$ is continuous. But something better is true:

1.2. Proposition. *If (π, V) is a unitary representation of G then the map*

$$G \times V \to V, \quad (g, v) \longmapsto \pi(g)v$$

is continuous.

This means that π is continuous in the more general sense.

Proof. Suppose g_0 and v_0 given. Then

$$\begin{aligned} \big\|\pi(xg_0)(v_0+u) - \pi(g_0)v_0\big\| &= \big\|\pi(xg_0)v_0 - \pi(g_0)v_0 + \pi(x)\pi(g_0)u\big\| \\ &\le \big\|\pi(xg_0)v_0 - \pi(g_0)v_0\big\| + \big\|\pi(x)\pi(g_0)u\big\| \\ &= \big\|\pi(x)\pi(g_0)v_0 - \pi(g_0)v_0\big\| + \|u\| \\ &< \varepsilon\,, \end{aligned}$$

if we choose a neighbourhood X of 1 in G such that

$$\|\pi(x)\pi(g_0)v_0 - \pi(g_0)v_0\| < \varepsilon/2$$

for all x in X and we choose $\|u\| < \varepsilon/2$.

There is no shortage of unitary representations:

1.3. Proposition. *If H is a unimodular closed subgroup of G, the right regular representation of G on $\mathrm{L}^2(H\backslash G)$ is unitary.*

Proof. That each R_g is unitary is immediate, so we must only verify the condition of continuity—given f in $\mathrm{L}^2(G)$ and $\varepsilon > 0$ we must find a neighbourhood X of 1 in G such that x in X implies $\|R_x f - f\| < \varepsilon$.

Because $\mathrm{L}^2(G)$ is the L^2-completion of $C_c(G)$, there exists a continuous function φ of compact support Ω such that $\|f - \varphi\| < \varepsilon/3$. Then also for any x in G $\|R_x f - R_x\varphi\| < \varepsilon/3$, so it remains only to find a compact neighbourhood X such that x in X implies $\|R_x\varphi - \varphi\| \le \varepsilon/3$.

For any $\eta > 0$ one can find a neighbourhood X of 1 such that (a) $\overline{\Omega}X \subseteq \Omega_*$ and (b) $|\varphi(gx) - \varphi(g)| < \eta$ everywhere if x lies in X. But then

$$
\begin{aligned}
\|R_x\varphi - \varphi\|^2 &= \int_G \big|\varphi(gx) - \varphi(g)\big|^2 \, dg \\
&= \int_{\Omega X^{-1}} \big|\varphi(gx) - \varphi(g)\big|^2 \, dg \\
&\le \eta^2 \cdot \mathrm{meas}(\Omega X^{-1}) \\
\|R_x\varphi - \varphi\| &\le \eta \sqrt{\mathrm{meas}(\Omega X^{-1})} \\
&= \varepsilon/3 \quad \text{if} \quad \eta\sqrt{\mathrm{meas}(\Omega X^{-1})} = \varepsilon/3 \,. \; \square
\end{aligned}
$$

2. Representation of measures

Let $M_c(G)$ be the space of bounded measures on G of compact support. It can be identified with the space of continuous linear functionals on the space $C(G, \mathbb{C})$ of all continuous functions on G. If μ_1 and μ_2 are two measures in $M_c(G)$ their convolution is defined by the formula

$$
\langle \mu_1 * \mu_2, f \rangle = \int_{G\times G} f(xy) \, d\mu_1 d\mu_2 \,.
$$

The identity in $M_c(G)$ is the Dirac δ_1 taking f to $f(1)$.

If G is assigned a Haar measure dx, the space $C_c(G)$ of continuous functions with compact support may be embedded in $M_c(G)$: $f \mapsto f\,dx$. The definition of convolution of measures then agrees with the formula for the convolution of two functions in $C_c(G)$:

$$
[f_1 * f_2](y) = \int_G f_1(x) f_2(x^{-1}y) \, dx
$$

The space of a unitary representation π becomes a module over $M_c(G)$ in accordance with the formula

$$
\pi(\mu)v = \int_G \pi(g)v \; d\mu \,.
$$

Since Hilbert spaces are isomorphic to their conjugate duals, this integral is defined uniquely by the specification that

$$
u \bullet \Big(\int_G \pi(g)v \; d\mu \Big) = \int_G (u \bullet \pi(g)v) \; d\mu
$$

for every u in V.

If Ω is the support of μ, then

$$
\|\pi(g)v\| = \|v\|
$$

for all g in Ω and

$$
\begin{aligned}
\big\|\pi(\mu)v\big\| &\le \int_\Omega \big\|\pi(g)v\big\| \, |d\mu| \\
&\le \Big(\int_\Omega |d\mu| \Big) \, \|v\| \,.
\end{aligned}
$$

Hence:

2.1. Proposition. *For every μ in $M_c(G)$ the operator $\pi(\mu)$ is continuous.*

For a given v in V the map $\mu \mapsto \pi(\mu)v$ from $M_c(G)$ to V is covariant with respect to the left regular representation, since

$$\begin{aligned} u \bullet \pi(L_g\mu)v &= \langle L_g\mu, u \bullet \pi(\bullet)v\rangle \\ &= \langle \mu, u \bullet \pi(g^{-1}\pi(\bullet)v\rangle \\ &= \langle \mu, \pi(g)u \bullet \pi(\bullet)v\rangle \\ &= \pi(g)u \bullet \pi(\mu)v \\ &= u \bullet \pi(g)\pi(\mu)v \ . \end{aligned}$$

The map $\mu \longmapsto \pi(\mu)$ is a ring homomorphims, since the composition of two operators $\pi(\mu_1)\pi(\mu_2)$ can be calculated easily to agree with $\pi(\mu_1 * \mu_2)$. For the left regular representation, $L_\mu f = \mu * f$.

I define a **Dirac function** on G to be a function f in $C_c^\infty(G, \mathbb{R})$ satisfying the conditions

(a) $f = h^\vee * h$, where $h \geq 0$;
(b) the integral $\int_G f(g)\,dg$ is equal to 1.

I recall that $h^\vee(g) = \overline{h}(1/g)$. The first condition implies that $f \geq 0$ (and is in particular real) , $f^\vee = f$.

A **Dirac sequence** is a sequence of Dirac functions f_n with support tending to 1.

2.2. Theorem. *If Ω is a finite set in V and $\{f_n\}$ a Dirac sequence on G then for some N*

$$\left\|\pi(f_n)v - v\right\| < \varepsilon$$

for all v in Ω, all $n > N$.

In other words, as measures they approach δ_1.

Part II. Compact operators

3. Compact subsets of L^2(H$\backslash$G)

The requirement for a unitary representation to be continuous has as consequence a stronger version of the basic continuity condition. Suppose (π, V) to be a unitary representation of G. Given v in V and $\varepsilon > 0$, choose X a neighbourhood of 1 in G so that $\left\|\pi(x)v - v\right\| < \varepsilon/3$ for x in X. Then for x in X, u in V

$$\begin{aligned} \pi(x)u - u &= \big(\pi(x)u - \pi(x)v\big) + \big(\pi(x)v - v\big) + (v - u) \\ \left\|\pi(x)u - u\right\| &\leq \left\|\pi(x)u - \pi(x)v\right\| + \left\|\pi(x)v - v\right\| + \|v - u\| \\ &\leq \left\|\pi(x)(u - v)\right\| + \left\|\pi(x)v - v\right\| + \|v - u\| \\ &\leq \|u - v\| + \left\|\pi(x)v - v\right\| + \|v - u\| \\ &< \|u - v\| + \varepsilon/3 + \|v - u\| \ . \end{aligned}$$

Therefore if we let U be the neighbourhood of v where $\|u - v\| < \varepsilon/3$ and $\|u - v\| < \varepsilon/3$, then

$$\left\|\pi(x)u - u\right\| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$$

I summarize: we can find for each v in V a neighbourhood U of v and a neighbourhood X of 1 in G such that $\left\|\pi(x)v - v\right\| < \varepsilon$ for all x in X. If Ω is a compact subset of V, we may cover it by a finite collection of U_i and take $X = \cap X_i$ so that this holds for all u in Ω and x in X. We have thus proved:

3.1. Proposition. *Suppose (π, V) to be a unitary representation of G. Given a compact subset Ω of V and $\varepsilon > 0$, there exists a neighbourhood X of 1 in G such that*

$$\big\|\pi(x)v - v\big\| < \varepsilon$$

for all v in Ω and x in X.

For spaces of functions on quotients of G, there is a kind of converse to this. A subset Ω of a Hilbert space V is **totally bounded** if for every $\varepsilon > 0$ the set Ω is covered by a finite number of translates of disks $\|v\| < \varepsilon$. In other words, a set is totally bounded when it is well approximated by finite sets. I also recall that in a Hilbert space a subset is relatively compact if and only if it is totally bounded.

3.2. Theorem. *Suppose H to be a unimodular closed subgroup of G with $H\backslash G$ compact, and let $V = \mathrm{L}^2(H\backslash G)$. A subset Ω of V is relatively compact if and only if*

(a) it is bounded;
(b) $\|R_x F - F\| \to 0$ as $x \to 0$, uniformly for $F \in \Omega$.

What (b) means is that for every $\varepsilon > 0$ there exists a neighbourhood X of 1 in G such that $\|R_x F - F\| < \varepsilon$ for all x in X, F in Ω.

Proof. The necessity of (a) is immediate; that of (b) follows from Proposition 3.1.

For sufficiency, I start with a related result about continuous functions on $H\backslash G$. Suppose Ω to be a set of functions in $C(H\backslash G)$ that are (a) bounded and (b) equicontinuous. This second condition means that for for any $\varepsilon > 0$ there exists some neighbourhood X of 1 in G with $|R_x F(g) - F(g)| < \varepsilon$ for all x in X, g in G.

Choose such an X for $\varepsilon/3$. Cover $H\backslash G$ by m sets $g_i X$. The image of Ω in $\mathbb{C}^m$ with respect to $f \mapsto \big(f(g_i)\big)$ is bounded, hence relatively compact. There therefore exist a finite number of F_j in Ω with the property that for every F in Ω there exists some F_j with $\big|F(g_i) - F_j(g_i)\big| < \varepsilon/3$ for all i. But then if $g = g_i x$

$$\begin{aligned} F(g) - F_j(g) &= F(g_i x) - F_j(g_i x) \\ &= \big(F(g_i x) - F(g_i)\big) + \big(F(g_i) - F_j(g_i)\big) + \big(F_j(g_i) - F_j(g_i x)\big) \\ |F(g) - F_j(g)| &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \\ &= \varepsilon\,. \end{aligned}$$

Since $H\backslash G$ is compact,

$$\|F\| \le \sqrt{\mathrm{meas}(H\backslash G)}\,\sup|F(g)|$$

so that Ω is totally bounded, hence relatively compact, in $\mathrm{L}^2(H\backslash G)$.

Now assume $\Omega \subset \mathrm{L}^2(H\backslash G)$ to satisfy (a) and (b). For a given φ in $C(G)$ conditions (a) and (b) imply that the functions $R_\varphi F$, $F \in \Omega$, form a relatively compact subset of $C(H\backslash G)$, hence of $\mathrm{L}^2(H\backslash G)$. But then by Theorem 2.2, for some choice of φ this set is within distance ε of Ω.

3.3. Corollary. *If $H \subseteq G$ are unimodular groups with $H\backslash G$ compact and f is in $C_c(G)$ then R_f is a compact operator.*

Proof. We have

$$\begin{aligned} R_f F(x) &= \int_G f(g)F(xg)\,dg \\ &= \int_G f(x^{-1}g)F(g)\,dg \\ &= \int_{H\backslash G} F(y)\,dy \int_H f(x^{-1}hy)\,dh \\ &= \int_{H\backslash G} K_f(x,y)F(y)\,dy \end{aligned}$$

where

$$K_f(x,y) = \int_H f(x^{-1}hy)\,dh\,.$$

This kernel is continuous. Since $H\backslash G$ is compact, it follows from this and Theorem 3.2 that R_f is compact.

Of course this is a simple case of a more general result which asserts that integral operators are Hilbert-Schmidt.

If $H\backslash G$ is not necessarily compact, we have the following variation, which generalizes a classical result concerning $\mathrm{L}^2(\mathbb{R})$.

3.4. Proposition. *Suppose H to be a unimodular closed subgroup of G. Let $V = \mathrm{L}^2(H\backslash G)$. A subset Ω of V is relatively compact if and only if*

- *(a) it is bounded;*
- *(b) $\|R_x F - F\| \to 0$ as $x \to 0$, uniformly for $F \in \Omega$.*
- *(c) there exists a sequence of compact subsets*

$$Y_0 \subseteq Y_1 \subseteq Y_2 \subseteq \cdots$$

exhausting $H\backslash G$ such that the L^2 norm of the restriction of F in Ω to the exterior of Y_k tends uniformly to 0 as $k \to \infty$.

Proof. The argument for the necessity of (a) and (b) is as in the previous result. For (c), if Ω is relatively compact and $\varepsilon > 0$ we can find a finite number of functions F_i within $\varepsilon/2$ of all of Ω. If we choose a compact X such that the restriction to the complement of X of each F_i has norm $< \varepsilon/2$, then the restriction of every F in Ω to the complement has norm $< \varepsilon$.

For sufficiency, suppose given $\varepsilon > 0$. Choose a compact Y such that the norm of F restricted to the complement of Y is $< \varepsilon/2$. Following the argument above, find a finite number of F_i approximating Ω within $\varepsilon/2$ on Y. But then off Y we have $\|F - F_i\| < \varepsilon$ as well.

4. Discrete sums of irreducible representations

Suppose (π, V) to be a unitary representation of G on the Hilbert space V. If f lies in $C_c(G)$ then $\pi(f)$ is certainly a bounded operator on V, and its adjoint is $\pi(f^\vee)$, where $f^\vee(g) = \overline{f}(g^{-1})$:

$$\pi(f^\vee)u \bullet v = u \bullet \pi(f)v\,.$$

A function f is called **self-adjoint** if $f = f^\vee$. Thus a Dirac function f is self-adjoint, and then so is the operator $\pi(f)$. If $f = h^\vee * h$ for a Dirac function h, it is also positive:

$$\pi(f)u \bullet u = \pi(h^\vee)\pi(h)u \bullet u = \big\|\pi(h)u\big\|^2\,.$$

A unitary representation (π, V) is **irreducible** if it contains no proper closed G-stable subspace. I'll call a subset of $C_c(G)$ *sufficiently dense* if it contains a Dirac sequence.

4.1. Theorem. *Suppose (π, V) to be a unitary representation of G on the (separable) Hilbert space V. If there exists in $C_c(G)$ a sufficiently dense subset U such that $\pi(f)$ is a compact operator for every f in U, then (a) π is a discrete sum of irreducible representations and (b) each occurs with finite multiplicity.*

Proof. It must be shown that (a) V is the Hilbert direct sum of irreducible G-stable subspaces V_i with (b) finite multiplicity. What (a) means more concretely that for every vector v in V and $\varepsilon > 0$ there exists some v_n in $\bigoplus_{i\le n} V_i$ with $\|v - v_n\| < \varepsilon$. What (b) means is that each isomorphism class occurs only a finite number of times.

If f is a Dirac function then $\pi(f)$ is a positive self-adjoint operator on V, and by assumption compact. Thus V is the Hilbert sum of the kernel $V_{f,0}$ of $\pi(f)$ and a sequence of eigenspaces V_{f,ω_i} of finite dimension, with $\omega_i > 0$. If there are an infinite number of eigenspaces then $\lim \omega_i = 0$.

Step 1. Suppose $v \neq 0$ in V, $\varepsilon > 0$. According to Theorem 2.2 we can find a positive Dirac function f with $\big\|\pi(f)v - v\big\| < \varepsilon$, but we can also write

$$v = v_0 + \sum v_i \ \text{ where } \ \pi(f)v_0 = 0, \quad \pi(f)v_i = \omega_i v_i \,.$$

Then

$$\varepsilon^2 > \big\|\pi(f)v - v\big\|^2 = \|v_0\|^2 + \sum (\omega_i - 1)^2 \|v_i\|^2 \,.$$

Either (a) the set of ω_i is finite, or (b) $\lim \omega_i = 0$. If it is finite

$$\Big\| v - \sum_i v_i \Big\|^2 = \|v_0\|^2 < \varepsilon^2 \,.$$

If it is infinite, we can choose n so large that $\omega_i < 1/2$ for $i > n$. Then $|1 - \omega_i|^2 > 1/4$ for $i > n$, and consequently

$$\Big\| v - \sum_1^n v_i \Big\|^2 = \|v_0\|^2 + \sum_{i>n} \|v_i\|^2 < 4\varepsilon^2 \,.$$

What I have shown is this: *If f is a Dirac function with $\|\pi(f)v - v\| < \varepsilon/2$, then there exists a vector u in a finite sum of eigenspaces $V_{f,\omega}$ with $\omega \neq 0$ such that $\|v - u\| < \varepsilon$.*

Thus the finite sums of eigenspaces V_{f,ω_i} of the $\pi(f)$ are dense in V.

Step 2. Suppose given a set $\{v_i\}$ of m vectors in V. The same reasoning applies to $m \cdot V$. According to Theorem 2.2, there exists some Dirac function f such that $\|\pi(f)v_i - v_i\| < \varepsilon/2$ for all v_i. But then by the previous step I can find vectors u_i in a finite sum of eigenspaces $V_{f,\omega}$ with $\omega \neq 0$, $\|v_i - u_i\| < \varepsilon$ for all i.

Step 3. The closed subspace $U \subseteq V$ is said to be **generated over** G by a subset A if either of these equivalent conditions hold: (a) U is the smallest G-stable space containing A or (b) if U is G-stable and the set of vectors in U perpendicular to $\pi(G)A$ is 0.

4.2. Lemma. *Suppose U to be generated over G by its eigenspace $U_{f,\omega}$. The map taking W to $W_{f,\omega}$ is an injection from G-stable closed subspaces of U to subspaces of $U_{f,\omega}$.*

Proof of Lemma. Suppose that U is closed and G-stable, but $U_{f,\omega} = U \cap V_{f,\omega} = 0$. Then U is the orthogonal sum of the subspaces $U_{f,\omega_i} = U \cap V_{f,\omega_i}$ with $\omega_i \neq \omega$, so $u \bullet v = 0$ for all v in $V_{f,\omega}$. But then since U is G-stable, $\pi(g)u \bullet v = u \bullet \pi(g^{-1})v = 0$ for all v in $V_{f,\omega}$ and by hypothesis $u = 0$.

Step 4. Let U be the subspace of V generated over G by one of the finite-dimensional subspaces $V_{f,\omega}$. Then according to the previous step U is the direct sum of a finite number of irreducible subspaces. The representations of G that occur there do not occur in $U^\perp$, since isomorphic representations of G have isomorphic eigenspaces for f.

Step 5. Now to conclude the proof of Theorem 4.1. Choose an orthogonal basis $\{v_i\}$ of V. Let $V_0 = 0$. We shall show inductively that we can find an increasing sequence of subspaces V_i of V such that (a) each V_i is a finite sum of irreducible G-spaces; (b) none of the irreducible representations occurring in $V^\perp \cap V_n$ occurs in V_{n-1}; (c) for each n there exists a finite set of vectors $u_i^{(n)}$ $(i \leq n)$ with $\big\|v_i - u_i^{(n)}\big\| < 1/2^n$ for all $i \leq n$.

For $n = 0$ there is nothing to prove. What happens for $n = 1$ already shows the basic pattern of argument. By the first step of this proof, I can find a Dirac function f and a vector u_1 in a finite sum of eigenspaces $V_{f,\omega}$ such that $\|v - u\| < 1$. Let V_1 be the subspace of V generated by this sum of eigenspaces. Then by the third step, the space V_1 is the direct sum of a finite number of irreducible G-stable subspaces, none of which occur in V_1. That finishes this case. For $n > 1$, use the same basic argument, but applying the multi-vector version of the second step.

Part III. Smooth vectors

5. Smooth vectors

Suppose now G to be a Lie group, $\mathfrak{g}$ its real Lie algebra.

Impose on Dirac functions the additional requirement that they be smooth.

Let (π, V) be a unitary representation of G. A vector v in V is said to be differentable if

$$\pi(X)v = \lim_{t\to 0} \left(\frac{\pi(\exp tX)v - v}{t} \right)$$

exists for all X in the Lie algebra $\mathfrak{g}$. The vector v is said to be in $V^{(1)}$ if the function $g \mapsto \pi(g)v$ lies in $C^1(G, V)$. We can continue on, and define the subspace $V^{(m)}$ to be the set of all v such that $\pi(g)v$ lies in $C^m(G, V)$. The image in $C^m(G, V)$ consists of all functions F such that $F(gx) = \pi(g)F(x)$.

Given two vectors u, v in V, the corresponding **matrix coefficient** is the function

$$\Phi_{u,v}(g) = \pi(g)u \bullet v$$

5.1. Proposition. *For u in $V^{(m)}$, v in V, the matrix coefficient $\Phi_{u,v}$ lies in $C^m(G, \mathbb{C})$, and*

$$R_X \Phi_{u,v} = \Phi_{\pi(X)u,v} \, .$$

5.2. Corollary. *For X, Y in $\mathfrak{g}$ and u in V^∞ we have*

$$\pi(X)\pi(Y)u - \pi(Y)\pi(X)u = \pi\big([X, Y]\big)u \, .$$

Proof. The Hahn-Banach theorem reduces this to the case of the action of $\mathfrak{g}$ on $C^\infty(G, \mathbb{C})$.

5.3. Corollary. *The operators $\pi(X)$ for X in $\mathfrak{g}$, acting on V^∞, determine a representation of $\mathfrak{g}$.*

We thus get an action of the enveloping algebra $U(\mathfrak{g})$ as well.

If u is in V^∞ then for every v in V the matrix coefficient $\pi(g)u \bullet v$ is a smooth function on G. The converse is also true:

5.4. Lemma. *A vector u is in V^∞ if and only if the matrix coefficient $\pi(g)u \bullet v$ is smooth for all v in V.*

Proof. An easy consequence of the identification of a Hilbert space with its double dual.

Choose a basis (α_i) of $\mathfrak{g}$. According to the Poincaré-Birkhof-Witt theorem, lexicographically ordered monomials

$$\alpha^p = \alpha_1^{p_1} \ldots \alpha_n^{p_n}$$

in the α_i give a basis (α^p) of $U(\mathfrak{g})$, the enveloping algebra of $\mathfrak{g}$. If

$$|p| = \sum p_i \, .$$

then the α^p with $|p| \le n$ form a basis of $U_n(\mathfrak{g})$. Define the norm

$$\|v\|_{(n)} = \sum_{|p|\le n} \|\alpha^p v\|$$

on $V^{(m)}$. There is a related topology defined on $C^m(G, V)$ by the semi-norms

$$\|F\|_{\Omega,(n)} = \sup_{g\in\Omega} \sum_{|m|\le n} \|\alpha^m F(g)\|,$$

for Ω a compact subset of G, making it a complete locally convex topological vector space. Since the image of $V^{(m)}$ in $C^m(G, V)$ consists of all F with $F(gx) = \pi(g)F(x)$, the image of $V^{(m)}$ is closed in this topology, so $V^{(m)}$ itself is complete, hence a Banach space.

A different choice of basis of $U(\mathfrak{g})$ will give a different norm, but all of these are equivalent. The space V^∞ is the intersection of all the $V^{(m)}$. and is a Fréchet space.

If f lies in $C_c^\infty(G)$ and v in V, then

$$\pi(f)v = \int_G f(g)\pi(g)\,dg$$

lies in V^∞. More precisely

$$\pi(X)\pi(f)v = \pi(L_X f)v$$

for every X in $U(\mathfrak{g})$. The subspace $V^{\mathcal{G}}$ spanned by all the $\pi(f)v$ is called the **Gårding subspace** of V. A famous and powerful result of [Dixmier-Malliavin:1978] asserts that $V^{\mathcal{G}} = V^\infty$, but that will not be used here.

5.5. Lemma. *The subspace $V^{\mathcal{G}}$ is dense in each $V^{(n)}$.*

Proof. Follows from the remark at the end of the previous section, since the space $V^{(n)}$ is a continuous representation of G.

So far, these definitions and results make sense and are valid, with only slight modification, for any continuous representation (π, V) of G on a quasi-complete topological vector space V. But unitary representations possess extra structure.

Suppose U to be a dense subspace of V, and T a symmetric linear operator defined on U. That means that $Tu \bullet v = u \bullet Tv$ for all u, v in V. The closure $\overline{T}$ of T is the linear operator defined on the completion of U with respect to the 'graph norm' $\|Tu\|^2 + \|v\|^2$ by the condition that it $u_n \to v$ and $Tu_n \to w$ then $\overline{T}v = w$. The **adjoint** T^* of T is the linear operator defined on the space of all v in V such that for some w in V we have

$$v \bullet Tu = w \bullet u$$

for all u in U, in which case $T^*v = w$. Formally, we can write

$$T^*v \bullet u = v \bullet Tu$$

whenever this makes sense. The operator T is **self-adjoint** when $T = T^*$, and (U, T) is **essentially self-adjoint** when $\overline{T} = T^*$. The importance of self-adjoint operators is that the spectral theorem applies to them. If an operator T is symmetric, then it has only real eigenvalues. If $\ker(T^* - cI) = \{0\}$ for a pair of distinct conjugate c then T is essentially self-adjoint. (For all this material, see Chapter VIII of [Reed-Simon:1970].)

5.6. Proposition. (Stone's Theorem) *Suppose (π, V) to be a unitary representation of G, X in $\mathfrak{g}$. The operator $(\pi(iX), V^{\mathcal{G}})$ is essentially self-adjoint, and the domain of its closure consists of all v such that*

$$\lim_{t\to 0} \frac{\pi(\exp(tX)v - v}{t}$$

exists in V.

Proof. I prove first the claim about self-adjointness. Let $T = \pi(X)$ on the domain $V^{\mathcal{G}}$. To prove essential self-adjointness of iT, according to a well known criterion it suffices to show that $\ker(T^* \pm I) = 0$. So suppose that φ lies in the domain of T^* with $(T^* - cI)\varphi = 0$ with $c = \pm 1$. This means that

$$\varphi \bullet (T - cI)v = 0$$

for every v in $V^{\mathcal{G}}$. For v in V^∞

$$\Phi(t) = \varphi \bullet \pi\big(\exp(tX)\big)v\,.$$

It is a smooth function of t and satisfies the ordinary differential equation

$$\Phi'(t) = c\,\Phi(t)\,.$$

which means that it is Ce^{ct}. But since π is unitary, $\Phi(t)$ must be bounded. Hence $\varphi = 0$.

For the remainder of the proof, see [Reed-Simon:1970], Theorems VIII.7-8.

A slightly different argument will prove:

5.7. Theorem. *Suppose (π, V) to be a unitary representation of G and Δ a symmetric elliptic element of $U(\mathfrak{g})$. The operator $\big(\pi(\Delta), V^{\mathcal{G}}\big)$ is essentially self-adjoint.*

The hypothesis means (a) the element of $U(\mathfrak{g})$ is invariant under the anti-automorphism that extends $X \mapsto -X$ on $\mathfrak{g}$ and (b) the image in the symmetric algebra has no non-trivial zeroes. This implies that it is elliptic as a differential operator on G. This result is from [Nelson-Stinespring:1959], and there is a short sketch of a simple proof at the end of §7 of [Nelson:1959]. However, the most elementary argument seems to be that in §3 of [Cartier:1966] (a report on Nelson's work), which I follow.

Proof. The operator $D = \pi(\Delta)$ is symmetric on $V^{\mathcal{G}}$. Also, $D\pi(f) = \pi(\Delta f)$ for any f in C_c^∞. To prove D essentially self-adjoint, it suffices to prove that the kernel of $D^* - cI$ has no kernel for non-real values of c. Thus, suppose

$$(D^* - cI)\varphi = 0$$

for some φ in the domain of D^*. This means that for every v in V, f in $C_c^\infty(G)$

$$\begin{aligned} 0 &= \varphi \bullet \big((D - cI)\pi(f)v\big) \\ &= \varphi \bullet \pi(L_{D-cI}f)v \\ &= \varphi \bullet \int_G (L_{D-cI}f)(g)\pi(g)v\,dg \\ &= \int_G (L_{D-cI}f)(g)\big(\varphi \bullet \pi(g)v\big)\,dg \end{aligned}$$

which means that when considered as a distribution the continuous function $\Phi(g) = \varphi \bullet \pi(g)v$ satisfies $(\Delta - cI)\Phi = 0$. But this is an elliptic differential equation, so by the regularity theorem for elliptic equations it must be smooth. Since it holds for all v, Lemma 5.4 implies that φ lies in V^∞. But since Δ is symmetric in $U(\mathfrak{g})$ the operator $\pi(\Delta)$ is symmetric on V^∞, either c is real or $\varphi = 0$.

6. Laplacians

Continue to assume (α_i) to be a basis of $\mathfrak{g}$. One important example of an elliptic Δ in $U(\mathfrak{g})$ is the sum

$$\Delta = -\sum \alpha_k^2\,,$$

which is the **Laplacian** corresponding to the basis (α_i). If G is semi-simple with $\mathfrak{k}$ the Lie algebra of a maximal compact subgroup K of G, let Ω be the Casimir element of $\mathfrak{g}$, $\Omega_{\mathfrak{k}}$ the Casimir of $\mathfrak{k}$. The sum $\Omega + 2\Omega_{\mathfrak{k}}$ is the Laplacian of a suitable basis.

Assume (π, V) to be a unitary representation of G. Let $D = \pi(\Delta)$. According to Theorem 5.7, it is essentially self-adjoint with respect to the domain $V^{\mathcal{G}}$. The domain of its closure, which I'll also call D, is the completion of $V^{\mathcal{G}}$ with respect to the 'graph norm' $\|v\|^2 + \|Dv\|^2$. All operators $\pi(\Delta^n)$ with domain $V^{\mathcal{G}}$ are symmetric and elliptic, hence essentially self-adjoint, and the closure of $\pi(\Delta^n)$ is the same as D^n, which is described usefully here:

6.1. Theorem. *Suppose (π, V) to be a unitary representation of G, Δ a Laplacian of $\mathfrak{g}$. The domain of $D^{\lceil n/2 \rceil}$ is contained in the space $V^{(n)}$.*

Here $\lceil r \rceil$ is the smallest integer $\geq r$. Thus $\lceil n/2 \rceil = k$ if $n = 2k$ and $k+1$ if $n = 2k+1$.

I'll take up the proof in a moment. As an immediate consequence:

6.2. Corollary. *The space V^∞ of smooth vectors is the same as the intersection of all the domains $\mathrm{Dom}(D^n)$.*

The corollary is from [Nelson:1959], but the Theorem itself is a slightly weaker version of a refinement apparently due to [Goodman:1970]. It is difficult for me to separate clearly the contributions of each, because Goodman's papers rely strongly on Nelson's, but proves things not explicitly stated by Nelson. Nelson's paper is difficult to follow, partly because the result we want is intricately interleaved with other difficult results, and partly because Nelson's notation is bizarre. (For the *cognoscenti*, I point out that his notation is not a context-free language. I suspect that this is the root of my trouble with it.) The only place that I know of where Nelson's argument has been disentangled is Chapter 1 of [Robinson:1991], but in places it, too, is a bit awkward. In spite of this complaint, I follow Robinson closely.

Let $A_k = \pi(\alpha_k)$. The proof reduces to:

6.3. Lemma. *For every n there exist constants a, $b > 0$ such that*

$$\|A_{i_1} \ldots A_{i_n} v\| \leq a \|D^{\lceil n/2 \rceil} v\| + b\|v\|$$

for all v in $V^{\mathcal{G}}$.

I'll start proving this in a moment. Why does this imply the theorem? Let $m = \lceil n/2 \rceil$, and suppose v in $\mathrm{Dom}(D^m)$. Since D^m is self-adjoint, we can find a sequence of v_n in $V^{\mathcal{G}}$ with $v_n \to v$ and $D^m v_n \to D^m v$. The Lemma implies that $\|A_{i_1} \ldots A_{i_n} v_n\|$ remains bounded for all i, so that the v_n form a Cauchy sequence in $V^{(n)}$, which is a Banach space, and hence they converge in $V^{(n)}$.

The prototype of the Theorem is the example of $\mathrm{L}^2(\mathbb{R})$, in which case the domains of the powers of the Euclidean Laplacian Δ are the Sobolev spaces, and the Theorem is a classic result not difficult to prove by use of the Fourier transform. I'll repeat the reasoning involved to prove the following result, which we'll need in the course of the proof later on:

6.4. Lemma. *Suppose Δ to be a Laplacian of $\mathfrak{g}$, (π, V) a unitary representation of G, D the self-adjoint extension of $(\pi(\Delta), V^{\mathcal{G}})$. If $0 \leq m \leq n$, the domain of D^n is contained in the domain of D^m.*

Proof. The cases $m = 0$, n are trivial. The projection form of the spectral theorem allows us to reduce the proof to this elementary fact:

CLAIM. *Assume $0 < m < n$. For every $a > 0$ there exists $b > 0$ such that $\lambda^m \leq a\lambda^n + b$ for all $\lambda \geq 0$.*

Proof of the claim. For $a, b > 0$ consider the function

$$f(\lambda) = a\lambda^n - \lambda^m \, .$$

We have $f(0) = b$, $f(\lambda) \sim a\lambda^n$ as $\lambda \to 0$, and

$$f'(\lambda) = an\lambda^{n-1} - m\lambda^{m-1} \, .$$

Therefore f has a single positive minimum at

$$\lambda_0 = \left(\frac{m}{na} \right)^{\frac{1}{n-m}} \, .$$

Then $f(\lambda) + b$ will be > 0 everywhere as long as $b > -f(\lambda_0)$.

Proof. The proof is in several steps. From now on, if A and B are operators taking $V^{\mathcal{G}}$ to itself, I'll write $C \preceq A$ if there exist $a, b \geq 0$ such that

$$\|Cv\| \leq a\|Av\| + b\|v\|$$

for all v in $V^{\mathcal{G}}$. In this terminology, the previous Lemma asserts that $D^m \preceq D^n$ for $0 \leq m \leq n$. What we want to prove is that $A_{i_1} \ldots A_{i_n} \preceq D^{\lceil n/2 \rceil}$. A trivial observation, but one that motivates much of the calculation to come, is that from $C \preceq D^k$ we can deduce, by applying Lemma 6.4, that $CD^\ell \preceq D^{k+\ell}$.

The cases $n = 1$ and $n = 2$ are special, and for them I follow Robinson. After that, it goes by induction on n and careful calculation of commutators.

Step 1. CLAIM. *For v in $V^{\mathcal{G}}$*

$$\|A_k v\| \leq \varepsilon\|Dv\| + (1/2\varepsilon)\|v\| \,.$$

Proof. We have

$$\begin{aligned}
\|A_k v\|^2 &= -v \bullet A_k^2 v \\
&\leq v \bullet Dv \\
&\leq \|\varepsilon\sqrt{2}Dv + (1/\varepsilon\sqrt{2})v\|^2/2 \\
\|A_k v\| &= \|\varepsilon\sqrt{2}Dv + (1/\varepsilon\sqrt{2})v\|/\sqrt{2} \\
&\leq \varepsilon\|Dv\| + (1/2\varepsilon)\|v\| \,. \blacksquare
\end{aligned}$$

Step 2. The next step is to show that all of $V^{(2)}$ is contained in the domain of D, and more precisely that the norm $\|v\|_{(2)}$ is bounded by $\|Dv\| + \|v\|$.

Define the structure constants of $\mathfrak{g}$:

$$[\alpha_i, \alpha_j] = \sum_k c_{i,j}^k \alpha_k \,.$$

Set

$$c = \sup_{1 \leq i,j,k \leq n} \left|c_{i,j}^k\right| \,.$$

CLAIM. *For v in $V^{\mathcal{G}}$ and all $\varepsilon > 0$*

$$\|A_i A_j v\| \leq (1+\varepsilon)\|Dv\| + (2n^2 c/\varepsilon)\|v\| \,.$$

Proof. This is Lemma 6.1 in [Nelson:1959]. I follow Robinson's calculation, which is somewhat more straightforward than Nelson's.

For v in $V^{\mathcal{G}}$

$$\begin{aligned}
\|A_i A_j v\|^2 &\leq \sum_k \|A_k A_j v\|^2 \\
&= \sum_k A_k A_j v \bullet A_k A_j v \\
&= \sum_k \Big(A_k A_j v \bullet A_j A_k v - A_k A_j v \bullet [A_j, A_k] v \Big) \\
&= \sum_k \Big(A_k A_j v \bullet A_j A_k v - A_k A_j v \bullet [A_j, A_k] v - A_j^2 v \bullet A_k^2 v \Big) - A_j^2 v \bullet Dv \\
&= -A_j^2 v \bullet Dv + \sum_k \Big(A_k A_j v \bullet A_j A_k v - A_j^2 v \bullet A_k^2 v \Big) - \sum_k \Big(A_k A_j v \bullet [A_j, A_k] v \Big) \,.
\end{aligned}$$

Now

$$\sum_k \Big(A_k A_j v \bullet A_j A_k v - A_j^2 v \bullet A_k^2 v = -\sum_k \Big(A_j A_k A_j v \bullet A_k v - A_k A_j A_j v \bullet A_k v$$
$$= -\sum_\ell c_{k,j}^\ell A_\ell A_j v, \bullet A_k v$$

and then deduce

$$\|A_i A_j v\|^2 \le -A_j^2 v \bullet Dv - \sum_{k,\ell} c_{k,j}^\ell \big(A_k A_j v \bullet A_\ell v + A_\ell A_j v \bullet A_k v \big) \, .$$

Let

$$C = \sup_{i,j} \|A_{i,j} v\| \, .$$

Then we get by the Cauchy-Schwarz inequality and the result of the first step:

$$C^2 \le C\|Dv\| + 2n^2 cC\big(\delta\|Dv\| + (1/2\delta)\|v\|\big)$$

for all $\delta > 0$. Set $\delta = \varepsilon/2n^2 c$, divide by C.

Step 3.

CLAIM. *In any ring*

$$T^n S = \sum_{k=0}^n \binom{n}{k} \operatorname{ad}_T^{n-k}(S) T^k \, .$$

Here $\operatorname{ad}_T(S) = [T, S] = TS - ST$. This is Lemma 1 of [Goodman:1969].

Proof. By induction on n. To start:

$$TS = (TS - ST) + ST = \operatorname{ad}_T(S) + ST \, ,$$

and then from the induction assumption

$$\begin{aligned} T^n S &= \sum_{k=0}^n \binom{n}{k} \operatorname{ad}_T^{n-k}(S) T^k \\ T^{n+1} S &= T^n \cdot TS \\ &= \sum_{k=0}^n \cdot \operatorname{ad}_T^{n-k}(TS) T^k \\ &= \sum_{k=0}^n \cdot \operatorname{ad}_T^{n-k}(\operatorname{ad}_T S + ST) T^k \\ &= \sum_{k=0}^n \cdot \binom{n+1}{k} \operatorname{ad}_T^{n+1-k}(S) T^k \, . \end{aligned}$$

Step 4.

CLAIM. *For all $k \ge 0$ there exist constants $c_{k,i}^p$ for all $p = (i_1, i_2, \ldots, i_{2k})$ such that*

$$\operatorname{ad}_D^k(A_i) = \sum c_{k,i}^p A^p \, ,$$

and constants $c^p_{k,i,j}$ *for all* $p = i_1 i_2 \ldots i_{2k+1}$ *such that*

$$\operatorname{ad}(D)^k A_i A_j = \sum c^p_{k,i,j} A^p \,.$$

This is just about obvious. The point is that the degree of $\operatorname{ad}^k_D(A_i)$ is at most $2k$, and that of $\operatorname{ad}^k_D(A_i A_j)$ is at most $2k+1$.

Step 5. Now I start on the main result, that

$$A_{i_1} \ldots A_{i_n} \preceq \begin{cases} D^k & n = 2k \\ D^{k+1} & n = 2k+1. \end{cases}$$

I have proved this when $n = 1$ and 2, and I now proceed by induction. In this step I'll handle the case $n = 2k+1$, and in the next $n = 2k+2$, assuming it to be true for $n \le 2k$.

By induction we have for suitable but varying a, b, etc. according to Lemma 6.4

$$\begin{aligned} \|A_{i_1} \ldots A_{i_{2k+1}} v\| &\le a\|D^k A_{i_{2k+1}} v\| + b\|A_{i_{2k+1}} v\| \\ &\le a\|D^k A_{i_{2k+1}} v\| + c\|Dv\| + d\|v\| \\ &\le a\|D^k A_{i_{2k+1}} v\| + e\|D^{k+1} v\| + f\|v\| \,. \end{aligned}$$

Setting $A = A_{i_{2k+1}}$, by Goodman's commutator formula the first term on the right can be written

$$D^k A v = AD^k + [D^k, A] = AD^k + \sum_{\ell=0}^{k-1} \binom{n}{\ell} \operatorname{ad}(D)^\ell A D^{k-\ell}$$

But now for the first term, applying the case $n = 1$:

$$AD^k \preceq D^{k+1}$$

and for the rest we apply induction and the formula for $\operatorname{ad}^k_D(A_i)$ to conclude.

Step 6. Now set $n = 2k+2$. Again we write according to Lemma 6.4

$$\begin{aligned} \|A_{i_1} \ldots A_{i_{2k+1}} A_{i_{2k+2}} v\| &\le a\|D^k A_{i_{2k+1}} A_{i_{2k+2}} v\| + b\|A_{i_{2k+1}} A_{i_{2k+2}} v\| \\ &\le a\|D^k A_{i_{2k+1}} A_{i_{2k+2}} v\| + b\|Dv\| + c\|v\| \\ &\le a\|D^k A_{i_{2k+1}} A_{i_{2k+2}} v\| + b\|D^{k+1} v\| + c\|v\| \,. \end{aligned}$$

It remains to show that

$$D^k A_{i_{2k+1}} A_{i_{2k+2}} \preceq D^{k+1} \,.$$

But here as before I write

$$D^k A_{i_{2k+1}} A_{i_{2k+2}} = A_{i_{2k+1}} A_{i_{2k+2}} D^k + \sum_{\ell=0}^{k-1} \operatorname{ad}^{k-\ell}_C (A_{i_{2k+1}} A_{i_{2k+2}}) D^\ell$$

Again apply induction and the earler formula for $\operatorname{ad}^k_D(A_i A_j)$. The proof of the Theorem is concluded. ▯

[Goodman:1970] proves the more precise result that for all n the domain of the fractional power of $D^{n/2}$ coincides with $V^{(n)}$.

Part IV. Admissibility

7. Admissibility

Suppose now G to be the group of $\mathbb{R}$-rational points on a reductive group defined over $\mathbb{R}$, K a maximal compact subgroup of G.

7.1. Theorem. *If (π, V) is an irreducible unitary representation of G and (σ, U) an irreducible representation of K, then*

$$\dim \mathrm{Hom}_K(U, V) \leq \dim U \, .$$

Equivalently, the representation σ occurs in π no more than it does in $\mathrm{L}^2(K)$. The reason one might expect this is in fact closely related to $\mathrm{L}^2(K)$. It turns out that every irreducible unitary representation of G may be embedded into some $\mathrm{Ind}(\rho \,|\, P, G)$, with ρ an irreducible representation of the minimal parabolic subgroup P. Frobenius reciprocity for $P \cap K$ and K then implies the Theorem. This is only heuristic reasoning, since the proof of the embedding theorem depends on at least a weak form of Theorem 7.1, but the proof below does depend on Frobenius reciprocity for such induced representations.

In common terminology, the subgroup K is a **large** subgroup of G. We'll see applications of this fundamental result in the last section of this essay.

In this section I'll lay out the proof of Theorem 7.1 up to the point where I need to apply a famous theorem about matrix algebras due to Amitsur and Levitski. This result is trivial for $\mathrm{SL}_2(\mathbb{R})$, so that the proof for that group will be complete at that point. In the next section I'll conclude the proof by discussing the theorem of Amitsur-Levitski, or at least a variant, which has independent interest.

The original proof of this seems to be due to Harish-Chandra. I follow the succinct account in [Atiyah:1988]. Another proof of a very slightly weaker result can be found in a long thread of results in [Borel:1972], with the final statement in §5.

Proof. There are several steps. Suppose (σ, U) to be an irreducible (finite-dimensional) representation of K, (π, V) a unitary representation of G, and let V_σ be the σ-component of V, which may be identified with

$$U \otimes_{\mathbb{C}} \mathrm{Hom}_K(U, V) \, .$$

It is also the image of the projection operator $p(\xi_\sigma)$, where

$$\xi_\sigma(k) = (\dim \sigma) \, \mathrm{trace} \, \sigma(k^{-1}) \, .$$

It is a therefore a closed K-stable subspace of V.

Step 1. The first step is to define the **Hecke algebra** $\mathcal{H}_\sigma$, a subring of $C_c(G)$ that acts on V_σ.

The function $\xi_\sigma(k)$ acts on any continuous representation (π, V) of K as the projection onto the σ-component V_σ of V.

Let $\mathcal{H}_\sigma = \mathcal{H}_\sigma(G/\!/K)$ be the **Hecke algebra** of functions f in $C_c(G, \mathbb{C})$ such that $\xi_\sigma * f = f * \xi_\sigma = f$. For example, if σ is the trivial representation, $\mathcal{H}_\sigma$ is the space of all functions on G invariant on left and right by K. It is a ring with convolution as multiplication. It has no unit, but if f_i is a Dirac sequence in $C_c(G)$, $f * \xi_\sigma$ will approach ξ_σ, which would be a unit if it were in $\mathcal{H}_\sigma$. Since $\pi(\xi_\sigma)$ is the K-projection onto V_σ, we have $\pi(f)V \subseteq V_\sigma$ for any f in $\mathcal{H}_\sigma$, and this determines a homomorphism from $\mathcal{H}_\sigma$ into the ring

$$\mathrm{End}(V_\sigma) = \text{ bounded operators on } V_\sigma \, .$$

Let H_σ be its image, and let $\mathcal{C}(H_\sigma)$ be the subring of all T in $\mathrm{End}(V_\sigma)$ that commute with all of H_σ.

7.2. Lemma. (Schur's Lemma) *Any bounded operator on V_σ that commutes with all operators in H_σ is a scalar multiplication.*

Proof. If f lies in $\mathcal{H}_\sigma$, then so does $\overline{f}^\vee$, since $\mathrm{trace}\,\sigma(k^{-1}) = \overline{\mathrm{trace}\,\sigma(k)}$. Furthermore, $\pi(\overline{f}^\vee)$ is the adjoint of $\pi(f)$. Thus H_σ is stable under adjoints, and so is $\mathcal{C}(H)$. If T lies in $\mathcal{C}(H)$ then so do $(T+T^*)/2$ and $(T-T^*)/2i)$, and

$$T = \frac{T+T^*}{2} + i\,\frac{T-T^*}{2i}\,,$$

so we may assume T to be self-adjoint. But then the projection operators associated to T by the spectral theorem also commute with H_σ. In order to prove that T is scalar, it suffices to prove that any one of these projection operators is the identity. So I may now assume T to be a projection operator. Let H_T be the image of T, and let U be the smallest G-stable subspace containing H_T. Since H_T is stable under $\mathcal{H}_\sigma$, the intersection of U with V_σ is H_T, and since π is irreducible this must be all of V_σ. Thus T is the identity operator.

Step 2. In finite dimensions, it is easy to see from the Lemma in the previous step that the image of $\mathcal{H}_\sigma$ is the entire ring of bounded operators on V_σ. In infinite dimensions, we can deduce only a slightly weaker fact:

7.3. Lemma. *If (π, V) is an irreducible unitary representation of G, the image of $\mathcal{H}_\sigma$ is dense in the ring of bounded operators on V_σ.*

The topology referred to here is the strong operator topology.

Proof. The image H_σ of $\mathcal{H}$ in $\mathrm{End}(V_\sigma)$ is a ring of bounded operators stable under the adjoint map. Since π is irreducible, the Lemma in the previous step implies its commutant is just the scalar multiplications. The operators in H_σ embed in turn into $\mathcal{CC}(H_\sigma)$, the operators which commute with $\mathcal{C}(H_\sigma)$. I have already pointed out that the idempotent ξ_σ is the limit of functions in $\mathcal{H}_\sigma$. Therefore this Lemma follows from the following rather more general result. If R is a set of bounded operators on a Hilbert space, let $\mathcal{C}(R)$ be the ring of operators that commute with all operators in R.

7.4. Lemma. (Von Neumann's double commutant theorem) *Suppose R to be a subring of the ring of bounded operators on a Hilbert space V which (a) is stable under the adjoint map $T \mapsto T^*$ and (b) contains the identity I in its strong closure. Then R is dense in the ring $\mathcal{CC}(R)$ of all operators that commute with $\mathcal{C}(R)$.*

Proof. I follow the short account at the very beginning of Chapter 2 in [Topping:1968]. It must be shown that if T commutes with all operators α in $\mathcal{C}(R)$, then T is in the strong operator closure of R within $\mathrm{End}(V)$.

I recall exactly what the Lemma means. The strong topology is defined by semi-norms

$$\|T\|_\sigma = \sup_{v\in\sigma} \|Tv\|$$

for finite subsets σ of V. An operator τ of $\mathrm{End}(V)$ is in the strong closure of a subset X if and only if every one of its neighbourhoods in the strong topology intersects X. Hence τ is in the strong closure of X if and only if for every finite subset σ of V and $\varepsilon > 0$ there exists a point x in X with $\|x(v) - \tau(v)\| < \varepsilon$ for all v in σ.

I do first the simplest case in which there is a single x in σ. Suppose T lies in $\mathcal{CC}(R)$. Let $\overline{Rx}$ be the closure of Rx in V. Since I is in the strong closure of R, it contains x. Since $\overline{Rx}$ is invariant with respect to R and R is stable under adjoints, its orthogonal complement is also stable under R. Therefore the projection p_x onto $\overline{Rx}$ satisfies the relations

$$p_x\, r\, p_x = r\, p_x$$

for every r in R, and also

$$p_x\, r^*\, p_x = r^* p_x\,.$$

If we take the adjoint of this we get

$$p_x\, r\, p_x = p_x r\,,$$

concluding that $p_x\, r = r\, p_x$, so p_x lies in $\mathcal{C}(R)$. By assumption, T commutes with p_x and hence also takes $\overline{Rx}$ into itself. But since x lies in $\overline{Rx}$, Tx is in $\overline{Rx}$.

If $\sigma = \{x_1, \ldots, x_n\}$, consider V^n, the orthogonal direct sum of n copies of V. The algebra $\mathrm{End}(V)$ acts on it component-wise. Let R_n be the diagonal image of R. Its commutant in this is the matrix algebra $M_n\big(\mathcal{C}(R)\big)$, and the double commutant is the diagonal image of $\mathcal{CC}(R)$. Apply the case $n = 1$ to this, and get r in R such that $\|r(x_i) - T(x_i)\| < \varepsilon$ for all i. This concludes the proof of Lemma 7.4.

Step 3. In this step, I show that the claim of the theorem is true for finite-dimensional representations of G. It is here where the assumption that G be a reductive algebraic group plays a role.

7.5. Lemma. *If G is the group of $\mathbb{R}$-rational points on any affine algebraic group defined over $\mathbb{R}$, the map from $C_c(G)$ to $\prod \mathrm{End}_{\mathbb{C}}(E)$, with the product over all irreducible finite-dimensional algebraic representations E of G, is injective.*

Proof. Fix an embedding of G into some $\mathrm{GL}(V)$. The polynomials in the coordinatees of its matrices separates points of G, so will be dense in $C(X)$ for any compact subset X of G, by the Stone-Weierstrass Theorem. But these polynomials are matrix entries for other representations of G.

7.6. Lemma. *For every irreducible finite-dimensional algebraic representation E of G we have*

$$\dim \mathrm{Hom}_K(U, E) \le \dim U\,.$$

The Iwasawa factorization asserts that $G = PK$, with $P = MN$ equal to the a minimal $\mathbb{R}$-rational parabolic subgroup of G. If $|A|$ is the connected component of the center of M, then $M = (M \cap K) \times |A|$, and $G = NAK$. Every irreducible finite-dimensional representation of G may be embedded into some $\mathrm{Ind}(\rho \,|\, P, G)$ with ρ irreducible. This, together with Frobenius reciprocity for K and G then conclude the proof.

Step 4. The last step is by far the most interesting. To summarize where we are: (a) the Hecke ring $\mathcal{H}_\sigma$ is dense in $\mathrm{End}(V_\sigma)$; (b) it injects into the product of endomorphism rings of finite-dimensional representations of G; (c) if V is irreducible and of finite dimension then

$$\dim V_\sigma \le (\dim \sigma)^2\,.$$

We want to conclude that the same inequality holds if V is unitary. This leads us to theory of **polynomial identities** for matrix algebras. In this section, I'll deal only with the case $G = \mathrm{SL}_2(\mathbb{R})$, which is elementary. In the next section I'll show how things go in general.

So assume in the rest of this section that $G = \mathrm{SL}_2(\mathbb{R})$. Apply Lemma 7.5. In this case $K = \mathrm{SO}(2)$. Its irreducible representations are all one-dimensional, and each occurs with multiplicity at most one in every irreducible finite-dimensional representation of G. Therefore $\mathcal{H}_\sigma$ is a commutative ring, and is isomorphic to $\mathbb{C}$. Since it is dense in $\mathrm{End}(V_\sigma)$, the subspace V_σ has dimension at most one. This concludes the proof of Theorem 7.1 for $\mathrm{SL}_2(\mathbb{R})$.

8. Polynomial identities

The proof that

$$\dim V_\sigma \le (\dim \sigma)^2$$

for $\mathrm{SL}_2(\mathbb{R})$ reduced to the observation that in this case $\dim \sigma$ was at most one, which implied that H_σ, the image of $\mathcal{H}_\sigma$ in $\mathrm{End}(V_\sigma)$, is commutative. In the general situation, we want to conclude that H_σ is a matrix algebra of dimension at most $(\dim \sigma)^2$.

Commutativity of a ring R means that $r_1 r_2 - r_2 r_1 = 0$ for all r_1, r_2 in R. It turns out that the matrix ring $M_n = M_n(\mathbb{C})$ satisfies a polynomial identity that in the case $n = 1$ reduces to commutativity, and which M_{n+1} does not satisfy. For each n define the **standard polynomial** $[x_1, \ldots, x_n]$ in associative but non-commuting variables:

$$[x_1, x_2, \ldots, x_n] = \sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}(\sigma) x_{\sigma(1)}\, x_{\sigma(2)} \ldots x_{\sigma(n)}\,.$$

For example

$$\begin{aligned}[x_1, x_2] &= x_1x_2 - x_2x_1 \\ [x_1, x_2, x_3] &= x_1x_2x_3 + x_2x_3x_1 + x_3x_1x_2 - x_2x_1x_3 - x_1x_3x_2 - x_3x_2x_1\,.\end{aligned}$$

Evaluated on a $\mathbb{C}$-algebra R, this polynomial is anti-symmetric, hence factors through $\bigwedge^n R$. If R has dimension $< n$ it must therefore evaluate identically to 0. In particular, for M_n the polynomial $[x_1, \ldots, x_{n^2+1}]$ must vanish identically. Let $p(n)$ be the least p such that $[x_1, \ldots, x_n]$ vanishes on M_n.

This number $p(n)$ has been known explicitly since 1950:

8.1. Proposition. (Amitsur-Levitski) *For all n, $p(n) = 2n$.*

Since the original proof in [Amitsur-Levitski:1950] this has been an almost inexhaustible source of interesting mathematics (see for example [Formanek:2010], who surveys many of these, and particularly [Kostant:1958]). I'll not prove it here, but follow [Atiyah:1988] in getting by with a weaker statement taken from Lemma 6.3.2 of [Herstein:1968].

8.2. Lemma. *The number $p(n) \ge p(n-1) + 2$.*

Proof. Embed M_{n-1} into M_n:

$$A \longmapsto \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}.$$

Let $p = p(n-1) - 1$. There exist matrices $A_1, \ldots, A_{n-1}$ such that $[A_1, \ldots, A_{p-1}] \ne 0$. If $e_{i,j}$ is the elementary matrix with a sole non-zero entry 1 in position (i, j), then for $1 \le k < n$ we have

$$\begin{aligned}[A_1, \ldots, A_p, e_{k,n}, e_{n,n}] &= [A_1, \ldots, A_p]e_{k,n}e_{n,n} \\ &= [A_1, \ldots, A_p]e_{k,n}\end{aligned}$$

But since $[A_1, \ldots, A_{p-1}]$ doesn't vanish, we can find k such that $[A_1, \ldots, A_{p-1}]e_{k,n} \ne 0$. Therefore $p(n) \ge p(n-1) + 2$. ▯

Now to conclude the proof of Theorem 7.1. If $n = \dim \sigma$ and $\mathcal{H}_\sigma$ embeds into the product of rings M_m with $m \le n$, the ring $\mathcal{H}_\sigma$ satifies the standard identity of degree $p(n)$. Therefore it cannot contain a matrix algebra of M_m with $m > n$. But it is dense in $\mathrm{End}(V_\sigma)$, so $\dim V_\sigma \le \dim \sigma$. ▯

9. Representations of the Lie algebra

Let $Z(\mathfrak{g})$ be the center of the universal enveloping algebra $U(\mathfrak{g})$. One of the consequences of the Proposition is:

9.1. Corollary. *Suppose (π, V) to be an irreducible unitary representation of G. Then*

(a) if v is a K-finite vector in V, there exists f in $C_c(G)$ such that $\pi(f)v = v$;
(b) its K-finite vectors are smooth;
(c) there exists a homomorphism

$$\zeta\colon\ Z(\mathfrak{g}) \to \mathbb{C}$$

such that $\pi(X) = \zeta(X)\, I$ for all X in $Z(\mathfrak{g})$.

Proof. According to Lemma 7.3, the image of the Hecke ring $\mathcal{H}_\sigma$ is dense in the finite-dimensional algebra $\mathrm{End}(V_\sigma)$, hencde equal to the whole of $\mathrm{End}(V_\sigma)$. In particular, it contains the identity operator. This implies (b), which implies (a), which in turn implies (c). ▯

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