Weyl's character formula and Lie algebra homology

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Suppose G to be a connected reductive algebraic group defined over F, an algebraically closed field of characteristic 0, and further:

$$\begin{aligned} B &= \text{a Borel subgroup} \\ N &= \text{the unipotent radical of } B \\ T &= \text{a maximal torus in } B \\ \overline{B} &= \text{the opposite of } B\text{, with } \overline{B} \cap B = T \\ \overline{N} &= \text{unipotent radical of } \overline{B} \\ \Sigma &= \text{roots of } G, T \\ \Sigma^{\pm} &= \text{positive and negative roots} \\ \Delta &= \text{corresponding set of simple roots} \\ W &= \text{the Weyl group}\,. \end{aligned}$$

In addition, define certain lattices and subsets:

$$\begin{aligned} L^\vee &= X_*(T) \\ &= \text{group of algebraic homomorphisms } \mathbb{G}_m \to T \\ L &= \mathrm{Hom}(X_*(T), \mathbb{Z}),\ \text{the dual of } L^\vee \\ X^*(T) &= \text{group of algebraic homomorphisms } T \to \mathbb{G}_m\text{, the weights of } T\,. \end{aligned}$$

There is a canonical map from L to $X^*(T)$—to λ corresponds the multiplicative character e^λ defined by the formula

$$e^\lambda(\mu^\vee(x)) = x^{\langle \lambda, \mu^\vee \rangle}\,,$$

which makes sense because all algebraic homomorphisms $\mathbb{G}_m \to \mathbb{G}_m$ are of the form $x \mapsto x^n$. This map is an isomorphism, as one can verify by assigning coordinates to T, which then becomes a product of copies of $\mathbb{C}^\times$. The point of not simply defining L to be $X^*(T)$ is that I use additive notation for L but multiplicative for $X^*(T)$. Thus $e^{\lambda+\mu} = e^\lambda e^\mu$.

There is a map $\alpha \mapsto \alpha^\vee$ from Δ to $L^\vee$, such that the matrix $(\langle \alpha, \beta^\vee \rangle)$ is the Cartan matrix associated to a positive definite W-invariant metric.

For any subset S of a lattice M, let M_S be the subgroup spanned by S. If M is a lattice, let $M^\vee$ be its dual $\mathrm{Hom}(M, \mathbb{Z})$. Thus

$$\begin{aligned} L_\Delta &= \text{submodule of } L \text{ spanned by } \Delta\text{, canonically isomorphic to } \mathbb{Z}^\Delta \\ (L_\Delta)^\vee &= \text{lattice dual to } L_\Delta \subseteq L \\ L_{\Delta^\vee} &= \text{submodule of } L^\vee \text{ spanned by the simple coroots } \Delta^\vee \\ (L_{\Delta^\vee})^\vee &= \text{lattice dual to } L_{\Delta^\vee} \subseteq L^\vee\,. \end{aligned}$$

In addition:

$$\begin{aligned} L^+_\Delta &= \text{integral cone in } L \text{ spanned by } \Delta \text{ (an obtuse cone)} \\ L^{++} &= \text{the cone of dominant weights} \\ &= \left\{ \lambda \in L \,\middle|\, \langle \lambda, \alpha^\vee \rangle \ge 0 \text{ for all } \alpha \in \Delta \right\}. \end{aligned}$$

If G is simply connected, $L^\vee = L_{\Delta^\vee}$, and if G is an adjoint group $L = L_\Delta$. The Weyl group W acts on both L and $L^\vee$, as well as all the other lattices defined here.

Lie algebras will be expressed in $\mathfrak{Fraktur}$ so that, for example, $\mathfrak{g}$ is the Lie algebra of G. To a coroot $\lambda^\vee$ corresponds an element h_λ of $\mathfrak{t}$. For each positive root λ, fix $x_\lambda \neq 0$ in the root space $\mathfrak{g}_\lambda$, and then y_λ in $\mathfrak{g}_{-\lambda}$ so that $[x_\lambda, y_\lambda] = h_\lambda$. For $\lambda < 0$, set $x_\lambda = y_{-\lambda}$ and then $y_\lambda = x_{-\lambda}$. The triple x_λ, h_λ, and y_λ span a copy of $\mathfrak{sl}_2$.

The **anti-dominant weights** in L are those λ such that $\langle \lambda, \alpha^\vee \rangle \le 0$ for all α in Δ. Associated to each anti-dominant weight λ is a unique irreducible representation (π^λ, V^λ) of G containing a one-dimensional eigenspace V^λ_λ on which T acts by e^λ and which is annihilated by $\overline{\mathfrak{n}}$. Let $V = V^\lambda$ temporarily. Projection identifies V_λ with $V/\mathfrak{n}V$.

The set of all characters μ of T with eigenspace $V_\mu \neq 0$ make up the **weights** of π. The weight λ is called the **lowest weight** of V since the weights of V are contained in the integral affine cone $\lambda + L^+_\Delta$. The set of weights is W-stable, and is contained in the convex hull of the W-orbit of λ. The intersection of the weight lattice with this convex hull is in fact, as I shall recall in Proposition 2.1, the intersection of all the W-transforms of $\lambda + L^+_\Delta$. Hermann Weyl's character formula identifies $\pi = \pi^\lambda$ as an element in the Grothendieck group of algebraic characters of T, which may be identified with the group algebra of L. In this essay I shall present a proof of this formula that originated with [Kostant:1961], using simplifications found in [Casselman-Osborne:1975] and [Vogan:1981]. The proof depends on Kostant's formula for the $\mathfrak{n}$-homology of representations of G. It is not the simplest or most straightforward proof of the character formula, but it has several virtues. Among other things, it serves as an introduction to things that are important elsewhere in representation theory, some of which are mentioned in this note. One application is to the characters of Harish-Chandra modules, which can be found in [Hecht-Schmid:1983].

The restriction of π to T is a sum of characters with multiplicities, but Weyl's formula does not give this expression directly. It is, rather, the quotient of two such expressions, one of which turns out to divide the other, and it is not easy to get from it a more explicit expression. To get a more explicit expression, one needs to know all of the characters of T that occur as well as their multiplicities. This can be produced by a recursive algorithm due to Freudenthal, and a more efficient version of this is found in [Moody-Patera:1982]. I'll explain these formulas as well.

What I say here is slightly different from what is commonly found in the literature, particularly in so far as I work with homology rather than cohomology of Lie algebras. I do this because in the theory of Harish-Chandra modules homology is the more natural choice. I include a brief introduction to Lie algebra homology in an appendix.

Contents

1. Weyl's formula

Let (ϖ_α) be the basis of $(L_{\Delta^\vee})^\vee$ dual to $\Delta^\vee$, and let

$$\rho = \sum_{\alpha \in \Delta} \varpi_\alpha \, .$$

It satisfies the equations $\langle \rho, \alpha^\vee \rangle = 1$ for all α in Δ, and is uniquely specified in $(L_{\Delta^\vee})^\vee$ by this condition. Similarly, let $(\widehat{\varpi}_\alpha)$ be the basis of $(L_\Delta)^\vee$ dual to Δ.

1.1. Lemma. *The element ρ is also half the sum of all positive roots.*

This at least makes sense, since $(L_{\Delta^\vee})^\vee$ contains Δ.

Proof. Let r be for the moment half the sum of all positive roots. Since the elementary reflection s_α $(\alpha \in \Delta)$ permutes the complement of α in the set Σ^+ of all positive roots and takes α to $-\alpha$, we have $s_\alpha r = r - \alpha$. Since

$$s_\alpha r = r - \langle r, \alpha^\vee \rangle \alpha$$

one deduces that $\langle r, \alpha^\vee \rangle = 1$ for all α in Δ. ∎

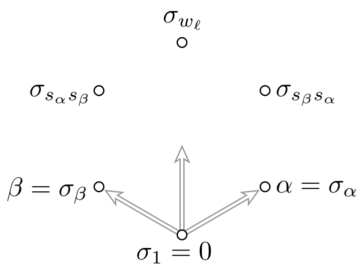
Let

$$\delta = e^\rho$$

and for each w in W let

$$\begin{aligned} \sigma_w &= \rho - w\rho \\ \delta_w &= e^{\sigma_w} \\ &= \delta \cdot w\delta^{-1} \\ &= e^{\rho - w\rho} \, . \end{aligned}$$

As we shall see in a moment, each σ_w lies in L even though ρ may not. Thus $\sigma_1 = 0$ and if w_ℓ is the longest element of W then $\sigma_{w_\ell} = 2\rho$, the sum of all positive roots. Here is what things look like for A_2:



More generally:

1.2. Lemma. *We have*

$$\sigma_w = \sum_{\substack{\lambda > 0 \\ w^{-1}\lambda < 0}} \lambda \, .$$

One consequence of this is that σ_w is in L_Δ.

Proof. We have

$$
\begin{aligned}
2\,(\rho - w\rho) &= \sum_{\lambda>0} (\lambda - w\lambda) \\
&= \sum_{\lambda>0} \lambda - \sum_{w^{-1}\lambda>0} \lambda \\
&= \sum_{\substack{\lambda>0 \\ w^{-1}\lambda>0}} \lambda + \sum_{\substack{\lambda>0 \\ w^{-1}\lambda<0}} \lambda - \sum_{\substack{\lambda>0 \\ w^{-1}\lambda>0}} \lambda - \sum_{\substack{\lambda<0 \\ w^{-1}\lambda>0}} \lambda \\
&= 2 \sum_{\substack{\lambda>0 \\ w^{-1}\lambda<0}} \lambda \,. \qquad \square
\end{aligned}
$$

There are several versions of Weyl's formula. Here is the one I shall prove:

1.3. Theorem. (Weyl's character formula) *For λ an anti-dominant weight in L*

$$
\pi^\lambda = \frac{\sum_w \operatorname{sgn}(w)\, e^{w\lambda}\, \delta_w}{\sum_w \operatorname{sgn}(w)\, \delta_w} \,.
$$

Here $\operatorname{sgn}(w)$ is the sign of the determinant of W acting on L. It is also $(-1)^{\ell(w)}$ (with $\ell(w)$ equal to the length of w as a minimal product of elementary reflections). This is to be interpreted as an identity in the group algebra of L, which is the Grothendieck group of T. The proof will take much of the rest of this paper. To make you feel at home, I'll give a couple of examples.

Example. Let $G = \mathrm{SL}_2(F)$, with T the subgroup of diagonal matrices, B of upper triangular ones. If I define the character

$$
x\colon \begin{bmatrix} a & 0 \\ 0 & 1/a \end{bmatrix} \longmapsto a \,,
$$

the anti-dominant weights are the x^{-n} for some $n \ge 0$. The corresponding representation π_n is that on homogeneous polynomials of degree n with coefficients in F. It has dimension $n+1$. Then as an element in the Grothendieck group of T we have

$$
\begin{aligned}
\pi_n &= x^{-n} + x^{-(n-2)} + \cdots + x^{n-2} + x^n \\
&= x^{-n}(1 + x^2 + \cdots + x^{2n}) \\
&= x^{-n} \cdot \frac{1 - x^{2n+2}}{1 - x^2} \\
&= \frac{x^{-n} - x^{n+2}}{1 - x^2} \,.
\end{aligned}
$$

It is curious that the form of Weyl's formula in the Theorem is not manifestly invariant under the Weyl group. But it can be so expressed, as we shall see in a moment. For this example, the invariant form of Weyl's formula is

$$
\frac{x^{-(n+1)} - x^{n+1}}{x^{-1} - x} \,,
$$

which is the ratio of two anti-invariant expressions.

Example. Let $G = \mathrm{SL}_3(F)$, and let B and T upper triangular and diagonal matrices in G. Define

$$
\begin{aligned}
x\colon\ a &\longmapsto a_1/a_2 \\
y\colon\ a &\longmapsto a_2/a_3
\end{aligned}
$$

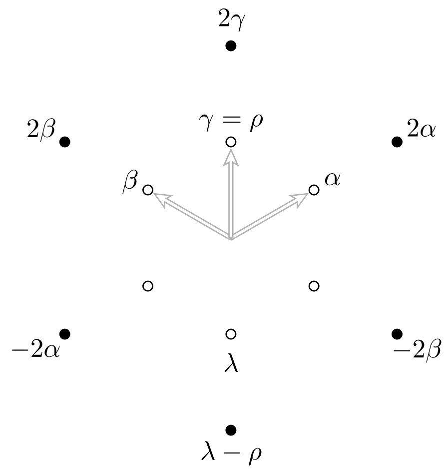
with α, β in Δ defined by

$$\begin{aligned} x &= e^{\alpha} \\ y &= e^{\beta}\,. \end{aligned}$$

Here

$$a = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}.$$

Then $\Delta = \{\alpha, \beta\}$, $\rho = \alpha + \beta$, and $\delta = xy$. Set $\lambda = -\rho$. This figure indicates what Weyl's formula gives us:



$$\begin{aligned} \pi^{-\rho} &= \frac{x^{-1}y^{-1} - x^{-1}y - xy^{-1} + x^3y + xy^3 - x^3y^3}{1 - y - x + x^2y + xy^2 - x^2y^2} \\ &= \frac{x^{-1}y^{-1}(1-x^2)(1-y^2)(1-x^2y^2)}{(1-x)(1-y)(1-xy)} \\ &= x^{-1}y^{-1}(1+x)(1+y)(1+xy) \\ &= x^{-1}y^{-1} + x^{-1} + y^{-1} + 2\cdot 1 + x + y + xy\,. \end{aligned}$$

As I have already remarked, it is not all that simple to go from Weyl's formula to an explicit expression as a sum of weights with multiplicities. We shall see later how this example generalizes nicely to all root systems.

The proof of Weyl's formula will be in several steps. I sketch here an outline of the argument, which will be filled in later on.

Step 1. Let V be the space of the representation π^{λ}. The homology groups $H_k(\mathfrak{n}, V)$ are the homology groups of the Koszul complex

$$\cdots \to \textstyle\bigwedge^{k+1}\mathfrak{n} \otimes V \to \bigwedge^{k}\mathfrak{n} \otimes V \to \bigwedge^{k-1}\mathfrak{n} \otimes V \to \cdots \to \mathfrak{n} \otimes V \to V\,,$$

whose definition I'll recall in the appendix. Since the Euler-Poincaré characteristic of a complex in the Grothendieck group of T-modules is the same as that of its homology, this says that

$$V \otimes \left(\sum_0^n (-1)^k \textstyle\bigwedge^k \mathfrak{n}\right) = \sum_0^n (-1)^k H_k(\mathfrak{n}, V)\,,$$

where $n = \dim\,\mathfrak{n}$. Dividing:

$$V = \frac{\sum_0^n (-1)^k H_k(\mathfrak{n}, V)}{\sum_0^n (-1)^k \bigwedge^k \mathfrak{n}}\,.$$

Step 2. In the Grothendieck group, $\bigwedge^k \mathfrak{n}$ is the sum of all products of k distinct positive roots. The denominator here is therefore

$$\prod_{\gamma>0} \left(1 - e^{\gamma}\right) .$$

To deduce Weyl's formula from the formula above, it therefore remains to prove the following two results:

1.4. Theorem. (Weyl's denominator formula) *In the Grothendieck group of T*

$$\prod_{\gamma>0} (1 - e^{\gamma}) = \sum_w \operatorname{sgn}(w)\, \delta_w .$$

This is a classic result due to Hermann Weyl, but in proving it (a bit later) I'll follow [Carter:1972], in which it is Theorem 10.1.8. It will fall out easily in the course of discussion of other matters.

The following is implicit in [Bott:1957] (in the proof of the Theorem of §15), but proved more directly in [Kostant:1961]. Incidentally, Bott mentioned the connection with Weyl's formula, but did not pursue his observation.

1.5. Theorem. *Suppose V to be the vector space of the representation π^λ with lowest weight λ. Then in the Grothendieck group of T*

$$H_k(\mathfrak{n}, V) = \bigoplus\nolimits_{\ell(w)=k} e^{w\lambda} \cdot \delta_w .$$

For example, T acts on $H_0(\mathfrak{n}, V^\lambda)$ by the character e^λ itself, and if $n = \dim(\mathfrak{n})$ it acts on $H_n(\mathfrak{n}, V^\lambda) = \bigwedge^n \otimes (V^\lambda)^{\mathfrak{n}}$ as the character $e^{w_\ell \lambda} \delta_{w_\ell}$.

Remark. There are several forms of the Weyl character formula. If one replaces w in the sums by ww_ℓ, where w_ℓ is the longest element in W, it becomes the formula one derives from $\mathfrak{n}$-cohomology instead of homology, with λ now a highest weight (as I assume it will be for the remainder of this section):

$$\pi^\lambda = \frac{\sum_w \operatorname{sgn}(w)\, e^{w\lambda}\, \delta_w^{-1}}{\sum_w \operatorname{sgn}(w)\, \delta_w^{-1}} = \frac{\sum_w \operatorname{sgn}(w)\, e^{w\lambda} e^{w\rho-\rho}}{\sum_w \operatorname{sgn}(w)\, e^{w\rho-\rho}} .$$

We can multiply both numerator and denominator by e^ρ, and this formula becomes

(1.6)
$$\pi^\lambda = \frac{\sum_w \operatorname{sgn}(w)\, e^{w(\lambda+\rho)}}{\sum_w \operatorname{sgn}(w)\, e^{w\rho}} ,$$

with λ again a highest weight. In the earlier example for SL_3 this gives us

$$\begin{aligned} \pi^{-\rho} &= \frac{x^{-2}y^{-2} - x^{-2} - y^{-2} + x^2 + y^2 - x^2y^2}{x^{-1}y^{-1} - x^{-1} - y^{-1} + x + y - xy} \\ &= \frac{x^{-2}y^{-2}(1 - x^2)(1 - y^2)(1 - x^2y^2)}{x^{-1}y^{-1}(1 - x)(1 - y)(1 - xy)} \\ &\dots\ . \end{aligned}$$

This is an especially elegant formula, since both top and bottom are anti-invariant with respect to the sign character of W. In fact, the denominator is a generator of anti-invariants in the Grothendieck group of T. Another interesting feature of this form is that it makes sense for any weight λ, and is skew-W-invariant in the sense that $\pi^\mu = \mathrm{sgn}(w)\pi^\lambda$ if $\mu+\rho = w(\lambda+\rho)$. This is consistent with the interpretation of π^λ as an Euler-Poincaré characteristic that one can derive from the version of Borel-Bott-Weil found in [Demazure:1976] and the fixed point theorem of [Bott:1987].

Another form, useful in understanding Macdonald's formula for unramified spherical functions of a $\mathfrak{p}$-adic group, is derived by splitting the formula into a sum, replacing the denominator according to Weyl's denominator formula, and dividing top and bottom by δ_w:

(1.7)
$$\pi^\lambda = \sum_w \frac{e^{w\lambda}}{\prod_{\gamma>0}(1-e^{-w\gamma})}\,.$$

which is again manifestly W-invariant. This is valid because

$$\begin{aligned}\frac{\mathrm{sgn}(w)\,\delta_w}{\prod_{\gamma>0}(1-e^{-\gamma})} &= \frac{\mathrm{sgn}(w)\prod_{\gamma>0,w^{-1}\gamma<0}e^{-\gamma}}{\prod_{\gamma>0}(1-e^{-\gamma})} \\ &= \frac{\mathrm{sgn}(w)}{\prod_{\gamma>0,w^{-1}\gamma<0}e^{-\gamma}\cdot\prod_{\gamma>0}(1-e^{-\gamma})} \\ &= \frac{1}{\prod_{\gamma>0}(1-e^{-w\gamma})}\,.\end{aligned}$$

Remark. If one evaluates Weyl's formula for a singular element of T, both numerator and denominator vanish. If this element is 1, the character is the dimension of the representation. Applying a version of l'Hôpital's rule, one deduces that

$$\dim V^\lambda = \prod_{\gamma>0}\frac{\langle\lambda+\rho,\gamma^\vee\rangle}{\langle\rho,\gamma^\vee\rangle}\,.$$

2. Geometry of the weight lattice

In this section I'll prove some facts about the geometry of the weight and root lattices that will be needed in later ones. As a by-product I'll prove the Weyl denominator formula and give a simple application. I begin with a discussion of W-invariant inner products, which we'll need here as well as in a later section.

INVARIANT INNER PRODUCTS. It is well known that if a root system is irreducible, there exists on the vector space spanned by the roots a W-invariant inner product that is unique up to a positive scalar multiple. We'll need later an explicit formula for it. The basic principle is that since the elementary reflections in W are orthogonal transformations, we must have

$$\begin{aligned}\langle\beta,\alpha^\vee\rangle &= 2\cdot\frac{\alpha\bullet\beta}{\alpha\bullet\alpha} \\ \langle\alpha,\beta^\vee\rangle &= 2\cdot\frac{\alpha\bullet\beta}{\beta\bullet\beta}\end{aligned}$$

so

$$\begin{aligned}\alpha\bullet\beta &= \frac{1}{2}\cdot(\alpha\bullet\alpha)\langle\beta,\alpha^\vee\rangle \\ \beta\bullet\beta &= 2\cdot\frac{\alpha\bullet\beta}{\langle\alpha,\beta^\vee\rangle}\,,\end{aligned}$$

for all α, β in Δ. Since the root system is irreducible and finite, its Dynkin diagram is a connected tree. If we start by specifying $\|\alpha\|^2$ at one α, this then determines what all $\beta\bullet\gamma$ are.

We need to extend this to an inner product on the lattice of weights. First I extend it to one on the lattice dual to the lattice spanned by the coroots $\alpha^\vee$. Let (ϖ_α) be the basis of this dual to the basis of simple coroots. These can be expressed as fractional linear combinations of the roots themselves. We'll need later an explicit form for this. The condition on the ϖ_α is that

$$\langle \varpi_\alpha, \beta^\vee \rangle = \begin{cases} 1 & \text{if } \alpha = \beta \\ 0 & \text{otherwise.} \end{cases}$$

We next want to find coefficients c_α^β such that

$$\varpi_\alpha = \sum\nolimits_\beta c_\alpha^\beta \, \beta \, .$$

To find the c_α^β we impose the conditions

$$\Big\langle \sum\nolimits_\beta c_\alpha^\beta \, \beta, \gamma^\vee \Big\rangle = \sum\nolimits_\beta c_\alpha^\beta \cdot \langle \beta, \gamma^\vee \rangle = \delta_\gamma^\beta \, .$$

This becomes the matrix equation

$$\varpi \cdot C = I$$

with

$$\varpi = (c_\alpha^\beta), \quad C = (\langle \alpha, \beta^\vee \rangle) \, .$$

In other words, C is the Cartan matrix, and ϖ is its inverse. Row α of ϖ contains the coefficients of ϖ_α as a linear combination of the simple roots β. Thus we can figure out the norm on all linear combinations of the ϖ_α, and get the matrix $(\varpi_\alpha \bullet \varpi_\beta)$. Finally, I define the inner product on all weights by the requirements that (a) on the ϖ it is as we have just seen, and (b) its radical is the annihilator of the coroots. This allows us to calculate:

$$\lambda \bullet \mu = \sum\nolimits_{\alpha,\beta} \langle \lambda, \alpha^\vee \rangle \langle \mu, \beta^\vee \rangle (\varpi_\alpha \bullet \varpi_\beta) \, .$$

CONVEX HULLS. For the moment, let V be $L_\Delta \otimes \mathbb{R}$, V^+ the closed real cone spanned by the α in Δ, and V^{++} that spanned by the dominant weights. Thus $L_\Delta^+ = L_\Delta \cap V^+$.

2.1. Proposition. *Suppose v to be in V^{++}. The convex hull of the W-orbit of v is the intersection of all the W-transforms of $v - V^+$.*

The intersection of this convex hull with V^{++} is the same as the intersection of $v - V^+$ with V^{++}.

Proof. First I prove that $w(v)$ lies in this cone by induction on the length of w. The case $\ell(w) = 0$ is trivial. Then suppose $w = s_\alpha x$ with $\ell(w) = \ell(x) + 1$, and assume that the claim is true for x. Then

$$w(v) = x(v) - \langle x(v), \alpha^\vee \rangle \alpha$$

but by assumption on the length of w we have $x^{-1}\alpha > 0$, which means that $\langle x(v), \alpha^\vee \rangle = \langle v, x^{-1}\alpha^\vee \rangle \geq 0$.

Now I want to show that if v is in the intersection of all the $w(v - V^+)$ it lies in the convex hull of the $w(v)$. Very generally, suppose X to be a finite subset of $\mathbb{R}^n$ space whose convex hull has a non-empty interior. Let C be the convex hull of X. Then an affine plane $f = 0$ with $f \leq 0$ on X is a face of C if and only if the surface $f = 0$ contains a subset of X whose affine support has codimension 1. If v is regular, then the hyperplanes $\langle x, \widehat{\varpi}_\alpha \rangle = \langle v, \widehat{\varpi}_\alpha \rangle$ satisfy this condition, and their W-transforms are therefore the faces of the convex hull of $W(v)$. Otherwise, a continuity argument will work.

Now for the second claim. Suppose that u lies in V^{++} and also in $v - V^+$. Then the same argument as above will show that $w(u)$ lies in $v - V^+$ for all w in W. Hence u lies in the intersection of all the $w \cdot (v - V^+)$. ∎

The continuity argument could be replaced by a reference to the explicit description of the faces of the convex hull in [Satake:1960] (or [Casselman:1982]).

The structure of the weights in the convex hull of $W \cdot \rho$ is relatively simple, and it is important in many applications. Recall that (ϖ_α) is the basis of L_Δ dual to the coroots, and for $S \subset \Delta^+$ let

$$\varpi_S = \sum_{\alpha \in S} \varpi_\alpha \,.$$

2.2. Lemma. *If $\lambda \neq \rho$ is a dominant weight in the convex hull of $W \cdot \rho$, it is singular.*

That is to say, there exists α in Δ such that $\langle \lambda, \alpha^\vee \rangle = 0$.

Proof. Suppose λ to be dominant, so that $\langle \lambda, \alpha^\vee \rangle \geq 0$ for all simple roots α. If it is regular, then we have the strongr condition $\langle \lambda, \alpha^\vee \rangle \geq 1$ for all simple α. But then λ lies in the cone $\rho + \Lambda^{++}$, which contradicts the requirement that it lie in $\rho - L_\Delta^+$. $\square$

2.3. Proposition. *The weights of the $\bigwedge^\bullet \mathfrak{n}$ are in the convex hull of the weights $\rho - w\rho$.*

If μ is one of these weights, then $\mu - \rho$ lies in the interior of a Weyl chamber if and only if $\mu = \rho - w\rho$.

Proof. For (a) I use some observations from Chapter 10 of [Carter:1972]. The weights of $\bigwedge^k \mathfrak{n}$ are the sums of k distinct positive roots. For $S \subseteq \Sigma^+$ define

$$\lambda_S = \sum_{\lambda \in S} \lambda \,.$$

Set $\rho_S = \rho - \lambda_S$ (so, for example, $\rho = \rho_\emptyset$). Then

$$\rho_S = (1/2)\Big(\sum_{\lambda \notin S} \lambda - \sum_{\lambda \in S} \lambda\Big) = (1/2)\Big(\sum_{\lambda > 0} \pm\lambda\Big)$$

and since W permutes all the roots,

$$w(\rho_S) = \rho_T$$

for some subset $T \subseteq \Sigma^+$. For trivial reasons, ρ_S also lies in $\rho - L_\Delta^+$. So ρ_S is also contained in each of the W-transforms of $\rho - L_\Delta^+$. The claim follows since the convex hull of the $w\rho$ is the intersection of all these W-transforms.

As for claim (b), if μ is one of these weights and $\mu - \rho$ lies in the interior of a Weyl chamber, then some $\nu = w(\mu - \rho)$ will lie in the interior of the positive chamber along with ρ. Apply Lemma 2.2. $\square$

More explicitly, some straightforward calculation will show that $w(\rho_S) = \rho_T$ where

$$T = \big\{\lambda > 0 \,\big|\, w^{-1}\lambda \in S \text{ or } w^{-1}\lambda \in -(\Sigma^+ - S)\big\} \,.$$

Here is another way to see what's going on. Let $\mathbb{P}(\Sigma)$ be the image of the roots in projective space. The expression

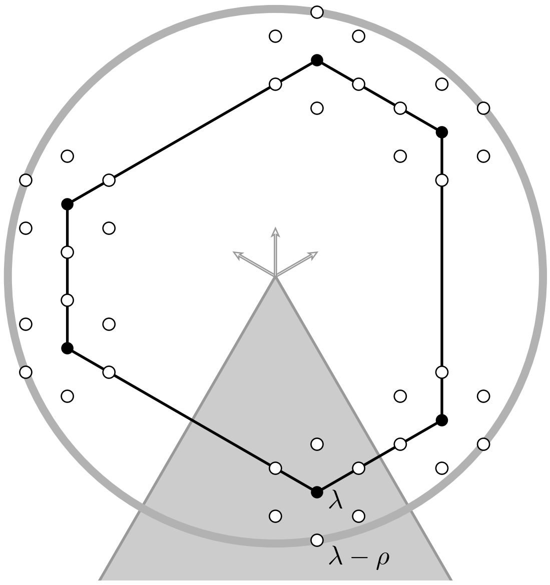
$$\rho_S = 1/2\Big(\sum\nolimits_{\lambda > 0} \pm \lambda$$

shows that the ρ_S correspond to sections of the projection from Σ to $\mathbb{P}(\Sigma)$.

There are some curious features of the action of W on the set of ρ_S, particularly the interaction with the sizes of the S. These arise in trying to understand the fine structure of Macdonald's formula for p-adic spherical functions. But there is no reason to say more about that here.

2.4. Corollary. *If $V = V^\lambda$, the weights $\mu = w(\lambda - \rho) + \rho$ are the only weights in $\bigwedge^\bullet \mathfrak{n} \otimes V$ with $\|\mu - \rho\| = \|\lambda - \rho\|$.*

Proof. The weights of V are all in the convex hull of λ. I'll leave the rest of the proof as an exercise, with the following figure as a hint:



In other words, it's an elementary fact about extremal points on convex polyhedra.

Proof of Weyl's denominator formula. Both sides of the formula are quasi-alternating. According to Proposition 2.3, all $\rho - \lambda_S$ except the $w\rho$ are fixed by some root reflection, hence cancel out in the expansion of the product.

For example, if $\mathfrak{g} = \mathfrak{sl}_3$ then this amounts to the equation

$$(1-x)(1-y)(1-xy) = 1 - (x + y + xy) + (xy + x^2y + xy^2) - x^2y^2 \,,$$

in which the two terms $\pm xy$ graciously cancel.

Exercise. *We have seen that all the weights $-\rho_S$ are in the convex hull of $W \cdot \rho)$ (which is the same as $W \cdot (-\rho)$). Use the character formula to show that*

$$\pi^{-\rho} = \sum_{S \subseteq \Sigma^+} \rho_S \,.$$

(Hint: Look at the example above with $G = \mathrm{SL}_3$.)

3. The Casimir element

In this section I'll recall how the the Casimir element of $Z(\mathfrak{g})$ acts on irreducible finite-dimensional representations of $\mathfrak{g}$.

DUALITY AND EUCLIDEAN NORMS. First, suppose for a moment that V is any real vector space with a Euclidean inner product $x \bullet y$. This gives rise to an associated isomorphism φ of V with its linear dual $\widehat{V}$, defined by the formula

(3.1) $$\langle \varphi(u), v \rangle = u \bullet v \,.$$

Since the inner product is symmetric, $\varphi^\vee = \varphi$.

There is an associated inner product on $\widehat{V}$ that can be characterized in any one of three ways.

3.2. Proposition. *Suppose given a Euclidean inner product on the vector space V with associated isomorphism*

$$\varphi\colon\ V \longrightarrow \widehat{V}\,.$$

Suppose (v_i) to be a basis of V, $(v_i^\bullet)$ its dual with respect to the inner product. The three following formulas for a Euclidean norm on $\widehat{V}$ are equivalent:

$$\begin{aligned}\|\widehat{v}\|^2 &= \|\varphi^{-1}(\widehat{v})\|^2 \\ &= \langle \widehat{v}, \varphi^{-1}(\widehat{v})\rangle \\ &= \sum \langle \widehat{v}, v_i\rangle\langle \widehat{v}, v_i^\bullet\rangle\,.\end{aligned}$$

This is a straightforward result, but useful to keep in mind. The second equation says that $\varphi^{-1}\colon \widehat{V} \to V$ is the isomorphism associated with the norm on $\widehat{V}$.

Proof. Take the first as definition. It suffices to prove the second and third for $\widehat{v} = \varphi_v$. The second holds since according to the definition (3.1)

$$\|\varphi(v)\|^2 = v \bullet v = \langle \varphi(v), v\rangle\,.$$

For the third, we may write

$$v = \sum (v \bullet v_i^\bullet)\, v_i$$

and then

$$\begin{aligned}\|\widehat{v}\|^2 &= \|v\|^2 \\ &= v \bullet \Big(\sum (v \bullet v_i^\bullet)\, v_i\Big) \\ &= \sum (v \bullet v_i)(v \bullet v_i^\bullet \\ &= \sum \langle \widehat{v}, v_i\rangle\langle \widehat{v}, v_i^\bullet\rangle\,. \qquad 0\end{aligned}$$

One consequence of this is that $\sum v_i \cdot v_i^\bullet$ is an element in the symmetric product $S^2(V)$ independent of the choice of basis.

THE KILLING FORM. For the moment, I assume $\mathfrak{g}$ to be semi-simple. The bilinear form

$$x \bullet y = \mathrm{trace}\,(\mathrm{ad}_x\, \mathrm{ad}_y)\,.$$

is non-degenerate as well as $\mathfrak{g}$-invariant. Because the sum of a positive and negative root is never a root, the decomposition

$$\mathfrak{g} = \mathfrak{t} \bigoplus\nolimits_{\lambda>0} (\mathfrak{g}_{-\lambda} \oplus \mathfrak{g}_\lambda)$$

is orthogonal. The Killing form is invariant also with respect to all automorphisms. Its restriction to $\mathfrak{t}$ is positive definite. The Weyl group acts orthogonally on both $\mathfrak{t}$ and its dual. This means that all reflections

$$s_\lambda\colon\ h \longmapsto h - \langle \lambda, h\rangle h_\lambda$$

are orthogonal or, equivalently, that for all h in $\mathfrak{t}$ and roots λ

$$\langle \lambda, h\rangle = 2\left(\frac{h \bullet h_\lambda}{h_\lambda \bullet h_\lambda}\right)\,.$$

But now according to Proposition 3.2 this implies that if φ is the map from $\mathfrak{t}$ to its dual associated to the Killing form, then

$$\varphi\colon\ \frac{2h_\lambda}{h_\lambda \bullet h_\lambda} \longmapsto \lambda\,.$$

Therefore by Proposition 3.2

$$\frac{2\langle \mu, h_\lambda\rangle}{h_\lambda \bullet h_\lambda} = \mu \bullet \lambda\,. \tag{3.3}$$

for any weight μ.

Another consequence of Proposition 3.2 is that for any weight λ and any choice of basis $\{h_i\}$ of $\mathfrak{t}$

$$\|\lambda\|^2 = \sum \langle \lambda, h_i\rangle\langle \lambda, h_i^\bullet\rangle\,. \tag{3.4}$$

The restriction of the Killing form to $\mathfrak{g}_{-\lambda} \oplus \mathfrak{g}_\lambda$ is non-degenerate. We'll need to know a bit more about it. Recall that I have chosen $x_\lambda \neq 0$ in $\mathfrak{g}_\lambda$. and y_λ in $\mathfrak{g}_{-\lambda}$ such that $[x_\lambda, y_\lambda] = h_\lambda$.

3.5. Lemma. *For any* λ

$$x_\lambda \bullet y_\lambda = \frac{h_\lambda \bullet h_\lambda}{2}\,. \tag{3.6}$$

Proof. Since the Killing form is invariant,

$$[x_\lambda, h_\lambda] \bullet y_\lambda + h_\lambda \bullet [x_\lambda, y_\lambda] = -2\, x_\lambda \bullet y_\lambda + h_\lambda \bullet h_\lambda = 0\,.$$ ∎

THE CASIMIR ELEMENT. By definition, the Casimir element $\mathfrak{C}$ of $U(\mathfrak{g})$ is the sum

$$C = \sum h_i h_i^\bullet + \sum_{\lambda\in\Sigma} x_\lambda x_\lambda^\bullet \tag{3.7}$$

with $x_\lambda^\bullet$ an element of $\mathfrak{g}_{-\lambda}$ such that $x_\lambda \bullet x_\lambda^\bullet = 1$. According to the previous Lemma we have in fact

$$x_\lambda^\bullet = \frac{2\,y_\lambda}{h_\lambda \bullet h_\lambda}\,. \tag{3.8}$$

Expressing

$$x_\lambda x_\lambda^\bullet = x_\lambda^\bullet x_\lambda + [x_\lambda, x_\lambda^\bullet]$$

for $\lambda < 0$, we get

$$C = \sum h_i h_i^\bullet + \sum_{\mu<0} [x_\mu, x_\mu^\bullet] + \sum_{\mu>0} (x_\mu x_\mu^\bullet + x_{-\mu}^\bullet x_{-\mu})\,,$$

which implies by (3.8) that

$$C = \sum h_i h_i^\bullet - \sum_{\mu>0} \frac{2\,t_\mu}{t_\mu \bullet t_\mu} + \text{something in } \mathfrak{n}\,U(\mathfrak{g})\,. \tag{3.9}$$

Since C is in the center of $U(\mathfrak{g})$, it acts as a scalar on any irreducible representation of $\mathfrak{g}$ of finite dimension. Suppose (ρ, V) to be a finite-dimensional representation with lowest weight λ. This means, as I have already mentioned, that the subspace of V annihilated by $\overline{n}$ is a one-dimensional eigenspace V_λ for $\mathfrak{t}$ with character λ. The embedding of V_λ into V induces an isomorphism with $V/\mathfrak{n}V$.

According to (3.4), the element $\sum h_i h_i^\bullet$ acts on V_λ^λ as multiplication by $\|\lambda\|^2$. According to (3.3) we can write

$$\begin{aligned}\sum_{\mu<0}[x_\mu, x_\mu^\bullet] &= -\sum_{\mu>0} \frac{2h_\mu}{h_\mu \bullet h_\mu} \\ &= -\sum_{\mu>0} \lambda \bullet \mu \\ &= -2\,\langle \lambda, \rho\rangle\,.\end{aligned}$$

Hence:

3.10. Lemma. *On the irreducible $\mathfrak{g}$-module V^λ with lowest weight λ, the Casimir $\mathfrak{C}$ acts as the scalar*

$$\|\lambda-\rho\|^2 - \|\rho\|^2\,.$$

4. Kostant's theorem

In this section I'll conclude the proof of Weyl's character formula by proving Kostant's Theorem (Theorem 1.5):

> *Suppose V to be the vector space of the representation π^λ with lowest weight λ. Then in the Grothendieck group of T*
>
> $$H_k(\mathfrak{n}, V) = \bigoplus\nolimits_{\ell(w)=k} e^{w\lambda}\cdot \delta_w\,.$$

The Lie algebra $\mathfrak{g}$ has a direct sum decomposition $\mathfrak{g} = \mathfrak{n}\oplus\mathfrak{t}\oplus\overline{\mathfrak{n}}$. Fix a basis Y_i of $U(\mathfrak{n})$ that are eigenvectors of $\mathrm{Ad}(T)$, T_j of $U(\mathfrak{t})$, and $\overline{Y}_k$ of $U(()\overline{\mathfrak{n}})$ also eigenvectors. By the theorem of Poincaré-Birkhoff-Witt, every X in $U(\mathfrak{g})$ may be expressed as a unique linear combination of products $Y_{i_m}T_{j_m}\overline{Y}_{k_m}$. If X lies in $Z(\mathfrak{g})$ then $\mathrm{Ad}(t)X = X$ for every t in T, which implies that $[\mathrm{Ad}(t)](Y_{i_m}) = [\mathrm{Ad}(t)]^{-1}(\overline{Y}_{i_m})$ for every m. As a consequence, if $\overline{Y}_{i_m}\neq 0$ then neither is Y_{k_m}. Hence:

4.1. Lemma. *If X lies in $Z(\mathfrak{g})$ there exists a unique $\lambda(X)$ in $U(\mathfrak{t})$ such that $X-\lambda(X)$ lies in $\mathfrak{n}\,U(\mathfrak{g})$.*

The map $X\mapsto\lambda(X)$ is called the **Harish-Chandra homomorphism**. I'll recall some of its properties in a moment.

Suppose for the moment that V is any $\mathfrak{g}$-module. Define an action of $Z(\mathfrak{g})$ on the Koszul complex $\bigwedge^\bullet\otimes V$, through its action on the second factor alone. Since $\mathfrak{n}$ is an ideal of $\mathfrak{b}$, there also exists a natural action of $\mathfrak{b}$ on the complex as well. As recalled in the appendix on homology, the induced action of $\mathfrak{n}$ on cohomology is trivial. Both these actions commute with the differentials of the Koszul resolution, hence induce actions of $Z(\mathfrak{g})$ and $U(\mathfrak{t})$ on $H_\bullet(\mathfrak{n}, V)$.

4.2. Theorem. *If V is any $U(\mathfrak{g})$-module, the action of $X\in Z(\mathfrak{g})$ on $H_\bullet(\mathfrak{n}, V)$ coincides with that of $\lambda(X)$.*

This is a result due to [Vogan:1981], which made a valuable simplification to the awkwardly formulated [Casselman-Osborne:1975]. I'll prove this now, explain a bit later what its consequences are for general $\mathfrak{g}$-modules V, and then finally show why it implies Kostant's theorem.

Proof. The assertion is immediate for $H_0(\mathfrak{n}, V)$. The proof will now proceed by induction on the $H_m(\mathfrak{n}, V)$. We can find a short exact sequence

$$0\to U\to W\to V\to 0$$

with W a projective module over $U(\mathfrak{n})$. Then $H_m(\mathfrak{n}, W) = 0$ for $m>0$ and the associated long exact sequence gives us

$$\begin{gathered} H_1(\mathfrak{n}, V)\hookrightarrow H_0(\mathfrak{n}, U) \\ H_m(\mathfrak{n}, V)\cong H_{m-1}(\mathfrak{n}, U)\end{gathered}$$

from which the Theorem follows. ∎

There are several ways to infer Kostant's theorem from this.

4.3. Lemma. *The character $w(\lambda-\rho)+\rho$ appears in $\bigwedge^k \mathfrak{n} \otimes V$ if and only if $\ell(w) = k$, and then with multiplicity one.*

Proof. Since $w(\lambda-\rho)+\rho = (\rho - w\rho) + \lambda$, this weight certainly occurs in $\bigwedge^k \mathfrak{n} \otimes V$. I leave the other direction as an exercise.(Take a look at the diagram just after the statement of Corollary 2.4). ∎

4.4. Proposition. *If (π, V) is the irreducible representation of $\mathfrak{g}$ with lowest weight λ, then $H = H_\bullet(\mathfrak{n}, V)$ is, as a module over $\mathfrak{t}$, the sum of eigenspaces H_μ, where μ ranges over a subset of the characters $w(\lambda-\rho)+\rho$.*

Proof. We know from Lemma 3.10 that the Casimir C acts on V as $\|\lambda-\rho\|^2 - \|\rho\|^2$, hence also on the complex $\bigwedge^\bullet \mathfrak{n} \otimes V$ as well as $H_\bullet(\mathfrak{n}, V)$ by the same scalar. But if μ is a weight of the action of T on $H_\bullet(\mathfrak{n}, V)$ then by Vogan's result we know also that C acts as $\lambda(C)$. But from (3.9) we know that

$$\lambda(C) = \sum h_\alpha h_\alpha^\bullet - \sum_{\mu>0} \frac{2\, t_\mu}{t_\mu \bullet t_\mu} = \|\mu\|^2 - 2(\mu \bullet \rho)\,.$$

Hence $\|\lambda-\rho\|^2 = \|\mu-\rho\|^2$, and we conclude by Corollary 2.4. ∎

Together, these imply Theorem 1.5 and conclude the proof of Weyl's character formula.

5. Vector bundles and Lie algebra cohomology

If (σ, V) is a finite-dimensional representation of B, the vector bundle $\mathcal{V}$ over G/B is $G \times^B V$, the quotient of $G \times V$ by the left B-action taking (g, v) to $(gb^{-1}, \sigma(b)v)$. If U is open in G/B, the space of (holomorphic) sections f over U of this bundle is may be identified with the space of (holomorphic) functions F on the inverse image of U in G with values in V such that $F(gb) = \sigma^{-1}(b)F(v)$, which is also the space of B-invariant functions from U to V. Formally, we have then a map from U to $U \times V$ that commutes with B, hence descends to the quotients:

$$f(uB) = \text{the image of } (u, F(u))\,.$$

The group G acts on the space of all sections over G/B, compatibly with its action on G/B.

6. Computing weight multiplicities

Let $\pi = \pi^\lambda$, $V = V^\lambda$ for some λ in L^{++}. I shall prove in this section a formula due to [Moody-Patera:1982] that will provide a reasonably efficient way to compute the dimensions of all the weight subspaces V_μ. It is an extension of a well known formula, due to Freudenthal, that is useful for computation by hand for cases of small rank, but not efficient enough for industrial production. The formula of Moody-Patera starts off from some version of Freudenthal's and this in turn depends on the case of $\mathfrak{sl}_2$, which I shall start with.

SL(2). Let

$$h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\,.$$

Here $[x, y] = h$, $[h, x] = 2x$, and $[\,t, y] = -2y$..

Suppose V to be a space on which $\mathfrak{sl}_2$ acts, with highest weight λ and highest weight vector v_0. For each $n > 0$, let $v_n = y^n \cdot v_0$. Thus

$$\begin{aligned} x \cdot v_0 &= 0 \\ h \cdot v_n &= (\lambda - 2n) v_n \\ y \cdot v_n &= v_{n+1} \,. \end{aligned}$$

Since all weights have multiplicity one in V, we must have $x \cdot v_n = c_n v_{n-1}$ for all $n \geq 0$ (taking c_{-1} to be 0). *What is c_n for $n > 0$?* This can be found by induction. We start with $c_0 = 0$. But then for $n > 0$

$$x \cdot v_n = x \cdot y \cdot v_{n-1} = [x, y] \cdot v_{n-1} + y \cdot x \cdot v_{n-1} = [x, y] \cdot v_{n-1} + c_{n-1} v_{n-1} = t \cdot v_{n-1} + + c_{n-1} v_{n-1} \,,$$

so that

$$c_n = (\lambda - 2(n-1)) + c_{n-1} \,.$$

This gives us

$$\begin{aligned} c_0 &= 0 \\ c_1 &= \lambda \\ c_2 &= (\lambda - 2) + c_1 \\ c_3 &= (\lambda - 4) + c_2 \\ &\dots \end{aligned}$$

leading to

$$c_n = \sum_{k=0}^{n-1} (\lambda - 2k) = \sum_{k=1}^{n} (\mu + 2k) \quad (\mu = \lambda - 2n)) \,.$$

This can be reformulated so as to be valid for any finite-dimensional representation of $\mathfrak{sl}_2$, irreducible or not. Let γ be the character of the diagonal matrices in $\mathfrak{sl}_2$ taking h to 2. We have

$$\pi(y)\pi(x) v_n = c_n v_n \,,$$

so that for each n

$$c_n = \operatorname{trace} \pi(x)\pi(y) \,|\, V_{\lambda - n\gamma}$$

as long as $V_{\lambda - n\gamma} \neq 0$. This formula is independent of the choice of vectors v_n. The formula for c_n can therefore also be written as

$$\operatorname{trace} \pi(x)\pi(y) \,|\, V_\mu = \sum_{k \geq 1} \langle \mu + k\gamma, h \rangle \dim V_{\mu + k\gamma} \,.$$

This makes sense, and remains valid, for an arbitrary representation (π, V) of $\mathfrak{sl}_2$., since it will be a direct sum of irreducible ones and the terms are additive. There is one last transformation. If we multiply both sides of this last equation by $2/\|t\|^2$ then by (3.3) we get

(6.1) $$\operatorname{trace} \pi(x)\pi(x^\bullet) \,|\, V_\mu = \sum_{k \geq 1} \big((\mu + k\gamma) \bullet \gamma\big) \dim V_{\mu + k\gamma} \quad (x \in \mathfrak{g}_\gamma) \,.$$

FREUDENTHAL'S FORMULAS. Now suppose $\mathfrak{g}$ arbitrary, (π, V) the representation of *highest weight* λ. Restricted to the copy of $\mathfrak{sl}_2$ associated to the root γ the last formula becomes

$$\operatorname{trace} \pi(x_\gamma)\pi(x_\gamma^\bullet) \,|\, V_\mu = \sum_{k \geq 1} ((\mu + k\gamma) \bullet \gamma) \dim V_{\mu + k\gamma} \,.$$

Recall from (3.7) that the Casimir operator is

$$\sum_{\alpha} t_\alpha t_\alpha^\bullet + \sum_{\gamma\in\Sigma^+} x_\gamma x_\gamma^\bullet \,.$$

Recall from Lemma 3.10 that it acts on V^λ as the scalar

$$\|\lambda+\rho\|^2 - \|\rho\|^2 \,.$$

We now have for each weight μ two different ways to evaluate the trace of the Casimir operator on V^λ_μ. On the one hand

$$\operatorname{trace} \pi(\mathfrak{C}) \,|\, V_\mu = \big(\|\lambda+\rho\|^2 - \|\rho\|^2\big) \dim V_\mu \,.$$

On the other, we can evaluate terms in its definition explicitly. By Proposition 3.2 the first part is

$$\sum_{\alpha} t_\alpha t_\alpha^\bullet \,|\, V_\mu = \|\mu\|^2 \dim V_\mu \,.$$

The second part is

$$\sum_{\gamma\in\Sigma} \operatorname{trace} \pi(x_\gamma x_\gamma^\bullet) \,|\, V_\mu = \sum_{\gamma\in\Sigma}\sum_{k\ge 1} \big((\mu+k\gamma)\bullet\gamma\big) \dim V_{\mu+k\gamma} \,.$$

Combining these, we deduce what I call the *raw* formula of Freudenthal:

6.2. Proposition. *For any weight μ of the representation on $V = V^\lambda$ with highest weight λ*

$$(\|\lambda+\rho\|^2 - \|\rho\|^2) \dim V_\mu = \|\mu\|^2 \dim V_\mu + \sum_{\gamma\in\Sigma} \sum\nolimits_{k\ge 1} ((\mu+k\gamma)\bullet\gamma) \dim V_{\mu+k\gamma} \,.$$

More practical formulas rely on two simple facts. For every root γ let

$$S_\gamma = \sum\nolimits_{k\ge 1} ((\mu+k\gamma)\bullet\gamma) \dim V_{\mu+k\gamma} \,.$$

Then

(a) if $w(\mu)=\mu$ then $S_{w(\gamma)} = S_\gamma$;
(b) $S_{-\gamma} = (\mu\bullet\gamma) + S_\gamma$.

For formula (a): if $w(\mu)=\mu$ then $\mu + w(\gamma) = w(\mu+\gamma)$ and

$$S_{w(\gamma)} = \sum\nolimits_{k\ge 1} \big((w(\mu+k\,\gamma)\bullet w(\gamma)\big) \dim V_{w(\mu+k\,\gamma)} = S_\gamma \,.$$

Now for formula (b). Since the root reflection s_γ preserves the λ-string through μ,

$$\sum\nolimits_{k\in\mathbb{Z}} \big((\mu+k\gamma)\bullet\gamma\big) \dim V_{\mu+k\gamma} = 0 \,.$$

Therefore

$$\begin{aligned} S_\gamma = \sum_{k\ge 1} \big((\mu+k\gamma)\bullet\gamma\big) \dim V_{\mu+k\gamma} &= -\sum_{k\le 0} \big((\mu+k\gamma)\bullet\gamma\big) \dim V_{\mu+k\gamma} \\ &= \sum_{k\ge 0} \big((\mu+k(-\gamma))\bullet(-\gamma)\big) \dim V_{\mu+k(-\gamma)} \\ &= -\mu\bullet\gamma + S_{-\gamma} \,. \end{aligned}$$

Property (a) leads immediately to the best known version of Freudenthal's formula:

6.3. Theorem. (Freudenthal) *For $V = V^\lambda$*

$$\big(\|\lambda+\rho\|^2 - \|\mu+\rho\|^2\big) \dim V_\mu = 2\sum_{\gamma>0}\sum_{k\ge 1} \big((\mu+k\gamma)\bullet\gamma\big) \dim V_{\mu+k\gamma}\,.$$

This can be used to compute $\dim V_\mu$ recursively, starting with $\dim V_\lambda = 1$. It may very well happen that $\nu = \mu + k\gamma$ is not dominant. In that case, there exists α with $\langle \nu, \alpha^\vee\rangle < 0$. In these circumstances $s_\alpha\nu$ has greater height than ν. A sequence of such reflections will produce an element in the fundamental domain, leaving the weight multiplicity invariant and continually raising heights. That is to say, we can find w in W such that $\mu + k\gamma$ lies in the fundamental domain $\mathcal{D}$. The multiplicity $\dim V_{w(\mu+k\gamma)}$ is the same as $\dim V_{\mu+k\gamma}$. Furthermore

$$(\mu+k\gamma)\bullet\gamma = w(\mu+k\gamma)\bullet w(\gamma)\,.$$

Induction therefore remains effective, especially if we have stored values of $\mu\bullet\gamma$ for all μ in $\mathcal{D}$ and positive roots γ.

It has one immediate and useful consequence, however. Let Ω be the set of weights of π, Ω^{++} that of dominant weights in Ω. The consequence is:

6.4. Corollary. *If $\mu \ne \lambda$ lies in Ω, then there exists a root $\gamma > 0$ such that $\mu + \gamma$ also lies in Ω.*

Proof. What the formula implies immediately is that both μ and some $\mu + k\gamma$ with $k \ge 1$ both lie in Ω. But the weights in a γ-string are the weights in a representation of a copy of $\mathfrak{sl}_2$, and there are therefore no gaps in it. So $\mu + \gamma$ is also in Ω. ∎

CONSTRUCTING THE DOMINANT WEIGHTS. [Moody-Patera:1982] came up with a formula that exploits symmetry in the formula of Proposition 6.2.

6.5. Lemma. *If $\mu \ne \lambda$ is in Ω^{++}, there exists a root $\gamma > 0$ such that $\mu + \gamma$ is also in Ω^{++}.*

The point is that one can construct all of Ω^{++} by starting with λ and descending into Ω^{++} from there, without ever leaving it.

Proof. The proof is constructive. Suppose $\mu \ne \lambda$ lies in Ω^{++}. By the first half of Corollary 6.4 there exists $\gamma > 0$ with $\mu + \gamma$ in Ω. If $\mu + \gamma$ is dominant, there is nothing more to prove.

Otherwise, suppose $\mu + \gamma$ not to be dominant. Then

$$\langle \mu+\gamma, \alpha^\vee\rangle = \langle \mu, \alpha^\vee\rangle + \langle \gamma, \alpha^\vee\rangle < 0$$

for some α in Δ. Since μ is dominant the first term is non-negative and hence $\langle \gamma, \alpha^\vee\rangle < 0$. Consider

$$s_\alpha(\mu+\gamma) = \mu + \gamma - \langle \mu+\gamma, \alpha^\vee\rangle\alpha\,.$$

The coefficient in the second term is positive. This reflection also lies in Ω. The sum $\mu+\gamma+\alpha$ also lies in Ω, because of convexity. Since $\langle \gamma, \alpha^\vee\rangle < 0$ and s_α takes all positive roots except α to positive roots, $s_\alpha(\gamma)$ is a positive root, and again by convexity $\gamma + \alpha$ is also a root. Thus we can replace γ by $\gamma + \alpha$, and repeat the argument. Sooner or later the repetition has to stop. ∎

THE FORMULA OF MOODY-PATERA. Suppose T for the moment to be any subset of Δ. The group W_T acts on the set Σ. It takes Σ_T into itself, and also takes $\Sigma^\pm - \Sigma^\pm_T$ into themselves. The map $\lambda \mapsto -\lambda$ induces a bijection of W_T orbits in $\Sigma^+ - \Sigma^+_T$ with those in $\Sigma^- - \Sigma^-_T$. The orbits in Σ_T are parametrized by length, so if $\mathcal{O}$ is an orbit ofW_T in Σ_T then $-\mathcal{O} = \mathcal{O}$. In each W_T-orbit $\mathcal{O}$ there will exist a unique $\xi = \xi_{\mathcal{O}}$ with $\langle \xi, \alpha^\vee\rangle \ge 0$ for $\alpha \in T$, since W_T is the Weyl group of the root system based on T. In the case $\mathcal{O}$ is contained in Σ_T it may happen that $\xi_{\mathcal{O}} < 0$, in which case I set $\gamma_{\mathcal{O}} = -\xi_{\mathcal{O}}$. Otherwise I set $\gamma_{\mathcal{O}} = \xi_{\mathcal{O}}$.

6.6. Theorem. *Suppose V to be the irreducible representation with highest weight λ, μ a weight of V, and T to be chosen so that W_T is the subgroup of W fixing μ. Then*

$$\left(\|\lambda+\rho\|^2 - \|\mu+\rho\|^2\right)\dim V_\mu = \sum_{\mathcal{O}} \|\mathcal{O}\| \sum_{k\ge 1} \left((\mu + k\gamma_{\mathcal{O}})\bullet\gamma_{\mathcal{O}}\right)\dim V_{\mu+k\gamma_{\mathcal{O}}} \,.$$

In this, the sum is over the orbits $\mathcal{O}$ of W_T in $\Sigma_T \cup \left(\Sigma^+ - \Sigma_T^+\right)$, and

$$\|\mathcal{O}\| = \begin{cases} |\mathcal{O}| & \text{if } \mathcal{O}\subset\Sigma_T \\ 2|\mathcal{O}| & \text{if } \mathcal{O}\subset\Sigma^+ - \Sigma_T^+. \end{cases}$$

Proof. Start with Proposition 6.2:

$$(\|\lambda+\rho\|^2 - \|\rho\|^2)\dim V_\mu = \|\mu\|^2 \dim V_\mu + \sum_{\gamma\in\Sigma}\sum_{k\ge 1}((\mu+k\gamma)\bullet\gamma)\dim V_{\mu+k\gamma}\,.$$

By property (a) above the sum

$$S_\gamma = \sum_{k\ge 1}((\mu+k\gamma)\bullet\gamma)\dim V_{\mu+k\gamma}$$

depends only on the W_T-orbit of γ, and we can express the sum on the right hand side as

$$\sum\nolimits_{\mathcal{O}} |\mathcal{O}| S_{\xi_{\mathcal{O}}}\,.$$

Here the sum is over all W_μ-orbits $\mathcal{O}$ in Σ.

There are three kinds of orbits: (a) those in Σ_T; (b) those in $\Sigma^+ - \Sigma_T^+$; (c) those in $\Sigma^- - \Sigma_T^-$.

Suppose $\mathcal{O}$ is of type (b). Then the corresponding orbit $-\mathcal{O}$ is of type (c), and according to property (b) for γ in $\mathcal{O}$ we have $S_{-\gamma} = \mu\bullet\gamma + S_\gamma$. Furthermore, if γ lies in in Σ_T ((i.e. is of type (a)) then $\mu\bullet\gamma = 0$, so the sum over orbits becomes the one in Theorem 6.6. $\square$

The formula of Moody-Patera is also an inductive formula for the weight multiplicities $\dim V_\mu$, but it is more efficient since there are fewer terms. In practice this works well, since representations for which there do not exist a large number of singular weights are unapproachable by any known algorithm.

One important point is if many multiplicities are to be calculated it is best to compute data for all possible $\mathcal{O}$ in advance. The same technique remarked on in connection with the original Freudenthal formula can be used for dealing with the case that $\nu = \mu + k\gamma_{\mathcal{O}}$ is not dominant.

[Bremner:1986] contains practical advice (regarding an implementation in the now forgotten but once well loved programming language Pascal). [Moody-Patera:1982] discusses some examples for $G = E_8$.

I note also that the norms $\|\mu\|^2$ and dot products $\mu\bullet\gamma$ can also be computed by descending induction on height:

$$(\mu-\gamma)\bullet\delta = (\mu\bullet\delta) - (\gamma\bullet\delta), \quad \|\mu-\gamma\|^2 = \|\mu\|^2 - 2(\mu\bullet\gamma) + (\gamma\bullet\gamma)\,,$$

given pre-computed values of $\gamma\bullet\delta$. So we compute the values of $\lambda\bullet\gamma$ for all $\gamma > 0$ directly, and then compute other values as we construct the dominant closure of λ.

7. *Appendix.* An introduction to the homology of Lie algebras

For the moment, suppose $\mathfrak{h}$ to be an arbitrary Lie algebra over a field F. If (F_m) is any resolution of the trivial $\mathfrak{h}$-module F by free $U(\mathfrak{h})$-modules, the homology of an $\mathfrak{h}$-module V is the homology of the complex $(F_m \otimes_{U(\mathfrak{h})} V)$. Any two free resolutions are homotopic, so this definition is independent of the resolution. There is one extremely convenient one, however, which I'll now explain.

If V is a module over $\mathfrak{h}$, its **Koszul sequence** is the sequence

$$\cdots \to \bigwedge^m \mathfrak{h} \otimes V \to \cdots \to \mathfrak{h} \otimes V \to V \to 0$$

with boundary maps $\partial_m \colon \bigwedge^m \mathfrak{h} \otimes V \to \bigwedge^{m-1} \mathfrak{h} \otimes V$, according to which

$$(h_m \wedge \ldots \wedge h_1) \otimes v$$

is mapped to

$$\sum_{1 \le i \le m} (h_m \wedge \ldots \wedge \widehat{h}_i \wedge \ldots \wedge h_1) \otimes (-1)^{i-1} h_i v$$
$$+ \sum_{1 \le i < j \le m} (h_m \wedge \ldots \wedge \widehat{h}_j \wedge \ldots \wedge \widehat{h}_i \wedge \ldots \wedge h_1 \wedge [h_j, h_i]) \otimes (-1)^{i+j} v \,.$$

The operator ∂ is well defined since linear maps with $\bigwedge^k \mathfrak{h}$ as source are equivalent to alternating multilinear maps from $\mathfrak{h} \times \cdots \times \mathfrak{h}$ (k factors). The composition ∂^2 vanishes, so the Koszul sequence is actually the Koszul complex. One could check this by hand, but this would be rather painful. Instead, I'll follow §23 of [Chevalley-Eilenberg:1948], and prove it as a consequence of a more basic fact, which will have other interesting consequences as well. The proof is in effect postponed.

As far as I can tell, that same paper is the original publication concerned with Lie algebra homology, although it deals with the dual theory of cohomology instead. The motivation for the definition of the cohomological version of ∂ is that it is the de Rham derivative restricted to differential forms that are left-invariant on a Lie group G (look at §§9–10 of [Chevalley-Eilenberg:1948]). The complex of $\mathfrak{g}$-coinvariants of compactly supported forms presumably gives rise to the definition of homology.

In the rest of this section, let

$$\Omega^k(V) = \bigwedge^k \mathfrak{h} \otimes V$$
$$n = \dim \mathfrak{h} \,.$$

The space $\Omega^k(V)$ is in the usual way a representation of $\mathfrak{h}$. This representation is derived from the one on tensor products. More specifically, h in $\mathfrak{h}$ takes ω to $\vartheta_h \omega$, with

$$\vartheta_h \colon (h_m \wedge \ldots \wedge h_1) \otimes v \longmapsto (h_m \wedge \ldots \wedge h_1) \otimes hv + \sum_{m \ge i \ge 1} (h_m \wedge \ldots \wedge [h, h_i] \wedge \ldots \wedge h_1) \otimes v \,.$$

Also define for h in $\mathfrak{h}$ the map

$$\wedge_h \colon \omega = (h_m \wedge \ldots \wedge h_1) \otimes v \longmapsto (h_m \wedge \ldots \wedge h_1 \wedge h) \otimes v \,.$$

The basic fact relating the three maps $\wedge_h$, ∂, and ϑ_h, and perhaps the basic fact about ∂, is this:

7.1. Proposition. *For h in $\mathfrak{h}$*

$$\vartheta_h = \partial \wedge_h + \wedge_h \partial \,.$$

This is called a **homotopy equation**, because such equations have their origin in formulas which say that homotopic maps give rise to the same map of homology groups of a topological space. In this case it will imply eventually that ϑ_h is homotopic to 0.

Proof. Let $\omega = (h_m \wedge \ldots \wedge h_1) \otimes v$. Then ∂ takes ω to

$$\sum_{m \ge i \ge 1} (-1)^{i-1} (h_m \wedge \ldots \wedge \widehat{h}_i \wedge \ldots \wedge h_1) \otimes h_i v \\ + \sum_{m \ge j > i \ge 1} (-1)^{i+j} (h_m \wedge \ldots \wedge \widehat{h}_j \wedge \ldots \wedge \widehat{h}_i \wedge \ldots \wedge h_1 \wedge [h_j, h_i]) \otimes v \,.$$

The composite $\partial \wedge_h$ takes ω to the sum

$$\begin{aligned} &(1a) \quad \textstyle\sum_{m \ge i \ge 1} (-1)^i (h_m \wedge \ldots \wedge \widehat{h}_i \wedge \ldots \wedge h_1 \wedge h) \otimes h_i v \\ &(1b) \quad + \; (h_m \wedge \ldots \wedge h_1) \otimes hv \\ &(1c) \quad + \; \textstyle\sum_{m \ge j > i \ge 1} (-1)^{i+j+2} (h_m \wedge \ldots \wedge \widehat{h}_j \wedge \ldots \wedge \widehat{h}_i \wedge \ldots \wedge h_1 \wedge h \wedge [h_j, h_i]) \otimes v \\ &(1d) \quad + \; \textstyle\sum_{m \ge j \ge 1} (-1)^{j+2} (h_m \wedge \ldots \wedge \widehat{h}_j \wedge \ldots \wedge h_1 \wedge [h_j, h]) \otimes v \,. \end{aligned}$$

Whereas $\wedge_h \partial$ takes it to

$$\begin{aligned} &(2a) \quad \textstyle\sum_{m \ge i \ge 1} (-1)^{i-1} (h_m \wedge \ldots \wedge \widehat{h}_i \wedge \ldots \wedge h_1 \wedge h) \otimes h_i v \\ &(2b) \quad + \; \textstyle\sum_{m \ge j > i \ge 1} (-1)^{i+j} (h_m \wedge \ldots \wedge \widehat{h}_j \wedge \ldots \wedge \widehat{h}_i \wedge \ldots \wedge h_1 \wedge [h_j, h_i] \wedge h) \otimes v \,. \end{aligned}$$

Adding these up, the terms (1a) and (2a) cancel, as do (1c) and (2b), leaving

(7.2) $$\sum_{m \ge i \ge 1} (h_m \wedge \ldots \wedge [h, h_i] \wedge \ldots h_1) \otimes v + (h_m \wedge \ldots h_1) \otimes hv = \vartheta_h(v) \,.$$ ▯

For example, $\wedge_h \partial$ maps

$$h_1 \otimes v \longmapsto h_1 v \longmapsto h \otimes h_1 v$$

while $\partial \wedge_h$ maps

$$h_1 \otimes v \longmapsto (h_1 \wedge h) \otimes v \longmapsto d\big((h_1 \wedge h) \otimes v\big) = h_1 \otimes hv - h \otimes h_1 v - [h_1, h] \otimes v$$

and the sum is

$$h_1 \otimes hv + [h, h_1] \otimes v \,.$$

This Proposition has a host of consequences. In dealing with them I follow, as I have already said, §23 of [Chevalley-Eilenberg:1948].

7.3. Lemma. *Suppose ω to lie in $\bigwedge^k \mathfrak{h}$. If $\omega \wedge h = 0$ for all h in $\mathfrak{h}$ then either $\omega = 0$ or $k = n$.*

Proof. An attractive way to put this is that for every h in $\mathfrak{h}$ the complex

$$0 \to F \xrightarrow{\wedge_h} \mathfrak{h} \xrightarrow{\wedge_h} \ldots \xrightarrow{\wedge_h} \textstyle\bigwedge^k \mathfrak{h} \xrightarrow{\wedge_h} \bigwedge^{k+1} \mathfrak{h} \xrightarrow{\wedge_h} \ldots \xrightarrow{\wedge_h} \bigwedge^n \mathfrak{h} \xrightarrow{\wedge_h} 0$$

is exact. I leave this as an exercise. ▯

This will enable in the next propositions a certain descending induction argument. Suppose F to be an array of maps

$$F_m \colon \Omega^m(V) \to \Omega^{m-k}(V)$$

commuting with all $\wedge_x$, for some $k > 0$ (taking $\Omega^k = 0$ for $k < 0$). In the cases to be considered, k will be 1 and 2. In the top dimension, $\wedge_x = 0$, so $\wedge_x F = F \wedge_x$ also vanishes on $\Omega^n(V)$. But since this is true for all x, Lemma 7.3 tells us that $F = 0$ on $\Omega^n(V)$. The same reasoning tells us by descending induction that $F = 0$ on all $\Omega^m(V)$.

The first result to which this argument will be applied is:

7.4. Corollary. *The Lie derivative ϑ_x commutes with ∂.*

Proof. I must prove that $\partial_m \vartheta_x - \vartheta_x \partial_m = 0$ for all m. The proof goes, as I have said, by descending induction on m. We need only verify:

$$\begin{aligned}\wedge_y(\vartheta_x\partial - \partial\vartheta_x) &= (\wedge_y\vartheta_x)\partial - (\wedge_y\partial)\vartheta_x \\ &= (\vartheta_x \wedge_y - \wedge_{[x,y]})\partial - (\vartheta_y - \partial\wedge_y)\vartheta_x \\ &= \vartheta_x(\wedge_y\partial) - wedge_{[x,y]}\partial - \vartheta_y\vartheta_x + \partial(\wedge_y\vartheta_x) \\ &= \vartheta_x(\vartheta_y - \partial\wedge_y) - \wedge_{[x,y]}\partial - \vartheta_y\vartheta_x + \partial(\vartheta_x \wedge_y - \wedge_{[x,y]}) \\ &= -(\vartheta_x\partial - \partial\vartheta_x)\wedge_y \ .\end{aligned}$$

Here $F_m = (-1)^m \wedge_y (\vartheta_x\partial_m - \partial_m\vartheta_x)$. In this calculation, I have applied the formula $\vartheta_x \wedge_y = \wedge_y \vartheta_x + \wedge_{[x,y]}$, which follows immediately from the definition of ϑ. ∎

The second:

7.5. Corollary. *In the Koszul complex $\partial^2 = 0$.*

Proof. Same trick. Applying the previous result, we have

$$\begin{aligned}\wedge_x\partial\partial &= (\vartheta_x - \partial\wedge_x)\partial \\ &= \vartheta_x\partial - \partial(\wedge_x\partial) \\ &= \partial\vartheta_x - \partial(\vartheta_x - \partial\wedge_x) \\ &= \partial\partial\wedge_x \ . \end{aligned}$$ ∎

The map from $\mathfrak{h} \otimes V$ to V takes $h \otimes v$ to hv, so the 0-th homology group of this complex is $V/\mathfrak{n}V$. In particular, if $V = U(\mathfrak{h})$ we get an extended complex

$$\cdots \to \bigwedge^m \mathfrak{h} \otimes U(\mathfrak{h}) \to \cdots \to \mathfrak{h} \otimes U(\mathfrak{h}) \to U(\mathfrak{h}) \to F \to 0$$

in which the last map from $U(\mathfrak{h})$ to F is the surjective **augmentation homomorphism** taking 1 to 1 and $\mathfrak{h}$ to 0.

7.6. Theorem. *If $V = U(\mathfrak{h})$, the Koszul complex is a resolution of F by free $U(\mathfrak{h})$-modules.*

I'll prove this in a moment, but first point out the consequence. Assuming the Theorem, the homology $H_\bullet(\mathfrak{h}, V)$ of any $\mathfrak{h}$-module V is therefore the homology of the complex

$$\cdots \to \big(\bigwedge^k \mathfrak{h} \otimes U(\mathfrak{h})\big) \otimes_{U(\mathfrak{h})} V \to \cdots \to \big(\mathfrak{h} \otimes U(\mathfrak{h})\big) \otimes_{U(\mathfrak{h})} V \to U(\mathfrak{h}) \otimes_{U(\mathfrak{h})} V \to 0$$

but because $U(\mathfrak{h}) \otimes_{U(\mathfrak{h})} V = V$, this may be identified with the Koszul complex of V. Hence:

7.7. Corollary. *The homology of the Koszul complex of V is $H_\bullet(\mathfrak{n}, V)$.*

Proof of the Theorem. It goes back to [Koszul:1950], and is presented clearly in [Cartan-Eilenberg:1956].

Step 1. First suppose $\mathfrak{h}$ to be abelian, say with basis $h_1, \ldots, h_n$. For $m \le n$ let $\mathfrak{h}_m$ be the subspace spanned by the h_i with $i \le m$, and let K_m for $m \le n$ be the Koszul complex of $\mathfrak{h}_m$. The embedding of $\mathfrak{h}_m$ into $\mathfrak{h}_{m+1}$ for all $m < n$ extends to an embedding of K_m into K_{m+1}. It is a summand—the projection π_{m+1} from K_{m+1}

to K_m is induced by the projection from $\mathfrak{h}_{m+1}$ to $\mathfrak{h}_m$ that evaluates all occurrences of x_{m+1} as 0. I claim that this embedding induces an isomorphism of homology, and I shall prove this by induction.

For $m = 1$ with $\mathfrak{h} = \mathfrak{h}_1$ we have the diagram

$$\begin{array}{ccccccc} 0 & \to & 0 & \longrightarrow & F & \to & 0 \\ 0 & & \downarrow & & \downarrow & & 0 \\ 0 & \to & \mathfrak{h} \otimes F[x] & \xrightarrow{x\otimes P \mapsto xP} & F[x] & \to & 0 \end{array}$$

in which the injection of the top row clearly induces an isomorphism of homology. This can be seen a bit more explicitly by using operators

$$\begin{aligned} &h_1 \colon F[x] \to \mathfrak{h} \otimes F[x], \quad P \mapsto x \otimes (P - P(0))/x\otimes \\ &h_2 = 0 \end{aligned}$$

Define h to be the array of maps (h_i). Then

$$\partial h + h\partial = \pi\,,$$

which means that h is a homotopy of K into its subcomplex F.

But now I do almost the same thing for $m > 1$ by defining homotopy operators retracting K_m onto K_{m-1} and applying induction.

Step 2. The claim for arbitrary Lie algebras reduces to that for abelian ones. One can filter the Koszul complex $K_{\mathfrak{h}}$ by total order: $(h_1 \wedge \ldots \wedge h_m) \otimes X$ has order $\le n+m$ if X lies in $U_n(\mathfrak{h})$. The Koszul differentials preserve this filtration, and by the Poincaré-Birkhoff-Witt theorem the graded complex is that of the abelianized $\mathfrak{h}$. But if the homology of the graded complex vanishes, so does that of the complex. ∎

Now suppose $\mathfrak{h}$ to be an ideal in the Lie algebra $\mathfrak{k}$, and that V is a module over $\mathfrak{k}$. The case $\mathfrak{k} = \mathfrak{h}$ is not without interest. Since $\mathfrak{h}$ is an ideal of $\mathfrak{k}$, the adjoint representation of $\mathfrak{k}$ makes $\mathfrak{h}$ an $\mathfrak{k}$-module, and therefore also each $\bigwedge^k \mathfrak{h}$. By the usual tensor product representation, $\mathfrak{k}$ then also acts on the terms in the Koszul complex of V:

$$\vartheta_k \colon\ (h_m \wedge \ldots \wedge h_1) \otimes v \longmapsto (h_m \wedge \ldots \wedge h_1) \otimes k\,v + \sum_{m \ge i \ge 1} (h_m \wedge \ldots \wedge [k, h_i] \wedge \ldots \wedge h_1) \otimes v\,.$$

From this one deduces several consequences:

7.8. Corollary. *For k in $\mathfrak{k}$, the map ϑ_k commutes with ∂ in the Koszul complex of $\mathfrak{h}$.*

Proof. By Corollary 7.4 this is true in the Koszul complex of $\mathfrak{k}$, but that for $\mathfrak{h}$ embeds into it. ∎

7.9. Corollary. *The action of $\mathfrak{h}$ on $H_\bullet(\mathfrak{h}, V)$ is trivial.*

Proof. This is because of the homotopy equation Proposition 7.1. ∎

The action of $\mathfrak{k}$ on $H_\bullet(\mathfrak{h}, V)$ therefore induces one of the quotient $\mathfrak{k}/\mathfrak{h}$.

7.10. Corollary. *For k in $\mathfrak{k}$ the operation ϑ_k commutes with ∂ in the Koszul complex of $\mathfrak{h}$.*

Proof. Because it does in the Koszul complex for $\mathfrak{k}$, and the Koszul complex for $\mathfrak{h}$ embeds into that. ∎

7.11. Proposition. *If*

$$0 \to U \to V \to W \to 0$$

is an exact sequence of $\mathfrak{k}$-modules, then the maps in the long exact homology sequence

$$\begin{aligned} \ldots \to H_m(\mathfrak{h}, U) \to H_m(\mathfrak{h}, V) \to H_m(\mathfrak{h}, W) \to H_{m-1}(\mathfrak{h}, U) \to \ldots \\ \to U/\mathfrak{h}U \to V/\mathfrak{h}V \to W/\mathfrak{h}W \to 0 \end{aligned}$$

are $\mathfrak{k}$-equivariant.

Proof. I leave this as an exercise. ∎

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