January 15, 1980

Dear Jim,

I see as I revise my earlier version that it was written on November 27, which seems to me years ago. Given what problems I had finishing it, I propose to write you in installments, if necessary, to explain, at least in summary, what I know about the relationship between embeddings into principal series and the behaviour at ∞ of matrix coefficients. I realize that you are most interested in real groups, but the results for p-adic groups are not only much simpler but historically motivated the real case, so I will discuss them first. In order to expedite things, I will send you these letters as they are written.

Throughout, let

$k =$ a local field

$G =$ the rational points on a reductive group defined over k

$K =$ a maximal compact in G, which perhaps I only use for k archimedean

$P_\emptyset =$ a minimal parabolic with factorization $P_\emptyset = M_\emptyset N_\emptyset$, $M_\emptyset$ stable under the Cartan involution if k is archimedean, compatible with K in the sense that $K \cap M_\emptyset$ is a maximal compact in $M_\emptyset$ otherwise

$A_\emptyset =$ the maximal split torus in $M_\emptyset$ if k non-archimedean, the connected component of the identity otherwise

$\Sigma =$ roots with respect to $A_\emptyset$

$\Sigma^+ =$ choice of positive roots determined by $P_\emptyset$

$\Delta =$ basis of Σ^+

$W =$ Weyl group

For $\Theta \subseteq \Delta$ let $P_\Theta \supseteq P_\emptyset$ be the usual standard parabolic subgroup.

Of course by restriction of scalars one may assume an archimedean k to be $\mathbb{R}$. In this case, let $\mathfrak{g}$, $\mathfrak{k}$ etc. be the complexified Lie algebras of G etc.

Let k be p-adic for a while. For $\epsilon > 0$ let

$$A_\emptyset(\epsilon) = \left\{ a \in A_\emptyset \,\middle|\, |\alpha(a)| < \epsilon \text{ for all } \alpha \in \Delta \right\}$$

Thus as $\epsilon \to 0$ one gets a sort of neighbourhood system at ∞ in one direction on $A_\emptyset$.

For convenience, for the moment let $P = P_\emptyset$, $A = A_\emptyset$ etc. By an admissible representation of G I shall always mean one which is finitely generated and admissible in the usual sense. If (π, V) is one, then as you know the Jacquet module (π_N, V_N) is the representation of M on the maximal N-trivial quotient of V. The assignment taking V to V_N is an exact functor. The space V_N has finite dimension, and is an admissible M-module.

If (σ, U) is any finite-dimensional representation of P, let $\mathrm{Ind}(\sigma \mid P, G)$ be the induced representation of G, normalized. It is

$$\{f \in C^\infty(G, U) \mid f(pg) = \sigma(p)\delta^{1/2}(p) f(g)\}$$

The group G acts on the right. The representation σ is necessarily trivial on N, and we have Frobenius reciprocity

$$\mathrm{Hom}_G(V, \mathrm{Ind}(\sigma)) \cong \mathrm{Hom}_M(V_N, \sigma\delta^{1/2})$$

for all smooth V. The isomorphism is obtained by composition with the P-map $\mathrm{Ind}(\sigma) \to U$, $f \mapsto f(1)$. This connects the Jacquet module with embeddings into principal series.

Let (π, V) be any G-admissible representation. Consider the matrix coefficient $\langle \pi(g)v, \widetilde{v} \rangle$ for v in V, $\widetilde{v}$ in $\widetilde{V}$ (the admissible dual of V). Since G is more or less $KA^{--}K$ for suitable K it suffices for most questions involving the asymptotic behaviour of matrix coefficients to choose g in A^{--}. The basic fact is that for a given v and $\widetilde{v}$ and ϵ small enough (depending on v, $\widetilde{v}$) $\langle \pi(a)v, \widetilde{v} \rangle$ is A-finite on $A^{--}(\epsilon)$. In other words, 'asymptotic to' means 'equals' on p-adic groups. Now any A-finite function on $A^{--}(\epsilon)$ has a unique A-finite extension to all of A; hence one may define a function $\langle \pi(a)v, \widetilde{v} \rangle_\infty$ on all of A by specifying that it agree with $\langle \pi(a)v, \widetilde{v} \rangle$ on $A^{--}(\epsilon)$ for ϵ small and that it be A-finite everywhere. One can then define what I a pairing $\langle \bullet, \bullet \rangle_N$ by the rule

$$\langle v, \widetilde{v} \rangle_N = \langle \pi(1)v, \widetilde{v} \rangle_\infty \ .$$

This pairing descends to a non-degenerate pairing between V_N and $\widetilde{V}_{N^{\mathrm{opp}}}$ which I call the **canonical pairing** between the two spaces. It induces an isomorphism of $\widetilde{V}_{N^{\mathrm{opp}}}$ with the contragredient of V_N.

These things are all explained in my notes on p-adic groups.

A picture might help.

...

Look at A additively. The region marked ... is A^{--} and the region ... is $A^{--}(\epsilon)$. If one goes along the walls of A^{--} one must look at Jacquet modules of larger parabolic subgroups. Of course for p-adic groups V_N might be 0, which is what happens for cuspidal representations. And what happens on $A^{--}(\epsilon)$ might not tell you anything about what happens in other directions. However, if V embeds into some $\mathrm{Ind}(\sigma \mid P, G)$ the module V_N with respect to the minimal parabolic P controls asymptotic behaviour everywhere, by an induction argument on Jacquet modules. Thus, for example, such a V is square-integrable precisely when $|\chi(a)| < \delta^{1/2}(a)$ for all a in $A^{--} - (Z_G \cap A)$, for all central characters χ of V_N.

What is the canonical pairing for the principal series? This breaks up into two questions: (1) If $I = \mathrm{Ind}(\sigma \mid P, G)$, how to describe I_N? (2) Given this, what is the canonical pairing?

I will answer these questions, in fact, when I is induced from a parabolic which is not minimal. Thus, choose $\Theta \subseteq \Delta$ and let $P_* = P_\Theta$ etc. Let W_* be the Weyl group of M_Θ, $[W_*\backslash W]$ the particularly good representatives of the cosets of W_Θ in W, with $w^{-1}\Theta > 0$. Thus G is the disjoint union of the cosets $P_\Theta w P$ (w in $[W_*\backslash W]$). Introduce on $W_*\backslash W$ the partial ordering by closure of cosets:

$$x \le y \Leftrightarrow \overline{G(x)} \subseteq \overline{G(y)} \ .$$

If σ_* is an admissible representation of M_*, there is the **Bruhat filtration** of $\mathrm{Ind}(\sigma_* \mid P_*, G)$ by the subspaces

$$I_w = \{f \text{ with support in } \bigcup_{x \ge w} G(x)\} \ .$$

We have $I_x \subseteq I_y$ when $x \ge y$, and $I_1 = I$. This filtration is also P-stable. Let $\mathcal{I}_w$ be the associated graded module:

$$\mathcal{I}_w = I_w / \sum_{x > w} I_x$$

The usual Knapp-Stein integral

$$\int_{w^{-1}N_* w\cap N\backslash N} f(wn)\,dn$$

induces an isomorphism of $(\mathcal{I}_w)_N$ with

$$w^{-1}(\sigma_{N\cap M_*}\delta_{P\cap M_*}^{-1/2})\,\delta_P^{1/2}$$

Here $\delta_\bullet$ means modulus character. If $P = P_*$, for example, this is just

$$(w^{-1}\sigma)\,\delta_P^{1/2}$$

These expressions make a little more sense if one notices that corresponding to the normalized induction is a normalized Jacquet module $V_N\,\delta_P^{-1/2}$, which I think was always used by Harish-Chandra. I'll use this from now on, so we rewrite the first expression as

$$w^{-1}\sigma_{N\cap M_*}$$

So we don't know the Jacquet module exactly, but we do have a filtration of it and an identification of the graded pieces. Of course if all these graded pieces are distinct, this filtration splits and becomes a direct sum. This happens generically. In general, however, there are non-trivial extensions here. This happens when there are poles of intertwining operators.

The contragredient of $\mathrm{Ind}(\sigma_*|P_*, G)$ is isomorphic, but not canonically, to $\mathrm{Ind}(\widetilde{\sigma}_*|P_*, G)$. The decomposition $G = \bigcup P_* wP$ gives also

$$G = \bigcup P_* w_{\ell,*} w w_\ell P^{\mathrm{opp}}$$

where w_ℓ is the longest element of W, $w_{\ell,*}$ that of W_*.

The map $w \mapsto w_{\ell,*}ww_\ell$ is an involution of $[W_*\backslash W]$ with itself, inverting the partial ordering.

Hence

$$G = \bigcup P_* wP^{\mathrm{opp}}$$

where gain the union is over $[W_*\backslash W]$. The space $I_{N^{\mathrm{opp}}}$ is therefore filtered by the same set, but with inverse ordering. Thus, for example, P_*P^{opp} is the open coset, where $P_* w_{\ell,*} w_\ell P$ is the old one. Knapp-Stein integrals again give a filtration of $\widetilde{I}_{N^{\mathrm{opp}}}$ as of I_N, the first a decreasing filtration, the second increasing. Now we know that I_N is the contragredient of $\widetilde{I}_{N^{\mathrm{opp}}}$, so we shouldn't be surprised by the way these dual filtrations work out. By the way, this is a different way of understanding one of Langlands' arguments in his paper on classification of representations of real reductive groups.

I should make this a little more precise, but it will take some time, and I'll postpone it until next time. I mean to include this in a revised version of my notes on p-adic representations. This line of reasoning gives in the end, for example, a new derivation of Macdonald's formula for spherical functions.

All of these things must be well known to Harish-Chandra in some form or another.

So we haven't gotten too far, but I'll send it off while I ponder.

Regards,

Bill C.

February 26, 1980

Dear Jim,

This is the second letter in this series. Again time has passed very quickly. This time there were some details to be worked out that required more concentration than came easily to me. We can probably thrash things out at Sally's conference in Chicago.

The first topic I want to look at is a precise version of the asymptotic behaviour of induced representations to which I alluded in the last letter. As I mentioned there I had never thought this out in detail and a satisfactory formulation took some thought. I hope what I have to say is comprehensible. I dwell on the p-adic case, even though I realize you want me to deal with $k = \mathbb{R}$, because a correct approach should work with little modification in both cases.

My notation will be pretty much what it was before, with a few small changes. The immediate ingredients are G and

$$\begin{aligned} P &= \text{minimal parabolic} \\ P_* &= P_\Theta \supseteq P \\ \sigma &= \text{admissible representation of } P_* \\ I &= \mathrm{Ind}(\sigma \,|\, P_*, G) \end{aligned}$$

I want to describe the behaviour at ∞ of matrix coefficients of I—or, equivalently, describe the canonical pairing on the Jacquet modules I_N, $\widetilde{I}_{N^{\mathrm{opp}}}$.

The contragredient $\widetilde{I}$ may be identified with $\mathrm{Ind}(\widetilde{\sigma}_* \,|\, P_*, G)$ once one has chosen a Haar measure on N_*^{opp}:

$$\langle f, \varphi \rangle = \int_{N_*} \langle f(n), \varphi(n) \rangle \, dn \ .$$

Fix such a measure once and for all. Explicitly the pairing comes from the composition

$$\mathrm{Ind}(\sigma \,|\, P_*, G) \otimes \mathrm{Ind}(\widetilde{\sigma}_* \,|\, P_*, G) \to \mathrm{Ind}(\delta_*^{1/2} \,|\, P_*, G) \to \text{ constants in } \mathrm{Ind}(\delta_*^{-1/2} \,|\, P_*, G)$$

where the first is $f \otimes \varphi \mapsto \langle f, \varphi \rangle$ and the second

$$\Phi \mapsto \int_{N_*^{\mathrm{opp}}} \Phi(n) \, dn$$

This last is G-covariant, so it may be written as

$$\Phi \mapsto R_w \Phi \mapsto \int_{N_*^{\mathrm{opp}}} R_w \Phi(n) \, dn$$

for each w in $[W_* \backslash W]$, which in turn is also

$$\int_{w^{-1} N_*^{\mathrm{opp}} w} \Phi(wn) \, dn$$

(measure on $w^{-1} N_*^{\mathrm{opp}} w$ determined by w-conjugation from the one on on N_*^{opp}).

Now I will modify my notation in the first letter slightly and define

$$I_w = \{f \in I \,|\, \mathrm{supp} f \subseteq \bigcup_{x \geq w} P_* x P\}$$

$$\widetilde{I}^0_w = \{\varphi \in \widetilde{I} \,|\, \mathrm{supp} \varphi \subseteq \bigcup_{x \ngeq w} P_* x P\}$$

We have $I_u \subseteq I_v$ if $u \geq v$ but $\widetilde{I}^0_u \subseteq \widetilde{I}^0_v$ if $u \leq v$. So $\{I_w\}$ filters I and $\{\widetilde{I}^0_w\}$ filters $\widetilde{I}$, but in the opposite direction.

For u in the space U of σ, let $\overline{u}$ be its image in $U_{M_* \cap N}$. Incidentally, in this letter I always use the normalized Jacquet module as I suggested in the first letter.

Define spaces

$$I^0_w = \sum_{x > w} I_x$$

$$\widetilde{I}_w = \{\varphi \in \widetilde{I} \,|\, \mathrm{supp} \varphi \subseteq \bigcup_{x \not> w} P_* x P^{\mathrm{opp}}\}$$

Then $I^0_w \subseteq I_w$, $\widetilde{I}^0_w \subseteq \widetilde{I}_w$. For f in I_w the restriction of f to $P_* w P$ has compact support modulo P_*, so the integral

$$\int_{w^{-1} P_* w \cap N \backslash N} \overline{f(wn)}\, dn = \Omega_w(f)$$

is well defined. Similarly

$$\widetilde{\Omega}_w(\varphi) = \int_{w^{-1} P_* w \cap N^{\mathrm{opp}} \backslash N^{\mathrm{opp}}} \overline{\varphi(wn)}\, dn \ .$$

where here $\overline{}$ means image in the Jacquet module. These integrals induce isomorphisms

$$(I_w / I^0_w)_N = w^{-1}(\sigma_{M_* \cap N})$$

$$(\widetilde{I}_w / \widetilde{I}^0_w)_{N^{\mathrm{opp}}} = w^{-1}(\widetilde{\sigma}_{M_* \cap N^{\mathrm{opp}}})$$

and thus identify the composition factors in the filtration of I_N, $\widetilde{I}_{N^{\mathrm{opp}}}$ corresponding to the Bruhat filtration of I, $\widetilde{I}$.

Theorem. *In the canonical pairing, $(I^0_w)_N$ is the annihilator of $(\widetilde{I})_{N^{\mathrm{opp}}}$, $(\widetilde{I}^0_w)_{N^{\mathrm{opp}}}$ that of $(I_w)_N$. The canonical pairing induces one of $(I_w/I^0_w)_N$ with $(\widetilde{I}_w/\widetilde{I}^0_w)_{N^{\mathrm{opp}}}$. This pairing is the same as the canonical pairing of $w^{-1}(\sigma_{M_* \cap N})$ and $w^{-1}(\widetilde{\sigma}_{M_* \cap N^{\mathrm{opp}}})$, using the isomorphisms written above.*

In other words, for a in A^{--} near 0

$$\delta^{-1/2}(a) \int_{w^{-1} N_* w} \langle f(wna), \varphi(wn) \rangle \, dn = \langle w^{-1}(\sigma_{M_* \cap N})(a) \Omega_w(f), \widetilde{\Omega}_w(\varphi) \rangle \ .$$

This holds for all σ, but for generic σ we can use the direct sum decomposition of the Jacquet module ands write

$$\langle R_a f, \varphi \rangle = \sum_{[W_* \backslash W]} \langle w^{-1}(\sigma_{M_* \cap N})(a) \Omega_w(f), \widetilde{\Omega}_w(\varphi) \rangle \ .$$

I laboured much over proving this, but I will postpone the proof to some later letter. When $w = 1$ this is essentially Langlands' argument in 'Classification ... ' The whole theorem is a formal version of that.

Next letter I hope to begin with $k = \mathbb{R}$.

Regards,

Bill C.

March 13, 1980

Dear Jim,

This is #3, where I begin to get down to real business. I have kept putting this stuff off because I hoped I could find a better way through the maze we seem to be in, but in fact this is probably not possible. The situation is intrinsically complicated, alas. I have been wondering in particular if you could get by with some version of Osborne's conjecture, now proven by Hecht ...

In this letter let $k = \mathbb{R}$. What I will first do is explain about Jacquet modules for real groups. I enclose a copy of my talk at Helsinki, which covers some of the territory I will cover here, but I understand things much better now than when I gave it.

To start with I work with admissible representations (π, V) of $(\mathfrak{g}, K)$ rather than G. They will always be finitely generated, hence of finite length. The obvious analogue of the Jacquet module in this case is $V/\mathfrak{n}V$, spanned by $\{\pi(\nu)v | v \in V, \nu \in \mathfrak{n}\}$. This is the maximal $\mathfrak{n}$-trivial quotient, and a module over $\mathfrak{m}$. (Here $\mathfrak{n} = \mathfrak{n}_\emptyset$, etc.) Also, $V/\mathfrak{n}V$ is finite-dimensional. The functor $V/\mathfrak{n}V$ is not exact, however, and this suggests at once that it is not the 'real' Jacquet module. Indeed, $V/\mathfrak{n}V$ is the same as $H_0(\mathfrak{n}, V)$, so that obstruction to exactness can even be measured by higher homology groups. We do have Frobenius reciprocity

$$\mathrm{Hom}_{(\mathfrak{g},K)}(V, \mathrm{Ind}(\sigma|P,G)) \cong \mathrm{Hom}_{(\mathfrak{p},K_P)}(V/\mathfrak{n}V, \sigma\,\delta_P^{1/2})$$

as long as σ is trivial on $\mathfrak{n}$. Here $K_P = P \cap K$, etc. Note however that this condition does not have the importance it has in the p-adic case, since it is not automatic. In general, all we can say of a finite-dimensional $\mathfrak{p}$-module is that $\mathfrak{n}$ acts nilpotently. This suggests that the correct Jacquet module involves all of the quotients $V/\mathfrak{n}^m V$ where $\mathfrak{n}^m V$ is the subspace spanned by

$$\pi(\nu_1)\pi(\nu_2)\dots\pi(\nu_m)v$$

with v in V, ν_i in $\mathfrak{n}$. This is the maximal quotient annihilated by $\mathfrak{n}^m$. No one of these has any clear superiority over the others, so we look at the projective limit

$$V_{[\mathfrak{n}]} = \varprojlim V/\mathfrak{n}^m V$$

This is a large space, of course. But not too large, because of a remark of Osborne's: since V is finitely generated and admissible, V is finitely generated as a module over $U(\mathfrak{n})$. This is what gives the finite-dimensionality of the quotients $V/\mathfrak{n}^m V$, and implies also a number of interesting things about the limit. Suppose for illustration that $\mathfrak{n} \cong \mathbb{R}^r$, for example as with an orthogonal group of rank 1. Then $U(\mathfrak{n})$ is a polynomial ring, and $V_{[\mathfrak{n}]}$ is the standard completion of the finitely generated module with respect to the ideal $(\mathfrak{n})$, and itself a module over the completion of $U(\mathfrak{n})$ with respect to this ideal, a formal power series ring. When $\mathfrak{n}$ is nilpotent, there is a similar theory of completions. In other words, whenever V is finitely generated over $U(\mathfrak{n})$, I a two-sided ideal of $U(\mathfrak{n})$, $\mathfrak{n}$ nilpotent, one can construct completions with respect to powers of I. When I is the augmentation ideal, the theory is elementary. The completed ring $U(\mathfrak{n})_{[\mathfrak{n}]}$ is Noetherian, and the completion functor is exact (a version of

Artin-Rees). In addition, the completion $V_{[\mathfrak{n}]}$ turns out to be a module over $U(\mathfrak{g})$ and also over the group P.

This module $V_{[\mathfrak{n}]}$ is even related to a well known sort of $U(\mathfrak{g})$-module, namely the Verma modules. To see how this happens, note that the projective limits of the $V/\mathfrak{n}^m V$ translates into a direct limit of their duals. This direct limit is the subspace $\widehat{V}^{[\mathfrak{n}]}$ of the linear dual of V (algebraic, no topology) annihilated by powers of $\mathfrak{n}$, and is a kind of Verma module. In face it is a quotient of some module $U(\mathfrak{g}) \oplus_{\mathfrak{p}} M$, with M a finite-dimensional P-module. So we have two large spaces $V_{[\mathfrak{n}]}$ and $\widehat{V}^{[\mathfrak{n}]}$ which are each others' duals. Both are modules over $(\mathfrak{g}, P)$.

The really interesting facts about these spaces has to do with the asymptotic behaviour of matrix coefficients. The formulation I am about to give owes something to a suggestion of Hecht. Let V be admissible, $U = \widetilde{V}$ its contragredient. The theory of differential equations satisfied by matrix coefficients asserts we have asymptotic expansions

$$\langle \pi(a)v, u\rangle \sim \sum_s \sum_m \sum_{N^\Delta} f_{s,m,n}\, \alpha^{s+n}(a) \log^m \alpha(a)$$

I realize here that I am assuming G semi-simple. No matter; for reductive G the facts are only a bit more complicated. The sets S and M depend only on the ideal of $Z(\mathfrak{g})$ annihilating V. The coefficients f depend on v and u.

We get from v in V a **formal series**

$$\begin{aligned} F(v) &= \sum f_{s,n,m}(v, \bullet)\alpha^{s+n} \log^m \alpha \\ &= \sum F_{s,m}(v)\alpha^s \log^m \alpha \end{aligned}$$

with coefficients $F_{s,m}$ in the linear dual of $\widetilde{V}$. Here s ranges over $S + N^\Delta$. For X in the root space $\mathfrak{g}_\gamma$

$$\begin{aligned} \langle \pi(a)\pi(X)v, u\rangle &= \langle \pi(\mathrm{Ad}(a)X)\pi(a)v, u\rangle \\ &= \gamma(a)\langle \pi(X)\pi(a)v, u\rangle \\ &= \gamma(a)\langle \pi(a), -\widetilde{\pi}(X)u\rangle \end{aligned}$$

Therefore $F(\pi(X)v) = \gamma(a)[F(v)\cdot(-\widetilde{\pi}(X)u)]$. This means that if X is in $U(\mathfrak{g})$ in $\mathfrak{n}^m$ then the first several terms of $F(v)$ vanish So as $n \to \infty$ the series for v in $\mathfrak{n}^m V$ has more and more terms vanishing. Similarly for u in $(\mathfrak{n}^{\mathrm{opp}})^m \widetilde{V}$. I.e. to every v in $V_{[\mathfrak{n}]}$ we associate a formal series

$$F(v) = \sum F_{s,m}(v)\alpha^s \log^m \alpha$$

with coefficients in the $\mathfrak{n}^{\mathrm{opp}}$-torsion dual of $\widetilde{V}$. I.e. this series lies in the Verma dual of $\widetilde{V}$ with respect to $\mathfrak{n}^{opp}$. Of course there is no question of convergence here, these really are a kind of formal power series. However, some elements of $V_{[\mathfrak{n}]}$ will correspond to finite series: those annihilated by some power of $\mathfrak{n}^{\mathrm{opp}}$. These series may be evaluated at 1 on A. This is entirely analogous to what happens for the p-adic case. So we get ihn this way a canonical map

$$V_{[\mathfrak{n}]}^{[\mathfrak{n}^{\mathrm{opp}}]} \to \text{ the linear dual of } (\widetilde{V})^{[\mathfrak{n}^{\mathrm{opp}}]}$$

This is extraordinarily complicated. And yet I think it is necessary if you want to understand the formula we will get eventually for the asymptotic behaviour of matrix coefficients.

Let me try to make it a bit more palatable by putting it a different context. Call a $\mathfrak{g}$-module $\mathfrak{p}$-finite if every vector in it is contained in a finite-dimensional subspace stable under $\mathfrak{p}$. The algebra $\mathfrak{n}$ acts nilpotently on every vector in such a space. For example, if V is an admissible representation over $(\mathfrak{g}, K)$ then $X = \widehat{V}^{[\mathfrak{n}]}$ is such a space, and of finite length over $U(\mathfrak{g})$, hence a quotient of some $U(\mathfrak{g}) \oplus_{\mathfrak{p}} M$, hence *finitely generated as a module over* $U(\mathfrak{n}^{\rm opp})$. We can now complete it with respect to powers of $\mathfrak{n}^{\rm opp}$! to get $X_{[\mathfrak{n}^{\rm opp}]}$. Then X consists of the $\mathfrak{n}$-finite vectors in this completion. But $X_{[\mathfrak{n}^{\rm opp}]}$ is also the dual of the $\mathfrak{n}^{\rm opp}$-torsion in $\widehat{X}$. This space $X^{[\mathfrak{n}^{\rm opp}]}$ is $\mathfrak{p}^{\rm opp}$-finite and of finite $\mathfrak{g}$-length, and X is the continuous dual of its completion with respect to $\mathfrak{n}$. In other words, we have a perfect duality between finitely generated $\mathfrak{P}p$-finite $\mathfrak{g}$-modules and finitely generated $\mathfrak{p}^{\rm opp}$-finite $\mathfrak{g}$-modules

$$X \to \widehat{X}^{[\mathfrak{n}^{\rm opp}]} \ .$$

So a map like the one above should not be too surprising, nor should this fact, which I conjectured many years ago and which Milič̌ić has proven: the pairing induced by the asymptotic behaviour of matrix coefficients induces an isomorphism. This seems to be a difficult, and fundamental, fact.

In particular the map from V to $V_{[\mathfrak{n}]}$ is an injection, which is also not obvious. This is related to the fact that every admissible representation embeds into principal series.

What comes next, as it did for p-adic groups, is seeing what happens for principal series, where we should be able to calculate everything explicitly, and in a natural way.

Cheers,

Bill C.

September 11, 1980

Dear Jim,

This is #4. In it I will fill up some holes in earlier letters (mainly in the p-adic part) and begin looking at the principal series of real groups. I will give no proofs at all here for real groups, but try to point out difficulties.

The first point here is an elementary one. In letter #1 I recalled the ordering on the set $[W_*\backslash W]$ related to closures of double cosets $P_*\backslash G/P$. When $W_* = \{1\}$ it is well known that this is the ordering defined combinatorially, and it is true but perhaps less well known that the same is true for $[W_*\backslash W]$. At any rate I include a proof here. (This plays an implicit role in the second letter, as I shall explain in a moment.)

Write $x \le_* y$ to mean $P_*xP \subseteq \overline{P_*yP}$ for x, y in $[W_*\backslash W]$. Write $x \le y$ for the combinatorial order. Then for any z in $[W_*\backslash W]$

$$\overline{Pw_{\ell,*}zP} = \overline{Pw_{\ell,*}P}\,\overline{PxP} = \overline{P_*zP}$$

So, applying the result for W we see that $x \le_* y \Leftrightarrow w_{\ell,*}x \le w_{\ell,*}y$. I recall that $w_{\ell,*}$ is the longest element of W_*. Also, from the combinatorial definition $x \le y$ implies $w_{\ell,*}x \le w_{\ell,*}y$. Furthermore

$$w_{\ell,*}x \le w_{\ell,*}y \Leftrightarrow w_{\ell,*}xw_\ell \ge w_{\ell,*}yw_\ell$$

However $z \mapsto w_{\ell,*}zw_\ell$ is stable on $[W_*\backslash W]$. Thus in turn $xw_\ell \ge yw_\ell$, $x \le y$. All in all

$$x \le_* y \Leftrightarrow w_{\ell,*}x \le w_{\ell,*}y \Leftrightarrow x \le y\ .$$

The point is the validity of something we used earlier:

Proposition. *The map $z \mapsto w_{\ell,*}zw_\ell$ is an order-inverting involution of $[W_*\backslash W]$.*

The second point is the proof of the fundamental formula for the canonical pairing in the first letter. I recall here what that formula was.

We are given the representation σ of P_* and are looking at the asymptotic behaviour of matrix coefficients of $\mathrm{Ind}(\sigma \,|\, P_*, G)$. Suppose f in $\mathrm{Ind}(\sigma)$ and φ in $\mathrm{Ind}(\widetilde{\sigma})$. We want to know that

$$\langle R_a f, \varphi\rangle = \sum_{[W_*\backslash W]} \langle w^{-1}(\sigma_{M_*\cap N})(a)\Omega_w(f), \widetilde{\Omega}_w(\varphi)\rangle$$

for $a \sim 0$. Since we know the relationship between asymptotic behaviour and the canonical pairing, and know in particular that the asymptotics only depends on the images of f and φ in their Jacquet modules, we can reduce this to special cases where f and φ have certain images in the Jacquet modules.

Convergence questions cause technical difficulties which I want to avoid. In fact, convergence need play no role, as it shouldn't since after all integrals on p-adic spaces are at best finite sums. Let me give one example which I shall use in a moment. I have mentioned that the pairing of $\mathrm{Ind}(\sigma \,|\, P_*, G)$ and $\mathrm{Ind}(\widetilde{(\sigma)} \,|\, P_*, G)$ is through the integrals

$$\int_{N_*^{\mathrm{opp}}} \langle f(n), \varphi(n) \rangle \, dn$$

But this involves convergence questions which I want to avoid. So now let me point out a technical way to deal with this: (1) The pairing can be defined in terms which involve no convergence, for example as an integral over K since $G = P_*K$. (2) For f and φ with support on $P_*N_*^{\mathrm{opp}}$ this pairing is (up to a positive constant) the integral over N_*^{opp} just above. Better, since the pairing is G-covariant: we can cover $P_*\backslash G$ by translates of N_*^{opp}, and on each one of these we have a similar integral formula for functions on those translates. The coset $P_*N_*^{\mathrm{opp}}$ is a neighbourhood of P_* in $P_*\backslash G$. The translation $P_*N_*^{\mathrm{opp}}w$ is a neighbourhood of P_*w for w in $[W_*\backslash W]$. So for f and φ with support on $P_*N_*^{\mathrm{opp}}w$ the pairing is

$$\int_{w^{-1}N_*^{\mathrm{opp}}w} \langle f(wn), \varphi(wn) \rangle \, dn$$

The point of this is that integrals over K are rather complicated, whereas integrals over unipotent groups are simple. So the best way to evaluate the pairing on an arbitrary pair f, φ is to write each of them as a sum of terms f_w, φ_w with support on $P_*n_*^{\mathrm{opp}}w$, and then use the integral just above. This decomposition is possible because of this simple remark about p-adic spaces: *if X is a p-adic space covered by a finite number of open subsets X_i then the space $C_c^\infty(X)$ is the sum of its subspaces $C_c^\infty(X_i)$.*

Something similar is valid for real groups, but with Schwartz spaces replacing functions of compact support.

Now what we want to prove is that for f with support in $\cup_{x \ge w} P_*xP^{\mathrm{opp}}$ and φ with support on $\cup_{x \le w} P_*xP^{\mathrm{opp}}$

$$\langle R_a f, \varphi \rangle = \langle w^{-1}(\sigma_{M_* \cap N})(a)\Omega_w(f), \widetilde{\Omega}_w(\varphi) \rangle$$

for $a \sim 0$. We shall prove this by a decomposition technique, and for this we need a geometric result which will play a role for both p-adic and real groups. It must be well known, but I am not aware of a reference.

Proposition. *We have*

$$\bigcup_{x \ge w} P_*xP = \bigcup_{x \ge w} P_*N_*^{\mathrm{opp}}x \quad (x \in [W_*\backslash W]) \, .$$

Proof. Keep in mind that $P_*N_*^{\mathrm{opp}} = P_*P^{\mathrm{opp}}$.

Step (1) It suffices to do this for $P_* = P$ minimal. This is because for w in $[W_*\backslash W]$

$$\bigcup_{x \ge w, x \in [W_*\backslash W]} P_*xP = \bigcup_{x \ge w} P_*xP = \bigcup_{x \ge w} PxP$$

To see this it suffices to prove that if x in W satisfies $x \ge w$ then every element of W_*x also satisfies it. This is equivalent to $x_0 \ge w$, where x_0 is the smallest element of W_*x. Now $x = sx_0$ with s in W_*. Since $w \le x$, there exists a reduced expression

$$x = s_1 s_2 \ldots s_n t_1 t_2 \ldots t_m, \quad s = \prod s_i, \quad x_0 = \prod t_j$$

where

$$w = s_{i_1} s_{i_2} \dots s_{i_k} t_{j_1} t_{j_2} \dots t_{j_\ell}$$

By Lemma 3.7 of Borel-Tits one may obtain a reduced expression for w in this way, and assume $\ell(w) = k + \ell$. But since w is in $[W_*\backslash W]$ and the s_i in W_*, $k = 0$. So w is a sub-product of t_j and $w \le x_0$.

Step (2) To prove the Lemma itself, with P minimal. The argument comes in two halves: (a) $PP^{\rm opp}w \subseteq \bigcup_{x\ge w} PxP$ and (b) $PwP \subseteq PP^{\rm opp}w$. For the first, note that

$$PP^{\rm opp}wP = Pw_\ell Pw_\ell wP \subseteq \bigcup_{y\le w_\ell} Pw_\ell yP = \bigcup_{y\ge w} PyP$$

since $u \ge v \Leftrightarrow w_\ell u \le w_\ell v$.

The second is because

$$PwPw^{-1}P^{\rm opp}w = PwPw^{-1}w_\ell Pw_\ell w = Pw_\ell Pw_\ell w$$

since $PuP\,PvP = PuvP$ whenever $\ell(uv) = \ell(u) + \ell(v)$.

Corollary.

$$P_*wP^{\rm opp} \cup \bigcup_{x\not\ge w} P_*xP^{\rm opp} = P_*P^{\rm opp}w \cup \bigcup_{x\not\ge w} P_*P^{\rm opp}x$$

So now what we have to do is look at the matrix coefficient when f has support on $P_*N^{\rm opp}x$ and φ has support on $P_*N_*^{\rm opp}y$, where x and y are in $[W_*\backslash W]$. But in this case the calculation is simple, and I leave it as an exercise.

* * *

Now let me turn to real groups. For p-adic groups I was able to define a filtration of principal series according to support on Bruhat closures. For real groups, this is not so simple. Consider as an example the representation of $SL_2(\mathbb{R})$ on the space of functions on $\mathbb{P}(\mathbb{R})$. The point ∞ is stable under the Borel group P of upper triangular matrices, and its complement is one large P-orbit. One candidate for a P-stable filtration might be the one whose bottom term consists of those functions vanishing at ∞. But for reasons similar to those leading me to reject $V/\mathfrak{n}V$ as a Jacquet module, this is not a good definition. Instead, following the reasoning I used there, we are led to define I_w to be the space of all smooth functions on $\mathbb{P}(\mathbb{R})$ vanishing of *infinite* order at ∞, and I_1 to be the quotient. The trouble with this is no K-finite function lies in I_w, so we must consider instead the whole space of smooth functions on $\mathbb{P}(\mathbb{R})$. We get an exact sequence

$$0 \to I_w^\infty \to I^\infty \to I_1^\infty \to 0$$

where the right hand map is that taking f to its Taylor series at ∞. The spaces I_w^∞ and I_1^∞ are not stable under G but they are stable under $U(\mathfrak{g})$. As a representation of N the space I_w is isomorphic to the Schwartz space of N, while I_1 is that of formal power series in $1/x$. The action of N on these spaces is simple.

The space I_1 is as a module over $\mathfrak{g}$ isomorphic to the Verma dual

$$\mathrm{Hom}_{U(\mathfrak{p})}(U(\mathfrak{g}), \mathbb{C})$$

where $\mathfrak{p}$ acts trivially on $\mathbb{C}$, on the left on $U(\mathfrak{g})$. It is the linear dual of the Verma module $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \mathbb{C}$. So things should look a bit familiar.

We can now filter the space $I^\infty_{[\mathfrak{n}]}$ as we did for p-adic groups. The point now is that our filtration gives although the filtration by $I_{w,\,1}$ is not one of K-finite functions, it does give a filtration of the $\mathfrak{n}$-completion of the K-finite functions.

Since we know that asymptotic behaviour factors through $\mathfrak{n}$-completions, we now do as before and discuss the asymptotic behaviour of smooth functions with support on various covering sets of $\mathbb{P}(\mathbb{R})$.

Thus, it seems to me plausible that the results we want have to pass through an analysis of $\mathfrak{n}$-completions and smooth representations of G, and even some relationship between the two.

Let me say a bit more. Suppose G arbitrary. Let σ be a finite-dimensional representation of P. Define $\mathrm{Ind}^\infty(\sigma \,|\, P, G)$. Frobenius reciprocity holds for smooth induction, too. But it has a topological qualification:

$$\mathrm{Hom}_{G,\mathrm{cont}}(V, \mathrm{Ind}^\infty(\sigma)) \cong \mathrm{Hom}_{P,\mathrm{cont}}(V, \sigma)$$

for any smooth G-representation V. (Here I will forget the $\delta^{1/2}$ factor.) We can write the right-hand side as

$$\mathrm{Hom}_P(\widehat{\sigma}, \widehat{V})$$

where now $\widehat{V}$ is the continuous dual of V. Since $\mathfrak{n}$ acts nilpotently on σ, this is the same as

$$\mathrm{Hom}_P(\widehat{\sigma}, \widehat{V}^{[\mathfrak{n}]})$$

Continuity matters disappear if we take V to be one of the canonical smooth representations defined by Wallach and me. Also, for I itself $I/\mathfrak{n}^m I$ is always finite-dimensional.

We seem to need now the fact that the $\mathfrak{n}$-completions of I and I^∞ are the same, and $(\mathfrak{g}, K)$ maps between K-finite representations extend continuously to smooth ones (more of Wallach and me).

So we now must look at the asymptotic behaviour of matrix coefficients for the smooth principal series. Whereas for K-finite functions the asymptotic series converge, this does happen here. We are in effect doing some simple calculus.

At any rate, the next step is to define a filtration I^∞ of any smooth principal series or, using Casselman-Wallach, representations induced from canonical smooth representations of parabolic subgroups. I will deal with the general case in the next letter, and here finish by looking at $SL_2(\mathbb{R})$.

We want to describe the matrix coefficient $\langle R_a f, \varphi \rangle$ where f is inInd$(\sigma \,|\, P, G)$ and φ in $\mathrm{Ind}(\sigma^{-1} \,|\, P, G)$ (σ a character). By techniques described above, we break φ into two pieces, each vanishing at infinity on two open subsets of G. Say, for example, that φ vanishes of infinite order along Pw. We can look at the convergent integral

$$\int_{N^{\mathrm{opp}}} f(na)\varphi(n)\, dn$$

where f has some kind of moderate growth along Pw. Here $a > 0$. The integral becomes

$$\int_{\mathbb{R}} \varphi(x) f(cx)\, dx$$

with φ in the Schwartz space, and we now take $c \to 0$. Expand

$$f(x) = \sum \frac{f^{(n)}(0)}{n!} x^n$$

at 0. Then our integral becomes formally

$$\sum c^n \, \frac{f^{(n)}(0)}{n!} \int x^n \varphi(x) \, dx$$

The coefficients here are like those defining intertwining operators, except that they come from distributions

$$f \mapsto f^{(n)}(0), \quad \varphi \mapsto \int_{\mathbb{R}} x^n \varphi(x) \, dx$$

which is $\mathfrak{n}$-nilpotent instead of $\mathfrak{n}$-invariant. When $n = 0$ they are the standard ones, however. Next time, more details. We are getting close.

Cheers,

Bill

October 9, 1980

Dear Jim,

This is #5. I continue with the discussion of asymptotics for real groups.

I begin by generalizing the calculation I did at the end of the last letter for arbitrary groups. Suppose $P_* = P_\emptyset = P$, σ a character of P, φ in $\mathrm{Ind}(\sigma^{-1}|P, G)$ with support on PP^{opp}.

Restricted to N^{opp} the function lies in $\mathcal{S}(\mathbb{R})$. The matrix coefficient is

$$\int_{N^{\mathrm{opp}}} f(xa)\varphi(x)\,dx = \int_{N^{\mathrm{opp}}} f(a\,a^{-1}xa)\varphi(x)\,dx = \sigma(a)\int_{N^{\mathrm{opp}}} f(a^{-1}xa)\varphi(x)\,dx$$

Express f near 1 on N^{opp} in terms of Taylor's series

$$f(x) \sim \sum_\Phi D_\Phi f(1)\Phi(x)$$

where the sum is over polynomials Φ on N^{opp}. Keep in mind that $U(\mathfrak{n}^{\mathrm{opp}})$ and the space of polynomials are in duality via $\langle D, \Phi\rangle = D\Phi(1)$. Choose Φ to be eigenvectors of A, D_Φ a dual basis. So we get an expansion

$$\sigma(a)\sum_\Phi (D_\Phi f)(1)\chi_\Phi(a^{-1})\int_{N^{\mathrm{opp}}} \Phi(x)\varphi(x)\,dx$$

The coefficients as functionals acting on f and φ are all $\mathfrak{n}^{\mathrm{opp}}$-finite.

In the more general case, nothing quite so explicit is possible, or at least very elegant. The best framework *seems* to be the one using by $\mathfrak{n}$-completion mechanism. I showed earlier that for finitely generated Harish-Chandra modules we expect an isomorphism

$$(V_{[\mathfrak{n}]})^{[\mathfrak{n}^{\mathrm{opp}}]} \to \widehat{U}^{[\mathfrak{n}^{\mathrm{opp}}]}$$

where $U = \widetilde{V}$, and now I want to add that I expect such an isomorphism to hold for each of the pieces in the Bruhat filtration of the smooth induced representations, where we extend σ to be the canonical smooth extension of the Harish-Chandra module constructed by Wallach and me. We expect this isomorphism to be explicit and natural on each piece.

Let me try to restate what I mean in more precise terms, and at the same time recall what I need.

Let (σ, U) be a 'reasonable' smooth representation of the parabolic subgroup P, $\widetilde{\sigma}$ a reasonable smooth contragredient. We want $\langle \sigma(p), u, \widetilde{u}\rangle$ to have nice asymptotic behaviour. Among other things σ should be nilpotent on $\mathfrak{n}$. Let X the quotient of $\mathrm{Ind}^\infty(\sigma \,|\, P, G)$ by the subspace of functions vanishing along P. This is algebraically $\mathrm{Hom}_{(\mathfrak{g},P)}(U(\mathfrak{g}), \sigma)$. let Y

be the space of φ in $\mathrm{Ind}(\widetilde{\sigma}\,|\,P,G)$ vanishing of infinite order along the complement of PP^{opp}. We can pair these two spaces and define

$$\langle R_a f, \varphi\rangle = \int_{N^{\mathrm{opp}}} \langle f(an), \varphi(n)\rangle \, dn$$

The asymptotics of this turn out to depend only on the image of f in X, thus inducing a map

$$(X_{[\mathfrak{n}]})^{[\mathfrak{n}^{\mathrm{opp}}]} \to \widehat{Y}^{[\mathfrak{n}^{\mathrm{opp}}]}$$

which, like the original canonical map is an isomorphism. This map can be given explicitly in terms of the canonical dual maps for σ, and that is what you (and I) want to know.

Let me go back again to $P = P_\emptyset$ to avoid topological questions. Then σ is finite dimensional. Here $X = X_{[\mathfrak{n}]}$. The subspace of $\mathfrak{n}^{\mathrm{opp}}$-torsion in X consists simply of the image of polynomials on N^{opp} with values in σ. That is to say we are mapping them to their Taylor's series at 1. But now such polynomials give rise canonically to elements of $\widehat{Y}^{[\mathfrak{n}^{\mathrm{opp}}]}$ as well, via integration:

$$\varphi \mapsto \int_{N^{\mathrm{opp}}} \langle \Phi(x), \varphi(x)\rangle \, dx$$

The theorem is now that *this natural map is the one given by asymptotics.*

The argument for proving this is the same as the one at the beginning of this section.

Now look at the general case. Suppose we are given the canonical map for σ. The space $(X_{[\mathfrak{n}]})^{[\mathfrak{n}^{\mathrm{opp}}]}$ is identical with polynomials on N^{opp} with values in $(U_{[\mathfrak{n}_\emptyset]})^{[\mathfrak{n}_\emptyset^{\mathrm{opp}}]}$, which we know maps canonically into $\widehat{\widetilde{U}}^{[\mathfrak{n}_\emptyset^{\mathrm{opp}}]}$, say via ι. Since $\mathfrak{n}^{\mathrm{opp}}$ is an ideal in $\mathfrak{n}_\emptyset^{\mathrm{opp}}$, $\mathfrak{n}_\emptyset^{\mathrm{opp}}$ acts nicely on this. Again the polynomial Φ corresponds to an element of $\widehat{Y}^{[\mathfrak{n}_\emptyset^{\mathrm{opp}}]}$, the one taking

$$\varphi \mapsto \int_{N^{\mathrm{opp}}} \langle \iota\Phi(x), \varphi(x)\rangle \, dx$$

A very similar formula always holds for every component of the Bruhat filtration. I find this independence of context very striking.

This must be almost what you want, although I have to confess your notation is sufficiently different from mine that I don't understand your notes so well.

Regards,

Bill

June 27, 1981

Dear Jim,

This is the last letter in the series. I don't see any more difficulties. The technical result we need, the decomposition of a Schwartz space into sums (a kind of partition of unity), turns out to be already known and indeed in the literature. It is a well known result of Łojasiewicz, and I found it referred to in a paper of Kashiwara's.

Here are some comments on your manuscript . . .

Regards,

Bill