AUTOMORPHIC FORMS AND A HODGE THEORY FOR CONGRUENCE SUBGROUPS OF $SL_2(\mathbb{Z})$

William Casselman
Vancouver

Suppose for the moment that Γ be any arithmetic subgroup of $SL_2(\mathbb{R})$. It acts discretely on the upper half-plane H by fractional linear transformations

$$z \to az + b/cz + d ,$$

and the quotient will be a non-singular Riemann surface. If the quotient $X = \Gamma\backslash H$ is compact then we have for its cohomology the usual Hodge theory:

$$H^0(X, \mathbb{C}) = \mathbb{C}$$

$$H^1(X, \mathbb{C}) = H^{1,0} \oplus H^{0,1} = \mathbb{C}^g \oplus \mathbb{C}^g$$

$$H^2(X, \mathbb{C}) = \mathbb{C}$$

where g is the genus of X. If X is not compact then it may be embedded into a compact, non-singular Riemann surface $\overline{X}$ by adding a finite number of points at infinity, called <u>cusps</u>:

$$X \cup \{\text{cusps}\} = \overline{X} .$$

The cohomology of X may be determined simply in terms of that of $\overline{X}$ by looking at the long exact sequence of relative cohomology associated to the embedding $X \to \overline{X}$:

$$H^0(X, \mathbb{C}) = \mathbb{C}$$

$$H^1(X, \mathbb{C}) = H^1(\overline{X}, \mathbb{C}) \oplus \mathbb{C}^{n-1}$$

$$H^2(X, \mathbb{C}) = 0 .$$

Here n is the number of cusps.

The cohomology $H^1(\overline{X}, \mathbb{C})$ is represented on X by cusp forms, holomorphic and anti-holomorphic, of weight 2. For the contribution of $\overline{X}$ to the cohomology of X, in other words, there is a perfectly good Hodge theory: yet another way of saying this is that $H^1(X, \mathbb{C})$ contains a subspace called cuspidal cohomology where cohomology is represented uniquely by harmonic forms. What I will be concerned with in this paper is understanding in a similar way the non-cuspidal cohomology - i.e. the term $\mathbb{C}^{n-1}$. It is clear that there must be some relationship with Eisenstein series of weight 2, which in this case also have dimension $(n-1)$. But then how to take account of the anti-holomorphic Eisenstein series, which are of course also harmonic?

Hodge theory on compact Riemannian manifolds involves something which is potentially much deeper, the analysis of the Laplace operator on C^∞ and even $\mathbb{L}^2$ forms. Similarly, what I have in mind is an analysis of the Laplace operator on X, which has a continuous spectrum when X is non-compact. The appropriate result of this analysis is a theorem resembling classical results of Paley-Wiener rather than an $\mathbb{L}^2$ decomposition. My account of this will take advantage of the opportunity to present a brief introduction to the theory of automorphic forms.

Nearly all of what I will say extends in some form to larger arithmetic groups, in particular to describe the cohomology of arithmetic subgroups of semi-simple rational groups of $\mathbb{Q}$-rank one. This completes work of Harder [1975a,b]. This extension has been joint work with

Birgit Speh. Details will appear sooner or later.

The right way to look at these matters, especially when larger arithmetic groups are concerned, is by means of representation theory. However, I have made this account as elementary as possible, and this subject will generally only lurk in the background. Also, I should add that the Hodge theory I suggest here is quite different from - much more analytical than - Deligne's version (as presented, say, in Griffiths-Schmid [1975]). In the context of quotients of general Hermitian symmetric spaces by discrete groups Γ, the relationship between the two will probably be interesting.

1. Tempered currents

The cohomology $H^{\bullet}(X, \mathbb{C})$ may be identified with the cohomology of the de Rham complex of C^{∞} forms on X, but for various technical reasons what I need to know is that it is also that of the complex of currents on X. Recall that a current is a continuous linear functional on the space of C_c^{∞} forms - see de Rham's book [1960]. I need to know also that the space of currents can be replaced by a certain subspace, that of tempered currents. This reduction is, as will be seen, a crucial step.

Let $\Omega \subseteq H$ be a region of the form $|x| \le a$, $y \ge b > 0$. A Schwartz function on Ω is a C^{∞} function $f\colon \Omega \to \mathbb{C}$ all of whose partial derivatives (in x and y) are $O(y^{-n})$ for all $n > 0$. Similarly a Schwartz form on Ω is one whose coefficients with respect to dx, dy or $dx \wedge dy$ are Schwartz functions.

If $\Gamma \subseteq SL_2(\mathbb{Q})$ is an arithmetic subgroup, let Γ_{∞} be the subgroup of all matrices in Γ of the form $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$. This will be generated by a single element $\begin{pmatrix} 1 & x_0 \\ 0 & 1 \end{pmatrix}$. A basis of neighborhoods of the cusp $i\infty$ is formed by the images in $\Gamma\backslash H$ of the sets $y > y_0$. A Schwartz function on this neighborhood is one which lifts back to a Schwartz function on the region $y > y_0$. If P is any other cusp, then some element g of $SL_2(\mathbb{Z})$ will transform it to $i\infty$. Coordinates (x,y) near P one obtains by pull-back will be called proper coordinates. A Schwartz function in the neighborhood of P is one which corresponds in this way to a Schwartz function on $g\Gamma g^{-1}\backslash H$ in the neighborhood of the cusp $i\infty$. A Schwartz function on all of $\Gamma\backslash H$ is one which is C^∞ on all of $\Gamma\backslash H$ and a Schwartz function in the neighborhood of each cusp. Similarly for a Schwartz form. I will write the space of all such functions on $X = \Gamma\backslash H$ as $S(X)$, that of all Schwartz forms as $\Omega_S^\cdot(X)$.

1.1. Theorem. The inclusion of $\Omega_c^\cdot(X)$ in $\Omega_S^\cdot(X)$ induces an isomorphism of cohomology.

Here Ω_c is the de Rham complex of C_c^∞ forms on X. In other words, the compactly supported cohomology of X may be identified with the cohomology of the Schwartz forms on X. Results like this were first noticed, I believe, by Borel.

One proof goes like this: compactify X by adding whole circles at each cusp, not just points: one point at $i\infty$, for example, for every vertical line $x = \text{const.}$ (modulo x_0). This gives the Borel-Serre compactification of X, which is a manifold with boundary. As a coordinate system on this boundary circle, use (x, y^{-1}): this gives neighborhoods of the component isomorphic to $(\mathbb{R}/\mathbb{Z}) \times [0,c)$ for various c. The Schwartz functions in this neighborhood are precisely those which are C^∞ up to and including the boundary circle, but vanish of infinite order along the boundary. Similarly for Schwartz forms, and other cusps as well, using proper coordinates. Thus the notion of Schwartz form here is a special case of the following definition: let M be any manifold with boundary ∂M. A Schwartz form on M is one which is C^∞ everywhere on M, up to and including the boundary, and vanishes of infinite order on the boundary. If M is locally $\{(x_1,\ldots,x_n) | x_n \geq 0\}$ then a C^∞ form on M is one obtained by restriction from $\mathbb{R}^n$. In this situation, both forms with support in the interior of M, which I will call $\Omega_c^{\bullet}(\mathrm{Int}\ M)$, and Schwartz forms are global sections of the fine sheaves on M defined by their germs. (The stalks of $\Omega_c^{\bullet}(\mathrm{Int}\ M)$ are null on the boundary.) Since $\Omega_c^{\bullet} \subseteq \Omega_S^{\bullet}$, the natural obvious generalization of 1.1 to arbitrary M is a consequence of a standard sheaf-theoretic result (see Godement [1958], Theorem 4.6.2 on p. 178) and thus:

1.2. Sublemma. *At the boundary of* M, *the cohomology of the complex of germs of Schwartz forms is null*.

This is an easy consequence of almost any proof of the Poincaré Lemma.

Dual to the complex $\Omega_c^{\bullet}$ on X is that of currents on X. Define the *tempered* currents $C^{\bullet}$ to be those currents which extend continuously to $\Omega_S^{\bullet}$. Since $\Omega_c^{\bullet}$ is dense in $\Omega_S^{\bullet}$, the extension must be unique. It is known that the cohomology of the complex of currents is the same as the ordinary cohomology of X, and dual to the cohomology of compact support. As a consequence of Theorem 1.1:

1.3. Corollary. *The inclusion of the complex of tempered currents in that of all currents induces an isomorphism of the cohomology of this complex with* $H^{\bullet}(X, \mathbb{C})$.

It may seem that replacing, say, the ordinary C^{∞} forms on X by such a large space is a regression, but it will turn out that this is not the case.

2. A decomposition of the Schwartz spaces

One way to obtain forms on $X = \Gamma\backslash H$ is by the construction of Eisenstein series. In the beginning we will need only the simplest version.

First I will formulate a technical observation about cusps. Consider the region $y \geq Y$ in $\mathcal{H}$ and its transform by an element $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, with $c \neq 0$. This image is the interior of a circle which touches the real axis at $g(\infty) = a/c$, and using the formula

$$y(g(z)) = y(z)|cz + d|^{-2}$$

it is not hard to see that the height of this circle is $(c^2 Y)^{-1}$.

Now let Γ be an arithmetic subgroup of $SL_2(\mathbb{Q})$. Its intersection with $SL_2(\mathbb{Z})$ has finite index in Γ, so that Γ preserves some lattice in $\mathbb{Z}^2$. Therefore some conjugate of Γ is contained in $SL_2(\mathbb{Z})$, and one may as well assume this to be the case for Γ itself. Hence for any $g \in \Gamma$ with $c \neq 0$, the image of the region $y \geq Y$ under g lies inside the region $y \leq Y^{-1}$.

Also, suppose P is some cusp of Γ not equivalent to $i\infty$ — i.e. in particular a rational number. Let C be a circle of height $Y > 0$ touching the real axis at P. Say $P = g(\infty)$ for some

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) .$$

Then for any $\gamma \in \Gamma$ the γ-transform of C is another circle in $\mathcal{H}$ touching the real axis. Since $\gamma = \gamma g \cdot g^{-1}$, the previous remarks apply to show that the height of γC is at most $c^2 Y$.

Now let ω be a C^∞ differential form on $\mathcal{H}$ satisfying these conditions: (1) it is invariant under translations and (2) its support lies in a band $0 < y_1 \leq y \leq y_2 < \infty$. If $\omega = \phi$ is a function, for example, these conditions amount to requiring that ϕ be the lift to $\mathcal{H}$ of a C_c^∞ function in $y \in (0, \infty)$. The Eisenstein series associated to ω is the form

$$E_\omega = \sum_{\Gamma_\infty \backslash \Gamma} \gamma^*(\omega) .$$

Here Γ_∞ is the subgroup $\begin{pmatrix} \pm 1 & * \\ 0 & \pm 1 \end{pmatrix}$ of Γ stabilizing the cusp $i\infty$. (It is slightly bigger than the Γ_∞ I defined earlier.) This sum makes at least formal sense because $\gamma^*(\omega) = \omega$ for $\gamma \in \Gamma_\infty$. It makes real sense because, as the geometrical observations above imply, only a finite number of terms are non-zero. The form E_ω is clearly Γ-invariant, and even of compact support modulo Γ.

Let P be some other cusp of Γ. Let N_P be the subgroup of unipotent elements of $SL_2(\mathbb{R})$ fixing P. If ω is a C^∞ form on H which is N_P-invariant and of compact support modulo N_P, we can define similarly an Eisenstein series associated to ω.

Assign to H the $SL_2(\mathbb{R})$-invariant measure coming from the form $y^{-2}dxdy$, and X the quotient measure. Define

$\Omega_2^\bullet$ = $\mathbb{L}^2$-forms on X

$\Omega_{2,Eis}^\bullet$ = closure in $\Omega_2^\bullet(X)$ of the subspace spanned by the compactly supported Eisenstein series associated to all cusps

$\Omega_{S,Eis}^\bullet$ = $\Omega_S^\bullet \cap \Omega_{2,Eis}^\bullet$

$\Omega_{2,cusp}^\bullet$ = orthogonal complement in $\Omega_2^\bullet$ of $\Omega_{2,Eis}^\bullet$

$\Omega_{S,cusp}^\bullet$ = $\Omega_S^\bullet \cap \Omega_{2,cusp}^\bullet$.

Then of course

$$(2.1) \quad \Omega_2^\bullet = \Omega_{2,Eis}^\bullet \oplus \Omega_{2,cusp}^\bullet$$

by definition. What is not at all obvious, however, is that there is a corresponding decomposition of the Schwartz forms as well. This is what will be proven in the rest of this section, except that for convenience of notation I will look only at functions, rather than forms, and will also assume Γ has one cusp (which I take to be $i\infty$). I will write $\mathbb{L}^2(X)$ instead of $\Omega_2^0(X)$, etc.

The elements of $\mathbb{L}^2_{cusp}$ have a useful characterization. Choose $\phi \in C_c^\infty(0,\infty)$. Then for $F \in \mathbb{L}^2(X)$,

$$(2.2) \qquad \langle F, E_\phi\rangle = \iint_{\Gamma\backslash H} F(z)E_\phi(z)y^{-2}dxdy$$

$$= \iint_{\Gamma\backslash H} F(z) \sum_{\Gamma_\infty\backslash\Gamma} \phi(y(\gamma(z)))y^{-2}dxdy$$

$$= \iint_{\Gamma_\infty\backslash H} F(z)\phi(y(z))y^{-2}dxdy$$

$$= \int_0^M dx\int_0^\infty F(x+iy)\phi(y)y^{-2}dy$$

$$= M\int_0^\infty F_0(y)\phi(y)y^{-2}dy$$

where

$$(2.3) \qquad F_0(y) = \frac{1}{M}\int_0^M F(x+iy)dx .$$

Here I take Γ_∞ to be generated by $\begin{pmatrix} \pm 1 & M \\ 0 & \pm 1 \end{pmatrix}$. Of course this makes sense only for almost all y. At any rate, $F \in \mathbb{L}^2_{cusp}$ if and only if this inner product (2.2) is null for all $\phi \in C_c^\infty(0,\infty)$: if and only if $F_0(y) = 0$.

The term F_0 is just one of several in the Fourier expansion of F:

$$(2.4) \qquad F(x+iy) = \sum F_n(y)e^{2\pi inx/M}$$

where

$$(2.5) \qquad F_n(y) = \frac{1}{M}\int_0^M e^{-2\pi inx/M}F(x+iy)dx.$$

The whole group $SL_2(\mathbb{R})$ acts on H, and the isotropy group of i is $K = SO(2)$. Hence functions on $\Gamma\backslash H$ may be identified with functions on $\Gamma\backslash SL_2(\mathbb{R})/K$, or with K-invariant functions on $\Gamma\backslash SL_2(\mathbb{R})$. Let f be a C_c^∞ function on $SL_2(\mathbb{R})$ which is bi-invariant with respect to K. Then for $F \in \mathbb{L}^2(\Gamma\backslash SL_2(\mathbb{R})/K)$ define

$$F*f(g) = \int_{SL_2(\mathbb{R})} F(gx)f(x)dx .$$

This lies again in $\mathbb{L}^2(\Gamma\backslash SL_2(\mathbb{R})/K)$, and is also C^∞. This technical remark allows us to prove

2.1. Lemma. <u>The intersection of</u> $C_c^\infty(X)$ <u>with</u> $\mathbb{L}^2_{cusp}(X)$ <u>is dense in</u> $\mathbb{L}^2_{cusp}(X)$.

Proof. Given $F \in \mathbb{L}^2(X)$ and Y large, let F_Y be the truncation of F in the neighborhood $y \geq Y$ of the cusp - that is to say $F_Y(x+iy) = 0$ for $y \geq Y$ while F and F_Y agree for $y < Y$ in X. As $Y \to \infty$, $F_Y \to F$ in $\mathbb{L}^2(X)$, clearly. Now given F_Y, choose functions $\phi \in C_c^\infty(K\backslash SL_2(\mathbb{R})/K)$ which are non-negative, of total integral one, and of support tending to K. Then $F_Y*\phi$ will be compactly supported in X and tend towards F_Y in $\mathbb{L}^2(X)$. If F lies in $\mathbb{L}^2_{cusp}(X)$ so does F_Y and $F_Y * \phi$, which proves the lemma.

This result yields a useful characterization of the decomposition of $F \in \mathbb{L}^2(X)$ according to (2.1). Say

(2.6) $$F = F_{cusp} + F_{Eis} .$$

Then (a) for every $\phi \in C^\infty_{c,cusp}(X)$

(2.7) $$\langle F,\phi\rangle = \langle F_{cusp},\phi\rangle$$

and (b) for every $\phi \in C^\infty_{c,Eis}(X)$

(2.8) $$\langle F,\phi\rangle = \langle F_{Eis},\phi\rangle .$$

It follows from Lemma 2.1 that these properties determine the decomposition (2.6). In other words, if $F_c \in \mathbb{L}^2_{cusp}(X)$ and $F_E \in \mathbb{L}^2_{Eis}$ such that (2.7)-(2.8) hold for F_c and F_E, respectively, then $F_c = F_{cusp}$ and $F_E = F_{Eis}$.

2.2 Theorem. The space $S(X)$ is the direct sum of $S_{cusp}(X)$ and $S_{Eis}(X)$. If $f \in S(X)$ and $f = f_{cusp} + f_{Eis}$ according to (2.1), then both f_{cusp} and f_{Eis} lie in $S(X)$.

Proof. It must be shown that $f \in S(X)$ implies $f_{cusp} \in S(X)$.

Since $f \in S(X)$, so is every $\Delta^n f$, where Δ is the Laplacian of H:

$$\Delta = y^2(\partial^2/\partial x^2 + \partial^2/\partial y^2) .$$

The differential operator Δ commutes with $SL_2(\mathbb{R})$ on H, and in particular with N_∞, so that if $\phi \in C^\infty_{c,cusp}$ so is every $\Delta^n\phi$. Similarly for $\phi \in C^\infty_{c,Eis}$. From this and the remarks about (2.6)-(2.8) it follows that the components of $\Delta^n f$ according to (2.1) are the same as the distributions $\Delta^n(f_{cusp})$, $\Delta^n(f_{Eis})$. Thus, let $F = f_{cusp}$. Then $F \in \mathbb{L}^2(X)$ and so are all the $\Delta^n F$. By Sobolev's lemma applied locally, since Δ is elliptic, F is at least a C^∞ function on X. Also, the constant term of F is null.

Let now Y be a large positive number. Let $f(y)$ be a C^∞ function on $(0,\infty)$ with the property that it is null for $y \le Y$, 1 for $y \ge Y+1$. Then the function $\Phi = f\cdot F$ satisfies these three conditions: (1) it is a C^∞ function on $\Gamma_\infty\backslash H$ with support in the region $y \ge Y$; (2) all $\Delta^n\Phi$ lie in $\mathbb{L}^2(\Gamma_\infty\backslash H)$; (3) $\Phi_0 \equiv 0$.

2.3. Proposition. Any function Φ satisfying these three conditions is, along with all its derivatives $(\partial^{n+m}/\partial x^m \partial y^n)\Phi$, rapidly decreasing at $i\infty$.

The proof will be long, but elementary. A similar result is true when Γ is allowed to be any arithmetic subgroup of a reductive algebraic group, but the only proof I know in this case uses a well known, relatively recent, result of Dixmier-Malliavin [1978] in representation theory. The proof I give here is close to arguments about Sobolev spaces, and this result of Dixmier-Malliavin can often be used to replace such arguments.

First I will show that Φ itself vanishes rapidly.

Express

$$\Phi = \sum \Phi_n(y)e^{2\pi inx/M} .$$

Then

$$\|\Phi\|^2 = \iint |\Phi(z)|^2 y^{-2}dxdy \tag{2.7}$$

$$= M\cdot \sum \int_0^\infty |\Phi_n(y)|^2 y^{-2}dy$$

and

$$\Delta\Phi = \sum y^2(\Phi_n'' - 4\pi^2n^2M^{-2}\Phi_n)e^{2\pi inx/M} . \tag{2.8}$$

This motivates the next step. Let $D = D_\lambda$ be the operator $\phi \mapsto y^2(\phi'' - \lambda^2)$ (for $\lambda \geq 0$) and consider a function $\phi(y)$ satisfying (a) $\phi \in C^\infty(0,\infty)$ with support on $(1,\infty)$; (b) all the $y^{-\ell}D^{\ell}\phi$ lie in $\mathbb{L}^2(0,\infty)$. For example, each Φ_n satisfies these, with $\lambda = 2\pi|n|/M$. Let $\psi = D\phi$. Then

$$\phi'' - \lambda^2\phi = y^{-2}\psi$$

or

$$\phi = c_1e^{\lambda y} + c_2e^{-\lambda y} - \frac{1}{2\lambda}\int_1^\infty e^{-\lambda|y-x|}\psi(x)x^{-2}dx. \tag{2.9}$$

for constants c_1, c_2. The integral, ϕ, and $c_2e^{-\lambda y}$ are all $\mathbb{L}^2$, so c_1 must be 0 and

$$c_2 = C = \frac{1}{2\lambda}\int_1^\infty e^{-\lambda x}\psi(x)x^{-2}dx . \tag{2.10}$$

The integral in (2.9) must now be estimated.

2.4 Lemma. Let $\lambda_0 > 0$ be given. There exist constants C_0, y_0 depending only on λ_0 such that

$$\int_1^y e^{-\lambda(y-x)}x^{-n}dx \le \frac{C_0}{\lambda y^n}$$

for all $\lambda \ge \lambda_0$, $y \ge y_0$.

This follows from integration by parts

$$\int_1^y e^{-\lambda(y-x)}x^{-n}dx = \frac{1}{\lambda}\left[\frac{1}{y^n} - e^{-\lambda(y-1)} + n\int_1^y e^{-\lambda(y-x)}x^{-(n+1)}dx\right]$$

together with

$$\int_1^y e^{-\lambda(y-x)}x^{-(n+1)}dx \le \frac{1}{y^n}$$

2.5 Lemma. For every integer $\ell > 0$ there exists a constant C_ℓ such that

$$|\phi(y)| \le \frac{C_\ell\|D^\ell\phi\|}{\lambda^{1/2+\ell}y^{2\ell-1}}$$

whenever $\lambda \ge \lambda_0$, $y \ge y_0$.

This comes from a calculation I will sketch in a moment. First I will show how the Proposition follows. Each Φ_n satisfies the above conditions with $\lambda = |2\pi n|/M \ge 2\pi/M$ since $\Phi_0 = 0$. So take $\lambda_0 = 2\pi/M$ Then for y_0 as above and $y \ge y_0$, and any ℓ,

$$|\Phi(x+iy)| \le \sum |\Phi_n(y)|$$

$$\le \frac{C_\ell}{y^{2\ell-1}} \sum_{n\neq 0} \frac{\|D^\ell_{2\pi|n|/M}\Phi_n\|}{(2\pi|n|)^{1/2+\ell}}$$

$$\le \frac{C_\ell}{y^{2\ell-1}} \|\Delta^\ell\Phi\| \left[\sum_{n\neq 0} \left(\frac{1}{2\pi|n|}\right)^{1+2\ell} \right]^{1/2}$$

(last by Cauchy-Schwarz). Thus Φ is of rapid decrease on $y \ge 1$.

As for the derivatives of Φ, let A be the operator $y\partial/\partial x$, B be $y\partial/\partial y$. Then A and B form a Lie algebra with $[B,A] = A$ and

$$\Delta = A^2 + B^2 - B .$$

As a consequence every operator of degree $\le k$ in the universal enveloping algebra of this algebra can be written as a sum of elements in one of the forms $A^m\Delta^n$ or $BA^m\Delta^n$ with these also of degree $\le k$. Now if Φ satisfies the hypotheses of the Proposition, so do all $\Delta^n\Phi$ as well. Since

$$-\langle\Delta\Phi,\Phi\rangle = \|A\Phi\|^2 + \|B\Phi\|^2$$

both $A\Phi$ and $B\Phi$ will lie in $\mathbb{L}^2$. Since $A\Phi$ is, so is $\partial\Phi/\partial x$, and since Δ and $\partial/\partial x$ commute, so is each $\partial^n\Phi/\partial x^n$. Since this is rapidly vanishing, so is $A^n\Phi$. But also from the remarks above, $B\Phi$ will satisfy the same conditions as Φ (consider $\Delta^n B\Phi = B\Delta^n\Phi - [B,\Delta^n]\Phi$). Hence every $A^m B^n\Phi$ will be of rapid decrease.

It remains to prove 2.5. Consider

$$\phi = Ce^{-\lambda y} - \frac{1}{2\lambda}\int_1^\infty e^{-\lambda|y-x|}\psi(x)x^{-2}dx$$

where C is given by (2.10). Applying Cauchy-Schwarz we see that

$$(2.11)\qquad |C| \le \|\psi\| \cdot \frac{1}{2\lambda}\left(\int_1^\infty e^{-2\lambda y} y^{-2} dy\right)^{1/2}$$

$$\le \|\psi\| \cdot \frac{1}{2\lambda} \cdot \frac{1}{\sqrt{2\lambda}} .$$

As for the integral, decompose it into two parts. By lemma 2.4,

$$(2.12)\qquad \left|\int_1^y e^{-\lambda(y-x)}\psi(x)x^{-2}dx\right| \le \|\psi\| \left(\int_1^y e^{-2\lambda(y-x)}x^{-2}dy\right)^{1/2}$$

$$\le \frac{C_2}{\lambda^{1/2}y}\|\psi\| .$$

Also, more easily from Cauchy-Schwarz,

$$(2.13)\qquad \left|\int_y^\infty e^{-\lambda(x-y)}x^{-2}\psi(x)dx\right| \le \|\psi\| \frac{1}{\lambda^{1/2}y} .$$

Combining (2.11)-(2.13) we see that there exists C_1 and y_0 such that

$$|\phi(y)| \le \frac{C_1\|D\phi\|}{\lambda^{3/2}y}$$

whenever $\lambda \ge \lambda_0$, $y \ge y_0$. The rest of the proof is a similar calculation, proceeding by induction from ℓ to $\ell+1$. ∎

This concludes the proof of Theorem 2.2. Incidentally, the Schwartz spaces may be given the obvious Fréchet norms, and the closed graph theorem (see, for example, Trèves [1967], p. 165 and p. 173) says that the decomposition is continuous. Thus dual to the decomposition above we have one of tempered distributions: if $A(X)$ is the continuous dual of $S(X)$ then $A(X) = A_{cusp}(X) \oplus A_{Eis}(X)$, where (1) A_{cusp} is the dual of S_{cusp} and the annihilator of S_{Eis}; (2) A_{Eis} is the dual of S_{Eis} and the annihilator of S_{cusp}. I do not know how to prove it in an elementary way, and will not attempt a proof here, but it is true that S_{cusp} is dense in A_{cusp}. This has as a consequence that <u>the Eisenstein series</u> E_ϕ, <u>as</u> ϕ <u>ranges over</u> $C_c^\infty(0,\infty)$, <u>are dense in</u> S_{Eis} (which does not quite seem to be a trivial fact).

I might add that I do not know whether or not $C_c^\infty(X)$ possesses a decomposition into cuspidal and Eisenstein components.

The whole of the above theory does, however, extend to Schwartz forms, and also to arbitrary arithmetic $\Gamma \subseteq SL_2(\mathbb{Q})$. Combining this with results of §1, we see that the cohomology of X comes in two pieces: (1) that of the cuspidal currents $C^\bullet_{cusp}$ and (2) that of the Eisenstein currents $C^\bullet_{Eis}$. The first component is represented uniquely by the harmonic cuspidal automorphic forms on X. It is the second that we are specifically concerned with here.

3. Harmonic analysis of $\mathbb{L}^2(X)$

The Eisenstein series defined earlier converged for simple reasons. Now something classical:

3.1. Lemma. <u>Suppose</u> $\sigma > 1$. <u>The series</u>

$$E_\sigma(z) = \sum_{\Gamma_\infty \backslash \Gamma} y(\gamma(z))^\sigma$$

<u>converges uniformly on compact subsets of</u> $\mathcal{H}$.

This is well known, but I will include a proof (which generalizes easily to higher dimensions). Assume that Γ has no torsion elements. This is true of some subgroup of finite index if not of Γ itself, so it is no serious restriction. Let U be a small (non-Euclidean) disc around i, invariant under $K = SO(2)$. Then for $g \in SL_2(\mathbb{R})$, $z = g(i)$, the set $U_z = gU$ depends only on z.

For $u \in U$, the function $(c,d) \mapsto |cu + d|^2$ is a positive definite quadratic form. Hence for a fixed u the ratio

$|cu+d|^2/(c^2+d^2)$ lies in a fixed neighborhood of 1 for all $(c,d) \neq (0,0)$. Since U is relatively compact, there exists $\lambda > 1$ with

$$\lambda^{-1} < \frac{|cu+d|^2}{c^2+d^2} < \lambda$$

for all $(c,d) \neq (0,0)$, $u \in U$. Also, of course, one may find $K > 1$ such that

$$K^{-1} < y(u) < K$$

for all $u \in U$. From these remarks it is straightforward to prove that

$$K^{-1}\lambda^{-1} < \frac{y(u)}{y(z)} < K\lambda$$

for all $u \in U_z$.

Now for a given z choose U small enough so that the sets $\{\gamma U_z | \gamma \in \Gamma\}$ are all disjoint. Choose Y so $y \geq Y$ on U_z. Then for $\gamma \in \Gamma - \Gamma_\infty$, $y \leq Y^{-1}$ on γU_z. If Γ_∞ is generated by $\begin{pmatrix} \pm 1 & M \\ 0 & \pm 1 \end{pmatrix}$ then some Γ_∞-translate of γU_z will lie in the band $|x| \leq M + Y^{-1}$. So

$$E_\sigma(z) = y(z)^\sigma + \sum_{\Gamma_\infty \backslash \Gamma - \Gamma_\infty} y(\gamma(z))^\sigma$$

and

$$|E_\sigma - y^\sigma| \leq (\text{meas } U)^{-1}(K\lambda)^{2\sigma} \iint_{\substack{y \leq Y^{-1} \\ |x| \leq M+Y^{-1}}} y^{\sigma-2}\,dx\,dy$$

which is finite if $\sigma > 1$.

A similar argument and the geometrical observations in §2 will prove further:

3.2. Proposition. <u>For any</u> $s \in \mathbb{C}$ <u>with</u> $\text{Re}(s) > 1$, <u>the series</u>

$$E_s(z) = \sum_{\Gamma_\infty \backslash \Gamma} y(\gamma(z))^s$$

converges to a real-analytic function on $X = \Gamma\backslash H$, with the properties

(1) In any region $y \geq Y$, $E_s(z) - y^s = O(y^{-(s-1)})$;

(2) In the neighborhood of any cusp of Γ not equivalent to $i\infty$, $E_s = O(y^{s-1})$;

(3) $\Delta E_s = s(s-1)E_s$.

The second property refers to the inside of any circle in H just touching the real axis at P.

The property (3) follows from the identity

$$\Delta y^s = s(s-1)y^s$$

together with the fact that Γ and Δ commute.

If Y is a large positive number and χ_Y is the characteristic function of $[Y,\infty)$ then let $E_{Y,s}$ be the Eisenstein series

$$E_{Y,s}(z) = \sum_{\Gamma_\infty \backslash \Gamma} \chi_Y(y(\gamma(z)))y(\gamma(z))^s$$

For any given z, only one term is non-zero. And in the region $y \geq Y_*$ it agrees with $\chi_Y \cdot y^s$ for large enough Y_*. One immediate consequence of 3.2 is that $E_s - E_{Y,s}$ always lies in $\mathbb{L}^2(X)$. This – as Colin de Verdière [1981] points out – and the equation $\Delta E_s = s(s-1)E_s$ are the characteristic properties of E_s. Indeed, he shows that it is almost trivial to deduce from this that E_s possesses an analytic continuation into the region $\mathrm{Re}(s) > 1/2$, $s \notin (1/2, 1]$. With only a little more work he goes on to imitate the classical theory of Sturm-Liouville operators on infinite intervals and to prove:

3.3. Theorem. The Eisenstein series continues meromorphically to all of $\mathbb{C}$. The constant term of E_s at the cusp $i\infty$ is of the form $y^s + c_\infty(s)y^{1-s}$, at a cusp P not equivalent to $i\infty$ of the form

$c_P(s)y^{s-1}$, <u>where the</u> $c_P(s)$ <u>have poles in the region</u> $\mathrm{Re}(s) \geq 1/2$ <u>at most on</u> $(1/2, 1]$.

I am not sure of the history of this result. There are many expositions, such as Kubota [1973], but the paper of Colin de Verdière is extraordinarily lucid, and recommended highly. Similar techniques probably extend to what might be called the rank-one situation for more general rational reductive groups (see Langlands [1966]). I will take this result as given, without further explanation, and proceed from it.

I will again assume for convenience from now on that Γ has but one cusp. Let $c(s) = c_\infty(s)$. The function E_{1-s} has the same Δ-eigenvalue as E_s and its constant term is $y^{1-s} + c(1-s)y^s$. Hence $c(1-s)E_s = E_{1-s}$, and $c(s)c(1-s) = 1$. The function E_s is real for real s, so $c(\overline{s}) = \overline{c(s)}$. Hence for $\mathrm{Re}(s) = 1/2$, $|c(s)| = 1$. (It is this sort of thing that is responsible for the lack of poles of $c(s)$ on $1/2 + i\mathbb{R}$ in general.)

The first step in the further analysis of E_s is a formula for the inner product of two <u>truncated</u> Eisenstein series. These already play a role in proving 3.3. What are they? Let Y be a large positive number. Then the <u>truncation</u> $T^Y E_s$ of E_s at Y in $y \geq Y$ is the difference between E_s and $[E_{Y,s} + c(s)E_{Y,1-s}]$. Roughly speaking, in other words, it is equal to E_s in the inside of X but equal to E_s less its constant term when $y \geq Y$. A slight modification of Proposition 2.3 shows that $T^Y E_s$ and all its derivatives are rapidly decreasing as $y \to \infty$. In particular, $T^Y E_s \in \mathbb{L}^2(X)$. Another way to define $T^Y E_s$:

$$\text{let} \quad M_s(y) = \begin{cases} -c(s)y^{1-s} & y \geq Y \\ y^s & y < Y . \end{cases}$$

Then $T^Y E_s$ is the Eisenstein series $E_{M(s)}$:

$$T^Y E_s(z) \quad = \quad \sum_{\Gamma_\infty \backslash \Gamma} M_s(y(\gamma(z))),$$

when this series converges, and its meromorphic continuation when not.

3.4. Proposition. *For* $s, t \in \mathbb{C}$ *with* $s-t \neq 0$, $s+t-1 \neq 0$,

$$\langle T^Y E_s , T^Y E_t \rangle \quad = \quad \frac{Y^{s+t-1} - c(s)c(t)Y^{1-s-t}}{s+t-1}$$

$$- \frac{c(s)Y^{-s+t} - c(t)Y^{s-t}}{s-t} \quad .$$

Proof. This is a variant of Green's Theorem, since both E_s and E_t are eigenfunctions of Δ. But the simplest way to prove it is to use a generalization of the calculation (2.2):

$$\begin{aligned}\langle T^Y E_s , T^Y E_t \rangle &= \langle E_s , T^Y E_t \rangle \\ &= \langle E_s , E_{M_t} \rangle \\ &= \langle E_{s,0} , M_t \rangle \,.\end{aligned}$$

This last is a sum of two integrals:

$$(3.1) \qquad \int_0^Y [y^s + c(s)y^{1-s}]y^{t-2}dy$$

and

$$(3.2) \qquad -\int_Y^\infty [y^s + c(s)y^{1-s}]c(t)y^{1-t-2}dy \,.$$

These make sense as long as $\mathrm{Re}(s+t) > 1$, $\mathrm{Re}(t) > \mathrm{Re}(s)$. The formula in (3.4) is an easy calculation.

3.5. Corollary. <u>When</u> $t = 1-s$ <u>but</u> $t \neq s$, <u>with</u> $\mathrm{Re}(s) = 1/2$

$$\langle T^Y E_s, T^Y E_t \rangle = 2 \log Y - c'(s)c(1-s) - \frac{c(s)Y^{1-2s} - c(1-s)Y^{2s-1}}{1-2s} .$$

Proof. Take a limit in 3.4. Setting $t = \overline{s}$ in these:

3.6. Corollary. <u>For</u> $s \notin \mathbb{R}$, <u>with</u> $\mathrm{Re}(s-1/2) = \sigma$ <u>and</u> $\tau = \mathrm{Im}(s) \neq 0$,

$$\|T^Y E_s\|^2 = \frac{1}{2\sigma}[Y^{2\sigma} - |c(s)|^2 Y^{-2\sigma}] - \frac{1}{2i\tau}[c(s)Y^{-2i\tau} - \overline{c(s)}Y^{2i\tau}]$$

<u>when</u> $\sigma > 0$ <u>and</u>

$$= [2 \log Y - c'(s)\overline{c(s)}] - \frac{1}{2i\tau}[c(s)Y^{-2i\tau} - \overline{c(s)}Y^{2i\tau}]$$

<u>when</u> $\sigma = 0$.

We will not need the case $\tau = 0$.

The left-hand side is always positive, of course. This translates to:

$$|\tau c + i\sigma X| \leq |sX|$$

or

$$\left|c + \frac{i\sigma X}{\tau}\right| \leq |X|\sqrt{1+\frac{\sigma^2}{\tau^2}} .$$

The first implies that in any region $0 < \sigma \leq \sigma_0$, $|\tau| \leq \tau_0$, τc is bounded. The second implies that in a region $0 < \sigma \leq \sigma_0$, $|\tau| \geq \tau_0 > 0$, c is bounded. Looking back at 3.6, this gives:

3.7. Proposition. (a) *The function* E_s *has no poles in the region* $\mathrm{Re}(s) \geq 1/2$ *except on* $(1/2, 1]$;

(b) *Any poles on* $(1/2, 1]$ *are simple*;

(c) *In any region* $1/2 \leq \sigma \leq \sigma_0$, $|\tau| \geq \tau_0 > 0$, c *is bounded*;

(d) *In any region* $1/2 < \sigma \leq \sigma_0$, $|\tau| \geq \tau_0 > 0$, *there exists a constant* C *with*

$$\|T^Y E_s\| \leq C/\sqrt{\sigma} .$$

The general principle behind (a) is that the behavior of $c(s)$ controls that of E_s, but in practice this is often a subtle matter. If σ is one of the poles of $c(s)$ in $(1/2, 1]$, then $\mathrm{Res}_\sigma E_s$ will be an eigenfunction of Δ with eigenvalue $\sigma(\sigma-1)$, and its constant term at $i\infty$ will be $(\mathrm{Res}_\sigma\, c(s)) y^{1-\sigma}$. This is square-integrable in the neighborhood of $i\infty$, and in fact $E_\sigma^* = \mathrm{Res}_\sigma E_s$ is square-integrable on all of X.

There is always a pole at $s = 1$, and E_1^* is a constant function. If Γ is arithmetic, or as we shall see later if it has only one cusp, then this is the only pole in $(1/2, 1]$. In the arithmetic case $c(s)$ may be evaluated explicitly in terms of Dirichlet L-functions (see for example p.46 of Kubota[1973]). In this case also the estimate 3.7(d) may be improved, but the best possible estimate is not yet known, and seems rather deep. These examples also show that where $\mathrm{Re}(s) < 1/2$ both $c(s)$ and E_s might be very badly behaved. This is not so important because of the functional equation for E_s.

Our next step will be to find the decomposition of $L^2_{Eis}(X)$ in terms of Eisenstein series. I follow Langlands [1966].

First I must recall some results of harmonic analysis on the multiplicative group of positive real numbers. Let ϕ be in $C_c^\infty(0,\infty)$. Its multiplicative Fourier transform is entire:

(3.3) $$\Phi(s) = \int_0^\infty x^{-s-1}\phi(x)\,dx \qquad (s \in \mathbb{C}) .$$

A theorem of Paley-Wiener characterizes completely such functions, but all we need to know is that $\Phi(\sigma+it)$, considered as a function of τ, lies in the Schwartz space, uniformly in σ. The function ϕ is computed in terms of Φ by the formula

(3.4) $$\phi(y) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\Phi(s)y^s\,ds$$

where σ is any real number. Given two functions $\phi, \psi \in C_c^\infty(0,\infty)$ with Fourier transforms Φ, Ψ, the product again lies in $C_c^\infty(0,\infty)$ and has Fourier transform $(2\pi i)^{-1}\Phi * \Psi$ (as can be seen by applying the inverse transform to the latter). In particular

(3.5) $$\int_0^\infty \phi(x)\psi(x)x^{-1}dx = \frac{1}{2\pi i}\Phi * \Psi(0)$$

$$= \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\Phi(t)\Psi(-t)\,dt .$$

I should remark here that the convolution I use here is defined by an integral over a <u>vertical</u> line. These formulae extend to a more general situation: suppose $\phi(x)$ is not assumed to vanish at $x = 0$, but only to be bounded on left half-lines. Then $\Phi(s)$ may be defined only for s with $Re(s) < 0$. All the above formulae still hold, however, as long as $\sigma < 0$. Equivalently, if ψ lies in this situation, one must assume $\sigma > 0$ in (3.5).

Continue to suppose $\phi \in C_c^\infty(0,\infty)$. Then at least formally it follows from 3.4 that

$$E_\phi = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\Phi(s)E_s\,ds .$$

This is easily justified for $\sigma > 1$ (where E_s behaves nicely). Taking constant terms, we have (for $\sigma > 1$)

$$E_{\phi,0}(y) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\Phi(s)[y^s + c(s)y^{1-s}]ds .$$

This can be rewritten as

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}[\Phi(s) + \bar{c}(1-s)\Phi(1-s)]y^s ds$$

where now $\sigma < 0$. In other words, the Fourier transform of $E_{\phi,0}$ is analytic for $\text{Re}(s) < 0$ and equal to $[\Phi(s) + c(1-s)\Phi(1-s)]$ in that region.

Now choose $\phi, \psi \in C_c^\infty(0,\infty)$ and consider the inner product $\langle E_\phi, E_\psi\rangle$. According to the calculation in (2.2),

$$\langle E_\phi , E_\psi\rangle = M\int_0^\infty \phi(y)E_{\psi,0}(y)y^{-2}dy$$

$$= M\int_0^\infty \phi(y)[E_{\psi,0}(y)y^{-1}]y^{-1}dy$$

which by (3.5) and an elementary shift

$$= \frac{M}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\Phi(s)[c(s)\Psi(s) + \Psi(1-s)]ds$$

with $\sigma > 1$. This contour, according to Theorem 3.3, may be moved to $\sigma = 1/2$. As this is done, a number of residues of $c(s)$ are encountered, say at $s_1, s_2, \ldots, s_n$. Let $c^*(s_i) = \text{Res}_{s_i} c(s)$. Then the inner product is the sum of

$$(3.6) \qquad \sum Mc^*(s_i)\Phi(s_i)\Psi(s_i)$$

and

$$\frac{M}{2\pi i}\int_{\sigma=1/2}\Phi(s)[\Psi(1-s) + c(s)\Psi(s)]ds \ .$$

Since $c(s)c(1-s) = 1$, this integral is also

$$\frac{M}{4\pi i}\int_{\sigma=1/2}[\Phi(s) + c(1-s)\Phi(1-s)][\Psi(1-s) + c(s)\Psi(s)]ds$$

$$= \frac{M}{4\pi i}\int_{\sigma=1/2}c(1-s)[\Phi(1-s) + c(s)\Phi(s)][\Psi(1-s) + c(s)\Psi(s)]ds \ .$$

If $\psi = \overline{\phi}$, these equations give

$$(3.7) \qquad \|E_\phi\|^2 = \sum [Mc^*(s_i)]|\Phi(s_i)|^2$$

$$+ \frac{M}{4\pi}\int_{-\infty}^{\infty}|\Phi(\tfrac{1}{2} - i\tau) + c(\tfrac{1}{2} + i\tau)\Phi(\tfrac{1}{2} + i\tau)|^2 d\tau \ .$$

How is this to be interpreted? Given any $f \in C_c^\infty(X)$ define its <u>Fourier-Eisenstein transform</u> $F(s)$ by the formula

$$(3.8) \qquad F(s) = \langle f, E_s\rangle \ .$$

It will be meromorphic in s, and satisfy the functional equation $F(s) = c(s)F(1-s)$ since $E_s = c(s)E_{1-s}$. Its residue at a pole s_i is $\langle f, E^*(s_i)\rangle = F^*(s_i)$. If $f = E_\phi$, then

$$F_\phi(s) = M[\Phi(1-s) + c(s)\Phi(s)]$$

and

$$F_\phi^*(s_i) = Mc^*(s_i)\Phi(s_i) \ .$$

All in all, (3.7) thus may be read that if $f = E_\phi$ then

(3.9) $$\|f\|^2 = \sum [Mc^*(s_i)]^{-1} |F_\phi^*(s_i)|^2 + \frac{1}{4\pi M}\int_{-\infty}^{\infty} |F_\phi(\tfrac{1}{2} + i\tau)|^2 d\tau,$$

while (3.6) may be read as

(3.10) $$<E_\phi , E_\psi> = \sum \Phi(s_i) F_\psi^*(s_i) + \frac{1}{2\pi i}\int_{\sigma=1/2} \Phi(s) F_\psi(s) ds \quad .$$

Since $|c(s)| = 1$, multiplication by $c(s)$ is an isometry of $\mathbb{L}^2(1/2 + i\mathbb{R})$ with itself, and the map $\Phi \mapsto \Lambda\Phi$ where

$$\Lambda\Phi(s) = \frac{1}{2}[\Phi(s) + c(s)\Phi(1-s)] ,$$

is a projection onto the closed subspace of F with $F(s) = c(s)F(1-s)$. Now $\mathbb{L}^2_{Eis}$ is by definition the closure of the E_ϕ with $\phi \in C_c^\infty(0,\infty)$ and $\mathbb{L}^2(1/2 + i\mathbb{R})$ is the closure of the functions $\Phi(s)$ which are Fourier transforms of such ϕ, so that the functions $\Lambda\Phi$ are dense in $\Lambda\mathbb{L}^2(1/2 + i\mathbb{R})$. The Fourier-Eisenstein transform $F(s)$ of E_ϕ is $(\Lambda\Phi)(1-s)$. Therefore Equation (3.8) implies

3.8 Theorem. *The map*

$$E_\phi \mapsto \{F^*(s_i), F(\tfrac{1}{2} + i\tau)\}$$

where F *is the Fourier-Eisenstein transform of* E_ϕ, *extends to an isometry of* $\mathbb{L}^2_{Eis}(X)$ *with*

$$\mathbb{C}^n \oplus \Lambda\mathbb{L}^2(\tfrac{1}{2} + i\mathbb{R})$$

where the norm on the latter is given by

$$\sum [Mc^*(s_i)]^{-1} |F^*(s_i)|^2 + \frac{1}{4\pi M} \int_{-\infty}^{\infty} |F(\tfrac{1}{2} + i\tau)|^2 d\tau \ .$$

The isomorphism given by this theorem is difficult to work with directly. However, it is easy to see that there is a very useful characterization of the Fourier-Eisenstein transform of $f \in \mathbb{L}^2_{Eis}(X)$. Say $\phi \in C_c^\infty$ with multiplicative Fourier transform Φ, and suppose $\{F^*(s_i), F(1/2 + i\tau)\}$ is the transform of f. Then from the definition it follows (see Equation (3.10)) that

$$(3.9) \qquad \langle f, E_\phi \rangle = \sum \Phi(s_i) F^*(s_i) + \frac{1}{2\pi i} \int_{\sigma=1/2} \Phi(s) F(s)\, ds \ .$$

Furthermore, the transform of f is determined by the condition that this holds for all $\phi \in C_c^\infty(0,\infty)$.

Even with this criterion, however, it seems not quite trivial to answer the following question: suppose $f \in C_c^\infty(X)$. Then $F(s) = \langle f, E_s \rangle$ is meromorphic in s, with poles only where the poles of $c(s)$ are. Is the restriction of F to $1/2 + i\mathbb{R}$ equal to the continuous part of its $\mathbb{L}^2$ Fourier-Eisenstein transform? This, and something slightly better, will be dealt with in the next section.

The inverse of the isomorphism above is not hard to describe explicitly. Suppose $F \in \mathbb{L}^2(1/2 + i\mathbb{R})$, and consider for $T > 0$ the function

$$f_T = \frac{1}{4\pi M i} \int_{1/2-iT}^{1/2+iT} F(1-s) E_s\, ds \ .$$

It is certainly a C^∞ function on X. Thus

$$\langle f_T , E_\phi \rangle = \frac{1}{4\pi Mi} \int_{1/2-iT}^{1/2+iT} F(1-s) \langle E_\phi , E_s \rangle ds$$

$$= \frac{1}{4\pi Mi} \int_{1/2-iT}^{1/2+iT} MF(1-s)[\Phi(1-s) + c(s)\Phi(s)]ds$$

$$= \frac{1}{4\pi i} \int_{1/2-iT}^{1/2+iT} \Phi(s)[F(s) + c(s)F(1-s)]ds$$

$$= \frac{1}{2\pi i} \int_{1/2-iT}^{1/2+iT} \Phi(s)\Lambda F(s)ds .$$

Hence, applying the criterion above, we see that the transform of f_T is $\Lambda F(s)$, truncated at $1/2 \pm iT$. Therefore as $T \to \infty$ the function f_T converges in $L^2(X)$ to a function f whose transform is ΛF.

One final remark: the formula (3.8) requires $c^*(s_i)$ to be positive at all s_i. However, we know that the poles of $c(s)$ are simple, so if $n \geq 2$ the signs of the $c^*(s_i)$ must alternate. Hence there can be only one pole.

4. A Paley-Wiener Theorem

What is to be done now is characterize the Fourier-Eisenstein transforms of functions in $S(X)$ - or, rather S_{Eis}.

Let $f \in S(X)$. Define

$$F(s) = \langle f, E_s \rangle .$$

Because f vanishes rapidly at $i\infty$ while E_s is of moderate growth, this is well defined whenever E_s is, and in fact it is meromorphic as a function of s. It has simple poles in the region $Re(s) \geq 1/2$, at most where the poles of $c(s)$ are. Let $F^*(s_i)$ be its residue at s_i, for $s_i \in (1/2, 1]$. (I am ignoring my last remark in §3.)

4.1 Lemma. *If* $f \in S_{Eis}(X)$ *then the* $F^*(s_i)$ *together with the restriction of* F *to* $1/2 + i\mathbb{R}$ *comprise the* $\mathbb{L}^2$ *transform of* f.

Proof. According to a remark made at the end of §2, I can find a sequence of $\phi \in C_c^\infty(0,\infty)$ with the E_ϕ converging to f in $S(X)$. Thus $F_\phi(s) \to F(s)$ for all $s \in \mathbb{C}$, where $F_\phi(s) = \langle E_\phi, E_s\rangle$. Similarly for the residues. This convergence is uniform on compact subsets in $\mathbb{C}$. But also by definition $F_\phi(s)$ converges in $\mathbb{L}^2(1/2 + i\mathbb{R})$ to the continuous $\mathbb{L}^2$-transform of f, and the $F_\phi^*(s_i)$ converge to the discrete part of the transform. It is easy to deduce the claim.

In view of this result, which says that the $\mathbb{L}^2$-transform of $f \in S_{Eis}$ is determined by the meromorphic function $F(s)$, I will call $F(s)$ the Fourier-Eisenstein transform of f, for every $f \in S(X)$. Of course if $f \notin S_{Eis}(X)$ then one cannot hope to recover f from F, but we will see that at least one can recover the component of f in $S_{Eis}(X)$.

If $f \in S(X)$ then so is every $\Delta^n f$. The transform of $\Delta^n f$ is $s^n(s-1)^n F(s)$, if F is the transform of f.

4.2. Theorem. *Suppose* $f \in S(X)$ *and for each* $n \geq 0$ *let* $F_n(s)$ *be the Fourier-Eisenstein transform of* $\Delta^n f$. *Then for every* $n \geq 0$:

(1) $F_n(s)$ *is meromorphic in all of* $\mathbb{C}$;

(2) $F_n(s) = c(s)F_n(1-s)$;

(3) *The poles of* $F_n(s)$ *in the region* $\mathrm{Re}(s) \geq 1/2$ *are among the poles of* $c(s)$, *and are simple*;

(4) *The restriction of* $F_n(s)$ *to* $\mathrm{Re}(s) = 1/2$ *is square-integrable*;

(5) *In any region* $|\tau| \geq \tau_0 > 0$, $0 < \sigma \leq \sigma_0$, $F_n(s) = O(1/\sqrt{\sigma})$.

Conversely, suppose given $f(s)$ *such that all the* $F_n(s) = s^n(s-1)^n F_0$

satisfy these conditions. *Then* $F(s)$ *is the Fourier-Eisenstein transform of some* $f \in S(X)$.

In condition (5), $\sigma = \mathrm{Re}(s) - 1/2$, $\tau = \mathrm{Im}(s)$.

The proof that every $F_n(s)$ satisfies (1)-(5) is simple, given what has been done already. The only slightly tricky point is (5). For this, write

$$E_s = T^Y E_s + R_s$$

where Y is a large positive number, $T^Y E_s$ is the truncation of E_s constructed in §3, and the remainder R_s will be null outside the region $y \geq Y$, and $y^s + c(s)y^{1-s}$ inside it. Then

$$F(s) = \langle E_s , f\rangle = \langle T^Y E_s , f\rangle + \langle R_s , f\rangle .$$

By 3.7(d), $\|T^Y E_s\| = O(1/\sqrt{\sigma})$ in the region $0 < \sigma \leq \sigma_0$, $|\tau| \geq \tau_0 > 0$, so that by Cauchy-Schwarz it only remains to show that $\langle R_s , f\rangle = O(1/\sqrt{\sigma})$ there. Because $c(s)$ is bounded in that region and R_s is invariant under translations $z \mapsto z+x$, it suffices to show that the integral

$$\int_Y^\infty y^{s-2} f_0(y)\,dy$$

is bounded in regions $|\sigma| \leq \sigma_0$. Since f is rapidly decreasing at $i\infty$, this is clear.

The proof of the converse is more difficult.

Given $F(s) = F_0(s)$ satisfying the conditions of the theorem, the candidate for f is more or less clear from remarks at the end of §3: define it to be the sum of

$$\lim_{T\to\infty} \frac{1}{4\pi M i} \int_{1/2-iT}^{1/2+iT} F(1-s) E_s \, ds$$

and

$$\sum[Mc^*(s_i)]^{-1}F^*(s_i)E^*_{s_i} \quad .$$

At the beginning, all we know is that f and indeed all the $\Delta^n f$ lie in $\mathbb{L}^2(X)$, and their $\mathbb{L}^2$-transforms are given by F. What we want to prove is that $f \in S(X)$. This is simply a condition on the behaviour of f near the cusp $i\infty$. Now near $i\infty$ the function f may be written as $f_0 + (f-f_0)$. Proposition 2.3 may be applied to see that $(f-f_0)$ and all its derivatives vanish rapidly at $i\infty$, so that all that remains is to show the same for the constant term f_0.

Note that so far we have used almost none of the properties (1)-(5).

What is f_0? It is the sum of

(4.1)
$$\lim_{T\to\infty} \frac{1}{4\pi i}\int_{1/2-iT}^{1/2+iT} F(1-s)[y^s + c(s)y^{1-s}]ds$$

and

(4.2)
$$\sum F^*(s_i)y^{1-s_i} \quad .$$

Note that the first is a function, call it $\phi(y)$, such that $\phi(y)y^{1/2}$ lies in $L^2(0,\infty)$ (with respect to the multiplicatively invariant measure dy/y). Because of the functional equation (2) for $F(s)$, the integral is the same as

$$\lim_{T\to\infty} \frac{1}{2\pi i}\int_{1/2-iT}^{1/2+iT} F(s)y^{1-s}ds \quad .$$

The aim now is to move the line of integration to the right. This is possible precisely because of condition (5) applied, say to $F_2(s)$, since $1/\sqrt{\sigma}$ is integrable on any interval $(0,\varepsilon)$. Therefore in fact we can keep moving the line of integration as far to the right as we want, except that we pick up some residues. These amount to

$$- \sum F^*(s_i) y^{1-s_i} ,$$

and of course cancel out the summand (4.2). So all in all, the constant term of f is equal to

$$\frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} F(s) y^{1-s} d|s| = O(y^{1-\sigma}).$$

This is true for all the F_n as well, so that f_0 and all the $\Delta^n f_0$ are rapidly vanishing at the cusp. But on constant terms Δ acts as $y^2 \partial/\partial y^2 = (y\partial/\partial y)^2 - (y\partial/\partial y)$, so that it is easy to see all the derivatives $(y\partial/\partial y)^n f_0$ vanish rapidly also. This is enough to get the same condition on the $(\partial/\partial y)^n f_0$.

I have talked only about functions, not forms of higher degree, and have assumed Γ has only one cusp in the last part of §3 and so far in §4, but similar techniques will apply without these restrictions.

Note that by starting with $f \in S(X)$ and reconstructing a function from $F(s)$, we obtain the Eisenstein component of f. This gives a new, more explicit, proof of the decomposition $S = S_{cusp} \oplus S_{Eis}$, but of course the key technical result is the same in both proofs.

At any rate, the consequence we will need in analyzing cohomology is this:

4.3. Corollary. On the Eisenstein component of $\Omega_S^{\cdot}$ each operator Δ^n acts injectively, with closed image and finite-dimensional cokernel.

This is a simple deduction from the generalization of 4.2, since Δ becomes multiplication by $s(s-1)$ on the Fourier-Eisenstein transform.

4.4. Theorem. The operator Δ is surjective on the Eisenstein component of the tempered currents.

This follows from 4.3 by duality.

It is either of 4.3 or 4.4 which is the basic contribution of analysis to the Hodge theory we are looking for.

5. Applications to cohomology

There will be two steps in which the Paley-Wiener theorem will contribute to analyzing cohomology. The first is:

5.1. Theorem. The cohomology of the complex made up of the Eisenstein component of tempered currents is the same as that of its subcomplex of currents annihilated by some power of the Laplacian.

This is the variant of the Hodge theorem I was referring to earlier. Note that it does not identify the cohomology with a subspace of the currents - it does not find unique representatives for the cohomology classes, but only allows a kind of reduction of the calculation. We will see later to what extent this is useful.

Proof. Let $C^{\bullet}[\Delta]$ be the subspace of tempered currents annihilated by some power of Δ. I claim that Δ induces an isomorphism of the quotient $C^{\bullet}/C^{\bullet}[\Delta]$ with itself. Injectivity is an immediate consequence of the definition of $C^{\bullet}[\Delta]$. Surjectivity follows from Theorem 4.4.

Consider the short exact sequence

$$0 \to C^{\bullet}[\Delta] \to C^{\bullet} \to C^{\bullet}/C^{\bullet}[\Delta] \to 0 .$$

The de Rham differential induces a differential on each of these complexes (note that Δ and d commute). The result we want will follow from the fact that the cohomology of $C^{\bullet}/C^{\bullet}[\Delta]$ must be trivial. This is a consequence of what has just been proved, together with the equation

$$\Delta = d\delta + \delta d$$

or

$$Id = d(\Delta^{-1}\delta) + (\Delta^{-1}\delta)d$$

on $C^{\bullet}/C^{\bullet}[\Delta]$, which says that $\Delta^{-1}\delta$ is a homotopy operator on this complex.

Note that from an analyst's point of view this proof is a bit peculiar, since the spaces in the short exact sequence we used are not topological vector spaces. The space $C^{\bullet}[\Delta]$ is in some sense an algebraic object.

Only in some weak sense, however, because the Paley-Wiener Theorem must be used again to identify it. Once again to illustrate the situation I will look at functions only. How can I describe the tempered distributions annihilated by some Δ^n? The answer is in terms of Eisenstein series. Note that since $(\Delta - s(s-1))^n(y^s \log^{n-1} y) = 0$, so also is $d^{n-1}/ds^{n-1}(E_s)$ annihilated by $(\Delta - s(s-1))^n$. To obtain those annihilated by Δ^n, we must set $s = 0$, $s = 1$. There are two problems, however: there is a pole of E_s at $s = 1$, and $s = 0$ lies in the "forbidden region" $\mathrm{Re}(s) < 1/2$. Neither of these is a serious difficulty, as we will see.

Suppose $P_1, \ldots, P_n$ are the distinct cusps of X, and say the corresponding M-numbers are $M_1, \ldots, M_n$. What I mean by this is that if N_i is the subgroup of unipotent elements fixing P_i, then M_i is the index of $\Gamma \cap N_i$ in $SL_2(\mathbb{Z}) \cap N_i$. Then it is easy to see that if $E_s^{(1)}, \ldots, E_s^{(n)}$ are the Eisenstein series associated to these cusps, the linear combination $\sum a_i E_s^{(i)}$ will have residue at $s = 1$ equal to the constant

$$\sum a_i M_i c^{(i)}*(1)$$

where $c^{(i)}*(1)$ is the residue of the corresponding c-function at $s = 1$. In particular, this linear combination will have no pole at $s = 1$ if and only if $\sum a_i M_i c^{(i)}*(1) = 0$. Call any such linear combinaof the $E_s^{(i)}$ <u>admissible</u>. One way to obtain functions annihilated by powers of Δ^n is to consider derivatives in s of such admissible combinations, evaluated at $s = 1$. Another way is by considering derivatives in s of any of the $(s-1)E_s^{(i)}$, again evaluated at $s = 1$. The second consequence of the Paley-Wiener Theorem we need is that <u>all tempered distributions in A_{Eis} annihilated by some power of Δ are obtained as linear combinations of these two types</u>. It is pretty clear, in fact, that for the second type, only one cusp need be taken into account.

Something similar holds for differential forms constructed from Eisenstein series.

I will not prove these assertions, except for a few remarks. The first is that the subspace of A_{Eis} annihilated by Δ^n is the dual of the quotient $S_{Eis}/\Delta^n S_{Eis}$, which one can see from Theorem 4.2 is clearly related to derivatives of Eisenstein series, and moreover is finite-dimensional. The second is that the point $s = 0$ plays no role because the functional equation for E_s says that the Eisenstein series and their derivatives obtained from the region $Re(s) < 1/2$ are obtained already from the region $Re(s) \geq 1/2$.

We come now to the final question: <u>how does one compute the cohomology of this rather large complex made up of Eisenstein series and their residues and derivatives</u>? One could in fact sort out the situation without too much trouble in the context we are dealing with, but it is here perhaps that it is most helpful to use representation theory.

The construction of Eisenstein series may be carried out for all weights of automorphic forms, not just functions or differential forms. One gets them all, in the region $Re(s) > 1$, by considering the series

$$\sum_{\Gamma_\infty \backslash \Gamma} y(\gamma(z))^s \varepsilon (cz+d)^n$$

where $\varepsilon(x) = x/|x|$ for $x \in \mathbb{C}^\times$, and (c,d) make up the bottom row of $\gamma \in \Gamma$, as well as the analogous series for all the cusps. One gets all automorphic forms in the Eisenstein component by meromorphically continuing these, at least as far as $\mathrm{Re}(s) = 1/2$, and taking residues and s-derivatives. In terms of representation theory the Eisenstein series themselves correspond to embeddings of principal series representations into the dual of the Schwartz space of $\Gamma\backslash SL_2(\mathbb{R})$, which may be defined much as $S(X) = S(\Gamma\backslash SL_2(\mathbb{R}))^K$ was. By a well known construction, the relative Lie algebra cohomology of these principal series (with respect to the pair $(\mathfrak{sl}_2, \mathfrak{so}_2)$) maps into the cohomology of X. Taking derivatives with respect to s amounts to mapping into the tempered distributions certain representations of G induced from non-irreducible representations of the subgroup $\begin{pmatrix} * & * \\ & * \end{pmatrix}$, related to the principal series. A variant of Shapiro's Lemma enables one to see easily what the cohomology amounts to.

I will be a little more precise. It is well known that by lifting forms on $\Gamma\backslash H$ to $\Gamma\backslash SL_2(\mathbb{R})$, one obtains an isomorphism of the de Rham complex of X with $C^\bullet(\mathfrak{g}, \mathfrak{k}, C^\infty(\Gamma\backslash G))$, where now $G = SL_2(\mathbb{R})$, $K = SO(2)$, and $\mathfrak{g}$ and $\mathfrak{k}$ are their Lie algebras. If we consider Schwartz forms, we obtain $S(\Gamma\backslash G)$ instead of $C^\infty(\Gamma\backslash G)$. And if we start with tempered currents on X, we obtain the space $A(\Gamma\backslash G)$ of tempered distributions on $\Gamma\backslash G$. The space $S(\Gamma\backslash G)$ possesses a G-stable decomposition $S_{cusp} \oplus S_{Eis}$, and so does $A(\Gamma\backslash G)$. The results at the beginning of this § translate to the statement that the Eisenstein component of the cohomology of X is the same as the $(\mathfrak{g},\mathfrak{k})$-cohomology of the subspace of elements of $A_{Eis}(\Gamma\backslash G)$ annihilated by some power of the Casimir operator C in the universal enveloping algebra of $\mathfrak{g}$.

Let B be the subgroup of elements

$$\begin{pmatrix} a & x \\ 0 & a^{-1} \end{pmatrix} \qquad (a \in \mathbb{R}^\times,\ x \in \mathbb{R})$$

in $SL_2(\mathbb{R})$. To $s \in \mathbb{C}$ is associated a character χ_s of B:

$$\begin{pmatrix} a & x \\ 0 & a^{-1} \end{pmatrix} \mapsto |a|^s .$$

The principal series representation of G associated to this is the representation π_s induced by this, the space of all C^∞ functions $f\colon G \to \mathbb{C}$ such that

$$f(bg) = \chi_s(b)f(g)$$

for $b \in B$, $g \in G$. The Eisenstein series E_s gives rise, when analytic at s, to an embedding $\pi_s \to A_{Eis}(\Gamma\backslash G)$. When it has a pole at s then the normalized Eisenstein series $(t-s)E_t$, evaluated at $t = s$, gives rise to a map $\pi_s \to A_{Eis}(\Gamma\backslash G)$ which will have a non-trivial kernel.

Associated also to s is a sequence of finite-dimensional representations $\chi_s^{(1)}, \chi_s^{(2)}, \ldots$ of B, obtained by letting B act by the right regular representation on the spaces spanned by $|a|^s$, $|a|^s \log|a|, \ldots, |a|^s \log^{n-1}|a|$. Let $|a|^s[\log|a|]$ be the union of all these spaces, of infinite dimension. Then since $|a|^s \log|a|$ is the derivative of $|a|^s$ with respect to s, the derivatives of normalized Eisenstein series give rise to maps from $\pi_{s,\log} = \mathrm{Ind}(|a|^s[\log|a|]|B,G)$ to $A_{Eis}(\Gamma\backslash G)$.

This construction can be carried out for all cusps as well. As a consequence of the Paley-Wiener theorem one can see that the space of all tempered Eisenstein distributions annihilated by some power of C is as a G-module isomorphic to the direct sum of $(n-1)$ copies of $\pi_{1,\log}$ and one copy of $\pi_{0,\log}$ (n = number of cusps). (The latter comes from the pole of E_s at 1.)

By Shapiro's Lemma,

$$H^{\cdot}(\mathfrak{g},\mathfrak{k},\pi_{s,\log}) \cong H^{\cdot}(\mathfrak{b},|a|^{s}[\log|a|]).$$

Hochschild-Serre give a spectral sequence converging to this with E_2-term

$$H^{\cdot}(\mathfrak{a},H^{\cdot}(\mathfrak{n},\mathbb{C}) \otimes |a|^{s}[\log|a|]).$$

Here $\mathfrak{b} = \mathfrak{a}+\mathfrak{n}$, $\mathfrak{n}$ the Lie algebra of unipotents, $\mathfrak{a}$ that of diagonal matrices. This is equal to 0 unless $s = 0$ or 1, and then, by an interesting calculation, gives

$$H^{m}(\mathfrak{g},\mathfrak{k},\pi_{0,\log}) = \begin{cases} \mathbb{C} & m = 0 \\ 0 & m \neq 0 \end{cases}$$

$$H^{m}(\mathfrak{g},\mathfrak{k},\pi_{1,\log}) = \begin{cases} \mathbb{C} & m = 1 \\ 0 & m \neq 1 \end{cases} .$$

Finally, the Eisenstein component of cohomology turns out to be what it has to be:

$$H^{0}_{Eis} = \mathbb{C}$$

$$H^{1}_{Eis} = \mathbb{C}^{n-1}$$

$$H^{2}_{Eis} = 0 .$$

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