**Review** of
*Spectral Decomposition and Eisenstein Series*
by C. Moeglin and J.-L. Waldspurger

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There were foreshadowings in work of Hecke and his student Reidemeister, but what we now call the analytic theory of Eisenstein series really began with work of Maass carried out just after the war and published in (Maass 1949). Maass defined formally the series

$$E_s(z) = \frac{1}{2} \sum_{\gcd(m,n)=1} \frac{y^s}{|mz+n|^{2s}}$$

for $z = x + iy$ in the upper half plane $\Im(z) > 0$. This series converges uniformly for $\Re(s) > 1$ to an eigenfunction of the Laplacian on the upper half-plane (with eigenvalue $\lambda_s = s(s-1)$) which is manifestly invariant under the group $SL_2(\mathbb{Z})$ of integral modular transformations. In particular it is invariant under the group of translations $z \mapsto z + n$, and hence has a Fourier expansion

$$E_s(z) = \sum_n E_{n,s}(y) e^{2\pi i n x}$$

where the Fourier coefficients, which are functions of $y$ alone, are expressed as

$$E_{n,s}(y) = \int_0^1 E_s(x+iy) e^{-2\pi i n x} \, dx \ .$$

These can be explicitly evaluated in terms of Riemann's $\zeta(s)$ and certain classical special functions, so that well known properties of $\zeta(s)$ lead to the meromorphic continuation of $E_s(z)$ with a simple pole at $s = 1$ and others contributed by the possible zeroes of $\zeta(2s)$ in the region $\Re(s) < 1/2$. The functional equation of $\zeta(s)$ leads to a functional equation

$$E_s(z) = c(s) E_{1-s}(z)$$

for some function $c(s)$ calculated explicitly in terms of $\zeta(s)$. Maass also defined Eisenstein series associated to Dirichlet characters and congruence subgroups, using Dirichlet $L$ functions instead of $\zeta(s)$. The significance of all these series is that since they are eigenfunctions of the non-Euclidean Laplacian on $\mathfrak{H}$, they are analogous to classical automorphic forms, and are now treated democratically with the classical ones. In the same paper Maass defined real-analytic automorphic forms more generally (which he called 'non-analytic' to distinguish them from complex-analytic ones). Just as for the classical holomorphic automorphic forms, one can define the cusp forms to be those automorphic forms orthogonal to the Eisenstein series, or equivalently those with vanishing constant term in the Fourier expansion at cusps. Roelcke examined Eisenstein series for more general discrete groups

commensurable with $SL_2(\mathbb{Z})$, and was able to show through spectral techniques that they could be defined for $s$ in the region $\Re(s) > 1/2$ except for the interval $(1/2, 1]$. This work was published in (Roelcke 1956) quite a while after the work was done. The point here is that constructing Eisenstein series is essentially a boundary value problem, with a few technical twists. The Laplacian $\Delta$ is negative definite, so that for $s$ in the region

$$R_{1/2} = \{\Re(s) > 1/2, s \notin (1/2, 1]\}$$

the eigenvalue $\lambda_s$ does not lie in the spectrum of $\Delta$. It is not difficult to see that for $s$ in $R_{1/2}$ we have the asymptotic behaviour $E_s \sim y^s$ for $y$ near $\infty$ in the sense that $E_s - y^s$ is square-integrable near $\infty$ (interpreting $y$ loosely here as a function on the quotient). If $\chi(y)$ is a test function on the quotient identically equal to 1 near $\infty$ and identically vanishing in the interior, then

$$(\Delta - \lambda_s)(E_s - \chi(y)y^s) = -(\Delta - \lambda_s)\chi(y)y^s$$

is then a smooth function of compact support on the quotient. This means that $E_s$ can be recovered from the solution to a simple boundary value problem for $s$ in $R_{1/2}$, and that in this region holomorphic continuation of $E_s$ is elementary.

Selberg worked on these questions also, but he was able to prove the meromorphic continuation as well as a functional equation over the whole complex plane. This was a much stronger and more difficult result. He himself did not publish at that time the details of this early work, but it has been well explained by others. Similar techniques, once the notion of cusp form had been explicated by Gelfand for a larger class of discrete subgroups, including all arithmetic subgroups of semi-simple groups defined over $\mathbb{Q}$, led to a similar theory of Eisenstein series for automorphic forms on the space $\Gamma\backslash G$ associated to maximal rational parabolic subgroups, and the spectral decomposition of the part of $L^2(\Gamma\backslash G)$ with a one-dimensional continuous spectrum. This part of the theory is from an analytic point of view reasonably well understood; particularly elegant expositions of the basic ideas, along slightly different lines (both following the classic work), can be found in (Colin de Verdière 1981) and (Efrat 1987). It was simple to extend this theory to adèle groups when they were introduced in the late 1950's.

The extension to Eisenstein series of higher rank and to the continuous spectrum of dimension greater than one, or in other words to automorphic forms associated to arbitrary rational parabolic subgroups, was more difficult. The decisive step was taken by Langlands around 1965, who produced a set of notes on the theory. These were widely distributed but not published until much later, in (Langlands 1976). There were two main parts to Langlands' theory. The first was the meromorphic continuation of Eisenstein series associated to cusp forms on the reductive factors of rational parabolic subgroups. The second applied these Eisenstein series to two further and intimately related problems: describing (1) the complete discrete spectra of arithmetic quotients $\Gamma\backslash G$ where $G$ is a reductive group and $\Gamma$ an arithmetic subgroup; (2) the complete spectral decomposition of $L^2(\Gamma\backslash G)$ in terms of the discrete spectra of reductive factors of rational parabolic subgroups.

Langlands' results were a major *tour de force*, and in the light of history they become all the more impressive. But his notes were difficult to read and in a few places the difficulty was extreme. The first part of the theory was not so inaccessible as the second, and was the main topic of the book (Harish-Chandra 1968), which appeared soon after Langlands' notes. This book, however, did not attempt to explain the spectral decomposition of Langlands' Chapter 7. The first attempt to do so was the book (Osborne & Warner 1981). This book was also extremely difficult to read, but it helped some mathematicians, at least, to penetrate slightly further into what Langlands had in mind.

A second step in the exposition of Langlands' theory was made in (Harder 1974) and (Morris 1982), who dealt with the entirely analogous but somewhat simpler case of groups defined over function field extensions of finite fields.

A third step in the exposition of Langlands' work is the book under review (which is subtitled *Une paraphrase de l'Écriture*). This includes complete proofs in nearly all cases, including both reductive groups defined over number fields and arithmetic function fields, as well as central extensions of these groups such as the metaplectic groups. It is not self-contained. It depends strongly on, among other things, the truncation theory of Arthur and Langlands which it does not itself attempt to explain, and occasionally refers to other places for the details of a proof. It does, however, attack the spectral decomposition directly and without hesitation, and does a reasonably clean job. The one exception to its completeness is that whereas Langlands and the book by Osborne & Warner deal with arbitrary arithmetic groups $\Gamma$, this one deals exclusively with adèle groups. This has both bad and good effects — it makes the underlying analysis a little less approachable, but it does get one further into the region where interesting applications lie. How complicated life in an adelic world can be is shown in an appendix treating relatively explicitly the case of $G_2$, the sequel to an interesting and more elementary appendix in Langlands' Springer lecture notes.

This book, as I have already indicated, is valuable, but it should probably not be expected to serve as an introduction to the subject. The main problem is that it proves things in the utmost generality straight off, without looking at simple examples first. For a beginning, the book (Kubota 1973) is still valuable. For a clearer proof of meromorphic continuation in the rank one case one should probably look first at (Colin de Verdière 1983) or (Müller 1996), next (Efrat 1987), before looking at the very general case dealt with in this book. The step from this to the meromorphic continuation of Eisenstein series in the higher rank case is in some sense a technicality, but it does not appear so simple in most expositions. The exposition in this book is not bad, but one still might want to look at Harish-Chandra's notes before tackling it.

As for the final few steps — constructing the residual spectrum and then the associated Eisenstein series — this book is probably the best, if not for some things the only, place to look. Undertaking this task is still not simple. Before one attempts to understand the argument, which is a simplification of Langlands' own, one would want to look at Langlands' article from the Boulder conference, where he discusses $SL_3(\mathbb{Z})$, as well as the brief survey (Arthur 1979) from the Corvallis conference.

For many applications of Eisenstein series, one does not have to understand the proofs. In fact, for many applications it might be more important to understand the technical details of truncation, which are used here but for which the reader is referred to the original papers of Arthur. In the future, however, one might very want to understand the construction of the residual spectrum, which in turn depends in a recursive way on the continuous spectra of lower rank groups. Ideally, one would be able to start with a description of the Eisenstein series associated to cusp forms for maximal proper rational parabolic subgroups of a given group, and then apply a straightforward algorithm to see what happens for the group itself. We are not apparently at this stage yet. This book explains necessary facts about the residue process used, but for help here the reader might want to look at (van den Ban & Schlichtkrull 1997), who try very hard to isolate what is needed in very general circumstances. A complicated but important example of how explicit calculations go can be found in the paper (Moeglin-Waldspurger 1989).

Perhaps the two most serious applications of Eisenstein series so far are the trace formula and the work (Franke 1997) on cohomology and automorphic forms. Also important is Werner Müller's proof of traceability in (Müller 1989), and it should not be forgotten that at the moment of their birth the residues of Eisenstein series at one point were used by Langlands to prove a large class of new cases of Weil's conjecture on Tamagawa numbers. None of these apparently requires understanding the most difficult parts of the theory of Eisenstein series. But in the long run, work on the conjectures of Arthur regarding the residual spectrum, as the authors point out, will probably be more demanding.

I do not believe that the story of Eisenstein series, even at the level at which this book is working, has come to its conclusion. I have mentioned above that in rank one situations Eisenstein series are the solution to a relatively simple boundary value problem. Many of the problems which Langlands had to confront, and which this book still confronts much as he did, would be drastically simplified if one understood how residual Eisenstein series were also the solution to a kind of boundary value problem. This can be done in a limited number of circumstances, and evidence suggests that it can always be done. Other, smaller, bits of the puzzle can be filled in right now. For example, one result of Langlands' work is that the Plancherel measure on the continuous spectrum is constant. I think it is fair to say that the underlying reasons for this are not seen directly in the argument of Langlands or of this book. Nonetheless it is not more than an interesting exercise in truncation theory to provide a direct argument.

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