The original version of this paper was published in *Automorphic forms, automorphic representations, and arithmetic*, the proceedings of a symposium held to honor Goro Shimura, and published by the American Mathematical Society, volume 66 in its series of symposia in pure mathematics. I have revised it to improve the exposition, and also to have its notation agree with my more recent conventions.

On the Plancherel measure for the continuous spectrum of the modular group

Bill Casselman
Department of Mathematics
University of B.C.
Vancouver, Canada
cass@math.ubc.ca

Let $\Gamma = SL_2(\mathbb{Z})$. In this paper I will give what is apparently a new proof of the basic Plancherel theorems on $\Gamma\backslash\mathcal{H}$. The same argument will work in much more varied circumstances.

Maass' Eisenstein series is defined by the formula

$$E_s(z) = \frac{1}{2} \sum_{(m,n)=1} \frac{y^s}{|mz+n|^{2s}}$$

for z in the upper half plane $\mathcal{H}$. If P is the group of upper triangular matrices in $\mathrm{SL}_2(\mathbb{R})$, this is also

$$\sum_{\Gamma\cap P\backslash\Gamma} \mathrm{IM}(\gamma(z))^s \ ,$$

since elements of $\Gamma \cap P$ leave the imaginary part of $z = x + iy$ invariant, and

$$\mathrm{IM}(g(z)) = \frac{\mathrm{IM}(z)}{|cz+d|^2}, \quad \text{if } g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \ .$$

This series is convergent for $\mathrm{RE}(s) > 1$, and continues meromorphically in s to all of $\mathbb{C}$. It is invariant under Γ and an eigenfunction of the Laplacian:

$$\Delta E_s = s(s-1)E_s \ .$$

Recall that the **constant term** of any Γ-invariant function F is the constant term in its Fourier expansion

$$F(x+iy) = \sum_n e^{2\pi nix} F_n(y) \ ,$$

which is

$$F_0(y) = \int_0^1 F(x+iy)\, dx \ .$$

The constant term of E_s is of the form $y^s + c(s)y^{1-s}$, where

$$c(s) = \frac{\xi(2s-1)}{\xi(2s)}, \quad \xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s) \ .$$

This function $c(s)$ also occurs in the functional equation

$$E_{1-s} = c(s)E_s \ .$$

The only properties of $c(s)$ that we shall need are:

- The function $c(s)$ is smooth on $\mathrm{RE}(s) = 1/2$.
- For $\mathrm{RE}(s) = 1/2$ it satisfies

$$|c(s)| = 1, \quad c(s)c(1-s) = 1\,.$$

The function E_s also satisfies the growth condition

$$E_s(z) = E_{s,0}(y) + O(y^{-N}) = y^s + c(s)y^{1-s} + O(y^{-N})$$

for all positive N and large y.

Similar properties hold for Eisenstein series associated to any arithmetic group of rank one, as well as Eisenstein series induced from cusp forms on maximal rational arabolic subgroups.

For $\varphi(s)$ smooth and of compact support on $\mathrm{RE}(s) = \sigma = 1/2$ I define two functions, the first on $(0, \infty)$ and the second on $\Gamma\backslash\mathcal{H}$:

$$\begin{aligned} F_\varphi(y) &= \frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} \varphi(s) y^s \, ds \\ E_\varphi(z) &= \frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} \varphi(s) E_s(z) \, ds\,. \end{aligned}$$

Its constant term of E_φ is

$$\begin{aligned} E_{\varphi,0}(y) &= \frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} \varphi(s)\big(y^s + c(s)y^{1-s}\big)\, ds \\ &= \frac{1}{2\pi i}\int_{\sigma=1/2} \big(\varphi(s) + c(1-s)\varphi(1-s)\big) y^s \, ds \\ &= \frac{1}{2\pi i}\int_{\sigma=1/2} \varphi_\#(s) y^s \, ds \\ &= F_{\varphi_*}(y)\,, \end{aligned}$$

where

$$\varphi_\#(s) = \big(\varphi(s) + c(1-s)\varphi(1-s)\big)$$

is twice the projection of φ onto the space of functions satisfying

$$\varphi_\#(1-s) = c(s)\varphi_\#(s)\,.$$

Thus for y large $E_\varphi(z)$ is equal to

$$F_{\varphi_*}(y) + O(y^{-N})$$

for all N.

The usual Fourier duality of Schwartz spaces on the multiplicative group $(0, \infty)$ guarantees that

$$F_{\varphi_*}(y) = O(y^{1/2} \log^{-N} y)$$

for all $N > 0$ and large y. Therefore for any t with $\mathrm{RE}(t) = 1/2$ the Fourier-Eisenstein integral

$$\langle E_\varphi, E_{1-t}\rangle = \int_{\Gamma\backslash\mathcal{H}} E_\varphi(z) E_{1-t}(z)\, \frac{dx\,dy}{y^2}$$

is absolutely convergent.

The first of the two major results of this paper is this:

Theorem 1. (Fourier inversion) *We have*

$$\begin{aligned}\langle E_\varphi, E_{1-t}\rangle &= \left(\frac{1}{2}\right)\int_0^\infty F_\#(y)\big(y^{1-t}+c(1-t)y^t\big)\,\frac{dy}{y^2}\\ &= \varphi_\#(t)\,.\end{aligned}$$

Here is the second:

Theorem 2. (Plancherel formula) *We have*

$$\langle E_\varphi, E_\psi\rangle_{\Gamma\backslash\mathcal{H}} = \left(\frac{1}{2}\right)\frac{1}{2\pi i}\int_{\sigma=1/2}\varphi_\#(s)\psi_\#(1-s)\,ds = \left(\frac{1}{2}\right)\langle F_{\varphi_\#}, F_{\psi_\#}\rangle_{(0,\infty)}\,.$$

Both are special cases of a formal equation

$$\langle E_\varphi, E_\Phi\rangle_{\Gamma\backslash\mathcal{H}} = \left(\frac{1}{2}\right)\langle\,\text{constant term of } E_\varphi,\ \text{constant term of } E_\Phi\rangle_{(0,\infty)}\,.$$

This theory (extended to include arbitrary arithmetic groups Γ of $\mathrm{SL}_2(\mathbb{Q})$) is due to Maass, Roelcke, and Selberg. The usual proof of these results (found, for example, in Langlands (1965) or §7.4 of Kubota (1973)) involves moving a contour integral from the domain of convergence of Eisenstein series to the critical line $\sigma = 1/2$. This will be avoided in the argument below, which considers only Eisenstein series parametrized by s on the critical line. The usual argument, I should remark, also proves the full Plancherel theorem for the continuous spectrum and the part of the discrete spectrum arising from residues of Eisenstein series, whereas the argument below says nothing directly about completeness. Another argument, however, like that in Casselman (1983), can be used in some circumstances to prove an even stronger completeness result starting with the one given here. In particular, the argument given here allows one to prove the Paley-Wiener theorem of Casselman (1983) without moving contours twice, first one way and then the other, as was done there.

The proof below, and the proofs I have in mind of all its generalizations, proceeds in three steps: (1) *truncation*; (2) *regularization*, which yields the Maass-Selberg formula, following Casselman (1993); (3) *ordinary Fourier analysis* (here on $(0,\infty)$). Of course the aim in all proofs of the Plancherel theorems for Eisenstein series is to reduce the calculation on $\Gamma\backslash\mathcal{H}$ to that on the multiplicative group. The only difference between various proofs is exactly how and where one does this. In particular, it is how this step is taken which is new in this paper. In the literature, when truncation is used to derive a Plancherel measure, it is followed by taking limits as the truncation parameter passes to infinity. What is apparently novel in this paper is that a well known device from harmonic analysis makes the limit process unnecessary.

The usual proof, as explained in Langlands (1965), generalizes to other rank one groups easily. It generalizes to groups of higher rank only with a great deal of trouble, since one has to be extremely careful of how one moves contours of integration around. This part of the theory is notoriously difficult, and the difficulty is only partly alleviated by the recent exposé in Moeglin-Waldspurger (1995). The proof I give below also generalizes immediately to other groups of rank one. Although the generalization to groups of higher rank is a little less obvious, it is still rather elementary. I will deal with this in another place. The proof I give below also seems to work without essential modification for real groups, p-adic groups, and symmetric spaces, but that, too, is another story.

The first section recalls well known facts, and the second contains the proofs of the Theorems.

I would like to take this opportunity to thank Goro Shimura for accepting me as a student one day many years ago in the hall of the old Fine Hall. Without his council, I might well have never taken up the subject of automorphic forms.

1. Truncation

A variant of Eisenstein series is formally adjoint to the constant term. For any function f on $(0, \infty)$ we can extend it to a function $\overline{f}$ on $\mathcal{H}$ invariant with respect to real translations $z \mapsto z + x$. The function $\overline{f}$ is invariant under $\Gamma \cap P$, where P is the group of upper triangular elements in $SL_2(\mathbb{R})$ and $\Gamma \cap P$ is thus the group of upper triangular elements

$$\pm \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$$

in Γ. Note that

$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$$

acts by translation on $\mathcal{H}$. We can then define, at least formally, the Eisenstein series

$$E_f(z) = \sum_{\Gamma \cap P \backslash \Gamma} \overline{f}(\gamma(z)) \ .$$

For $f = y^s$ this gives Maass' series. A formal manipulation gives

$$\langle E_f, F \rangle_{\Gamma \backslash \mathcal{H}} = \langle f, F_0 \rangle_{(0,\infty)} \ .$$

For $T > 0$, define the operator C^T on $(0, \infty)$ to be multiplication by the characteristic function $\chi_{(T,\infty)}$, and Λ^T to be multiplication by $\chi_{(0,T)}$. Thus for any f on $(0, \infty)$ we have an orthogonal decomposition

$$f = \Lambda^T f + C^T f \ .$$

Now we combine these constructions to define analogous operations on $\Gamma \backslash \mathcal{H}$. Assume $T > 1$. For F on $\Gamma \backslash \mathcal{H}$, let f be its constant term. Then for $T > 0$, $C^T f$ has support on (T, ∞). We can consider its Eisenstein series in turn, which I define to be $C^T F$:

$$C^T F = E_{C^T f} \ .$$

It is characterized by these two conditions: (a) it is Γ-invariant; (b) in the fundamental domain $|x| \leq 1/2$, $|z| \geq 1$

$$C^T F = \begin{cases} 0 & y < T \\ F_0(y) & \text{otherwise.} \end{cases}$$

Define the **truncation** of F at T to be the difference

$$\Lambda^T F = F - C^T F \ .$$

Because the image of

$$\mathcal{H}_T = \{\textsc{im}(z) \geq T\}$$

under γ in Γ is either $\mathcal{H}_T$ itself (if γ lies in $\Gamma \cap P$) or a circle in the region $\{\textsc{im}(z) < 1\}$ (otherwise), the decomposition

$$F = \Lambda^T F + C^T F$$

is orthogonal whenever orthogonality makes sense. Both operators Λ^T and C^T are idempotent and self-adjoint:

$$C^T C^T = C^T, \quad \Lambda^T \Lambda^T = \Lambda^T, \quad \langle C^T F_1, F_2 \rangle = \langle F_1, C^T F_2 \rangle, \quad \langle \Lambda^T F_1, F_2 \rangle = \langle F_1, \Lambda^T F_2 \rangle \ .$$

For functions F satisfying some mild growth and smoothness conditions—when F is of **uniform moderate growth** on $\Gamma \backslash \mathcal{H}$—the function $\Lambda^T F$ is rapidly decreasing at ∞.

The integral E_s has uniform moderate growth. Thus

$$E_s = \Lambda^T E_s + C^T E_s$$

is an orthogonal sum at least formally, and $\Lambda^T E_s$ is rapidly decreasing at ∞. In particular it is square-integrable on $\Gamma\backslash\mathcal{H}$.

The **Maass-Selberg formula** asserts that

$$\langle \Lambda^T E_s, \Lambda^T E_{1-t}\rangle = \frac{T^{s-t} - c(s)c(1-t)T^{-(s-t)}}{s-t} + \frac{c(s)T^{1-s-t} - c(1-t)T^{s+t-1}}{1-s-t} \ .$$

In Casselman (1992) this is derived from a curious regularization of integrals on $\Gamma\backslash\mathcal{H}$, which allows one to start with the formal identity

$$\int_{\Gamma\backslash\mathcal{H}} E_s E_{1-t} \, \frac{dx\,dy}{y^2} = 0$$

which can be justified for generic values of s and t. In this scheme, the Maass-Selberg formula is expressed in the form

$$\begin{aligned}\langle \Lambda^T E_s, \Lambda^T E_{1-t}\rangle &= \int_0^T \left(y^s + c(s)y^{1-s}\right)\left(y^{1-t} + c(1-t)y^t\right) dy/y^2 \\ &= \int_0^T \left(y^{s-t} + c(s)y^{1-s-t} + c(1-t)y^{s+t-1} + c(s)c(1-t)y^{-s+t}\right) dy/y\end{aligned}$$

where the right hand side is evaluated by analytic continuation, writing

$$\int_0^T y^s\, dy/y = \frac{T^s}{s} \ .$$

Since truncations are rapidly decreasing at infinity, the right hand side of the Maass-Selberg formula cannot, and does not, have any poles, even though its individual terms do.

2. Fourier analysis

Because of orthogonality, we can write

$$\begin{aligned}E_\varphi &= \Lambda^T E_\varphi + C^T E_\varphi \\ \langle\, E_\varphi, E_{1-t}\,\rangle &= \langle\, \Lambda^T E_\varphi, E_{1-t}\,\rangle + \langle C^T E_\varphi, E_{1-t}\rangle \\ &= \langle\, \Lambda^T E_\varphi, \Lambda^T E_{1-t}\,\rangle + \langle C^T E_\varphi, C^T E_{1-t}\rangle \\ &= \left\langle \frac{1}{2\pi i}\int_{\sigma=1/2} \varphi(s)\Lambda^T E_s\, ds, \Lambda^T E_{1-t} \right\rangle + \langle C^T E_\varphi, C^T E_{1-t}\rangle \ .\end{aligned}$$

The point now is that each of these two terms has an interpretation purely in terms of harmonic analysis on $(0,\infty)$.

The second term is

$$\langle C^T \cdot \ \text{constant term of } E_\varphi,\ \text{constant term of } E_{1-t}\rangle_{(0,\infty)} = \langle C^T F_{\varphi_\#}, y^{1-t} + c(1-t)y^t\,\rangle_{(0,\infty)} \ .$$

As for the first term, we can bring the integral outside the coupling because the integrand is square-integrable. We then have from Maass-Selberg

$$\begin{aligned}&\Big\langle \frac{1}{2\pi i}\int_{\sigma=1/2}\varphi(s)\Lambda^T E_s\,ds, \Lambda^T E_{1-t}\Big\rangle\\ &\qquad = \frac{1}{2\pi i}\int_\sigma \varphi(s)\langle \Lambda^T E_s, \Lambda^T E_{1-t}\rangle\,ds\\ &\qquad = \frac{1}{2\pi i}\int_\sigma \varphi(s)\left[\frac{T^{s-t}-c(s)c(1-t)T^{-(s-t)}}{s-t}+\frac{c(s)T^{1-s-t}-c(1-t)T^{s+t-1}}{1-s-t}\right]ds\ .\end{aligned}$$

Since the integrand is holomorphic in s and t we can also write it as

$$\lim_{\epsilon\to 0}\frac{1}{2\pi i}\int_{|s-t|\ge\epsilon,|s+t-1|\ge\epsilon}\varphi(s)\left[\frac{T^{s-t}-c(s)c(1-t)T^{-(s-t)}}{s-t}+\frac{c(s)T^{1-s-t}-c(1-t)T^{s+t-1}}{1-s-t}\right]ds\ .$$

But now we can separate terms in the integral to get this to be the limit as $\epsilon\to 0$ of the sum

$$\begin{aligned}&\frac{1}{2\pi i}\int_{|s-t|\ge\epsilon}\varphi(s)\,\frac{T^{s-t}}{s-t}\,ds-\frac{1}{2\pi i}\,c(1-t)\int_{|s-t|\ge\epsilon}\varphi(s)c(s)\,\frac{T^{t-s}}{s-t}\,ds\\ &\qquad+\frac{1}{2\pi i}\int_{|s+t-1|\ge\epsilon}\varphi(s)c(s)\,\frac{T^{1-s-t}}{1-s-t}\,ds-\frac{1}{2\pi i}\,c(1-t)\int_{|s+t-1|\ge\epsilon}\varphi(s)\,\frac{T^{s+t-1}}{1-s-t}\,ds\ .\end{aligned}$$

We can make a change of variables of $1-s$ for s in two of these and reassemble to make this

$$\lim_{\epsilon\to 0}\ -\frac{1}{2\pi i}\int_{|s-t|\ge\epsilon}\varphi_\#(s)\,\frac{T^{-(t-s)}}{t-s}\,ds-\frac{1}{2\pi i}\,c(1-t)\int_{|1-t-s|\ge\epsilon}\varphi_\#(s)\,\frac{T^{-(1-t-s)}}{1-t-s}\,ds\ .$$

How can this be interpreted? At this point I recall some elementary Fourier analysis on $(0,\infty)$. Recall that we are given on $(0,\infty)$ the measure dy/y^2. The Fourier transform is therefore

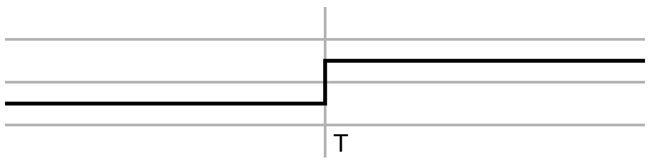
$$\widehat{f}(s)=\int_0^\infty f(y)y^{1-s}\,dy/y^2$$

for $\mathrm{RE}(s)=1/2$. The inverse transform is

$$\frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty}F(s)y^s\,ds\ .$$

Define sgn_T to be the function

$$\mathrm{sgn}_T=\begin{cases}\ \ 1/2 & \text{if } y>T\\ -1/2 & \text{otherwise}\end{cases}$$



Its Fourier transform is the **principal value** of T^{-s}/s, which is the distribution

$$\langle\mathcal{PV}(T^{-s}/s),\varphi\rangle=\lim_{\epsilon\to 0}\frac{1}{2\pi i}\int_{|s|\ge\epsilon}\varphi(s)\left(\frac{T^{-s}}{s}\right)ds\ .$$

Recalling how convolution and multiplication relate to each other under the Fourier transform, we see that

$$\begin{aligned}\lim_{\epsilon\to 0}\ -\frac{1}{2\pi i}\int_{|s-t|\ge\epsilon}\varphi_{\#}(s)\,\frac{T^{-(t-s)}}{t-s}\,ds-\frac{1}{2\pi i}\,c(1-t)\int_{|1-t-s|\ge\epsilon}\varphi_{\#}(s)\,\frac{T^{-(1-t-s)}}{1-t-s}\,ds\\ =-\langle\mathrm{sgn}_T F_{\varphi_{\#}},y^{1-t}+c(1-t)y^t\rangle_{(0,\infty)}\ .\end{aligned}$$

In other words, Fourier duality on $(0,\infty)$ proves this extremely useful if technical result, which is really the main result of this paper:

Theorem 3. *For φ of compact support on* $\mathrm{RE}(s)=1/2$

$$\langle\Lambda^T E_\varphi,E_{1-t}\rangle_{\Gamma\backslash\mathcal{H}}=-\langle\mathrm{sgn}_T\,F_{\varphi_{\#}},y^{1-t}+c(1-t)y^t\rangle_{(0,\infty)}\ .$$

Our original expression therefore becomes

$$\begin{aligned}\langle-\mathrm{sgn}_T F_{\varphi_{\#}},y^{1-t}+c(1-t)y^t\rangle+\langle C^T F_{\varphi_{\#}},y^{1-t}+c(1-t)y^t\rangle\\ =(1/2)\,\langle F_{\varphi_{\#}},y^{1-t}+c(1-t)y^t\rangle\\ =(1/2)\,\big(\varphi_{\#}(t)+c(1-t)\varphi_{\#}(1-t)\big)\\ =\varphi_{\#}(t)\end{aligned}$$

since $-\mathrm{sgn}_T+\chi_{(T,\infty)}$ is the constant $1/2$.

From Theorem 1 we can easily evaluate the inner product

$$\langle E_\varphi,E_\psi\rangle=\left(\frac{1}{2}\right)\langle F_{\varphi_{\#}},F_{\psi_{\#}}\rangle\ .$$

Keep in mind that $F_{\varphi_{\#}}$ is the constant term of E_φ, $F_{\psi_{\#}}$ that of E_ψ.

The function sgn_T is a multiplicative translation of $\mathrm{sgn}=\mathrm{sgn}_1$. The function sgn has exactly two important properties we require for this proof to work:

$$\begin{aligned}-\mathrm{sgn}+\chi_{(1,\infty)}&=1/2\\ \widehat{\mathrm{sgn}}&=\mathcal{PV}\left(\frac{1}{s}\right)\ .\end{aligned}$$

I leave it for now as interesting exercise in Euclidean truncation to figure out what function will replace it for root systems of rank >1.

References

1. W. Casselman, 'Automorphic forms and a Hodge theory for congruence subgroups of $SL_2(\mathbb{Z})$', in *Lie group representations II*, Lect. Notes in Math. **1041**, Springer, 1983.

2. W. Casselman, 'Extended automorphic forms in the upper half plane', *Mathematische Annalen* **296** (1992), 755–762.

3. T. Kubota, *Elementary theory of Eisenstein series*, Kodansha and Wiley, 1983.

4. R. P. Langlands, 'Eisenstein series', 235–252 in *Proc. Symp. Pure Math.* **IX**, A. M. S., 1965.

5. C. Moeglin and J.-L. Waldspurger, *Spectral decomposition and Eisenstein series*, Cambridge, 1995.