Geometric rationality of Satake compactifications

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Throughout this paper, except in a few places, let

$$\begin{aligned} G &= \text{ the } \mathbb{R}\text{-rational points on a reductive group defined over } \mathbb{R} \\ K &= \text{ a maximal compact subgroup} \\ X &= \text{ the associated symmetric space} \end{aligned}$$

Let (π, V) be a finite-dimensional algebraic representation of G. Eventually G will be assumed semi-simple and π irreducible. If V contains a vector v with the property that K is the stabilizer of v, I will call (π, V, v) a **spherical representation**. Let $[v]$ be the image of v in $\mathbb{P}(V)$. The closure of the image of the G-orbit of $[v]$ in $\mathbb{P}(V)$ is then a G-covariant compactification $\overline{X}_\pi$ of the symmetric space X. Such compactifications were first systematically examined in Satake (1960a), and they are called **Satake compactifications**.

Suppose G to be defined over $\mathbb{Q}$ and Γ an arithmetic subgroup. In a second paper Satake (1960b) a procedure was given to obtain compactifications of the arithmetic quotient $\Gamma\backslash X$ from certain of these compactifications $\overline{X}_\pi$, by adjoining to X certain **rational boundary components** in $\overline{X}_\pi$. It was not clear at that time which (π, V, v) were **geometrically rational** in the sense that one could use them to construct such compactifications of $\Gamma\backslash X$, but Satake did formulate a useful criterion for geometric rationality in terms of the closures of Siegel sets in X, and verified that this criterion held for several classical arithmetical groups. Satake's geometrical conditions were reformulated more algebraically in Borel (1962), where it was shown very generally that (π, V, v) is geometrically rational if π is irreducible and $\mathbb{Q}$-rational. A little later it was shown in Baily-Borel (1966) that certain compactifications of Hermitian symmetric spaces X (those now usually called Baily-Borel compactifications) were also geometrically rational, even when not $\mathbb{Q}$-rational. It seems, however, to have remained an unsolved problem since then to formulate a simple necessary and sufficient criterion for geometrical rationality. In this paper I will give a reasonably useful result of this kind, involving the real and rational Galois indices of the group G, for the case of irreducible representations.

The literature on compactifications dealing with these questions, aside from the early papers already mentioned, is sparse. One exception is the book Ash et al. (1975), in which the complicated case-by-case argument of Baily and Borel was replaced by a more direct proof involving rational homogeneous cones. Another exception is Zucker (1983), in which the general subject of geometric rationality was broached perhaps for the first time since the original work. It was perhaps there that the question of whether or not one could find a criterion involving the Galois indices was first raised, although it was not a topic with which that paper was directly concerned. It will become apparent that I am indebted to Zucker's paper for several valuable suggestions. In order to avoid confusion, however, I should point out that Zucker's discussion of rationality matters (in §3 of Zucker (1983)) is somewhat obscure and, as far as I can tell, in error (particularly his Proposition (3.3)).

Aside from the criterion for geometric rationality, there are a few other points in this paper perhaps worth calling attention to. (1) Satake's original construction was for semi-simple groups G. It started with an arbitrary irreducible (π, V) (no K-fixed vector), and then considered the associated representation of G on the space of Hermitian forms on V, among which lies a positive definite one fixed by K. Things are in fact greatly simplified if one looks instead directly at spherical triples. It is possible also to use **projective spherical triples**, where v is only assumed to be an eigenvector of K, but this does not seem to add much, and makes things a bit more complicated. (2) The boundary components in Satake's compactifications are all symmetric spaces associated to semi-simple quotients of Levi factors of certain parabolic subgroups of G. In an initial version of this paper I looked at the case of reducible representations, when the boundary components possess abelian factors as well. Allowing this, one obtains a larger class of compactifications than the ones Satake found (the **reductive** as opposed to **semi-simple** ones). The most interesting example is probably the one used by Goresky-Harder-MacPherson (1994), obtained by collapsing the unipotent fibres at infinity in the Borel-Serre compactification. I have abandoned this idea in the present paper because I was not entirely satisfied with certain technically awkward points. I hope to return to it in a subsequent paper. (3) Another direction for generalization would be to consider certain compactification of other homogeneous quotients G/H. Among the most interesting of these are Oshima's $G \times G$-covariant compactification of G. (4) I am not sure that compactifications arising from finite-dimensional representations are ultimately the natural ones to consider. Satake's procedure for obtaining compactifications of symmetric spaces has much to recommend it, especially its simplicity. The topological structure of these compactifications is not difficult to understand. But in fact the compactifications one obtains are semi-algebraic spaces, and in various applications it this extra structure is useful. Moreover, one can also use the same technique to obtain $G(\mathbb{C})$-covariant completions of the algebraic varieties $G(\mathbb{C})/K(\mathbb{C})$. The problem is that as semi-algebraic or algebraic completions the varieties one gets by Satake's procedure leave much to be desired. This is pointed out in Vust (1990), where he constructs all normal G-covariant completions of $G(\mathbb{C})/K(\mathbb{C})$, and shows that Satake compactifications are not generally normal. If one wants compactifications of G/K with a structure finer than topological, one should probably look at real points on Vust's varieties, or at least use some of Vust's techniques.

The basic technique in this paper is to look at the image of split real tori in compactifications of X. These are looked at on their own in §§1–3. At the end of §3 there is an observation about the classification of regular polyhedra. In §4 I put results from §3 in a slightly more general context, with future work in mind. Finite dimensional representations of G and compactifications of X are examined in §§5–6. In §7 I introduce the Galois index of Borel and Tits, in §8 I formulate and prove the main results, and in §9 I look at a few examples.

Although the subject of this paper is not directly related to work of Roger Richardson's, it seems to me very likely that eventually the most elegant answer to questions raised here will in fact depend on the results of Richardson-Springer (1990) and Richardson-Springer (1993) on the structure of $B_{\mathbb{C}}\backslash G_{\mathbb{C}}/K_{\mathbb{C}}$, and for this reason it is perhaps not inappropriate to dedicate it to his memory. I also wish to thank I. Satake and A. Borel for helpful comments when I started out on this project.

1. Convex polyhedra

This section will formulate some simple results about the geometry of convex polyhedra. Proofs are straightforward and will be left as exercises. Suppose V to be a vector space over $\mathbb{R}$, V^* its dual. Suppose C to be a compact convex polyhedron in V^*.

If H is a hyperplane intersecting C and all of C lies on one side of H then $H \cap C$ is a closed face of C. If H is the hyperplane $\lambda = c$ and the side containing C is $\lambda \le c$ then this face is the subset of C where λ achieves its maximum value c on C. I call this the **maximum locus** of λ on C.

Conversely, if λ is any vector in V, its maximum locus is a closed face F of C. The value of λ on this face is a constant $\langle \lambda, F \rangle$, and on points not on F is strictly less.

Each closed face F of C therefore determines a subset σ_F of V, the set of λ such that F is the maximum locus of λ.

Lemma 1.1. *Suppose F to be a closed face of C.*
(a) The vector λ lies in σ_F if and only if (i) λ takes a constant value $\langle \lambda, F \rangle$ on F, and (ii) for any x in C but not in F

$$\langle \lambda, x \rangle < \langle \lambda, F \rangle .$$

(b) The vector λ lies in the closure of σ_F if and only if (i) λ takes a constant value $\langle \lambda, F \rangle$ on F, and (ii) for any x in C

$$\langle \lambda, x \rangle \le \langle \lambda, F \rangle \ .$$

Proof. The only slightly non-trivial point is the sufficiency of these conditions for the closure. Suppose λ to satisfy them, and let μ be an arbitrary element of σ_F. For all $t > 0$ the sum $\lambda + t\mu$ lies in σ_F. ◻

The subset σ_F is conical, which is to say that λ lies in σ_F if and only if $c\lambda$ lies in it, for any $c > 0$. It is certainly convex. It is also polyhedral. We can describe its faces explicitly:

Proposition 1.2. *Suppose E and F are faces of C with F contained in the closure of E. For any u in the interior of E and v in the interior of F, the closure of σ_E is the maximum locus of $u - v$ in the closure of σ_F. The closure of σ_F is the union of these σ_E.*

Proof. We must show that (•) $u - v$ is constant, in fact 0, on σ_E, and that (•) for λ in $\overline{\sigma}_F$ but not in $\overline{\sigma}_E$,

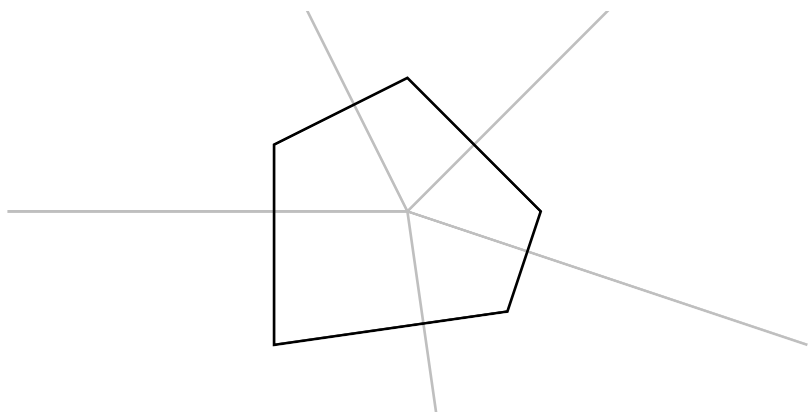
$$\langle u - v, \lambda \rangle < 0 \ .$$

The first follows immediately from the definition of σ_E since F is contained in the closure of E and λ is constant on E. As for the second, suppose that λ lies in σ_F and that $\langle u, \lambda \rangle = \langle v, \lambda \rangle$. Since λ lies in $\overline{\sigma}_F$, λ is at most $\langle \lambda, v \rangle$ everywhere on C, and since u is in the interior of E, λ must be constant on E. But then its value on E is the same as its value on F, and by the previous result λ must lie in the closure of σ_E.

The last assertion follows from Lemma 1.1. ◻

A compact convex polyhedron C therefore determines a **dual** partition of V into conical convex polyhedral subsets σ_F, one for each face F of C. The face C itself corresponds to the linear space of functions constant on all of C, and vertices of C correspond to open polyhedral cones.

In the following picture, the edges of the σ_F are in light gray, the faces of C in black.



Several compact convex subsets C of V^* may determine the same partition of V. What in fact determines the partition is the local configuration at each of the vertices of C, not its global structure. In fact, a consequence of the previous result is that *if v is a vertex of C then the structure of the open cell σ_v is completely determined by the cone $c(u-v)$ as c ranges over all $c \geq 0$ and u ranges over all of C*. This cone will have one face for each closed face of C meeting v. Two compact convex polyhedra in V^* will determine the same partition of V precisely when the cones attached to each vertex are the same.

2. Compactifications of torus orbits

This section will formulate simple results relating homogeneous spaces of real algebaric tori and convex polyhedra. Again, proofs will generally be left as exercises.

Let A be the group of real points on a split torus defined over $\mathbb{R}$, A^{conn} its connected component. Let (π, V) be an algebraic representation of A and v a vector in V.

The vector v may be expressed as a sum of eigenvectors

$$v = \sum_\alpha v_\alpha$$

where the α are $\mathbb{R}$-rational characters of A. The representation π induces an action of A on $\mathbb{P}(V)$ as well. Let $[v]$ be the image of v in $\mathbb{P}(V)$, $A_{\mathfrak{C}}$ its isotropy subgroup—the subgroup of A acting by scalar multiplication on v. The closure of the A^{conn}-orbit of $[v]$ will be an A^{conn}-covariant compactification of $A^{\mathrm{conn}}/A_{\mathfrak{C}}^{\mathrm{conn}}$. In this section I shall describe the structure of this closure.

Define

$$\begin{aligned} \mathfrak{chars}(v) &= \{\alpha \mid v_\alpha \neq 0\} \\ \mathfrak{C} = \mathfrak{C}(v) &= \text{ the convex hull of } \mathfrak{chars}(v) \text{ in } X^*(A) \otimes \mathbb{R} \end{aligned}$$

where $X^*(A)$ is the lattice $\mathrm{Hom}(A, \mathbb{G}_m)$ of rational characters of A. Thus $\mathfrak{C}$ is a compact convex polyhedron of dimension equal to that of $A/A_{\mathfrak{C}}$.

Recall that the lattice $X_*(A) = \mathrm{Hom}(\mathbb{G}_m, A)$ is dual to $X^*(A)$, where the duality is expressed by the formula

$$t^{\langle \lambda, \alpha \rangle} = \alpha(\lambda(t))$$

for t in $\mathbb{R}^\times$. As suggested by the discussion in the previous section, to each closed face F of $\mathfrak{C}$ associate the subset σ_F of all λ in $X_*(A) \otimes \mathbb{R}$ with the property that F is the maximum locus of λ. That is to say for each λ in σ_F the values

$$\langle \lambda, F \rangle = \langle \lambda, \alpha \rangle$$

are all the same for α in F, and if β is any other vertex of $\mathfrak{C}$ then

$$\langle \lambda, \beta \rangle < \langle \lambda, F \rangle \ .$$

The σ_F make up the dual convex polyhedral decomposition of $X_*(A) \otimes \mathbb{R}$ described in §1. All the σ_F are invariant under translation by $X_*(A_{\mathfrak{C}}) \otimes \mathbb{R} = \sigma_{\mathfrak{C}}$. Whereas the λ in σ_F are characterized by the property that F is the maximum locus of λ, an element in the closure of the polyhedral cone σ_F will be constant on F, and this constant will be the maximum value on $\mathfrak{C}$, but it may also have the same value on a larger face. From §1:

Lemma 2.1. *The closure of σ_F consists of all λ such that λ takes its maximum value $\max_{\mathfrak{C}} \lambda$ on F. If $E \subseteq F$ then $\sigma_F \subseteq \overline{\sigma}_E$.*

In other words, the partition of $X_*(A) \otimes \mathbb{R}$ is dual to that of $\mathfrak{C}$ into faces.

If F is any closed face of $\mathfrak{C}$ then define

$$v_F = \sum_{\alpha \in F} v_\alpha \ .$$

Lemma 2.2. *The coweight λ lies in σ_F if and only if*

$$[\pi(\lambda(t))v] \to [v_F]$$

as $t \to \infty$.

Proof. This follows from the calculation

$$\begin{aligned}
\pi(\lambda(t))v &= \sum_{\alpha \in \mathfrak{chars}(v)} t^{\langle \lambda, \alpha \rangle} v_\alpha \\
&= t^\ell \sum_{\alpha \in \mathfrak{chars}(v)} t^{\langle \lambda, \alpha \rangle - \ell} v_\alpha \ .
\end{aligned}$$

If $\ell = \langle \lambda, F \rangle$ then all terms vanish as $t \to \infty$ except the ones where $\langle \lambda, \alpha \rangle = \ell$. ◻

In particular v_F lies in the closure of the A^{conn}-orbit of $[v]$. What does the whole closure look like?

Let $\mathcal{E}$ be the extremal vertices of $\mathfrak{C}$. If α is any character in $\mathfrak{chars}(v)$ then

$$\alpha = \sum_{\mathcal{E}} r_\epsilon \epsilon \quad (r_\epsilon \ge 0,\ \sum r_\epsilon = 1)$$

If $r_\epsilon = m_\epsilon/n$ with a common denominator n then we have homogeneous relations

$$c_\alpha^n = \prod c_\epsilon^{m_\epsilon}$$

These relations persist in projective space. Since all points in the orbit itself have non-zero coordinates, and any point in projective space must have at least one non-zero coordinate, any point in the projective closure must lie in at least one of the affine subsets where $c_\epsilon \ne 0$. The same argument shows more generally that if a point $[u]$ in the orbit has $c_\alpha \ne 0$ for some α in the interior of a face, then $c_\alpha \ne 0$ for all α in the face. Thus the set of α with $c_\alpha \ne 0$ must coincide with a whole face.

I claim now that if this face is F, then $[u]$ lies in the orbit of $[v_F]$. If γ is a vertex of F then $[u]$ lies in the affine subset of projective space where $c_\gamma \ne 0$. We can normalize projective coordinates by dividing by c_γ, which amounts to multiplying all coordinates on the orbit of v by γ^{-1}. So the representation of v is in these affine coordinates

$$v = \sum v_\alpha$$

where now γ is the trivial character. There will be a finite number of rational linear relations among the characters occurring among these coordinates, and we can therefore find a finite number of non-negative integers $m_{i,\alpha}$ and sets P_i, N_i such that the relations

$$\prod_{\alpha \in P_i} c_\alpha^{m_{i,\alpha}} = \prod_{\alpha \in N_i} c_\alpha^{m_{i,\alpha}}, \quad c_\alpha > 0$$

characterize vectors in the A^{conn}-orbit of v. The coordinates of the vector u must also satisfy these relations, and all $c_\alpha \ne 0$ for α in F. But that implies that u lies in the orbit of v_F.

In other words:

Lemma 2.3. *The closure of the A^{conn}-orbit of $[v]$ is the union of the A^{conn}-orbits of the $[v_F]$ as F ranges over the closed faces of $\mathfrak{C}$. The orbit of $[v_F]$ is contained in the closure of the orbit of $[v_E]$ if and only if F is contained in E.*

The orbit of $[v_F]$ is isomorphic to A/A_F where A_F is the subgroup of A with the property that all α in F are equal on A_F:

$$A_F = \cap_{\alpha,\beta\in F} \operatorname{Ker} \alpha\beta^{-1}$$

The subgroup A_F is also the **linear support** of σ_F, that is to say the smallest subtorus of A such that $X_*(A_F)\otimes\mathbb{R}$ contains σ_F. Let γ_F be the character of A_F which is the common restriction to A_F of the characters in F. The vector space spanned by the v_α with α in F can be characterized as the eigenspace of V for the torus A_F with respect to the character γ_F.

Let η_F be the set of characters $\alpha\beta^{-1}$ with $\alpha \in \mathfrak{chars}(v)$, $\beta \in F$. It generates a semi-group of $X^*(A)$ determining an affine covariant embedding of $A/A_{\mathfrak{C}}$ whose

A-orbits are indexed by the closed faces of $\mathfrak{C}$ containing F. The image is exactly the closure of $[v]$ in the open affine subset of $\mathbb{P}(V)$ where the the $c_\alpha \neq 0$ for α in F. Abstractly the closure of $[v]$ is obtained by patching together these affine spaces as F ranges over the faces of $\mathfrak{C}$. (This sort of argument is standard in Kempf et al. (1973).) The algebraic structure of the closure of $[v]$ in $\mathbb{P}(V)$ depends explicitly on the semi-groups η_F rather than the hull $\mathfrak{C}$ itself. This conforms with remarks at the end of §1.

Corresponding to the polyhedra σ_F in $X_*(A) \otimes \mathbb{R}$ are subsets of A^{conn} which are in some sense their **spans**. If F is a face of the cone $\mathfrak{C}$ then define S_F to be the subset of a in A^{conn} such that all values of $\alpha(a)$ for α in F are equal to some real constant c_a, while $\alpha(a) < c_a$ for α not in F. Thus for λ in σ_F and $t > 1$, $\lambda(t)$ lies in S_F. Then $\pi(a^n)v \to v_F$ as $n \to \infty$. The sets S_F partition A^{conn}. It is simple to prove:

Proposition 2.4. *The closure of $S_F[v]$ in $\mathbb{P}(V)$ is the union of the $S_F[v_E]$ as E ranges over all faces of $\mathfrak{C}$ containing F.*

Corollary 2.5. *The A^{conn}-orbits intersected by $S_F[v]$ in $\mathbb{P}(V)$ are those containing the limits*

$$\lim_{t\to\infty} [\pi(\lambda(t))v]$$

for λ in $\overline{\sigma}_F$.

3. The convex hulls of Coxeter group orbits

In this section, let (W, S) be an arbitrary finite Coxeter group. For each s, t in S let $m_{s,t}$ be the order of st.

Fix a realization (π, V) of (W, S). That is to say, π is a representation of W on the finite dimensional real vector space V in which elements of S act by reflections. There exists a positive definite metric invariant under W, and there exists a set of vectors $\{\alpha_s\}$ in V, indexed by S, such that the angle between α_s and α_t is $\pi - \pi/m_{s,t}$ for each s, t in S.

Let Δ be the set of α_s. The **Coxeter graph** of (W, S) is the graph with nodes indexed by S or equivalently Δ, in which α_s and α_t are linked by an edge labelled by $m_{s,t}$ when $m_{s,t} > 2$. (By default, an unlabelled edge has $m_{s,t} = 3$.)

In V the region

$$C^{++} = \{v \in V \mid \langle \alpha, v\rangle > 0 \text{ for all } \alpha \in \Delta\}$$

is an open fundamental domain for W. For each $\Theta \subseteq \Delta$ define

$$C^{++}_\Theta = \{v \mid \langle \alpha, v\rangle = 0 \text{ for } \alpha \in \Theta, \langle \alpha, v\rangle > 0 \text{ for } \alpha \notin \Theta\}\ .$$

Thus $C^{++} = C^{++}_\emptyset$, and the closure of C^{++} is the disjoint union of the C^{++}_Θ. We have similar sets $\widehat{C}^{++}_\Theta$ in $\widehat{V}$.

Fix for the moment χ in $\widehat{V}$, lying in the closure of $\widehat{C}^{++}$. It will lie in a unique $\widehat{C}^{++}_\Theta$. Let $\delta = \delta_\chi$ be the complement of Θ in the set of nodes in the Coxeter graph. Equivalently

$$\delta_\chi = \Delta - \{\alpha_s \mid s\chi = \chi\}\ .$$

A subset $\kappa \subseteq \Delta$ is said to be δ-**connected** if every one of its nodes can be connected to an element of δ by a path inside itself. Given a δ-connected set κ, its δ-complement $\zeta(\kappa)$ is the set of α in the complement of κ which (a) are not in δ and (b) are not connected to κ by an edge in the Coxeter graph. Define its δ-**saturation**

$$\omega(\kappa) = \kappa \cup \zeta(\kappa) \ .$$

More generally, if θ is any set of nodes in the graph, define

$$\begin{aligned} \kappa(\theta) &= \text{ the largest } \delta\text{-connected subset of } \theta \\ \zeta(\theta) &= \zeta(\kappa(\theta)) \\ \omega(\theta) &= \kappa(\theta) \cup \zeta(\theta) \end{aligned}$$

If $\kappa = \emptyset$ then ζ is the complement of δ. By construction, the two groups W_κ and W_ζ commute with each other.

If an element of θ has an edge linking it to $\kappa(\theta)$ then it lies in $\kappa(\theta)$ itself, and if doesn't then it lies in $\zeta(\theta)$. Hence

$$\kappa(\theta) \subseteq \theta \subseteq \omega(\theta) \ .$$

If κ is any δ-connected subset of Δ and ξ is any subset of Δ then $\kappa(\xi) = \kappa$ if and only if

$$\kappa \subseteq \xi \subseteq \omega(\kappa) \ .$$

The sets $\omega(\kappa)$ one gets as κ ranges over all δ-connected subsets of Δ I call the δ-**saturated** subsets of Δ. The correspondence between κ and ω is bijective.

The following is implicit in work of Satake and Borel-Tits (cf. Lemma 5 of Satake (1960a) and §12.16 of Borel-Tits (1965)). The proof is essentially theirs as well.

Theorem 3.1. *Suppose χ to lie in the closure of $\widehat{C}^{++}$, and let $\delta = \delta_\chi$. The map taking κ to the convex hull F_κ of $W_\kappa \cdot \chi$ is a bijection between the δ-connected subsets of S and the faces F of the convex hull of $W \cdot \chi$ such that σ_F meets $\overline{C}^{++}$.*

Every face of the convex hull of $W \cdot \chi$ is W-conjugate to exactly one of these faces. The face corresponding to the empty set $\emptyset$ is χ itself. The dimension of the face F_κ is the cardinality of κ. An increasing chain of δ-connected subsets corresponds to an increasing chain of faces.

Proof. I begin with a simple observation. Suppose κ to be a δ-connected subset of Δ, $\omega = \omega(\kappa)$. Since s fixes χ and commutes with W_κ if s lies in $\zeta = \zeta(\kappa)$, the group W_ζ fixes all of $W_\kappa \cdot \chi$ and

$$W_\kappa \cdot \chi = W_\omega \cdot \chi$$

so that for any $\theta \subseteq \Delta$, the orbit $W_\theta \cdot \chi$ is the same as $W_{\kappa(\theta)} \cdot \chi$.

If φ lies in $\widehat{V}$ then we can write it as $\sum c_\alpha \alpha$. The **support** of φ is the set $\mathrm{supp}(\varphi)$ of α with $c_\alpha \neq 0$.

For the duration of the proof, for each w in W let

$$\Sigma_w = \mathrm{supp}(\chi - w\chi) \ .$$

It is simple to calculate Σ_w in terms of a reduced expression for w. If $w = s$ lies in S we have (with $\alpha = \alpha_s$)

$$\begin{aligned} s\chi &= \chi - \langle \chi, \alpha^\vee \rangle \alpha \\ \chi - s_\alpha \chi &= \langle \chi, \alpha^\vee \rangle \alpha \end{aligned}$$

and the coefficient on the right is always non-negative. It is positive if α lies in δ, otherwise 0. Therefore

$$\Sigma_s = \begin{cases} \{\alpha_s\} & \alpha_s \in \delta \\ \emptyset & \text{otherwise} \end{cases}$$

Inductively, we write

$$\chi - w\chi = \sum c_\gamma \gamma$$

with all $c_\gamma \geq 0$ and Σ_w a δ-connected subset of Δ. We then look at $s_\alpha w > w$. Since $w^{-1}\alpha > 0$

$$\begin{aligned} w\chi &= \chi - \sum c_\gamma \gamma \\ s_\alpha w\chi &= w\chi - \langle w\chi, \alpha^\vee \rangle \alpha \\ &= \chi - \sum c_\gamma \gamma - \langle \chi, w^{-1}\alpha^\vee \rangle \alpha \\ \chi - s_\alpha w\chi &= \sum c_\gamma \gamma + \langle \chi, w^{-1}\alpha^\vee \rangle \alpha \end{aligned}$$

so that again the coefficients are always non-negative. Furthermore either (a) $\Sigma_{s_\alpha w} = \Sigma_w$ or (b) α does not lie in Σ_w and $\Sigma_{s_\alpha w}$ is the union of Σ_w and $\{\alpha\}$. In the second case

$$\langle w\chi, \alpha^\vee \rangle = \langle \chi, \alpha^\vee \rangle - \sum c_\gamma \langle \gamma, \alpha^\vee \rangle$$

where all the non-zero terms in the sum are non-positive. Since the total sum is non-zero, either α lies in δ or α is linked by an edge in the Coxeter diagram to an element in Σ_w. Either way we see that $\Sigma_{s_\alpha w}$ is again δ-connected.

Summarizing this argument:

Lemma 3.2. *The set Σ_w is always δ-connected. If $w = xy$ with $\ell(w) = \ell(x) + \ell(y)$ then $\Sigma_y \subseteq \Sigma_w$.*

The argument shows also that if w lies in W_κ then $\Sigma_w \subseteq \kappa$. The converse is also true:

Lemma 3.3. *If κ is a δ-connected subset of Δ then the character $w\chi$ lies in $W_\kappa \cdot \chi$ if and only if Σ_w is contained in κ.*

Proof. For the new half, let $w = s_n s_{n-1} \ldots s_1$ be a reduced expression for w. Let $w_i = s_i \ldots s_1$ for each i, $w_0 = 1$. If k is the smallest such that $w_k\chi$ doesn't lie in $W_\kappa \cdot \chi$, then $\epsilon = w_{k-1}\chi$ does lie in $W_\kappa \cdot \chi$, but $s_k\epsilon$ doesn't. Then s_k isn't in W_ω, and $s_k\epsilon \neq \epsilon$. Since

$$s_k\epsilon = \epsilon - n_{\alpha_k}\alpha_k$$

with $n_{\alpha_k} > 0$, the support of $\chi - w_k\chi$ contains α_k. The same is true of $\chi - w\chi$ since support increases along a chain. ◻

Lemma 3.4. *Every δ-connected subset κ of Δ is the support of some $\chi - w\chi$.*

Proof. We prove this by induction on the size of the δ-connected subset. It is clear for single elements in δ.

Otherwise, suppose that κ is δ-connected and $\chi - x\chi$ has support κ. Suppose then that α does not lie in κ and that $\kappa \cup \alpha$ is δ-connected. Either there is an edge in the Coxeter graph connecting α to κ or α itself lies in δ.

If $x\chi = \chi - \sum n_\beta \beta$ where all $n_\beta > 0$ for β in κ, then

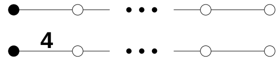
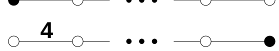
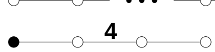
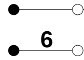
$$\begin{aligned} s_\alpha x\chi &= s_\alpha\chi - \sum n_\beta s_\alpha \beta \\ &= s_\alpha\chi - \sum n_\beta[\beta - \langle \beta, \alpha^\vee\rangle\alpha] \\ s_\alpha x\chi - \chi &= (s_\alpha\chi - \chi) - \sum n_\beta\beta + \sum n_\alpha\langle\beta, \alpha^\vee\rangle\alpha \end{aligned}$$

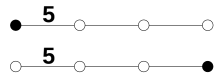
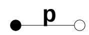
and all of the coefficients in the last sum are non-positive. If α itself lies in δ then $s_\alpha\chi$ has support α, so the support of the whole expression is the union of κ and α. If not, then the first term vanishes, but at least one term in the sum is actually negative. ◻

Lemma 3.5. *For λ in C_θ^{++} the subset of $W\cdot\chi$ where λ achieves its maximum is $W_{\kappa(\theta)}\cdot\chi$.*

This is immediate from Lemma 3.3. The main Theorem now follows from it and Lemma 3.4, which guarantees that the sets $W_\kappa\cdot\chi$ are all distinct.

The main Theorem of this section is curiously relevant to the classification of regular polyhedra. It follows by induction that the symmetry group of any regular polyhedron is a Coxeter group. Since all vertices are W-conjugate every regular polyhedron must be the convex hull of a single vector. Since all faces of a given dimension must also be W-conjugate, the δ-connected sets must form a single ascending chain. This means that δ itself must be a single node at one end of the Coxeter graph and that the Coxeter graph has no branching. Also, an automorphism of the Coxeter graph induces an isomorphism of corresponding polyhedra. Therefore *the regular polyhedra correspond to isomorphism classes of pairs Δ, α where Δ is the Coxeter graph of a finite Coxeter group, without branching, and α is an end-node in Δ.* This is of course the usual classification, which I exhibit in these terms in the following table. Keep in mind that the dimension in which the figure is embedded is the number of nodes. The marked end denotes the vertex. Thus, in the first line for H_3 we list the vertex, then the pentagonal face, on the dodecahedron, whereas in the next line, reading from right to left, we meet the vertex and then a triangular face.

System	Marked Coxeter diagram	Figure
A_n $(n \geq 2)$		tetrahedron
B_n $(n \geq 2)$		cube
		octahedron
F_4		
G_2		hexagon
H_3		dodecahedron
		icosahedron

H_4		
I_p ($p = 5, p \geq 7$)		polygon

4. Satake partitions

Continue the notation of the last section. In particular, χ is an element of V^* lying in the closure of $\widehat{C}^{++}$, and $\delta = \delta_\chi$. In this section I shall describe in more detail the partition of V dual to the convex hull of $W \cdot \chi$.

I shall begin in a little more generality. I define a **Satake partition** of V to be any partition of V into convex polyhedral cones σ which is geometric in the sense that any open face of one of these cones is also a subset of the partition, and which is also W-invariant. According to Proposition 1.2, the partition of V dual to the convex hull of any W-orbit of a finite set of points in V^* is a Satake partition.

Define a **Weyl cell** in V to be a W-transform of some C_θ^{++}. The partition of V into Weyl cells is a Satake partition I call the **Weyl partition**.

Suppose now and for a while that that σ is one of the cells in an arbitrary Satake partition. Since the partition is W-invariant, for any given w in W exactly one of two possibilities is true: (a) $w\sigma$ and σ are disjoint, or (b) $w\sigma = \sigma$. Define W_σ to be the group of w in W with $w\sigma = \sigma$. Define the **centre** of σ to be the elements of σ fixed by W_σ. For any x in σ define

$$\Pi_\sigma x = \frac{1}{\# W_\sigma} \sum_{W_\sigma} w\, x$$

to be the average of the transforms of x by elements of W_σ. Because of convexity, $\Pi_\sigma x$ lies also in σ, and in particular the **centre** C_σ of σ is not empty. If $wx = x$ for w in W and x in C_σ then $w\sigma \cap \sigma \neq \emptyset$ so w has to be in W_σ. In other words, for any x in C_σ

$$W_\sigma = \{w \in W \mid wx = x\}$$

which proves:

Lemma 4.1. *There exists a unique Weyl cell containing the centre of σ.*

For $\theta \subseteq \Delta$ define

$$\Pi_\theta x = \frac{1}{\# W_\theta} \sum_{W_\theta} w\, x$$

to be the average over W_θ.

Lemma 4.2. *For $\psi \subseteq \theta \subseteq \Delta$ and x in C_θ^{++} the projection $\Pi_\psi x$ lies in C_ψ^{++}.*

Proof. For x in C_θ^{++}

$$\Pi_\psi x = x - \sum_{\theta - \psi} c_\alpha \alpha^\vee$$

with $c_\alpha \geq 0$. But then for $\beta \in \Delta - \psi$

$$\langle \beta, \Pi_\psi x \rangle = \langle \beta, x \rangle - \sum c_\alpha \langle \beta, \alpha^\vee \rangle\ .$$

Since by assumption the first term is positive and the rest are non-negative, the sum is positive. ◻

Lemma 4.3. *If C is any Weyl cell intersecting σ, then the closure of C contains the centre C_σ.*

Proof. The interior of the segment from x to $\Pi_\sigma x$ cannot cross any root hyperplane. ◻

If σ intersects $\overline{C}^{++}$, suppose that κ is minimal such that σ intersects C_κ^{++}. This will be unique, since a segment from C_θ^{++} to C_ψ^{++} will cross $C_{\theta\cap\psi}^{++}$. I call it the **linear support** of σ. The centre of σ will intersect a unique Weyl cell, which according to Lemma 4.3 will be contained in C_κ^{++}. It will therefore be C_ω^{++} for some set $\omega \supseteq \kappa$ which I call the **central support** of σ.

Lemma 4.4. *Suppose that σ intersects $\overline{C}^{++}$. Let κ be the linear support of σ and ω its central support. Then*

$$\sigma \cap \overline{C}^{++} = \bigcup_{\kappa\subseteq\theta\subseteq\omega} \sigma \cap C_\theta^{++}$$

Proof. A simple consequence of Lemma 4.2. ◻

Lemma 4.5. *Suppose that σ intersects $\overline{C}^{++}$ and let ω be its central support. then*

$$\sigma = W_\omega(\sigma \cap \overline{C}^{++}) \; .$$

Proof. If w in W takes C to C_θ^{++} then $\overline{C}$ and $\overline{C}_\theta^{++}$ must both contain C_ω^{++}. But then $wC_\omega^{++} = C_\omega^{++}$, which in turn means that w must lie in W_ω. ◻

Suppose that κ is a δ-connected subset of Δ, and let F be the face of the convex hull of $W{\cdot}\chi$ spanned by $W_\kappa{\cdot}\chi$. We know from the previous section that the intersection of σ_F with $\overline{C}^{++}$ is the union of the C_θ^{++} with θ ranging over all subsets of Δ with $\kappa \subseteq \theta \subseteq \omega(\kappa)$. Lemma 4.5 implies:

Proposition 4.6. *If κ is a δ-connected subset of Δ, $\omega = \omega(\kappa)$, and F is the face of the convex hull of $W{\cdot}\chi$ spanned by $W_\kappa{\cdot}\chi$ then σ_F is the union of the transforms of the C_θ^{++} by elements of W_ω as θ ranges over all subsets of Δ such that*

$$\kappa \subseteq \theta \subseteq \omega \; .$$

It can also be characterized as the union of all Weyl cells C in the linear subspace spanned by the cell C_κ^{++} with $\overline{C}$ containing C_ω^{++}.

This result shows that the partition dual to a W-orbit has the property that all its cells are unions of Weyl cells. Suppose that, conversely, we are given a Satake partition $\{\sigma\}$ with this property. If σ is an open cell in this partition then it must contain at least one Weyl chamber as an open subset. We can transform this cell by an element of W so that this chamber is just C^{++}. Let δ be the central support of this σ. Lemma 4.4 then shows that the intersection of σ and $\overline{C}^{++}$ consists of all C_θ^{++} with $\theta \subseteq \delta$. In other words, the open cell of the partition containing C^{++}

is the same as the open cell of the Satake partition associated to δ. The geometric condition implies that in fact the two partitions are the same.

Proposition 4.7. *If Σ is a Satake partition of V whose cells are unions of Weyl cells, then there exists a subset δ of Δ such that Σ is the Satake partition associated to δ.*

In other words, there is a natural bijection between Satake partitions of this sort and subsets of Δ. We know then that the structure of the partition can be described in terms of the apparatus of δ-connected sets. Can we understand arbitrary Satake partitions in terms of similar combinatoric data? Lemma 4.4, Lemma 4.5, and Proposition 4.7 suggest a start.

5. Parabolic subspaces

In this section, let

$$\begin{aligned} G &= \text{ a complex reductive group} \\ B = AN &= \text{ a Borel subgroup of } G \\ \Delta &= \text{ the basic roots associated to the choice of } B \end{aligned}$$

For each $\theta \subseteq \Delta$ let $P_\theta = M_\theta N_\theta$ be the associated parabolic subgroup containing $B = P_\emptyset$, so that the Lie algebra $\mathfrak{n}_\theta$ is the sum of positive root spaces $\mathfrak{n}_\alpha$ with α not a linear combination of elements of θ. The split centre A_θ of M_θ is the intersection of the kernels $\ker(\alpha)$ as α ranges over θ. Let C_θ^{++} be the subset of $X_*(A_\emptyset)$ comprising λ with $\langle\alpha, \lambda\rangle = 0$ for α in θ, $\langle\alpha, \lambda\rangle > 0$ for α in $\Delta - \theta$. The Lie algebra of M_θ is spanned by $\mathfrak{a}_\emptyset$ and the root spaces $\mathfrak{g}_\alpha$ with $\langle\alpha, \lambda\rangle = 0$ for λ in C_θ^{++} and the Lie algebra of N_θ is spanned by the $\mathfrak{g}_\alpha$ with $\langle\alpha, \lambda\rangle > 0$ for λ in C_θ^{++}.

Let W be the Weyl group of G with respect to A, S the reflections associated to elements of Δ. The set Δ may be identified with the nodes of the Coxeter graph of (W, S).

Fix a finite-dimensional representation (π, V) of G. A **parabolic subspace** of V is one of the form $\mathrm{Fix}(N)$ where N is the unipotent radical of some parabolic subgroup of G. I recall that

$$\mathrm{Fix}(N) = \{v \in V \mid \pi(n)v = v \text{ for all } n \in N\} .$$

Suppose that π is irreducible with highest weight χ. The set of all weights of π is contained in the convex hull of the Weyl orbit $W \cdot \chi$. Let $\delta = \delta_\pi$ be the set of roots α in Δ such that $s_\alpha \chi \neq \chi$. Equivalently, δ_π is the complement in Δ of the set θ such that P_θ is the stabilizer of the line through the highest weight vector of π. For example, $\delta = \emptyset$ when π is the trivial representation of G, and $\delta = \Delta$ itself when the highest weight is regular. Recall that a δ-saturated subset ω is a subset of the nodes of the Dynkin diagram of G with this property: let κ be the union of connected components of ω containing elements of δ. Then the complement of ω in Δ is made up of exactly those nodes of Δ which do not lie in κ, and which either lie in δ or possess an edge in common with an element of κ.

Theorem 5.1. *Suppose (π, V) to be an irreducible finite dimensional representation of G with highest weight χ. Suppose that χ lies in C_θ^{++} and let $\delta = \Delta - \theta$.*

Then for any δ-saturated subset ω of Δ the subspace $\mathrm{Fix}(N_\omega)$ is the sum of weight spaces with weights in the convex hull of $W_\omega \cdot \chi$. This space is the same as $\mathrm{Fix}(N_\theta)$ for any θ with $\omega(\theta) = \omega$. It is an irreducible representation of M_ω.

Proof. Let V_ω be the direct sum of the weight spaces with weights in the convex hull of $W_\omega \cdot \chi$. It follows from the remarks above about C_θ^{++} and Lemma 2.1 that $\pi(\nu)v = 0$ if ν lies in $\mathfrak{n}_\theta$ and v in V_ω. Since $N_\omega \subseteq N_\theta$ we have

$$V_\omega \subseteq \mathrm{Fix}(N_\theta) \subseteq \mathrm{Fix}(N_\omega) \ .$$

The space V_ω is stable under P_ω, hence a representation of M_ω, as is $\mathrm{Fix}(N_\omega)$. Both have a unique highest weight vector, namely the highest weight vector of V itself, and are hence irreducible and identical. So the inclusions above are all equalities. ◻

Another way of phrasing this is to define a **saturated parabolic subgroup** of G (with respect to π) to be a conjugate of some P_ω with ω saturated. The Theorem amounts to the assertion that *the map $P = MN \mapsto \mathrm{Fix}(N)$ is a bijection between saturated parabolic subgroups of G and parabolic subspaces of V.*

Recall that if A is a torus in G and λ in $X_*(A)$ then P_λ is the parabolic subgroup of G corresponding to the sum of eigenspaces $\mathfrak{g}_\alpha$ with $\langle \alpha, \lambda \rangle \ge 0$. Its unipotent radical N_λ has as Lie algebra the sum of eigenspaces with $\langle \alpha, \lambda \rangle > 0$. The proof above also leads to the following result.

Proposition 5.2. *Suppose A to be any torus in G, λ in $X_*(A)$. Let π be an irreducible representation of G, let F be the face of the convex hull of the weights of π restriced to A where λ takes its maximum value, and let V_F be the sum of weight spaces associated to F. Then $V_F = \mathrm{Fix}(N_\lambda)$. This is also $\mathrm{Fix}(N)$ if $P = MN$ is the minimal saturated subgroup containing P_λ.*

How does M_ω act on V_ω? The Dynkin diagram of M_ω is the sub-diagram of the Dynkin diagram of G corresponding to the nodes in ω. The group M_ω is isogeneous to a product of a torus and semi-simple groups G_κ, G_ζ whose diagrams are the sub-diagrams corresponding to κ and ζ. Since N_κ acts trivially on V_ω, so do all its conjugates in M_ω, and in particular by elements of W_ω. The group A_ω acts by scalars on V_ω. Therefore the kernel of the representation of M_ω on V_ω contains G_ζ, and this representation factors through a representation of G_κ.

6. Compactifications of G/K

Again suppose G to be the group of $\mathbb{R}$-valued points on a semi-simple group defined over $\mathbb{R}$. Let $P_\emptyset$ be a minimal real parabolic subgroup with unipotent radical $N_\emptyset$, $M_\emptyset$ the quotient $P_\emptyset/N_\emptyset$, $A_\emptyset$ the maximal split real torus in $P_\emptyset$ invariant under the involution associated to the maximal compact subgroup K.

A **spherical representation** of G is a triple (π, V, v) where v in V is fixed by K.

Irreducible spherical representations arise in a simple manner. If χ is a rational character of $P_\emptyset$ then the space

$$\mathrm{Ind}(\chi \mid P_\emptyset, G) = \{ f \in C^\infty(G) \mid f(pg) = \chi(p)\ f(g) \text{ for all } p \in P_\emptyset \}$$

is a smooth representation of G with respect to the right regular action. If (π, V) is the irreducible finite dimensional representation with lowest weight χ then the map from V to $V/\mathfrak{n}_\emptyset V$ is χ-covariant, hence induces by Frobenius reciprocity a G-map from V to $\mathrm{Ind}(\chi \mid P_\emptyset, G)$. If χ is trivial on $P_\emptyset \cap K$ then since $G = PK$ the space of K-invariant functions in $\mathrm{Ind}(\chi \mid P_\emptyset, G)$ has dimension 1. It is a theorem of Helgason that the finite-dimensional representation contains the space of K-invariants. The converse is also true: any irreducible spherical representation has the property that its lowest weight is a character of $P_\emptyset$ trivial on $P_\emptyset \cap K$. (A strictly algebraic proof of all these assertions can be found in Vust (1974).)

To summarize:

Proposition 6.1. *The irreducible spherical representations of G are the irreducible finite dimensional representations with highest weights which are characters of $P_\emptyset$ trivial on $P_\emptyset \cap K$.*

Corollary 6.2. *If (π, V, v) is a spherical representation of G then it is defined over $\mathbb{R}$, and it possesses a highest weight whose stabilizer in G is a real parabolic subgroup of G.*

Proof. Since $P_\emptyset/N_\emptyset(P_\emptyset \cap K)$ is split over $\mathbb{R}$, its algebraic characters are all real. The stabilizer of a highest weight stabilized by $P_\emptyset$ must be defined over $\mathbb{R}$ and contain $P_\emptyset$, hence be a real parabolic subgroup of G. ◻

In the terminology of Borel-Tits (1965), the irreducible spherical representations of G are **strongly rational** over $\mathbb{R}$.

Proposition 6.3. *If (π, V, v) is an irreducible spherical representation of G then the highest weight of v with respect to $A_\emptyset$ is the restriction to $A_\emptyset$ of the highest weight of V.*

Proof. Since π is irreducible, $V = U(\mathfrak{g})v$. Because $\mathfrak{g} = \mathfrak{n}_\emptyset + \mathfrak{a}_\emptyset + \mathfrak{k}$, $U(\mathfrak{g})v = U(\mathfrak{n}_\emptyset)U(\mathfrak{a}_\emptyset)v$. ◻

Corollary 6.4. *If (π, V, v) is an irreducible spherical representation of G then the extremal weights of the restriction to $A_\emptyset$ of the weights of V are the same as the exremal $A_\emptyset$-weights of v.*

Suppose that (π, V, v) is a spherical triple, non-trivial on each simple factor of G. We can embed $X = G/K$ into $\mathbb{P}(V)$ according to the recipe

$$x \mapsto \iota(x) = \pi(g)v, \quad (x = gK) .$$

What does the closure $\overline{X}_\pi$ of $X_\pi = \iota(X)$ look like?

If P is a parabolic subgroup of G and A the maximal split torus in the centre of a Levi component of P, define A^{++} to be the subset of a in A with the property that all its eigenvalues on the Lie algebra of P are ≥ 1, and let $\overline{A}^{++}$ be the closure of A^{++}.

The Cartan decomposition asserts that $G = K\overline{A}_\emptyset^{++}K$, and implies that the closure $\overline{X}_\pi$ is the same as the K-orbit of the closure of $A_\emptyset^{++}[v]$, and also of the closure of $A_\emptyset[v]$. According to results of §2 this closure is obtained in the following way: for any face F of the convex hull of the $A_\emptyset$-weights of v, let v_F be the sum of the components of v corresponding to characters in F. The closure of $A_\emptyset[v]$ is the union of $A_\emptyset$-orbits of the $[v_F]$.

What are the faces of this convex hull?

It will be shown in the next section that the extremal characters of the restriction of π to $A_\emptyset$ are the $W_\mathbb{R}$-transforms of the restriction of the highest weight, and according to Proposition 6.3 this is also the set of extremal $A_\emptyset$-weights of v. It will also be shown in the next section that the weight space corresponding to a face of this hull is stablized by a π-saturated parabolic subgroups of G which is real.

Suppose P to be a real π-saturated parabolic subgroup of G. Let V_P be $\mathrm{Fix}(N_P)$, π_P the representation of P on π_P, which factors through P/N_P. If ι is the Cartan involution of G fixing the elements of K, let M_P be $P \cap P^\iota$, the unique Levi subgroup of P stable under ι. Let A_P be the maximal split torus contained in M_P, a in A_P^{++}. Then

$$v_P = \lim_{n\to\infty} \pi(a^n)v$$

lies in V_P, and since π_P is irreducible, (π_P, V_P, v_P) is a spherical triple for M_P. If P corresponds to the face F of the convex hull of the $A_\emptyset$-weights of v, then v_P is the same as v_F. The natural embedding of V_P into V induces an embedding of $\mathbb{P}(V_P)$ into $\mathbb{P}(V)$. Let L_P be the projective kernel of the representation of M_P on V_P, G_P the quotient M_P/L_P. The embedding of $\mathbb{P}(V_P)$ into $\mathbb{P}(V)$ induces one of $X_P = G_P/K_P$ into X, where K_P is the image of $K \cap P$ in G_P. The symmetric space X_P is contained in the closure of X, and called a **boundary component** of X. The transverse structure of a neighbourhood of X_P in $\overline{X}$ is related to the group L_P, which I call the **link** group.

Proposition 6.5. *The closure of X in $\mathbb{P}(V)$ is the union of its boundary components. If A is a split torus in G, λ in $X_*(A)$, then the limit*

$$\lim_{t\to\infty} \lambda(t)x_0$$

lies in X_P if P is the smallest saturated parabolic subgroup of G containing P_λ.

The parabolic subgroup P_λ is the one with the property that its Lie algebra is spanned by the eigenvectors of $\lambda(t)$ $(t > 1)$ with eigenvalues ≤ 1.

7. Galois indices

Let k be a subfield of $\mathbb{C}$, G a semi-simple group defined over k. Let $P_{\emptyset,k}$ be a minimal parabolic subgroup of G, A_k a maximal k-split torus in $P_{\emptyset,k}$.

Choose a Borel subgroup $B_\mathbb{C}$ in $G_\mathbb{C}$ and let $A_\mathbb{C}$ be a maximal torus contained in it. Let $\Delta_\mathbb{C}$ be the corresponding basis of roots of the pair $\mathfrak{g}_\mathbb{C}$, $\mathfrak{a}_\mathbb{C}$. The maximal parabolic subgroups of $G_\mathbb{C}$ containing $B_\mathbb{C}$ are parametrized by maximal proper subsets of $\Delta_\mathbb{C}$, or equivalently by their complements, which are singletons. If τ is an automorphism of $\mathbb{C}/k$ and P is a maximal proper parabolic subgroup of $G_\mathbb{C}$ then P^τ is conjugate to a unique maximal proper parabolic subgroup of G containing B. This induces an action of $\mathrm{Aut}(\mathbb{C}/k)$ on the complex Dynkin diagram which Borel-Tits (1965) call the $*$ action (see also §2.5 of Tits (1966)). It factors through the Galois group of the algebraic closure of k in $\mathbb{C}$, and does not depend on the particular choice of data.

We may assume that $B_\mathbb{C} \subseteq P_{\emptyset,k}$, $A_k \subseteq A_\mathbb{C}$. Restriction of roots determines a map

$$\rho_{\mathbb{C}/k}\colon \Delta_\mathbb{C} \to \Delta_k \cup \{0\} .$$

The **anisotropic kernel** of $\rho_{\mathbb{C}/k}$ is the inverse image $\Delta^0_{\mathbb{C}/k}$ of $\{0\}$. It is stable under $\mathrm{Aut}(\mathbb{C}/k)$, as is its complement. Galois orbits in the complement are exactly the inverse images in $\Delta_{\mathbb{C}}$ of single elements of Δ_k. The Galois action and the anisotropic kernel together make up the **index** of G_k. Tits explains in §2.5 of Tits (1966) how the relative root system can be reconstructed from the index.

If $k = \mathbb{R}$, there is a standard way of coding the index in the complex Dynkin diagram. The nodes of the anisotropic kernel are coloured black, and the remaining white nodes interchanged by conjugation are linked together. We shall see some examples later on.

For a subset $\theta \subseteq \Delta_k$ define

$$\epsilon_{\mathbb{C}/k}(\theta) = \rho^{-1}(\theta) \cup \Delta^0_{\mathbb{C}/k} \ .$$

Such sets are exactly the subsets of $\Delta_{\mathbb{C}/k}$ stable under $\mathrm{Aut}(\mathbb{C}/k)$ containing $\Delta^0_{\mathbb{C}/k}$, and the parabolic subgroups they parametrize are the k-rational parabolic subgroups containing $P_{\emptyset,k}$.

For any subset θ of Δ_k let $C^{++}_{k,\theta}$ be the corresponding wall of the closed positive chamber in $X_*(A_k)$. The definitions imply immediately:

Proposition 7.1. *If θ is a subset of Δ_k and the element λ of $X_*(A_k)$ lies in $C^{++}_{k,\theta}$ then its image in $X_*(A_{\mathbb{C}})$ lies in $C^{++}_{\mathbb{C},\psi}$ where $\psi = \epsilon_{\mathbb{C},k}(\theta)$.*

If we are given a Satake partition of $X_*(A_{\mathbb{C}})$ with the property that its cells are unions of Weyl cells, then the partition of A_k induced from inclusion in $A_{\mathbb{C}}$ will also be such a Satake partition. According to Proposition 4.7 it is therefore determined by a subset δ_k of Δ_k. The significance of this is that $C^{++}_{k,\theta}$ is in the same cell of the partition of $X_*(A_{\mathbb{C}})$ as $C^{++}_{k,\kappa}$ if κ is the union of connected components in θ containing elements of δ_k. There is a simple criterion to determine the set δ_k in terms of the index. Let $\kappa^0 = \kappa^0_{\mathbb{C}/k}$ be $\kappa(\Delta^0_{\mathbb{C}/k})$, and let

$$\delta_{\mathbb{C}/k} = \begin{array}{l} \text{the nodes of } \Delta - \Delta^0_{\mathbb{C}/k} \text{ which are either in } \delta_{\mathbb{C}} \text{ or connected by an edge} \\ \text{to an element of } \kappa^0_{\mathbb{C}/k}. \end{array}$$

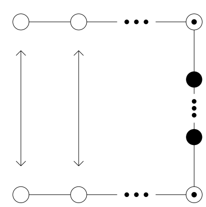
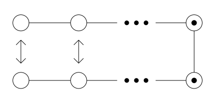
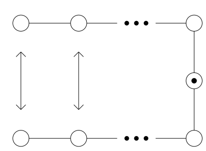
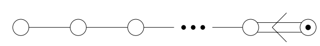
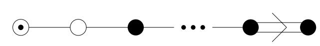
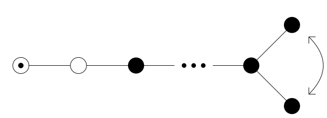
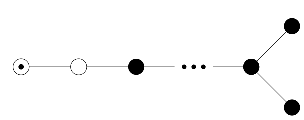
Corollary 7.2. *A node α in Δ_k lies in δ_k if and only if it is the image under restriction of an element of $\delta_{\mathbb{C}/k}$.*

Proof. The set δ_k is the complement of $\omega_k(\emptyset)$. This is the largest set θ such that $C^{++}_{k,\theta}$ lies in the same partition cell as $C^{++}_{k,\emptyset}$. By the previous result, this is the partition cell containing $C^{++}_{\Delta^0}$, which contains, in addition to $\Delta^0 = \Delta^0_{\mathbb{C}/k}$, all nodes not in $\delta_{\mathbb{C}}$ and not connected to κ^0 by an edge. ◻

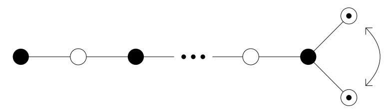
A strongly k-rational representation of G is one where the stabilizer of some highest weight is a k-rational parabolic subgroup. When $\delta_{\mathbb{C}}$ arises from a strongly k-rational representation of G then it has no nodes inside $\Delta^0_{\mathbb{C}/\mathbb{R}}$, and it is invariant under the Galois group. It is therefore equal to the inverse image of δ_k. In these circumstances, if θ is any subset of Δ_k then the smallest saturated subset of Δ containing it is the same as the inverse image of the smallest saturated subset of Δ_k with respect to the subset δ_k.

Suppose that π is strongly k-rational. It follows from the remarks above and from Proposition 5.2 that the convex hull of the weights of π restricted to A_k has its vertices the W_k-transforma of the highest weight, and that the stabilizers of its faces are the parabolic subgroups of G associated to saturated subsets of Δ_k. (This observation was required in the previous section for $k = \mathbb{R}$.)

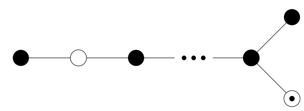
For some groups G, the symmetric space $X = G/K$ possesses a G-invariant Hermitian structure (this is explained nicely in §1.2 of Deligne (1979)). In these cases, X may be embedded as a bounded symmetric domain in a complex vector space, and its closure in this vector space will be a G-stable compactification of X. It is called a **Baily-Borel** compactification of X. It is not associated to a spherical triple (not even for $PGL_2(\mathbb{R})$), but it is topologically equivalent to one which is. In the following pictures, I exhibit the index data for Baily-Borel compactifications (compare the diagrams in Deligne (1979)). Nodes in δ are dotted, compact nodes (making up the anisotropic kernel of $\mathbb{C}/\mathbb{R}$) are black.

System	Type	Marked index diagram
$AIII$	$SU(p,q) \quad (q \geq p+2)$	
	$SU(p,p+1)$	
	$SU(p,p)$	
CI	$Sp(2n)$	
BI	$SO(2,p) \quad (p \text{ odd })$	
DI	$SO(2,p) \quad (p \equiv 2\ (4))$	
	$SO(2,p) \quad (p \equiv 0\ (4))$	

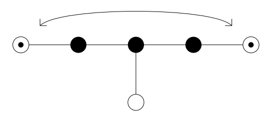
DIII $SO^*(2n)$ (n odd)



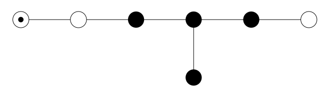
$SO^*(2n)$ (n even)



EIII



EVII



8. Arithmetic quotients

Suppose G to be defined over $\mathbb{Q}$. Let x_0 be the point in X fixed by K. If P is a parabolic subgroup of G and A a maximally split torus in the centre of a Levi component then I shall call (P, A) a **parabolic pair**. If (P, A) is a parabolic pair, let Σ_P be the eigencharacters of A acting on the Lie algebra of the unipotent radical N of P, and define for $T > 0$ the set $A^{++}(T)$ to be

$$A^{++}(T) = \left\{ a \in A \,\middle|\, |\alpha(a)| > T \text{ for all } \alpha \in \Sigma_P \right\} .$$

If $(P < A)$ is a $\mathbb{Q}$-rational parabolic pair, Ω is a compact subset of P, and $T > 0$, the **Siegel set** associated to these data is the image of

$$\mathfrak{S}(P, A, \Omega, T) = \Omega A^{++}(T)$$

in $X = G/K$. The main result of reduction theory, in its simplest form, is that (a) the arithmetic quotient $\Gamma\backslash X$ is covered by a finite number of Siegel sets with respect to minimal $\mathbb{Q}$-rational parabolic subgroups, and (b) the covering of X by the Γ-transforms of this finite collection of Siegel sets is locally finite. In other words, we can assemble a sort of fundamental domain for Γ from Siegel sets. We can choose one set for each Γ-conjugacy class of minimal $\mathbb{Q}$-rational parabolic subgroups.

Fix a spherical triple (π, V, v), and let $\overline{X} = \overline{X}_\pi$ be the corresponding Satake compactification of X. For brevity let $\Delta^0 = \Delta^0_{\mathbb{C}/\mathbb{Q}}$ and similarly for κ^0.

A boundary component X_P of $\overline{X}$ is called **geometrically rational** if it satisfies these two conditions:

(GR1) Its stabilizer P is a $\mathbb{Q}$-rational parabolic subgroup of G.

(GR2) Its link group L_P is isogeneous to the product of a rational group and a compact one.

These conditions guarantee that the image of $\Gamma \cap P$ in M_P is arithmetic, and acts discretely on X_P. I shall say that the compactification $X \hookrightarrow \overline{X}$ itself is geometrically rational if the boundary components met by the closure of every Siegel set are all geometrically rational.

If $X \hookrightarrow \overline{X}$ is geometrically rational, the **rational boundary components** are those intersected by the closure of a Siegel set. In these circumstances, define X^* to be the union of X and all of the rational boundary components. The conditions guarantee that each of these components X_P is a symmetric space of some semi-simple group defined over $\mathbb{Q}$, which will be the product of G_P and some possibly trivial compact factor. Assign to X^* the topology characterized by the condition that a set is open if and only if its intersection with the closure of every Siegel set in $\overline{X}$ is open. Combining the main result of Satake (1960b) with a result of Borel (1962), we see that Γ acts discretely on X^* and that the quotient $\Gamma\backslash X^*$ is both compact and Hausdorff.

What properties of π determine whether or not $X \hookrightarrow \overline{X}_\pi$ is geometrically rational or not? The first step is to decide what boundary components are intersected by Siegel sets. Suppose (P, A) is a $\mathbb{Q}$-rational parabolic pair with P minimal $\mathbb{Q}$-rational. Choose parabolic pairs $(P_\mathbb{C}, A_\mathbb{C})$, $(P_\mathbb{R}, A_\mathbb{R})$ with

$$P \subseteq P_\mathbb{R} \subseteq P_\mathbb{C}, \quad A \supseteq A_\mathbb{R} \supseteq A_\mathbb{C} \, .$$

Let $\delta = \delta_\pi$ in $\Delta_\mathbb{C}$. If θ is a subset of $\Delta_\mathbb{Q}$ then for each λ in $X^{++}(A_\theta)$ the limit of $\lambda(t)x_0$ lies in X_{P_ω} where $\omega = \omega(\epsilon_{\mathbb{C}/\mathbb{Q}}(\theta))$. According to Proposition 5.2 each of the boundary components intersected by the closure of $A^{++}(T)$ is of this form. Since each one of the saturated parabolic subgroups contains P, these are also the boundary components intersected by any Siegel set for P.

Lemma 8.1. *If (P, A) is a $\mathbb{Q}$-rational parabolic pair and P is a minimal $\mathbb{Q}$-rational parabolic subgroup of G then the boundary components met by the closure of a Siegel set $\mathfrak{S}(P, A, \Omega, T)$ are those whose stabilizer is P_ω, where ω is of the form $\omega(\epsilon_{\mathbb{C}/\mathbb{Q}}(\theta))$.*

Therefore, in order that (GR1) hold it is necessary and sufficient that

($\bullet$) *for every $\theta \subseteq \Delta_\mathbb{Q}$ the group P_ω is $\mathbb{Q}$-rational, where $\omega = \omega(\epsilon_{\mathbb{C}/\mathbb{Q}}(\theta))$.*

This condition amounts to a finite number of conditions, but the number of conditions grows rapidly with the $\mathbb{Q}$-rank, so that it is not entirely practical.

What does condition (GR1) amount to for $\theta = \emptyset$? Since $\epsilon_{\mathbb{C}/\mathbb{Q}}(\emptyset)$ is just $\Delta^0_{\mathbb{C}/\mathbb{Q}}$, ω in this case will be the union of $\Delta^0_{\mathbb{C}/\mathbb{Q}}$ and those roots outside $\Delta^0_{\mathbb{C}/\mathbb{Q}}$ which are not in $\Delta_{\mathbb{C},\delta}$ and not connected to $\kappa^0_{\mathbb{C}/\mathbb{Q}}$ by a single edge. Since this contains $\Delta^0_{\mathbb{C}/\mathbb{Q}}$, it will parametrize a $\mathbb{Q}$-rational parabolic subgroup if and only if the elements in it which are not in $\Delta^0_{\mathbb{C}/\mathbb{Q}}$ are Galois invariant, or equivalently if and only if its complement $\delta_{\mathbb{C}/\mathbb{Q}}$ in $\Delta_\mathbb{C} - \Delta^0_{\mathbb{C}/\mathbb{Q}}$ is Galois invariant. I recall that this complement consists of elements of $\Delta_\mathbb{C} - \Delta^0_{\mathbb{C}/\mathbb{Q}}$ which are either in the set $\delta_\mathbb{C}$ or connected to $\kappa^0_{\mathbb{C}/\mathbb{Q}}$ by a single edge. In order that the stabilizer of every minimal rational boundary component be $\mathbb{Q}$-rational, it is thus necessary and sufficient that $\delta_{\mathbb{C}/\mathbb{Q}}$ (defined in §7 by these two conditions) be Galois invariant. In fact:

Theorem 8.2. *Condition (GR1) holds if and only if $\delta_{\mathbb{C}/\mathbb{Q}}$ is Galois invariant.*

Proof. We need to show sufficiency of this condition. It remains to show that under the assumption, if θ is a Galois invariant subset of $\Delta_{\mathbb{C}}$ containing $\Delta^0_{\mathbb{C}/\mathbb{Q}}$, then $\omega(\theta)$ is Galois invariant.

Suppose that α lies in $\kappa(\theta)$. There exists a path inside θ connecting α to an element of δ. Either this path lies entirely inside $\Delta^0_{\mathbb{C}/\mathbb{Q}}$, or it doesn't. In the first case, α lies in $\kappa^0_{\mathbb{C}/\mathbb{Q}}$. In the second, either the path goes through an element of $\Delta - \Delta^0$ which lies in δ, or it passes back into Δ^0 without meeting such an element. In the second case it passes through an element of $\Delta - \Delta^0$ which is connected to κ^0 by a single edge. In either of the last two cases, it is connected to an element of $\delta_{\mathbb{C}/\mathbb{Q}}$ inside θ. Conversely, if it is connected inside θ to an element of $\delta_{\mathbb{C}/\mathbb{Q}}$, then it lies in $\kappa(\theta)$. As a consequence

($\bullet$) *the set $\kappa(\theta) - \kappa^0$ is Galois invariant.*

We shall use this observation a bit later on.

The set $\zeta(\theta)$ is made up of the elements of $\Delta_{\mathbb{C}}$ which are not connected to $\kappa(\theta)$ by a single edge, and $\omega(\theta)$ is the union of $\kappa(\theta)$ and $\zeta(\theta)$. Since θ contains Δ^0 it only has to be shown that if α lies in $\omega(\theta)$ and the complement of Δ^0 then any Galois transform also lies in $\omega(\theta)$.

If α lies in $\kappa(\theta) - \Delta^0$ then we have just seen that any Galois transform lies also in the same set. So it remains to be seen that if α lies in $\zeta(\theta)$ but not in Δ^0 then any Galois transform lies in the same set. If β is a transform of α which does not lie in $\zeta(\theta)$ then there exists an edge from β to an element of $\kappa(\theta)$. If this edge leads to an element of $\kappa(\theta) - \kappa^0$ we get a contradiction, while if it leads to an element of κ^0 then β lies in $\delta_{\mathbb{C}/\mathbb{Q}}$ and we again get a contradiction. ◻

Corollary 8.3. *If G is quasi-split, then a compactification*

$$X \hookrightarrow \overline{X}_\pi$$

is geometrically rational precisely when it is rational.

Proof. In this case the anisotropic kernel is empty and condition (GR2) is automatic. ◻

The situation for condition (GR2) is similar—that is to say, the condition for it to hold with respect to minimal boundary components, which is clearly necessary, is also sufficient for it to hold in general. For the minimal boundary component, the factor G_P has Dynkin diagram κ^0. Condition (GR2) means that this must be a rational group up to a compact factor, which means in turn that the Galois orbit of κ^0 must be the union of κ^0 and a set of compact nodes.

Theorem 8.4. *Assuming that condition (GR1) is valid, condition (GR2) holds if and only if the Galois orbit of κ^0 in the index is the union of κ^0 and a set of compact nodes.*

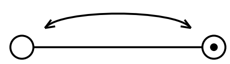
Proof. This is clear, since condition (GR1) implies, as the proof of the last result showed, that for any θ parametrizing a $\mathbb{Q}$-rational subgroup, $\kappa(\theta)$ is the union of a Galois invariant subset and κ^0. ◻

If $X \hookrightarrow \overline{X}$ is a geometrically rational compactification, then the rational boundary components coincide with those whose stabilizers are $\mathbb{Q}$-rational, and the stabilizers in this case coincide with the rational parabolic subgroups parametrized by saturated subsets of $\Delta_{\mathbb{Q}}$.

9. Some examples

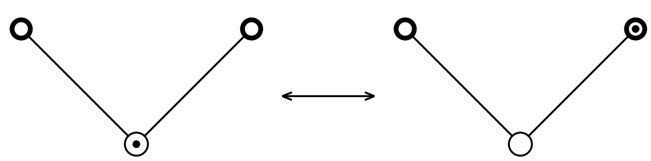
Fix a real quadratic extension F of $\mathbb{Q}$.

• Let G be the unitary group of an isotropic Hermitian form of dimension three over F, which is a $\mathbb{Q}$-rational group isomorphic over F to $SL_3(F)$. Over $\mathbb{Q}$ it is quasi-split. Its Dynkin diagram and index are



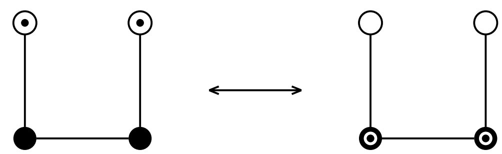
Over $\mathbb{R}$ the group is split. If we pick as δ either one of the two nodes (say the one marked by a dot in the centre), condition (GR1) is not satisfied. On the other hand this example satisfies Assumption 1 (quasi-rationality) of §3.3 in Zucker (1983), and this example seems to show that his Proposition 3.3.(ii) is false. It is not clear to me what the significance of Zucker's notion of quasi-rationality is.

• Let Q be an anisotropic quadratic form over F of dimension four, arising from a quaternion algebra over F which is split at both real embeddings of F into $\mathbb{R}$. Let G be the orthogonal group of the quadratic form $H \oplus Q$, where H is hyperbolic space. The group G may be considered by restriction of scalars as a group over $\mathbb{Q}$. The index of G is shown by the following figure:



Here the anisotropic kernel of G over $\mathbb{Q}$ consists of the top four nodes. Conjugation over $\mathbb{Q}$ just interchanges the two components. If δ is made up of the left bottom node and the right top node, then (GR1) is valid but (GR2) is not.

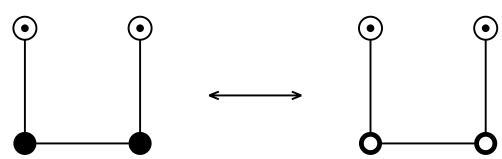
• Let K be a quadratic imaginary extension of F obtained by adjunction from one over $\mathbb{Q}$. Let Q be a Hermitian form of dimension three over K, anisotropic at one real embedding of F and of signature $(2, 1)$ at the other. Let G be the unitary group of of $H \oplus Q$, where H is the hyperbolic Hermitian form, considered as a group over $\mathbb{Q}$. Its index is



where the anisotropic kernel over $\mathbb{Q}$ consists of the four bottom nodes. Conjugation over $\mathbb{Q}$ interchanges the two components, and complex conjugation interchanges the branches inside each component.

Let δ be the marked nodes. Then the compactification of each real factor of G is its Baily-Borel compactification, and conditions (GR1) and (GR2) are both valid. All other Baily-Borel compactifications can be dealt with in a similar case by case analysis, somewhat different from the one in Baily-Borel (1966). What one needs for this are the diagrams in §7 and the observation in §3.1.2 of Tits (1966) about induced indices.

This compactification does not arise from a rational representation, and the boundary components have a different structure from those of the compactification associated to the similar rational representation whose figure is



although the saturated parabolic subgroups are the same for both cases. In the second case, the minimal boundary components are just points, while in the first they are non-trivial symmetric spaces.

I have not attempted a classification of all semi-simple groups over $\mathbb{Q}$ in order to try to find all geometrically rational Satake compactifications. This example illustrates that the first step is probably to break them up into families, each family having the same set of saturated $\mathbb{Q}$-rational parabolic subgroups, but differing in the structure of its boundary components.

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