Schwartz Analysis and Intertwining Distributions

by

Xinyu Liu

B.Sc., University of Science and Technology of China, 2007
M.Sc., University of Science and Technology of China, 2011

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

in

The Faculty of Graduate and Postdoctoral Studies

(Mathematics)

THE UNIVERSITY OF BRITISH COLUMBIA

(Vancouver)

December 2017

Abstract

In this dissertation, we combine the work of A. Aizenbud and D. Gourevitch on Schwartz functions on Nash manifolds, and the work of F. du Cloux on Schwartz inductions, to develop a toolbox of Schwartz analysis. We then use these tools to study the intertwining operators between parabolic inductions, and study the behavior of intertwining distributions on certain open subsets. Finally we use our results to give new proof of results in [15] on irreducibilities of degenerate principal series and minimal principal series.

Lay Summary

Parabolic inductions and intertwining operators between them are of great importance in representation theory. The structure of parabolic inductions is still mysterious, and only sporadic results have been obtained.

Our final goal is to find a unified approach to study intertwining operators between parabolic inductions, by using only the structures of the quotient spaces and the information about the representations to be induced.

This dissertation shows the very first step of the entire framework. We combine the work of A. Aizenbud, D. Gourevitch and F. du Cloux, develop an algebraic toolbox, and use the results to study the irreducibility of unitary parabolic inductions. We give a new proof of results on irreducibilities of degenerate principal series and minimal principal series in [15].

Preface

This dissertation is the original, unpublished work of the author, Xinyu Liu.

Table of Contents

List of Figures

Acknowledgements

My deep gratitude goes to my advisor Prof. Bill Casselman, for introducing me to many beautiful areas of representations theory, especially the subject studied in this dissertation. While working on this dissertation, a lot of difficulties were overcome after discussion with him. The crucial idea to consider torsion subspaces also belongs to him. In a word, this thesis would never exist without his encouragement and financial support in the last six years.

I would like to thank the members of my PhD committee: Prof. Julia Gordon and Prof. Lior Silberman, for creating a friendly and encouraging atmosphere.

I also thank the secretaries in the main office, especially Lee Yupitun, Roseann Kinsey, Marlowe Dirkson, Ann Artuso and Jessica Trat. Their hard work made my study and life in UBC much easier.

Finally I want to thank my parents for their support and understanding of my choice to study mathematics for so many years.

Chapter 1

Introduction.

1.1 A Brief Summary of the Thesis

1.1.1 Goal of the Thesis

The long-term goal of our work is to study intertwining operators between two parabolic inductions, by using the tools of Schwartz analysis developed in [1], [2], [20] and Chapter 4 of this thesis.

In this thesis, to simplify the problem, we will work under the following basic setting:

- The Lie groups studied in the thesis, are *the real point groups of connected reductive linear algebraic groups defined over* $\mathbb{R}$.
- We will only work on *smooth inductions of unitary representations*, namely for a unitary representation (τ, V) of a parabolic subgroup P, we study the smooth induction $C^\infty\mathrm{Ind}_P^G(\tau \otimes \delta_P^{1/2})$, which is infinitesimally equivalent to the ordinary (Hilbert) normalized induction $\mathrm{Ind}_P^G\tau$. And temporarily we only study the *irreducibilities* of such representations.

Many sporadic results have been obtained for irreducibilities of parabolic inductions. Most of them follow Mackey's machinery, and prove the irreducibility by finding upper bound of certain multiplicities. Some other people use pure algebraic tools especially for degenerate principal series. Our aim is to find a unified approach to study intertwining operators, by only using the geometric structure on the group (or flag manifolds) and results in pure representation theory. As the reader can see in the main body of this thesis (Chapter 5—9), we prove irreducibilities by geometric/algebraic tricks, combined with some fundamental results in representation theory (e.g. Frobenius reciprocity, Schur lemma etc). The only unusual result we have used is Shapiro's Lemma, but we only use the special case on 0th cohomology, which is an extension of the Frobenius reciprocity.

In this thesis, we first develop a toolbox of Schwartz analysis on algebraic groups. Then we study intertwining operators as intertwining distributions, classify them by their supports, and study their local behavior by restricting the distributions to open neighbourhoods of double cosets. In the local study of intertwining distributions, we combine the tools of transverse derivatives, torsion subspaces and Shapiro's Lemma. In the current version of thesis, we are able to prove the results analogous to Theorem 7.2 and 7.4 in [15], about irreducibilities of minimal principal series and degenerate principal series, for the above class of Lie groups.

In the following three subsections 1.1.2, 1.1.3 and 1.1.4, we give a sketchy outline of the thesis.

1.1.2 Distribution Analysis (Chapter 5)

The central idea of our study is to realize intertwining operators (between two parabolic inductions) as "intertwining distributions on Schwartz induction spaces", and study the local properties of such distributions. The strong confinement from the local properties will tell us when there is no non-scalar intertwining distributions (operators), thus the unitary parabolic inductions are irreducible. This method also partially tells us what the intertwining distributions look like on certain open subsets. One object of our future study is to answer how to the intertwining distributions on the entire group.

Intertwining Distributions (5.1, 5.2, 5.3)

Let $\mathbf{G}$ be a connected reductive linear algebraic group defined over $\mathbb{R}$, let $\mathbf{P}, \mathbf{Q}$ be two parabolic $\mathbb{R}$-subgroups of $\mathbf{G}$, let G, P, Q be the corresponding real point groups. Let (σ_1, V_1) be a Harish-Chandra representation of P (see 2.3), and (σ_2, V_2) be a Harish-Chandra representation of Q. Let $C^\infty\mathrm{Ind}_P^G\sigma_1$ (resp. $C^\infty\mathrm{Ind}_Q^G\sigma_2$) be the space of (unnormalized) smooth induction of σ_1 (resp. σ_2) from P (resp. Q) to G. Since the $\mathbf{P}, \mathbf{Q}$ are parabolic subgroups, the quotient manifolds $P\backslash G, Q\backslash G$ are compact, and the above two smooth inductions equal to the corresponding Schwartz inductions (see 4.6 for the term "Schwartz inductions"): $C^\infty\mathrm{Ind}_P^G\sigma_1 = \mathcal{S}\mathrm{Ind}_P^G\sigma_1, C^\infty\mathrm{Ind}_Q^G\sigma_2 = \mathcal{S}\mathrm{Ind}_Q^G\sigma_2$. For simplicity, we denote them by $I = \mathcal{S}\mathrm{Ind}_P^G\sigma_1, J = \mathcal{S}\mathrm{Ind}_Q^G\sigma_2$, and let $\mathrm{Hom}_G(I, J)$ be the space of intertwining operators from I to J.

To study the intertwining operators in $\mathrm{Hom}_G(I, J)$, we first apply Frobenius reciprocity, i.e. the isomorphism:

$$\mathrm{Hom}_G(I, J) \xrightarrow{\sim} \mathrm{Hom}_Q(I, V_2), \quad T \mapsto \Omega_e \circ T.$$

Thus we identify the space $\mathrm{Hom}_G(I, J)$ of intertwining operators, with the space $\mathrm{Hom}_Q(I, V_2)$ of Q-equivariant V_2-valued distributions on $I = \mathcal{S}\mathrm{Ind}_P^G\sigma_1$. We call such distributions **intertwining distributions**. In this thesis, we will work on the case when $\mathbf{Q} = \mathbf{P}, \sigma_2 = \sigma_1$.

Self-Intertwining Distributions and Irreducibility (5.4)

Let $\mathbf{P} = \mathbf{P}_\Theta$ be a parabolic $\mathbb{R}$-subgroup of $\mathbf{G}$ corresponding to a subset Θ of the base of restricted roots, with real point group $P = P_\Theta$, let (σ, V) be a Harish-Chandra representation of P, and let $I = \mathcal{S}\mathrm{Ind}_P^G\sigma$ $(= C^\infty\mathrm{Ind}_P^G\sigma)$ be the Schwartz (smooth) induction space. We are interested in the space $\mathrm{Hom}_G(I, I)$ of self-intertwining operators on I.

In particular, when $\sigma = \tau \otimes \delta_P^{1/2}$ where τ is a unitary representation on V, the I is irreducible if and only if $\mathrm{Hom}_G(I, I) = \mathbb{C}$. Actually the I is infinitesimally equivalent to the normalized unitary (Hilbert) parabolic induction $\mathrm{Ind}_P^G\tau$ (see 2.3.6). The I and $\mathrm{Ind}_P^G\tau$ are irreducible/reducible simultaneously. And they are irreducible if and only if $\mathrm{Hom}_G(I, I) = \mathrm{Hom}_G(\mathrm{Ind}_P^G\tau, \mathrm{Ind}_P^G\tau) = \mathbb{C}$. The main body of the thesis (Chapter 6 to 9), are devoted to the study of irreducibility of I. By the above discussion, we can show the irreducibility of I (equivalently $\mathrm{Ind}_P^G\tau$), by showing the space $\mathrm{Hom}_P(I, V)$ of intertwining distributions is one dimensional.

The Supports and Maximal Double Cosets (5.2, 5.3, 5.4)

We consider self-intertwining distributions and keep the above setting on $G, P = P_\Theta, \tau, \sigma$. Let $D \in \mathrm{Hom}_P(I, V)$ be an intertwining distribution. We will show in 5.2 that every intertwining distribution has a well-defined support, denoted by $\mathrm{supp}D$, and it is a (real) Zariski closed union of (P, P)-double cosets. Obviously, the intertwining distributions corresponding to scalar intertwining operators have their supports contained in P.

Actually for any algebraic subgroup H of P, the $\mathrm{supp}D$ is also a union of (P, H)-double cosets. In particular, for any subset $\Omega \subset \Theta$, let $\mathbf{P}_\Omega$ be the corresponding standard parabolic $\mathbb{R}$-subgroup with P_Ω be its real point group. Then the support $\mathrm{supp}D$ is also a union of (P, P_Ω)-double cosets. There are finitely many (P, P_Ω)-double cosets on G, parameterized by the set $[W_\Theta\backslash W/W_\Omega]$ of minimal representatives (see 3.3), and they are ordered by the closure order (see 3.3) by the real Zariski topology.

For each $\Omega \subset \Theta$, and a representative $w \in [W_\Theta\backslash W/W_\Omega]$, we denote by

$G_w^\Omega = PwP_\Omega$ the corresponding (P, P_Ω)-double coset, and let

$$G_{\geq w}^\Omega = \coprod_{PwP_\Omega \subset \overline{PxP_\Omega}} PxP_\Omega$$

$$G_{>w}^\Omega = G_{\geq w}^\Omega - G_w^\Omega$$

These two are Zariski open subsets of G, and G_w^Ω is closed in $G_{\geq w}^\Omega$ with open complement $G_{>w}^\Omega$. For the (P, P_Ω)-stable subsets $G_{\geq w}^\Omega, G_{>w}^\Omega, G_w^\Omega$, we denote the corresponding local Schwartz inductions (see 4.6.2) by

$$I_{\geq w}^\Omega = \mathcal{S}\mathrm{Ind}_P^{G_{\geq w}^\Omega}\sigma, \quad I_{>w}^\Omega = \mathcal{S}\mathrm{Ind}_P^{G_{>w}^\Omega}\sigma, \quad I_w^\Omega = \mathcal{S}\mathrm{Ind}_P^{G_w^\Omega}\sigma.$$

They are smooth P_Ω-representations under the right regular P_Ω-actions. We have the inclusions (of P_Ω-representations): $I_{>w}^\Omega \subset I_{\geq w}^\Omega \subset I$ for each $\Omega \subset \Theta$ and $w \in [W_\Theta \backslash W / W_\Omega]$.

Since there are finitely many (P, P_Ω)-double cosets on G, *one can choose a (non-unique) maximal* (P, P_Ω)*-double coset* (under the closure order) contained in $\mathrm{supp}D$, say G_w^Ω. Then the restriction of the intertwining distribution $D \in \mathrm{Hom}_P(I, V)$ to the subspace $I_{\geq w}^\Omega$ is *non-zero*, and the restriction of D to the subspace $I_{>w}^\Omega$ is *zero*.

Restricting the Intertwining Distributions (5.5)

Let $D \in \mathrm{Hom}_P(I, V)$ be an intertwining distribution with its support denoted by $\mathrm{supp}D$. There are two special cases: (1) the $\mathrm{supp}D$ is contained in P (the identity double coset PeP); (2) the $\mathrm{supp}D$ is not contained in P. We will study these two cases separately.

Suppose $\mathrm{supp}D$ is contained in P. Let $I_{>e} = \mathcal{S}\mathrm{Ind}_P^{G-P}\sigma$ be the local Schwartz induction of σ from P to the open complement $G - P$, then $I_{>e}$ is a P-subrepresentation of I, and D vanishes on this subspace. Therefore $D : I \to V$ factor through the quotient space $I/I_{>e}$, and let $\overline{D}$ be the corresponding map $\overline{D} : I/I_{>e} \to V$. Obviously $\overline{D}$ is still P-equivariant when the quotient is endowed with the quotient P-action. Its adjoint map $\overline{D}^* : V' \to (I/I_{>e})'$ is also a P-equivariant continuous linear map. Since V and $I/I_{>e}$ are nuclear hence reflexive, we have the following linear isomorphism:

$$\{D \in \mathrm{Hom}_P(I, V) : \mathrm{supp}D \subset P\} \xrightarrow{\sim} \mathrm{Hom}_P(V', (I/I_{>e})'), \quad D \mapsto \overline{D}^*.$$

Suppose $\mathrm{supp}D$ is not contained in P. Then *for each subset* $\Omega \subset \Theta$, as above we can find a *maximal* (P, P_Ω)*-double coset contained in* $\mathrm{supp}D$, *say* G_w^Ω for some $w \in [W_\Theta \backslash W / W_\Omega]$. Since G_w^Ω is maximal in the support, the restricted distribution $D^{\Omega}_{\geq w} : I^{\Omega}_{\geq w} \to V$ is P_{Ω}-equivariant, nonzero and vanishes on the subspace $I^{\Omega}_{>w}$. Similar to the above discussion, it factor through the quotient $I^{\Omega}_{\geq w}/I^{\Omega}_{>w}$, and the corresponding map is denoted by $\overline{D}^{\Omega}_{\geq w} : I^{\Omega}_{\geq w}/I^{\Omega}_{>w} \to V$. This map is also P_{Ω}-equivariant and non-zero. By the same argument as above, the correspondence $D \mapsto (\overline{D}^{\Omega}_{\geq w})^*$ gives a bijection:

$$\{D : G^{\Omega}_{w} \text{ is maximal in supp} D\} \xrightarrow{\sim} \mathrm{Hom}_{P_{\Omega}}(V', (I^{\Omega}_{\geq w}/I^{\Omega}_{>w})') - \{0\}$$

Combining the above two cases, to show the irreducibility of I, we just need to:

- Show the $\mathrm{Hom}_{P}(V', (I/I_{>e})') = \mathbb{C}$.
- Find a subset $\Omega \subset \Theta$, and for all $w \in [W_{\Theta}\backslash W/W_{\Omega}]$ and $w \neq e$, show the $\mathrm{Hom}_{P_{\Omega}}(V', (I^{\Omega}_{\geq w}/I^{\Omega}_{>w})') = \{0\}$.

1.1.3 Organization of the Thesis

By the above discussion, we are required to study the local behavior of intertwining distributions, namely the dual quotients $(I^{\Omega}_{\geq w}/I^{\Omega}_{>w})'$ (the $(I/I_{>e})'$ is the special case when $w = e$), and the spaces $\mathrm{Hom}_{P_{\Omega}}(V', (I^{\Omega}_{\geq w}/I^{\Omega}_{>w})')$, for all $\Omega \subset \Theta$ and $w \in [W_{\Theta}\backslash W/W_{\Omega}]$.

General Attempts

We have made progress in the following three steps:

- **Step 1: Express the dual quotients $(I^{\Omega}_{\geq w}/I^{\Omega}_{>w})'$ as transverse derivatives.** The main result in the first step is summarized as Theorem 10.1, namely, the $(I^{\Omega}_{\geq w}/I^{\Omega}_{>w})'$ is isomorphic to the space of distributions on $I^{\Omega}_{\geq w}$ supported in $G^{\Omega}_{>w}$, and the latter space consists of transverse derivatives of distributions on G^{Ω}_{w}. More precisely the following map is an isomorphism:

  $$(I^{\Omega}_{w})' \otimes U(\mathfrak{t}^{\Omega}_{w}) \to (I^{\Omega}_{\geq w}/I^{\Omega}_{>w})',$$

  where the $\mathfrak{t}^{\Omega}_{w}$ is the transverse subalgebra $\overline{\mathfrak{n}}_{\Omega} \cap w^{-1}\overline{\mathfrak{n}}_{P}w$.

- **Step 2: Study the torsion subspace on $(I^{\Omega}_{\geq w}/I^{\Omega}_{>w})' \simeq (I^{\Omega}_{w})' \otimes U(\mathfrak{t}^{\Omega}_{w})$.** To make the spaces $\mathrm{Hom}_{P_{\Omega}}(V', (I^{\Omega}_{\geq w}/I^{\Omega}_{>w})')$ computable, we need the special properties on V'. The following two special cases caught our attention:

(1) **If (τ, V) is P-irreducible and unitary**, then the $N_P = N_\Theta$ acts trivially on V and V'. If we let $\Omega = \Theta$, then an arbitrary map $\Phi \in \mathrm{Hom}_{P_\Theta}(V', (I^\Theta_{\geq w}/I^\Theta_{>w})')$ has its image in the $\mathfrak{n}_\Theta$-invariant subspace of $(I^\Theta_{\geq w}/I^\Theta_{>w})'$. Therefore we have

$$\mathrm{Hom}_{P_\Theta}(V', (I^\Theta_{\geq w}/I^\Theta_{>w})') = \mathrm{Hom}_{M_\Theta}(V', [(I^\Theta_{\geq w}/I^\Theta_{>w})']^{[\mathfrak{n}_\Theta{}^1]}),$$

where the $[(I^\Theta_{\geq w}/I^\Theta_{>w})']^{[\mathfrak{n}_\Theta{}^1]}$ is the $\mathfrak{n}_\Theta$-invariant subspace. *However, computing the $\mathfrak{n}_\Theta$-invariant subspace on*

$$(I^\Theta_{\geq w}/I^\Theta_{>w})' \simeq (I^\Theta_w)' \otimes U(\mathfrak{t}^\Theta_w)$$

is infeasible since the tensor product is not a module tensor product. We do not have a good algebraic description of the $\mathfrak{n}_\Theta$-invariant subspace $[(I^\Theta_{\geq w}/I^\Theta_{>w})']^{[\mathfrak{n}_\Theta{}^1]}$. To compromise, we compute the $\mathfrak{n}_\Theta$-torsion subspace $[(I^\Theta_{\geq w}/I^\Theta_{>w})']^{[\mathfrak{n}_\Theta{}^\bullet]}$, and we have the following inclusion:

$$\mathrm{Hom}_{P_\Theta}(V', (I^\Theta_{\geq w}/I^\Theta_{>w})') \hookrightarrow \mathrm{Hom}_{M_\Theta}(V', [(I^\Theta_{\geq w}/I^\Theta_{>w})']^{[\mathfrak{n}_\Theta{}^\bullet]}).$$

(2) **If V is $\mathfrak{n}_\emptyset$-torsion** (e.g. V is finite dimensional or certain discrete series), then the $\mathfrak{n}_\emptyset$ acts nilpotently on V and V'. In particular, the V' equals to its $\mathfrak{n}_\emptyset$-torsion subspace, and an arbitrary $\Phi \in \mathrm{Hom}_{P_\emptyset}(V', (I_{\geq w}/I_{>w})')$ has its image in the $\mathfrak{n}_\emptyset$-torsion subspace of $(I_{\geq w}/I_{>w})'$. Therefore we have

$$\mathrm{Hom}_{P_\emptyset}(V', (I_{\geq w}/I_{>w})') = \mathrm{Hom}_{M_\emptyset}(V', [(I_{\geq w}/I_{>w})']^{[\mathfrak{n}_\emptyset{}^\bullet]}).$$

The above two special cases urge us to compute the torsion subspaces on the dual quotients. To deal with all cases simultaneously, we consider the $\mathfrak{n}_\Omega$-torsion subspaces on $(I^\Omega_{\geq w}/I^\Omega_{>w})' \simeq (I^\Omega_w)' \otimes U(\mathfrak{t}^\Omega_w)$ for all $\Omega \subset \Theta$. The main result in the second step is summarized as Theorem 10.3, namely we have:

$$[(I^\Omega_w)' \otimes U(\mathfrak{t}^\Omega_w)]^{[\mathfrak{n}^\bullet_\Omega]} = [(I^\Omega_w)']^{[\mathfrak{n}^\bullet_\Omega]} \otimes U(\mathfrak{t}^\Omega_w).$$

- **Step 3: The $\mathfrak{n}_\Omega$-torsion subspace on $(I^\Omega_w)'$.** The above two steps require us to find the M_Ω-action on the torsion subspace $[(I^\Omega_w)']^{[\mathfrak{n}^\bullet_\Omega]}$ or more precisely on the annihilators $[(I^\Omega_w)']^{[\mathfrak{n}^k_\Omega]}$ for all $k \geq 0$. This step is comprised by the following three steps:

(1) First we use the annihilator-invariant trick to show

$$[(I_w^\Omega)']^{[\mathfrak{n}_\Omega^k]} = H^0(\mathfrak{n}_\Omega, (I_w^\Omega \otimes F_k^\Omega)'),$$

where $F_k^\Omega = U(\mathfrak{n}_\Omega)/(\mathfrak{n}_\Omega^k)$ is a finite dimensional representation of N_Ω (or even P_Ω).

(2) Second we show the local Schwartz induction I_w^Ω is isomorphic to another Schwartz induction space:

$$I_w^\Omega \simeq \mathcal{S}\mathrm{Ind}_{P_\Omega \cap w^{-1}Pw}^{P_\Omega} \sigma^w$$

where $\sigma^w = \sigma \circ \mathrm{Ad}w$ is the twisted representation of σ.

(3) Third we use the tensor product trick to show

$$I_w^\Omega \otimes F_k^\Omega \simeq \mathcal{S}\mathrm{Ind}_{P_\Omega \cap w^{-1}Pw}^{P_\Omega} (\sigma^w \otimes F_w^\Omega).$$

In sum, we have

$$[(I_w^\Omega)']^{[\mathfrak{n}_\Omega^k]} \simeq H^0(\mathfrak{n}_\Omega, \mathcal{S}\mathrm{Ind}_{P_\Omega \cap w^{-1}Pw}^{P_\Omega} (\sigma^w \otimes F_w^\Omega)),$$

and we just need to study the M_Ω-action on it. Currently, we are only able to compute the above space for $\Omega = \emptyset$ by using Shapiro's lemma, and we will explain the reason below.

The Reason to Consider the Case $\Omega = \emptyset$

In the above step 3, we meet the two main obstacles for further study:

1. First we need to find the explicit M_Ω-action on the tensor product

$$[(I_w^\Omega)']^{[\mathfrak{n}_\Omega^\bullet]} \otimes U(\mathfrak{t}_w^\Omega).$$

However, this is not a tensor product of M_Ω-representations for some of the Ω (including Θ itself), as one can see the $t_w^\Omega = \overline{\mathfrak{n}}_\Omega \cap w^{-1}\overline{\mathfrak{n}}_P w$ is not stable under the M_Ω-conjugation. This will make the M_Ω-action on $[(I_w^\Omega)']^{[\mathfrak{n}_\Omega^\bullet]} \otimes U(\mathfrak{t}_w^\Omega)$ very complicated. Fortunately, when $\Omega = \emptyset$, the $\mathfrak{t}_w^\emptyset = \overline{\mathfrak{n}}_\emptyset \cap w^{-1}\overline{\mathfrak{n}}_P w$ is $M_\emptyset$-stable and the tensor product $(I_w')^{[\mathfrak{n}_\emptyset{}^\bullet]} \otimes U(\mathfrak{t}_w^\emptyset)$ is indeed a tensor product of $M_\emptyset$-representations.

2. Second we are only able to compute the space of invariant distributions

$$H^0(\mathfrak{n}_\Omega, \mathcal{S}\mathrm{Ind}_{P_\Omega \cap w^{-1}Pw}^{P_\Omega} (\sigma^w \otimes F_w^\Omega))$$

for the case $\Omega = \emptyset$. This is because the quotient $P_\emptyset \cap w^{-1}Pw\backslash P_\emptyset$ is $N_\emptyset$-transitive, and we can apply Shapiro's lemma to compute the above 0th invariant space. *But in general the quotient $P_\Omega \cap w^{-1}Pw\backslash P_\Omega$ is not N_Ω-transitive, and Shapiro's lemma does not apply to the general case.*

For the above two reasons, we are only able to obtain results of irreducibilities, by applying the above three steps, to the case when Ω is the empty set. This case already includes the interesting cases of degenerate principal series and minimal principal series.

To make the thesis clear and neat, in the main body of the thesis (Chapter 6—8, we choose Ω to be the empty set, show the results in the above three steps, and in Chapter 9 we apply them to reproduce the Theorem 7.2a and 7.4 in [15]. We put the general case of Ω in the Chapter 10, since we are currently unable to apply our techniques to obtain useful results.

1.1.4 Main Body of the Thesis (Chapter 6—9)

Currently, we are only able to prove results on irreducibility, by using our general results in the special case when Ω is the empty set. In the main body of the thesis (Chapter 6—9), we will only consider the case $\Omega = \emptyset$. We simply drop all superscript Ω, and use the notations $G_{\geq w}, G_{>w}, G_w, I_{\geq w}, I_{>w}, I_w$. We will study the dual-quotients $(I_{\geq w}/I_{>w})'$ for all $w \in [W_\Theta\backslash W]$, and show the irreducibility of I by showing:

$$\mathrm{Hom}_P(V', (I/I_{>e})') = \mathbb{C}$$
$$\mathrm{Hom}_{P_\emptyset}(V', (I_{\geq w}/I_{>w})') = \{0\}, \quad \forall w \neq e, w \in [W_\Theta\backslash W]$$

under certain conditions on σ (or τ). The three general steps shown in the last subsection correspond to the chapters 6, 7 and 8. And in the Chapter 9, we use the results in the three steps (three chapters) to reproduce analogous results of Theorem 7.2a and 7.4 in [15].

Transverse Derivatives (Chapter 6)

The first step is to write elements in the dual-quotient $(I_{\geq w}/I_{>w})'$ as transverse derivatives. The space $(I_{\geq w}/I_{>w})'$ is exactly the kernel of the following restriction map of scalar distributions:

$$\mathrm{Res}_w : I'_{\geq w} \to I'_{>w},$$

i.e. the elements in $(I_{\geq w}/I_{>w})'$ are exactly the scalar distributions on $I_{\geq w}$ that vanish on the subspace $I_{>w}$. Similar to the classical results on Euclidean spaces, we will show such distributions are given by transverse derivatives of distributions on G_w (see 6.1.1 for detailed explanation):

Theorem (Theorem 6.1)**.** *The natural map*

$$I'_w \otimes U(\mathfrak{n}_w^-) \to \mathrm{Ker}(\mathrm{Res}_w)$$

given by $U(\mathfrak{n}_w^-)$-derivatives of distributions on I_w, is an isomorphism (of $M_\emptyset$-representations). Here $\mathfrak{n}_w^- = w^{-1}\overline{\mathfrak{n}}_P w \cap \overline{\mathfrak{n}}_\emptyset$ is the transverse tangent space of G_w in $G_{\geq w}$.

The idea to consider transverse derivatives is not new, and is adopted in many previous works, e.g. [15], [31]. However the expression of distributions as transverse derivatives in these reference are on narrow neighbourhoods and are not explicit enough. In particular, their expressions give no clue on how to find the distributions D from its restrictions. The expression in the Theorem 6.1 is in a neat algebraic form. More importantly, our expressions are global on each open $G_{\geq w}$, which will make the globalization of local distributions easier than the previous works.

Torsion Subspaces (Chapter 7)

The second crucial idea is to consider the $\mathfrak{n}_\emptyset$-torsion subspace

$$[(I_{\geq w}/I_{>w})']^{[\mathfrak{n}_\emptyset{}^\bullet]} = [I'_w \otimes U(\mathfrak{n}_w^-)]^{[\mathfrak{n}_\emptyset{}^\bullet]}$$

on the dual-quotient $(I_{\geq w}/I_{>w})'$, where $\mathfrak{n}_\emptyset$ is the nilpotent radical of the standard minimal parabolic subalgebra. We will explain the reason to consider the torsion subspace at the beginning of Chapter 7. Briefly speaking, in many cases (including the minimal principal series), the image of a map $\Phi \in \mathrm{Hom}_{P_\emptyset}(V', (I_{\geq w}/I_{>w})')$ has its image in the $\mathfrak{n}_\emptyset$-torsion subspace.

The space $(I_{\geq w}/I_{>w})' \simeq \mathrm{Ker}(\mathrm{Res}_w) \simeq I'_w \otimes U(\mathfrak{n}_w^-)$ is actually a $U(\mathfrak{g})$-module thus also a $U(\mathfrak{n}_\emptyset)$-module. The I'_w is also a $U(\mathfrak{n}_\emptyset)$-module, however the tensor product $I'_w \otimes U(\mathfrak{n}_w^-)$ is not a tensor product of $\mathfrak{n}_\emptyset$-modules. This makes the $\mathfrak{n}_\emptyset$-torsion subspace on $I'_w \otimes U(\mathfrak{n}_w^-)$ very hard to compute. Surprisingly, we have the following main theorem of Chapter 7:

Theorem (Theorem 7.1)**.** *The $\mathfrak{n}_\emptyset$-torsion subspace on*

$$(I_{\geq w}/I_{>w})' \simeq \mathrm{Ker}(\mathrm{Res}_w) \simeq I'_w \otimes U(\mathfrak{n}_w^-)$$

is given by

$$[I'_w \otimes U(\mathfrak{n}_w^-)]^{[\mathfrak{n}_\emptyset{}^\bullet]} = [I'_w]^{[\mathfrak{n}_\emptyset{}^\bullet]} \otimes U(\mathfrak{n}_w^-).$$

This Chapter is the most subtle and technical part of the entire thesis. The above main theorem of Chapter 7 is prove by a combination of geometric and algebraic tricks. It is also one of the innovative part, since no other references have studied the torsion subspaces.

Application of Shapiro's Lemma (Chapter 8)

The $\mathfrak{n}_\emptyset$-torsion subspace

$$[(I_{\geq w}/I_{>w})']^{[\mathfrak{n}_\emptyset{}^\bullet]} \simeq [I'_w]^{[\mathfrak{n}_\emptyset{}^\bullet]} \otimes U(\mathfrak{n}_w^-)$$

is $M_\emptyset$-stable where $M_\emptyset$ is the Levi factor of the minimal parabolic $P_\emptyset$. Moreover, the right-hand-side is a tensor product of $M_\emptyset$-representations. To study the $M_\emptyset$-structure on the torsion subspace $[(I_{\geq w}/I_{>w})']^{[\mathfrak{n}_\emptyset{}^\bullet]}$, we need to study the $M_\emptyset$-action on the component $[I'_w]^{[\mathfrak{n}_\emptyset{}^\bullet]}$, or more precisely on the annihilators $[I'_w]^{[\mathfrak{n}_\emptyset{}^k]}$ for all $k \in \mathbb{Z}_{>0}$. This is the main object in Chapter 8, and it consists of the following steps:

(1) **The annihilator-invariant trick**: we will show (see (8.1)):

$$[I'_w]^{[\mathfrak{n}_\emptyset{}^k]} \simeq H^0(\mathfrak{n}_\emptyset, (I_w \otimes F_k)')$$

where F_k is the finite dimensional quotient $\mathfrak{n}_\emptyset$-module $U(\mathfrak{n}_\emptyset)/(\mathfrak{n}_\emptyset{}^k)$.

(2) We will show the $I_w = \mathcal{S}\mathrm{Ind}_P^{G_w}\sigma$ is isomorphic to the following Schwartz induction (Lemma 8.9)

$$I_w \simeq \mathcal{S}\mathrm{Ind}_{N_\emptyset \cap w^{-1}Pw}^{N_\emptyset}\sigma^w,$$

where $\sigma^w = \sigma \circ \mathrm{Ad}w$ is the twisted representation of $N_\emptyset \cap w^{-1}Pw$ on V.

(3) **The tensor product trick**: let (η_k, F_k) be the finite dimensional (conjugation) representation of $N_\emptyset$ on F_k. By combining the above two steps, we will show the following isomorphism (Lemma 8.20):

$$I_w \otimes F_k = (\mathcal{S}\mathrm{Ind}_{N_\emptyset \cap w^{-1}Pw}^{N_\emptyset}\sigma^w) \otimes F_k \simeq \mathcal{S}\mathrm{Ind}_{N_\emptyset \cap w^{-1}Pw}^{N_\emptyset}(\sigma^w \otimes \eta_k|_{N_\emptyset \cap w^{-1}Pw}).$$

(4) Combining all the above steps, the kth annihilator $[I'_w]^{[\mathfrak{n}_\emptyset{}^k]}$ is isomorphic to

$$H^0(\mathfrak{n}_\emptyset, [\mathcal{S}\mathrm{Ind}_{N_\emptyset \cap w^{-1}Pw}^{N_\emptyset}(\sigma^w \otimes \eta_k|_{N_\emptyset \cap w^{-1}Pw})]').$$

We will apply the **Shapiro's Lemma** to this $\mathfrak{n}_\emptyset$-invariant space and show it is isomorphic to

$$H^0(\mathfrak{n}_\emptyset \cap w^{-1}\mathfrak{p}w, V' \otimes F'_k)$$

with the $\mathfrak{n}_\emptyset \cap w^{-1}\mathfrak{p}w$ acts on $V' \otimes F'_k$ by the representation $\widehat{\sigma^w} \otimes \widehat{\eta_k}$.

(5) We will show the $M_\emptyset$ acts on the

$$[I'_w]^{[\mathfrak{n}_\emptyset{}^k]} \simeq H^0(\mathfrak{n}_\emptyset \cap w^{-1}\mathfrak{p}w, V' \otimes F'_k)$$

by the representation $\widehat{\sigma^w} \otimes \widehat{\eta_k} \otimes \gamma_w$ where γ_w is a character as in (8.22).

In sum, the main result in this chapter is to find the explicit $M_\emptyset$-action on the annihilators $[I'_w]^{[\mathfrak{n}_\emptyset{}^k]}$, namely the Lemma 8.27.

Application of the Main Results (Chapter 9)

We can combine the main results in Chapter 6, 7, 8, to give a new proof of Theorem 7.2a and Theorem 7.4 in [15] for linear algebraic Lie groups.

Let (τ, V) be a irreducible unitary representation (not necessarily finite dimensional), and $\sigma = \tau \otimes \delta_P^{1/2}$. The $I = \mathcal{S}\mathrm{Ind}_P^G \sigma$ (equivalently the normalized induction $\mathrm{Ind}_P^G \tau$) is irreducible if and only if $\mathrm{Hom}_G(I, I) = \mathbb{C}$.

First we can show:

Theorem (Theorem 9.8)**.** *Let $\sigma = \tau \otimes \delta_P^{1/2}$ be as above. Then the intertwining distributions with supports contained in P are all obtained from scalar intertwining operators by Frobenius reciprocity.*

From this theorem, we see for unitary (normalized) parabolic inductions, the non-trivial intertwining distributions have their supports containing non-identity double cosets. In particular, for the reducible unitary parabolic inductions, the interesting phenomenon occurs on non-identity double cosets.

By the above theorem, to show the irreducibility of I, we just need to show $\mathrm{Hom}_{P_\emptyset}(V', (I_{\geq w}/I_{>w})') = \{0\}$ for all $w \in [W_\Theta \backslash W]$ and $w \neq e$.

If (τ, V) is finite dimensional, then the V and V' are $\mathfrak{n}_\emptyset$-torsion. Then we have

$$\mathrm{Hom}_{P_\emptyset}(V', (I_{\geq w}/I_{>w})') = \mathrm{Hom}_{M_\emptyset}(V', [(I_{\geq w}/I_{>w})']^{[\mathfrak{n}_\emptyset{}^\bullet]}).$$

By combining the main theorems in Chapter 6 and 7, we see

$$\mathrm{Hom}_{P_\emptyset}(V', (I_{\geq w}/I_{>w})') = \mathrm{Hom}_{M_\emptyset}(V', [I'_w]^{[\mathfrak{n}_\emptyset{}^\bullet]} \otimes U(\mathfrak{n}_w^-)).$$

Also since V' is finite dimensional, there are positive integers k and n (large enough) such that

$$\mathrm{Hom}_{M_\emptyset}(V', [I'_w]^{[\mathfrak{n}_\emptyset{}^\bullet]} \otimes U(\mathfrak{n}_w^-)) = \mathrm{Hom}_{M_\emptyset}(V', [I'_w]^{[\mathfrak{n}_\emptyset{}^k]} \otimes U_n(\mathfrak{n}_w^-)).$$

By the Lemma 8.27, we see the $M_\emptyset$ acts on the $[I'_w]^{[\mathfrak{n}_\emptyset{}^k]} \otimes U_n(\mathfrak{n}_w^-)$ by the representation $\widehat{\sigma^w} \otimes \widehat{\eta_k} \otimes \gamma_w \otimes U_n(\mathfrak{n}_w^-)$, which is further expressed as

$$\widehat{\tau^w} \otimes w^{-1}\delta_P^{-1/2} \otimes \widehat{\eta_k} \otimes \gamma_w \otimes U_n(\mathfrak{n}_w^-).$$

Meanwhile, the $M_\emptyset$ acts on V' by the representation $\widehat{\tau}\otimes\delta_P^{-1/2}$. By comparing these two $M_\emptyset$-actions, we obtain the following analogue of Theorem 7.4 and Theorem 7.2a in [15]:

Theorem (Theorem 9.6)**.** *If the representation τ is regular in Bruhat's sense (see Definition 9.4), then the above two $M_\emptyset$-actions are not equal for all $w \neq e, k > 0, n \geq 0$, hence*

$$\mathrm{Hom}_{P_\emptyset}(V', (I_{\geq w}/I_{>w})') = \{0\}, \quad \forall w \in [W_\Theta \backslash W], w \neq e.$$

Therefore the parabolic induction I is irreducible.

Theorem (Theorem 9.15)**.** *If $P = P_\emptyset$ is the minimal parabolic subgroup, and for all $w \in W$ the representation τ^w is not equivalent to τ, then the above two $M_\emptyset$-actions are not equal for all $w \in W, w \neq e, k > 0, n \geq 0$, hence*

$$\mathrm{Hom}_{P_\emptyset}(V', (I_{\geq w}/I_{>w})') = \{0\}, \quad \forall w \in W, w \neq e.$$

Therefore the I is irreducible.

The Last Chapter of the Thesis

In the Chapter 10, we will sketch the proof of the general results mentioned in 1.1.3, i.e. the generalization of the Theorem 6.1, Theorem 7.1 and some of the results in Chapter 8. We also discuss some topics we are currently working on.

1.2 History and Motivation

The parabolic inductions are of great importance in representation theory, in the following sense:

- It is an important method to construct new (infinite dimensional) representations from representations of smaller groups.
- The subrepresentation theorem, i.e. all irreducible representations occurs as a sub or quotient representations in certain parabolic inductions. In some sense, they essentially contains all information about representations of a group.

- In the character theory, the characters of parabolic inductions form a "dual basis" of the characters of irreducible representations.

We summarize some previous study on irreducibility of parabolic inductions, which motivate our work in this dissertation.

1.2.1 The Work of Bruhat [15]

In [15], Bruhat defined the induced representations of Lie groups, and studied the intertwining forms between two induced representations following the scheme of Mackey. More precisely, in the above thesis he developed the theory of quasi-invariant distributions on Lie groups, and applied it to the study of intertwining numbers between two induced representations. Then he applied the results to show certain sufficient conditions of irreducibilities for the principal series (non-degenerate and degenerate), induced from parabolic subgroups of *connected semisimple Lie groups.*

In a word, the work of Bruhat gives us a method to *estimate* the intertwining numbers, i.e. it gives upper bound of the intertwining numbers between two induced representations ([15] p160 Theorem 6.1 and p171 Theorem 6.3). In particular, it can partially answer the following questions (under strong conditions):

- When is a unitary induction irreducible? ([15] p177 Theorem 6.5)
- When are two induced representations disjoint?

However, it has the following constraint:

- The intertwining forms introduced in [15] is different from intertwining operators (see the following subsubsection), and the intertwining operators are of more interests to us.
- The dual of a unitary induction is isomorphic to the induction of the dual. However, for non-unitary representations of a subgroup, this may not be true. Hence Bruhat's method does not apply to the irreducibility of non-unitary inductions.
- The most important aspect is, the work in [15] only gives *estimation* of intertwining forms, and it cannot answer what exactly are the intertwining forms (operators).

These cannot deny the importance of [15], because it is the starting work on induced representations of Lie groups and their irreducibilities. The crucial ideas, such as reducing the problem into a distribution analysis, and the idea to study double cosets separately (also the initial form of "Bruhat decomposition"), are adopted in most of the subsequent works. In the following part of this subsection, we will give a brief review of [15].

Remark 1.1. In the entire subsection 1.2.1, we assume all representation spaces are, at least, *locally convex Hausdorff topological vector spaces.* And later in the thesis we will work on the so-called Harish-Chandra representations, and explain why it is sufficient to deal with such representations. Of course for unitary representations there is no ambiguity since representation spaces are all Hilbert spaces.

Intertwining Operators and Intertwining Forms (Paragraphe 1-6)

The main object studied in [15] is the intertwining forms rather than intertwining operators:

Definition 1.2 ([15] p156)**.** Let G be a Lie group, and for $i = 1, 2$ let (π_i, U_i) be two representations of G.

- A continuous linear map $T : U_1 \to U_2$ is called an **intertwining operator (from U_1 to U_2)**, if $\pi_2(g) \circ T = T \circ \pi_1(g)$ for all $g \in G$. Let

  $$\begin{aligned} \mathfrak{I}(U_1, U_2) &= \text{the space of intertwining operators } U_1 \to U_2 \\ I(U_1, U_2) &= \text{the dimension of } \mathfrak{I}(U_1, U_2) \end{aligned}$$

- A separately continuous bilinear form $B : U_1 \times U_2 \to \mathbb{C}$ is called an **intertwining form (between U_1 and U_2)**, if $B(\pi_1(g)u, \pi_2(g)v) = B(u, v)$ for all $u \in U_1, v \in U_2$. Let

  $$\begin{aligned} \mathfrak{i}(U_1, U_2) &= \text{the space of intertwining forms between } U_1, U_2 \\ i(U_1, U_2) &= \text{the dimension of } \mathfrak{i}(U_1, U_2) \end{aligned}$$

  The $i(U_1, U_2)$ is called the **intertwining number between π_1, π_2**.

If U_1, U_2 are *unitary representations*, then one has ([15] p156)

$$\mathfrak{I}(U_1, U_2) = \mathfrak{I}(U_2, U_1), \text{ and } I(U_1, U_2) = I(U_2, U_1).$$

However for non-unitary representations, the above equalities are false. (see [15] p156 comment 24.) To overcome this non-symmetric property of intertwining operators, Bruhat introduce the above notion of intertwining forms and numbers, which is obviously symmetric:

$$\mathfrak{i}(U_1, U_2) = \mathfrak{i}(U_2, U_1) \text{ and } i(U_1, U_2) = i(U_2, U_1),$$

for general representations U_1, U_2 (not only for unitary representations). And Bruhat studied the space of intertwining forms instead of intertwining operators in his thesis.

Remark 1.3. For *unitary* representation (π, U) of G, the π is irreducible if and only if $I(U,U) = 1$. However if (π, U) is a general representation, then $I(U,U) = 1$ only implies π is *indecomposable*, but it may not be irreducible. In [15], Bruhat studied the irreducibilities *only for unitary inductions*. As we can see below, for unitary representations, there is no essential difference between intertwining operators and forms.

Remark 1.4. For $i = 1, 2$, let (π_i, U_i) be two representations of G, and let $(\widehat{\pi_2}, \widehat{U_2})$ be the contragredient representations of U_2. Then one has the isomorphism between vector spaces

$$\mathfrak{i}(U_1, U_2) \simeq \mathfrak{I}(U_1, \widehat{U_2})$$

and their dimensions are thus equal:

$$i(U_1, U_2) = I(U_1, \widehat{U_2}).$$

Remark 1.5. Let (σ, V) be a unitary representation of a subgroup P of G, and let $\mathrm{Ind}_P^G \sigma$ be the *normalized* induction of σ, and we know the $\mathrm{Ind}_P^G \sigma$ is also unitary. The $\mathrm{Ind}_P^G \sigma$ is irreducible, if and only if $I(\mathrm{Ind}_P^G \sigma, \mathrm{Ind}_P^G \sigma) = 1$. By the above discussion, it is irreducible if and only if $i(\mathrm{Ind}_P^G \sigma, \widehat{\mathrm{Ind}_P^G \sigma}) = 1$.

As a basic property of unitary inductions, one has $\widehat{\mathrm{Ind}_P^G \sigma} \simeq \mathrm{Ind}_P^G \widehat{\sigma}$, hence the irreducibility of $\mathrm{Ind}_P^G \sigma$ is equivalent to

$$i(\mathrm{Ind}_P^G \sigma, \mathrm{Ind}_P^G \widehat{\sigma}) = 1.$$

Hence the irreducibility problem of *unitary parabolic inductions* is included as a special case of the study of intertwining forms between two parabolic inductions. And Bruhat showed the irreducibilities of $\mathrm{Ind}_P^G \sigma$ by showing the upper bound of $i(\mathrm{Ind}_P^G \sigma, \mathrm{Ind}_P^G \widehat{\sigma})$ is 1.

Minimal Principal Series (Paragraphe 7, §1 − 5)

Bruhat proved the following theorem for principal series (induced from minimal parabolic subgroups):

Theorem ([15] p193 Theorem 7.2)**.** *Let G be a* connected semisimple Lie group*, $P_\emptyset$ be a minimal parabolic subgroup of G with Langlands decomposition $P_\emptyset = {}^\circ M_\emptyset A_\emptyset N_\emptyset$.*

(1) *Let (σ, V) be an irreducible unitary finite dimensional representation of $P_\emptyset$ (which has to be trivial on $N_\emptyset$ and is extended from a irreducible finite dimensional representation of ${}^\circ M_\emptyset A_\emptyset$). If for all* non-identity element *$w \in W$ of the Weyl group, the representation σ and $w\sigma$ of ${}^\circ M_\emptyset A_\emptyset$ are not equivalent, then the (normalized) parabolic induction $\mathrm{Ind}_{P_\emptyset}^G \sigma$ is irreducible.*

(2) *Let $(\sigma_1, V_1), (\sigma_2, V_2)$ be two irreducible finite dimensional representations of $P_\emptyset$, then the parabolic inductions $\mathrm{Ind}_{P_\emptyset}^G \sigma_1$ and $\mathrm{Ind}_{P_\emptyset}^G \sigma_2$ are equivalent, if and only if there exists a $w \in W$ such that the ${}^\circ M_\emptyset A_\emptyset$-representations σ_1 and $w\sigma_2$ are equivalent.*

We will prove the analogue of part (1) for algebraic Lie groups in Chapter 9.

Degenerate Principal Series (Paragraphe 7, §6)

Let P_Θ be a real parabolic subgroup of G corresponding to a subset Θ of the base of restricted roots, with Langlands decomposition $P_\Theta = {}^\circ M_\Theta A_\Theta N_\Theta$. Let (τ, V) be an irreducible unitary finite dimensional representation of P_Θ, then it is of the form

$$\tau = \tau_{{}^\circ M_\Theta} \otimes \chi_\Theta \otimes 1$$

where $\tau_{{}^\circ M_\Theta}$ is the restriction of τ to ${}^\circ M_\Theta$ (it is a irreducible unitary representation of ${}^\circ M_\Theta$ on V), χ_Θ is the restriction of τ to the split component A_Θ (it is a unitary character of A_Θ) and 1 means the trivial representation of the unipotent radical N_Θ.

The restriction of τ to the $A_\emptyset$ is in general not a single character, but a finite set of characters of $A_\emptyset$. Let

$$\mathrm{Spec}(A_\emptyset, \tau) = \text{the } A_\emptyset\text{-spectrum on } (\tau, V).$$

Then this is a finite set of unitary characters of $A_\emptyset$. Let $\chi \in \mathrm{Spec}(A_\emptyset, \tau)$ be an arbitrary element, then every character in $\mathrm{Spec}(A_\emptyset, \tau)$ are of the form $w\chi$ for some $w \in W_\Theta$.

Definition 1.6 (**Regularity of** $\mathrm{Spec}(A_\emptyset, \tau)$)**.** Consider the following condition

$$\text{(Reg-1):} \qquad w\chi \neq \chi, \quad \forall w \in W - W_\Theta \tag{1.1}$$

If a character $\chi \in \mathrm{Spec}(A_\emptyset, \tau)$ satisfies the above condition (Reg-1), then we say χ is **regular**.

It is easy to see if one character in $\mathrm{Spec}(A_\emptyset, \tau)$ is regular, then all characters in $\mathrm{Spec}(A_\emptyset, \tau)$ are regular.

Bruhat has shown the regularity is a sufficient condition for irreducibility:

Theorem ([15] p203 Theorem 7.4)**.** *Let (τ, V) be an irreducible unitary finite dimensional representation of P_Θ, and $\chi \in \mathrm{Spec}(A_\emptyset, \tau)$ be a $A_\emptyset$-character occurring on τ. If χ is regular (in the sense of Definition 1.6), then the (normalized) parabolic induction $\mathrm{Ind}_{P_\Theta}^G \tau$ is irreducible.*

We will reproduce this theorem for algebraic Lie groups in Chapter 9.

1.2.2 Some Other Related Works

We introduce some related works on irreducibilities of unitary parabolic inductions. Most of them are on inductions of finite dimensional unitary representations (unitary characters or induction from minimal parabolic subgroups). Our short-term goal is to reproduce these works as much as possible, then we have the confidence to attack the infinite dimensional (τ, V).

The Work of Wallach on Minimal Principal Series

We introduce work of N. Wallach on minimal principal series of split groups.

The connected complex semisimple Lie groups are all complex algebraic groups, and by restriction of scalar we can study them as real points groups of connect semisimple algebraic groups defined over $\mathbb{R}$.

Let G be a connected complex semisimple Lie group, with B a Borel subgroup (minimal parabolic). Assume T is a maximal complex torus of G, such that $B = TN$ where N is the unipotent radical of B. The T is isomorphic to a complex torus, hence its irreducible representations are characters. Wallach has proved the following strong result:

Theorem (Theorem 4.1 on page 112 of [40])**.** *Let G be a connected complex semisimple Lie group, with B a minimal parabolic subgroup. Let $B = TN$ be its Levi decomposition, and χ be a unitary character of T. Then the normalized (Hilbert) parabolic induction $\mathrm{Ind}_B^G \chi$ is irreducible.*

Let $\mathbf{G} = \mathbf{SL}_n$, and $G = \mathbf{SL}(n, \mathbb{R})$. Let B be the subgroup of upper triangular matrices in G, T be the subgroup of diagonal matrices in B, N be the unipotent radical of B. The T decomposes into

$$T = S \times A$$

where

$$S = \{\mathrm{diag}(m_1, \ldots, m_n) : m_i = \pm 1, m_1 m_2 \cdots m_n = 1\}$$
$$A = \{\mathrm{diag}(a_1, \ldots, a_n) : a_i > 0, a_1 a_2 \cdots a_n = 1\}$$

Let $m = (m_1, \ldots, m_n)$ be a generic element in S, and let ϵ_0 be the trivial character on S: $\epsilon_0(m) = 1, \forall m \in S$. For $i = 1, 2, \ldots, n-1$, let ϵ_i be the character $\epsilon_i(m) = m_i$. And let $\epsilon_n = \epsilon_1 \epsilon_2 \cdots \epsilon_{n-1}$. Then every nontrivial unitary characters of S are of the form $\epsilon_{i_1} \cdots \epsilon_{i_r}$ for some $1 \leq i_1 < i_2 < \ldots < i_r \leq n-1$.

Theorem (Theorem 5.1 on page 113 of [40])**.** *Let $G = \mathbf{SL}(n, \mathbb{R})$ and let P be the minimal parabolic subgroup of upper triangular matrices.*

- *If n is odd, then for any unitary character χ of T, the unitary (Hilbert) principal series $\mathrm{Ind}_B^G \chi$ is irreducible.*
- *If n is even, and let $\xi = \epsilon_{i_1} \cdots \epsilon_{i_r}$ for some $1 \leq i_1 < \ldots < i_r \leq n-1$ and $r \neq n/2$, and let ν be a unitary character of A, then the unitary principal series $\mathrm{Ind}_B^G(\xi \otimes \nu)$ is irreducible.*
- *If n is even, and let $\xi = \epsilon_{i_1} \cdots \epsilon_{i_r}$ for some $1 \leq i_1 < \ldots < i_r \leq n-1$ and $r = n/2$, and let ν be a unitary character of A, then the unitary principal series $\mathrm{Ind}_B^G(\xi \otimes \nu)$ is reducible, and is a direct sum of two irreducible constituents.*

Remark 1.7. The results on complex groups are not reachable now. However we are working on the group $\mathbf{SL}(n, \mathbb{R})$ and trying to reproduce the above Theorem on $\mathbf{SL}(3, \mathbb{C})$, and find the intertwining operators studied in [37].

Degenerate Principal Series for Symplectic Groups

There are numerous works on the irreducibilities of degenerate principal series. Some of them use pure algebraic method, by computing the K-types on the inducted representations. But algebraic methods cannot tell us what the intertwining operator is.

Among the works on degenerate principal series, the works in [25] and [22], [21] catch our attention.

Let

$$G = \mathbf{Sp}(2n+2, \mathbb{C}) = \{g \in \mathbf{GL}(2n+2, \mathbb{C}) : {}^t g J g = J\}$$

where J is the matrix with all anti-diagonal entries equal to 1, and all other entries equal to 0. Let M be the subgroup

$$M = \{ \begin{pmatrix} a & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & a^{-1} \end{pmatrix} : a \in \mathbb{C}^{\times}, A \in \mathbf{Sp}(2n, \mathbb{C})\}$$

Then M is the Levi factor of the maximal parabolic subgroup P. Let χ be a unitary character of the torus

$$A = \{ \begin{pmatrix} a & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & a^{-1} \end{pmatrix} : a \in \mathbb{C}^{\times}\},$$

and we extend it trivially to the entire M. The unitary induction $\mathrm{Ind}_P^G\chi$ form the unitary degenerate principal series when χ run through all unitary characters of A. K. Gross has proved the following result

Theorem (Theorem 7 on page 422 of [25])**.** *If χ is not the trivial character of A, the* $\mathrm{Ind}_P^G\chi$ *is irreducible. If χ is the trivial character, then* $\mathrm{Ind}_P^G\chi$ *is reducible with two irreducible constituents.*

Similar for the complex symplectic groups, T. Farmer has proved an analogous result. Let $G = \mathbf{Sp}(2n+2, \mathbb{R})$ and replace all the above subgroups P, M, A by subgroups with real matrices, then one has

Theorem (Theorem on page 411 of [22])**.** *If the χ is a non-trivial character, then* $\mathrm{Ind}_P^G\chi$ *is irreducible. If χ is the trivial character, then* $\mathrm{Ind}_P^G\chi$ *is reducible with two constituents.*

Remark 1.8. Actually both Gross and Farmer have computed the commuting algebra $\mathrm{Hom}_G(\mathrm{Ind}_P^G\chi, \mathrm{Ind}_P^G\chi)$, and shown they are generated by certain Mellin transformations on the groups. This motivate us to reproduce these intertwining operators by using our tools of Schwartz analysis.

Parabolic Inductions of Unipotent Representations

We do not have many examples on the irreducibilities of parabolic inductions of infinite dimensional representations. Among them, we want to mention the work of Barbasch and Vogan ([4]).

Let G be a complex semisimple Lie group, and $\mathfrak{g}$ be its Lie algebra. Let O be a nilpotent orbit in $\mathfrak{g}$, and assume it is special with its dual orbit ${}^LO$ even. Let $\overline{A}(O)$ be the Lusztig quotient group defined in (4.4c) of

[4]. The irreducible representations of the finite group $\overline{A}(O)$ parameterize a finite packet of irreducible representations of G, called **special unipotent representations** (see Theorem III on page 46 of [4]). For each irreducible representation π of $\overline{A}(O)$, let X_π be the corresponding special unipotent representations.

Assume the orbit O is induced from a nilpotent orbit $O_{\mathfrak{m}}$ of a Levi subalgebra $\mathfrak{m}$ (thus the O is special), and assume ${}^L O$ is even. Then the Lusztig quotient $\overline{A}(O)$ has a subgroup $\overline{A}^{\mathfrak{m}}(O)$. This group is not defined thoroughly in [4], and it is claim to be the same as the group considered in the book [32] of Lusztig.

In the last section of [3], the authors claim the following result

Theorem. *Let O, $\mathfrak{m}$, $\overline{A}^{\mathfrak{m}}(O)$ be as above. Then for each irreducible representation π of $\overline{A}^{\mathfrak{m}}(O)$, there is an irreducible representation $X_\pi^{\mathfrak{m}}$ of the Levi subgroup M, such that the parabolic induction $\mathrm{Ind}_P^G X_\pi^{\mathfrak{m}}$ has all its irreducible subquotients isomorphic to a representation X_η in the special unipotent packet corresponding to O. For each irreducible representation η of the Lusztig quotient $\overline{A}(O)$, the corresponding special unipotent representation X_η occurs with multiplicity*

$$[\eta|_{\overline{A}^{\mathfrak{m}}(O)} : \pi].$$

The representations $X_\pi^{\mathfrak{m}}$ are characterized as follows:

(1) they have the infinitesimal characters $(w\lambda_O, \lambda_O)$ (where λ_O is defined in (1.15b) in [4]).

(2) their left and right annihilators (in $U(\mathfrak{g})$) are maximal among all irreducible representations with the same infinitesimal characters.

This theorem is prove by matching the characters of the representations, and there is no clue how to find the intertwining operators. It is even unknown whether the representations $X_\pi^{\mathfrak{m}}$ are unitary.

For classical groups, Barbasch has shown in [3] that the index of $\overline{A}^{\mathfrak{m}}(O)$ in $\overline{A}(O)$ is either 1 or 2, hence the induced representation $\mathrm{Ind}_P^G X_\pi^{\mathfrak{m}}$ has at most two irreducible factors in the composition series.

1.3 Reading Guide and Key Points of the Chapter 2, 3 and 4

The entire thesis is divided into two parts: the foundation part consists of Chapter 2, 3 and 4. The main part consists of Chapter 5, 6, 7, 8, 9. The main part is already summarized at the beginning of the thesis. Here we write a short reading guide for the foundation part.

We recall necessary notions and results, and omit all details by giving precise references. The Chapter 2, 3 recall some basic knowledge which may be well-known to the reader, while the stuff in Chapter 4 are some not well-known works from [1], [2] and [20], and our own works.

1.3.1 Chapter 2

The Chapter 2 is short review on three topics: algebraic Lie groups, topological vector spaces and representations of Lie groups. The section 2.1 and 2.3 are well-known and we keep them in the shortest form. The notions of NF/DNF-spaces are not so well-known and the reader need to pay attention to the section 2.2.

- The Lie groups studied in the thesis are **real point groups of connected reductive linear algebraic groups defined over** $\mathbb{R}$. The section 2.1 is a short summary of the structure theory on algebraic groups and their real point groups. We mainly follow the book [10] of Borel.

- We will mainly work on a particular class of topological vector spaces: nuclear TVS or more precisely NF/DNF-spaces. All terms and results could be found in Treves' [36], and for the properties of NF/DNF-spaces, we follow the appendix in [18]. The NF/DNF-spaces are very similar to finite dimensional vector spaces, they behave well under algebraic construction (e.g. strong dual and tensor products are exact), and these good algebraic properties in crucial to build the distribution theory in Chapter 4.

- The representations studied in the thesis are **Harish-Chandra representations**. The section 2.3 is a short review of the notion of Harish-Chandra representations introduced in [17]. We also give a quick review of the notions of smooth inductions, Hilbert inductions and Frobenius reciprocity.

1.3.2 Chapter 3

The Chapter 3 consists of three parts:

- In 3.1 and 3.2, we recall basic notions in real algebraic geometry. We follow the [6] in section 3.1, and the [1], [2] in section 3.2.

The section 3.1 is a quick review of affine real algebraic varieties. Typical examples of affine real algebraic varieties are $\mathbf{G}(\mathbb{R})$ for algebraic group $\mathbf{G}$ and double cosets on $\mathbf{G}(\mathbb{R})$ of algebraic subgroups.

In 3.2, we recall the notions of (affine) Nash manifolds, on which the theory of Schwartz functions is built in [1]. We basically follow the definitions from [1]. The nonsingular affine real algebraic varieties is the most important class of affine Nash manifolds, and we will only consider affine real algebraic varieties in the thesis.

- In 3.3, we study the double coset decomposition on the $\mathbf{G}(\mathbb{R})$. The important concepts are anti-actions, closure order, and the notations $G_{\geq i}, G_{>i}$.

- In 3.4, we study the torsion subspace of tensor product module over a Lie algebra. This is a pure algebraic section, and it could be read independently. The reader can skip the last four subsections (from 3.4.3 to 3.4.7), and they will not affect the reading.

  Let $\mathfrak{h}$ be a complex Lie algebra, and $\mathcal{M}_1, \mathcal{M}_2$ be two (left) $\mathfrak{h}$-modules and $\mathcal{M}_1 \otimes \mathcal{M}_2$ be their tensor product modules, we prove the following results:

  - $\mathcal{M}_1^{[\mathfrak{h}^\bullet]} \otimes \mathcal{M}_2^{[\mathfrak{h}^\bullet]} \subset (\mathcal{M}_1 \otimes \mathcal{M}_2)^{[\mathfrak{h}^\bullet]}$.
  - If one of them, say $\mathcal{M}_2$ is finite dimensional and $\mathfrak{h}$-torsion, i.e. $\mathcal{M}_2^{[\mathfrak{h}^\bullet]} = \mathcal{M}_2$, then the above inclusion is equality:

  $$\mathcal{M}_1^{[\mathfrak{h}^\bullet]} \otimes \mathcal{M}_2^{[\mathfrak{h}^\bullet]} = (\mathcal{M}_1 \otimes \mathcal{M}_2)^{[\mathfrak{h}^\bullet]}.$$

  These results sounds elementary but still not found in reference. We include them for the reader to see how much harder the Chapter 7 is.

1.3.3 Chapter 4

Some parts of this chapter is also innovative, and not in any reference, but it is not hard to derive all results in this chapter from the knowledge in [1], [2] and [20].

The Chapter 4 has two parts:

- In section 4.1, 4.2, 4.3 and 4.4, we recall the work of Aizenbud-Gourevitch in [1] and [2]. In these sections, the geometric objects are affine Nash manifolds.

- In 4.2, we recall the notion of Schwartz $\mathbb{R}$-valued functions on affine Nash manifolds, and crucial properties on the spaces of Schwartz functions (e.g. NF-space, cosheaf structure etc). The notions and results are generalized to $\mathbb{C}$-valued Schwartz functions without difficulty. We will only consider complex valued Schwartz functions (or even vector valued Schwartz functions in complex TVS), and the analogous properties are summarized in Lemma 4.21.
- In 4.3, we generalize the definitions of scalar valued Schwartz functions to E-valued Schwartz functions where E is a NF-space. The E-valued Schwartz function spaces have similar properties as scalar valued case, and we summarize the analogous properties as Proposition 4.30.
- In 4.4, we study the strong dual spaces of Schwartz function spaces and call elements in them **Schwartz distributions**. The crucial properties of Schwartz distribution spaces are summarized in Lemma 4.32, Lemma 4.35, Lemma 4.36. And the sheaf property of Schwartz distributions lead to the following crucial result: the space of Schwartz distributions supported in a closed Nash submanifold is independent of the neighbourhood (Lemma 4.38). This result plays an important role in Chapter 6.

- In section 4.5 and 4.6, we will work on nonsingular affine real algebraic varieties. Such varieties is a special class of affine Nash manifolds that we will only study in the thesis. On nonsingular affine real algebraic varieties, the Schwartz functions defined in [1] coincide with the Schwartz functions defined in [20]. Currently B. Elazar and A. Shaviv have generalized the work of [1] to non-affine/non-smooth real algebraic varieties.
  - In 4.5, we first restrict the cosheaf of Schwartz functions and sheaf of Schwartz distributions to the Zariski topology on nonsingular affine real algebraic varieties, and call them pseudo-cosheaf/pseudo-sheaf on the Zariski topology. We show that a Schwartz distribution has a well-defined support under the Zariski topology. When the spaces of Schwartz functions are built on algebraic groups, these spaces have the right regular actions on right stable subvarieties.
  - In 4.6, we combine the notion of (local) Schwartz inductions introduced in [20], with the properties of Schwartz functions devel-

oped in [1]. More precisely, the (local) Schwartz induction spaces are defined to be the images of Schwartz function spaces under certain mean value integration map. In particular, the Schwartz induction spaces are NF-spaces and behave well under the Zariski P-topology. Therefore we can develop a set of crucial properties similar to the Proposition 4.30 on Schwartz functions, and the distribution analysis on Schwartz inductions is very much similar to the analysis on Schwartz functions. Note that the Schwartz induction from parabolic subgroups are exactly smooth inductions since the base quotient manifold is compact.

1.4 Convention on Notation

We give a short index of notations frequently used in the thesis.

Algebraic Groups and Lie Groups

We will fix the notations of algebraic groups and Lie groups in 2.1. In the entire thesis, we follow the "notation-choosing rules" in Remark 2.1.

From Chapter 6, we use the following notations of subgroups

$$\begin{aligned} N_w &= \dot{w}^{-1}\overline{N}_P\dot{w} \\ N_w^+ &= \dot{w}^{-1}\overline{N}_P\dot{w} \cap N_\emptyset \\ &= N_w \cap N_\emptyset \\ N_w^- &= \dot{w}^{-1}\overline{N}_P\dot{w} \cap \overline{N}_\emptyset \\ &= N_w \cap \overline{N}_\emptyset \end{aligned}$$

Topological Vector Spaces

Let V, V_1, V_2 be topological vector spaces (TVS for short)

$$\begin{aligned} V' &= \text{the strong topological dual of a TVS } V \\ V^* &= \text{the algebraic dual of } V \\ L(V_1, V_2) &= \text{the space of continuous linear maps from } V_1 \text{ to } V_2 \\ V_1 \mathbin{\widehat{\otimes}} V_2 &= \text{the completed tensor product of } V_1 \text{ and } V_2 \end{aligned}$$

Representation Theory

Let (σ, V) be a continuous representation of a Lie group G, then we denote by $(\widehat{\sigma}, V')$ the contragredient (dual) representation.

For two representations $(\sigma_1, V_1), (\sigma_2, V_2)$ of a Lie group, we denote by

$$\mathrm{Hom}_G(V_1, V_2) = \mathrm{Hom}_G(\sigma_1, \sigma_2)$$

the space of intertwining operators from (σ_1, V_1) to (σ_2, V_2).

Let P be a real parabolic subgroup of G (see 2.1), and (τ, V) be a Hilbert representation of P, then

$$\begin{aligned} C^\infty\mathrm{Ind}_P^G\tau &= \text{the smooth parabolic induction} \quad \text{(Definition 2.30)} \\ \mathrm{Ind}_P^G\tau &= \text{the normalized Hilbert parabolic induction} \quad \text{(Definition 2.33)} \\ \mathcal{S}\mathrm{Ind}_P^G\tau &= \text{the Schwartz induction} \quad \text{(Definition 4.57)} \end{aligned}$$

Lie Algebras and Modules (Section 3.4 and Chapter 7)

All real Lie algebras are denoted by fraktur letters with a subscript 0, e.g.$\mathfrak{g}_0$, and its complexification is denoted by the same letter with the subscript removed, e.g. $\mathfrak{g}$.

For a complex Lie algebra $\mathfrak{h}$, let

$$\begin{aligned} U(\mathfrak{h}) &= \text{the universal enveloping algebra} \\ (\mathfrak{h}^k) &= \text{the two-sided ideal generated by } k\text{-products of element in } \mathfrak{h} \\ U_n(\mathfrak{h}) &= \text{the finite dimensional subspace of } U(\mathfrak{h}) \\ &\quad\ \text{spanned by } i\text{-products of elements in } \mathfrak{h}, i \leq n \end{aligned}$$

Let $\mathcal{M}$ be a left $\mathfrak{h}$-module (left $U(\mathfrak{h})$-module), we let

$$\begin{aligned} \mathcal{M}^{[\mathfrak{h}^k]} &= \text{the annihilator of the ideal } (\mathfrak{h}^k), \forall k \geq 0 \\ \mathcal{M}^{[\mathfrak{h}^\bullet]} &= \bigcup_{k \geq 0} \mathcal{M}^{[\mathfrak{h}^k]} \\ &= \text{the (left) } \mathfrak{h}\text{-torsion submodule of } \mathcal{M} \end{aligned}$$

Similarly for $\mathfrak{M}$ be a right $\mathfrak{h}$-module (right $U(\mathfrak{h})$-module), we let

$$\begin{aligned} \mathfrak{M}_{[\mathfrak{h}^k]} &= \text{the annihilator of the ideal } (\mathfrak{h}^k), \forall k \geq 0 \\ \mathfrak{M}_{[\mathfrak{h}^\bullet]} &= \text{the (right) } \mathfrak{h}\text{-torsion submodule of } \mathfrak{M} \end{aligned}$$

Lie Algebra Elements as Vector Fields (Chapter 7)

Let G be a Lie group, and $\mathfrak{g}_0$ be its abstract (real) Lie algebra. An element $X \in \mathfrak{g}_0$ could be regarded as a left invariant vector field on G, denoted by X^L, and as a right invariant vector field on G, denoted by X^R. For a point $g \in G$, let X_g^L (resp. X_g^R) be the tangent vector of X^L (resp. X^R) in the tangent space T_gG at g. We only need these notations in Chapter 7.

Double Cosets on G

Let $\mathbf{G}$ be a connected reductive linear algebraic groups defined over $\mathbb{R}$, let $\mathbf{P}_\Theta, \mathbf{P}_\Omega$ be two standard parabolic $\mathbb{R}$-subgroups corresponding to subsets Θ, Ω of the base. Let W_Θ, W_Ω be the corresponding parabolic Weyl subgroups. We use the following notations (from [16]):

$$\begin{aligned}[][W/W_\Omega] &= \{w \in W | w\Theta \subset \Sigma^+\} \\ [W_\Theta \backslash W] &= \{w \in W | w^{-1}\Theta \subset \Sigma^+\} \\ [W_\Theta \backslash W/W_\Omega] &= [W/W_\Omega] \cap [W_\Theta \backslash W]\end{aligned}$$

The $[W_\Theta \backslash W/W_\Omega]$ is the set of minimal representatives (see 3.3), and it is in one-to-one correspondence with the (P_Θ, P_Ω)-double cosets in G.

By abuse of notation, we use w to denote both the element in W and a fixed representative of it in G. For every $w \in [W_\Theta \backslash W/W_\Omega]$, we denote by G_w^Ω the double coset $P_\Theta w P_\Omega$. Under the closure order defined in 3.3, we let

$$\begin{aligned} G_{\geq w}^\Omega &= \coprod_{P_\Theta w P_\Omega \subset \overline{P_\Theta x P_\Omega}} P_\Theta x P_\Omega \\ G_{>w}^\Omega &= G_{\geq w}^\Omega - G_w^\Omega \end{aligned}$$

The $G_{\geq w}^\Omega$ and $G_{>w}^\Omega$ are Zariski open in G, while the G_w^Ω is closed in $G_{\geq w}^\Omega$.

In particular, *when $\Omega = \emptyset$ (empty set), we simply drop all superscript.*

Schwartz Function Spaces and Schwartz Inductions

Let X be a nonsingular real affine algebraic variety (see 3.1), or more generally an affine Nash manifold (see 3.2), let E be a nuclear Fréchet space. Then we denote by

$$\begin{aligned} \mathcal{S}(X, \mathbb{R}) &= \text{the space of } \mathbb{R}\text{-valued Schwartz functions on } X \\ \mathcal{S}(X, \mathbb{C}) &= \text{the space of } \mathbb{C}\text{-valued Schwartz functions on } X \\ \mathcal{S}(X, E) &= \text{the space of } E\text{-valued Schwartz functions on } X \end{aligned}$$

Let P be a real parabolic subgroup of G, σ be a nuclear Fréchet representation of P, which is of moderated growth, let Y be a left P-stable subvariety of G, then we let

$$\mathcal{S}\mathrm{Ind}_P^Y\sigma = \text{the local Schwartz induction space} \quad \text{(Definition 4.64)}.$$

In particular, for $P = P_\Theta$, and $\Omega \subset \Theta$, we use the following simplified notations for local Schwartz inductions:

$$\begin{aligned} I_w^\Omega &= \mathcal{S}\mathrm{Ind}_P^{G_w^\Omega}\sigma \\ I_{\geq w}^\Omega &= \mathcal{S}\mathrm{Ind}_P^{G_{\geq w}^\Omega}\sigma \\ I_{>w}^\Omega &= \mathcal{S}\mathrm{Ind}_P^{G_{>w}^\Omega}\sigma \end{aligned}$$

As before, *when* $\Omega = \emptyset$ *(empty set), we omit the superscript* Ω, and for each $w \in [W_\Theta \backslash W]$, we let

$$\begin{aligned} I_w &= I_w^\emptyset \\ I_{\geq w} &= I_{\geq w}^\emptyset \\ I_{>w} &= I_{>w}^\emptyset \end{aligned}$$

Chapter 2

Basic Setting

Summary of This Chapter

In this chapter, we unify the terminology and notations, and give precise reference for the key results without giving proof.

- In 2.1, we recall the basic notions and results on linear algebraic groups. The Lie groups studied in this thesis are *real points of connected reductive linear algebraic groups defined over* $\mathbb{R}$.
- In 2.2, we recall the basic notions of nuclear Fréchet (NF)-spaces and their strong dual (DNF-spaces). These topological vector spaces form a good category similar to the category of finite dimensional vector spaces, and they behave well under algebraic constructions, e.g. the strong dual and tensor product functors are exact, and tensor products commute with strict direct limit. These good properties are crucial in the study of Schwartz functions/inductions and distributions (which will be introduced in Chapter 4).
- In 2.3, we recall the definition of Harish-Chandra representations introduced in [17], and the main object of the thesis—smooth parabolic inductions.

2.1 Groups Studied in the Thesis

The groups we will study in this thesis are groups of real points of connected reductive linear algebraic groups defined over $\mathbb{R}$.

For algebraic groups, we follow the terms and notations from the following book and papers of Borel: [7], [8], [11], [10]. In particular, *all algebraic groups are treated in the classical sense, and regarded as complex algebraic varieties*. In terms of group schemes, the algebraic groups studied in this thesis are $\mathbb{R}$-group schemes, and we identify them with their $\mathbb{C}$-rational points, and study the groups of $\mathbb{R}$-rational points of them.

- The 2.1.1 is on the relative theory of reductive linear algebraic groups defined over $\mathbb{R}$. We recall the relative root datum, the structure of parabolic $\mathbb{R}$-subgroups and relative Bruhat decompositions on such groups.

- The 2.1.2 is on Lie groups. The real points groups of connected reductive linear algebraic groups are of Harish-Chandra class. The algebraic structure theory coincide with the Lie group structure theory on such groups.

2.1.1 Algebraic Groups in this Thesis

Let $\mathbf{G}$ be a connected reductive linear algebraic group defined over $\mathbb{R}$. Let $\mathbf{G}(\mathbb{R})$ be the group of $\mathbb{R}$-rational points of $\mathbf{G}$, then $\mathbf{G}(\mathbb{R})$ has a smooth Lie group structure.

We denote the Lie group $\mathbf{G}(\mathbb{R})$ by G, and let

$$\mathfrak{g}_0 = \mathrm{Lie} G$$

be its (real) Lie algebra. Let $\mathfrak{g}^{\mathbb{C}}$ be the algebraic Lie algebra of the $\mathbf{G}$, then $\mathfrak{g}_0$ is a real form of $\mathfrak{g}^{\mathbb{C}}$.

Remark 2.1. *Throughout the entire thesis, we stick to the following rules of notations*: for an algebraic group (defined over $\mathbb{R}$) denoted by a boldface letter, (e.g. $\mathbf{H}$),

- its (complex) algebraic Lie algebra is denoted by fraktur letter with superscript $\mathbb{C}$ (e.g. $\mathfrak{h}^{\mathbb{C}}$);

- its Lie group of real points is denoted by the same uppercase letter (e.g. H);

- the Lie algebra of the real point Lie group is denoted by fraktur letter with a subscript 0 (e.g. $\mathfrak{h}_0$);

- the complexification of $\mathfrak{h}_0$ is denoted by the same fraktur letter without subscript (e.g. $\mathfrak{h}$).

Relative Roots, Relative Weyl Group

Let

$$\begin{aligned}
\mathbf{S} &= \text{a maximal } \mathbb{R}\text{-split torus of } \mathbf{G} \\
\Sigma &= \text{the set of } \mathbf{S}\text{-roots} \\
&\quad\ \text{(relative roots)} \\
\mathbf{N}_{\mathbf{G}}(\mathbf{S}) &= \text{the normalizer of } \mathbf{S} \\
\mathbf{Z}_{\mathbf{G}}(\mathbf{S}) &= \text{the centralizer of } \mathbf{S} \\
W &= \mathbf{N}_{\mathbf{G}}(\mathbf{S})/\mathbf{Z}_{\mathbf{G}}(\mathbf{S}) \\
&= \text{the relative Weyl group}
\end{aligned}$$

For each $\alpha \in \Sigma$, let $\mathfrak{g}_\alpha^{\mathbb{C}}$ be the α-root subspace on $\mathfrak{g}^{\mathbb{C}}$. In general (unlike the absolute theory), the $\mathfrak{g}_\alpha^{\mathbb{C}}$ is neither one dimensional, nor a Lie subalgebra. If $2\alpha \notin \Phi$, then $\mathfrak{g}_\alpha^{\mathbb{C}}$ is a Lie subalgebra of $\mathfrak{g}^{\mathbb{C}}$. If $2\alpha \in \Phi$, then $\mathfrak{g}_\alpha^{\mathbb{C}} + \mathfrak{g}_{2\alpha}^{\mathbb{C}}$ is a subalgebra of $\mathfrak{g}^{\mathbb{C}}$. Let

$$\Phi_{red} = \{\alpha \in \Phi : 2\alpha \notin \Phi\}$$

be the subset of reduced roots. For each $\alpha \in \Phi_{red}$, let $(\alpha) = \{k\alpha \in \Phi | k \in \mathbb{Z}_+\}$ and

$$\mathfrak{g}_{(\alpha)}^{\mathbb{C}} = \begin{cases} \mathfrak{g}_\alpha^{\mathbb{C}} & \text{if } \alpha \in \Phi_{red} \\ \mathfrak{g}_\alpha^{\mathbb{C}} + \mathfrak{g}_{2\alpha}^{\mathbb{C}} & \text{if } \alpha \notin \Phi_{red} \end{cases} \tag{2.1}$$

Then one has the Lie algebra decomposition ([10] p231 §21.7):

$$\mathfrak{g}^{\mathbb{C}} = \mathrm{Lie}\mathbf{Z}_{\mathbf{G}}(\mathbf{S}) \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha^{\mathbb{C}} = \mathrm{Lie}\mathbf{Z}_{\mathbf{G}}(\mathbf{S}) \oplus \bigoplus_{\alpha\in\Phi_{red}} \mathfrak{g}_{(\alpha)}^{\mathbb{C}}. \tag{2.2}$$

For each $\alpha \in \Phi$, there exists a unique closed connected unipotent $\mathbb{R}$-subgroup $\mathbf{U}_{(\alpha)}$ normalized by $\mathbf{Z}_{\mathbf{G}}(\mathbf{S})$ with Lie algebra $\mathfrak{g}_{(\alpha)}^{\mathbb{C}}$ ([10] p232 Proposition 21.9). It is the unipotent subgroup directly spanned by absolute root subgroups $\mathbf{U}_\beta$ such that $\beta \in \Phi^{\mathbb{C}}$ restricted to $\mathbf{S}$ equals to α.

Standard Parabolic $\mathbb{R}$-subgroups

We fix a minimal parabolic $\mathbb{R}$-subgroup $\mathbf{P}_\emptyset$ containing $\mathbf{S}$, it determines the following datum on the relative root system:

$$\begin{aligned}
\Sigma^+ &= \text{the relative roots occurring in the Lie algebra } \mathfrak{p}_\emptyset{}^{\mathbb{C}} \\
\Delta &= \text{the base of } \Sigma \text{ determined by } \Sigma^+ \\
\Sigma^- &= -\Sigma^+ \\
S &= \{s_\alpha : \alpha \in \Delta\}
\end{aligned}$$

The Σ^+ is a positive system of Σ which determines the base Δ, and the S is a generator set of W.

The parabolic $\mathbb{R}$-subgroups containing $\mathbf{P}_\emptyset$ are called **standard** parabolic $\mathbb{R}$-subgroups, and they are in one-to-one correspondence with subsets of Δ. For each subset $\Theta \subset \Delta$, let

$$\begin{aligned}
\mathbf{S}_\Theta &= (\bigcap_{\alpha\in\Theta} \mathrm{Ker}\alpha)^\circ \\
\mathbf{M}_\Theta &= \mathbf{Z}_\mathbf{G}(\mathbf{S}_\Theta) \\
\mathbf{N}_\Theta &= \langle \mathbf{U}_{(\alpha)} : \alpha \in \Sigma^+ - \langle\Theta\rangle\rangle \\
W_\Theta &= \langle s_\alpha : \alpha \in \Theta\rangle
\end{aligned}$$

Then the $\mathbf{P}_\Theta = \mathbf{P}_\emptyset \cdot W_\Theta \cdot \mathbf{P}_\emptyset$ is a standard parabolic $\mathbb{R}$-subgroup with Levi $\mathbb{R}$-factor $\mathbf{M}_\Theta$, unipotent radical $\mathbf{N}_\Theta$, and Levi decomposition $\mathbf{M}_\Theta \ltimes \mathbf{N}_\Theta$. The W_Θ is the relative Weyl group $W(\mathbf{M}_\Theta, \mathbf{S})$ of $\mathbf{M}_\Theta$.

In particular when $\Theta = \emptyset$, the parabolic $\mathbb{R}$-subgroup of $\mathbf{G}$ corresponding to $\emptyset \subset \Delta$ is exactly the $\mathbf{P}_\emptyset$ we fixed at the beginning, and this is why we use the subscript $\emptyset$. And the $\mathbf{S}_\emptyset = \mathbf{S}$, $W_\emptyset = \{e\}$ (identity).

The Relative Bruhat decomposition

Let $\mathbf{G}(\mathbb{R}), \mathbf{P}_\emptyset(\mathbb{R}), \mathbf{N}_\mathbf{G}(\mathbf{S})(\mathbb{R})$ be the groups of $\mathbb{R}$-rational points on $\mathbf{G}, \mathbf{P}_\emptyset$ and $\mathbf{N}_\mathbf{G}(\mathbf{S})$, and $S = \{s_\alpha : \alpha \in \Delta\}$ be the set of reflections of $\mathbb{R}$-simple roots. Then the quadruple

$$(\mathbf{G}(\mathbb{R}), \mathbf{P}_\emptyset(\mathbb{R}), \mathbf{N}_\mathbf{G}(\mathbf{S})(\mathbb{R}), S)$$

is a Tits system ([10] p236 Theorem 21.15).

Let Θ, Ω be two subsets of Δ, and $\mathbf{P}_\Theta, \mathbf{P}_\Omega$ the corresponding standard parabolic $\mathbb{R}$-subgroups, and W_Θ, W_Ω be the corresponding (parabolic) subgroups of W. By the same remark on p22 of [13], one has the double cosets decomposition of $\mathbf{G}(\mathbb{R})$:

$$\mathbf{G}(\mathbb{R}) = \coprod_{w\in[W_\Theta\backslash W/W_\Omega]} \mathbf{P}_\Theta(\mathbb{R}) w \mathbf{P}_\Omega(\mathbb{R}). \tag{2.3}$$

Here we still adopt the notation of the representative sets on page 7 of [16]:

$$\begin{aligned}
[W/W_\Omega] &= \{w \in W | w\Omega \subset \Sigma^+\} \\
[W_\Theta\backslash W] &= \{w \in W | w^{-1}\Theta \subset \Sigma^+\} \\
[W_\Theta\backslash W/W_\Omega] &= [W/W_\Omega] \cap [W_\Theta\backslash W]
\end{aligned}$$

2.1.2 G as a Lie Group

Let $\mathbf{G}$ be a connected reductive linear algebraic group defined over $\mathbb{R}$, and $G = \mathbf{G}(\mathbb{R})$ be the Lie group corresponding to the group of real rational points. We assume $\mathbf{G}$ has nonzero semisimple $\mathbb{R}$-rank, i.e. the $\mathbb{R}$-rank of $\mathcal{D}\mathbf{G}$ is nonzero, to make sure $\mathbf{G}$ has proper parabolic $\mathbb{R}$-subgroups. *For* $\mathbf{G}$ *and its* $\mathbb{R}$*-subgroups, we follow the notation-choosing rules in Remark 2.1.*

The G has the following properties:

1. G has finite number of connected components (under the Euclidean topology).

2. Its Lie algebra $\mathfrak{g}_0 = \mathrm{Lie}G$ is reductive.

3. Let G° be its identity component, then the derived subgroup $\mathcal{D}(G^\circ)$ has finite center and is closed in G°.

4. The G is of inner type, i.e. $\mathrm{Ad}G \subset \mathrm{Int}\mathfrak{g}$ where $\mathfrak{g} \simeq \mathrm{Lie}\mathbf{G}$.

Hence the G is a **Lie group of Harish-Chandra class** in the sense of [28] p105-106. Also, the G is a **real reductive Lie group** in the sense of [12].

Cartan Decompositions and Restricted Roots

Since G has finitely many components, every compact subgroup is contained in a maximal one. Every maximal compact subgroup meets all connected components, and they are conjugated under G°. There exists a **global Cartan decomposition on** G: G is diffeomorphic to a product $K \times V$ where K is an arbitrary maximal compact subgroup and V is a vector subgroup. The corresponding involution $\Theta : k \cdot v \mapsto k \cdot v^{-1}$ is called a **global Cartan involution of** G ([33] section 3).

The differential $\theta = d\Theta$ of the global Cartan involution is called the **local Cartan involution** on the Lie algebra $\mathfrak{g}_0$, which gives the local Cartan decomposition

$$\mathfrak{g}_0 = \mathfrak{g}_0(\theta, 1) \oplus \mathfrak{g}_0(\theta, -1)$$

where $\mathfrak{g}_0(\theta, \pm 1)$ are the ± 1-eigenspaces of the involution θ. The eigenspace $\mathfrak{g}_0(\theta, +1)$ is exactly the Lie algebra $\mathfrak{k}_0$ of K.

For the -1-eigenspace, a subspace of $\mathfrak{g}_0(\theta, -1)$ is a subalgebra of $\mathfrak{g}_0$ if and only if it is abelian. All maximal abelian subalgebra of $\mathfrak{g}_0(\theta, -1)$ are conjugate under K°.

Let $\mathfrak{a}_0$ be a maximal abelian subalgebra of $\mathfrak{g}_0(\theta, -1)$. The adjoint action of $\mathfrak{a}_0$ on $\mathfrak{g}_0$ gives weight space decomposition. For $\lambda \in \mathfrak{a}_0^*$ (real dual), one has the weight space $\mathfrak{g}_{0\lambda} = \{X \in \mathfrak{g}_0 | [Z, X] = \lambda(Z)X, \forall Z \in \mathfrak{a}_0\}$. The λ that $\mathfrak{g}_{0\lambda} \neq \{0\}$ are called **$\mathfrak{a}_0$-roots** or **restricted roots** on $\mathfrak{g}_0$. Let

$$\begin{aligned} \Sigma(\mathfrak{g}_0, \mathfrak{a}_0) &= \text{the set of } \mathfrak{a}_0\text{-roots on } \mathfrak{g}_0 \\ W(\mathfrak{g}_0, \mathfrak{a}_0) &= N_{K^\circ}(\mathfrak{a}_0)/Z_{K^\circ}(\mathfrak{a}_0) \\ &= \text{the restricted Weyl group} \end{aligned}$$

Then the $\Sigma(\mathfrak{g}_0, \mathfrak{a}_0)$ form a root system in the vector space it spans with $W(\mathfrak{g}_0, \mathfrak{a}_0)$ as its Weyl group.

Restricted Root System vs Relative Root System

As in 2.1.1, by fixing a maximal $\mathbb{R}$-split torus $\mathbf{S}$ of $\mathbf{G}$, one has the relative root system $\Sigma = \Sigma(\mathbf{G}, \mathbf{S})$ and the relative Weyl group $W = W(\mathbf{G}, \mathbf{S})$ determined by $\mathbf{S}$.

As above, by fixing a Cartan involution Θ on G (θ on $\mathfrak{g}_0$), and a maximal abelian subalgebra $\mathfrak{a}_0$ in $\mathfrak{g}_0(\theta, -1)$, one has the restricted root system $\Sigma(\mathfrak{g}_0, \mathfrak{a}_0)$ and the restricted Weyl group $W(\mathfrak{g}_0, \mathfrak{a}_0)$ determined by $\mathfrak{a}_0$.

We can choose the $\mathbf{S}$, Θ *and* $\mathfrak{a}_0$ *to be compatible.* More precisely, let S be the Lie group of the $\mathbf{S}(\mathbb{R})$. The Lie algebra $\mathfrak{s}_0$ of S (which is also the Lie algebra of A) is conjugate to the $\mathfrak{a}_0$ we fixed above. In particular, *we can choose* $\mathbf{S}$ *and* $\mathfrak{a}_0$ *such that the* $\mathfrak{s}_0 = \mathfrak{a}_0$. In this case, the $A_\emptyset = \exp \mathfrak{a}_0$ is exactly the identity component S°.

The restriction of the differential $\alpha \mapsto d\alpha \mapsto d\alpha|_{\mathfrak{a}_0}$ thus gives a bijection between the relative root system and restricted root system:

$$\begin{aligned} \Sigma(\mathbf{G}, \mathbf{S}) &\to \Sigma(\mathfrak{g}_0, \mathfrak{a}_0) \\ \alpha &\mapsto d\alpha|_{\mathfrak{a}_0} \end{aligned} \tag{2.4}$$

By abuse of notation, we use the same α to express both the algebraic root in $\Sigma(\mathbf{G}, \mathbf{S})$ and the corresponding restricted root in $\Sigma(\mathfrak{g}_0, \mathfrak{a}_0)$. The above bijection also induces an isomorphism between the relative and restricted Weyl groups: $W(\mathbf{G}, \mathbf{S}) \simeq W(\mathfrak{g}_0, \mathfrak{a}_0)$.

Real Parabolic Subgroups.

A subgroup of G is called a **(real) parabolic subgroup of** G if it is the real points of a parabolic $\mathbb{R}$-subgroup of $\mathbf{G}$. In particular, we call $P = \mathbf{P}(\mathbb{R})$ a **minimal (real) parabolic subgroup of** G, if the $\mathbf{P}$ is minimal parabolic $\mathbb{R}$-subgroup of $\mathbf{G}$.

As in 2.1.1 we have fixed a minimal parabolic $\mathbb{R}$-subgroup of $\mathbf{G}$ containing $\mathbf{S}$, denoted by $\mathbf{P}_{\emptyset}$, then the $P_{\emptyset} = \mathbf{P}_{\emptyset}(\mathbb{R})$ is a minimal parabolic subgroup of G. Similar to the algebraic parabolic subgroups, we call a (real) parabolic subgroup P of G **standard**, if the corresponding $\mathbf{P}$ is standard, i.e. containing the $\mathbf{P}_{\emptyset}$. Thus by fixing the minimal parabolic $\mathbb{R}$-subgroup $\mathbf{P}_{\emptyset}$, the standard parabolic $\mathbb{R}$-subgroups of $\mathbf{G}$ are in one-to-one correspondence with the standard (real) parabolic subgroups of G, and the correspondence is $\mathbf{P} \mapsto \mathbf{P}(\mathbb{R})$. And the standard (real) parabolic subgroups of G are also parameterized by subsets of Δ.

For a standard parabolic $\mathbb{R}$-subgroup $\mathbf{P} = \mathbf{P}_{\Theta}$, its unipotent radical $\mathbf{N}_{\mathbf{P}} = \mathbf{N}_{\Theta}$ and a Levi $\mathbb{R}$-factor $\mathbf{M}_{\mathbf{P}} = \mathbf{M}_{\Theta}$ are given in 2.1.1. The two factors are defined over $\mathbb{R}$, and the semi-direct product induces the semi-direct product decomposition on $\mathbf{P}(\mathbb{R})$: $\mathbf{P}_{\Theta}(\mathbb{R}) = \mathbf{M}_{\Theta}(\mathbb{R}) \ltimes \mathbf{N}_{\Theta}(\mathbb{R})$. Following the notation-choosing rules in Remark 2.1, we have the Lie group semi-direct product:

$$P_{\Theta} = M_{\Theta} \ltimes N_{\Theta}.$$

And we call this decomposition the **Levi decomposition of** P_{Θ}, and call the M_{Θ} a **Levi factor of** P_{Θ} and N_{Θ} the **unipotent radical of** P_{Θ}.

The Langlands Decomposition of P_{Θ}

The $\mathbb{R}$-torus $\mathbf{S}_{\Theta} = (\bigcap_{\alpha\in\Theta} \mathrm{Ker}\alpha)^{\circ}$ is a subtorus of $\mathbf{S}$. The Lie group identity component $A_{\Theta} = \mathbf{S}_{\Theta}(\mathbb{R})^0$ is a subgroup of $A_{\emptyset} = \mathbf{S}(\mathbb{R})^0$. (Note that $\mathbf{S} = \mathbf{S}_{\emptyset}$). It is connected, simply connected and isomorphic to direct products of $\mathbb{R}_{>0}$. It is abelian and the its exponential map is a smooth diffeomorphism.

The A_{Θ} has a unique complement inside the group M_{Θ}, namely the

$$^{\circ}M_{\Theta} := \bigcap_{\chi\in X(M_{\Theta})} \mathrm{Ker}|\chi|,$$

here $X(M_{\Theta}) = \mathrm{Hom}_{cont}(M_{\Theta}, \mathbb{R}^{\times})$ is the group of continuous homomorphisms from M_{Θ} to $\mathbb{R}^{\times}$. The $^{\circ}M_{\Theta}$ is in general not connected, but is a real reductive Lie group or a group of Harish-Chandra class. If Θ is the empty set, the $^{\circ}M_{\Theta}$ is a compact group.

And the M_{Θ} is a direct product of the two subgroups:

$$M_{\Theta} = A_{\Theta} \times {}^{0}M_{\Theta}.$$

The decomposition

$$P_{\Theta} = {}^{0}M_{\Theta} \times A_{\Theta} \ltimes N_{\Theta}$$

is called the **Langlands decomposition** of the real parabolic subgroup P_Θ.

The A_Θ and ${}^\circ M_\Theta$ are not real point groups of algebraic groups. But we still follow the notation-choosing rules in Remark 2.1, and let $\mathfrak{a}_{\Theta 0} = \mathrm{Lie} A_\Theta$ (real Lie algebra) and $\mathfrak{a}_\Theta$ be its complexification.

2.2 NF-Spaces and DNF-Spaces

We recall some crucial properties of nuclear Fréchet spaces and their dual. For basic terms and results, we follow the book [36]. Most of the materials about NF-spaces and DNF-spaces are from the short survey in the Appendix A of [18].

2.2.1 Basic Notions on Topological Vector Spaces

In this thesis, the term **topological vector space(s)** is abbreviated as **TVS**. *If not otherwise stated, all TVS are assumed to be locally convex and Hausdorff.*

- A TVS is **Fréchet** ([36] p85), if it is locally convex, metrizable and complete.
- A locally convex Hausdorff TVS E is **nuclear**, if for every continuous seminorm p on E, there is a continuous seminorm q such that $p \leq q$ and the canonical map $\hat{E}_q \to \hat{E}_p$ is nuclear ([36] p479 Definition 47.3).

Definition 2.2. A nuclear Fréchet TVS is called an **NF-space** for short, and the strong dual of an NF-space (with the strong topology) is called a **DNF-space**.

The strong dual of a NF-space is also nuclear ([36] p523 Proposition 50.6), but almost never Fréchet.

Homomorphisms and Topological Exact Sequences of TVS

Let E, F be two TVS, and $\phi : E \to F$ be a continuous linear map. As in linear algebra, it factor through the linear map between vector spaces:

$$\phi_0 : E/\mathrm{Ker}(\phi) \to F.$$

which is called the **map associated with** ϕ. With the quotient topology on $E/\mathrm{Ker}(\phi)$, the ϕ_0 is continuous. It has a naive linear inverse $\phi_0^{-1} : \phi(E) \to E/\mathrm{Ker}(\phi)$ since ϕ_0 is a linear isomorphism onto the $\phi(E)$, but this inverse

ϕ_0^{-1} need not to be continuous. The ϕ_0^{-1} is continuous if and only if ϕ_0 is an open map, or equivalently the quotient topology on $\phi(E)$ from ϕ_0 is the same as the induced topology from F.

Definition 2.3 ([36] p35 Definition 4.1)**.** A continuous linear map $\phi : E \to F$ is called a **homomorphism of TVS**, (or **strict morphism**) if its associated map ϕ_0 is open.

Lemma 2.4 ([36] p170 Theorem 17.1)**.** *Let E, F be two metrizable and complete TVS. Then every surjective continuous linear map $\phi : E \to F$ is a homomorphism.*

Lemma 2.5 ([34] p77 Corollary 1)**.** *Let E, F be two Fréchet spaces and $\phi : E \to F$ be an injective continuous linear map. Then ϕ is a homomorphism if and only if its image $\phi(E)$ is closed in F.*

Definition 2.6. Let E_1, E_2, E_3 be three TVS, then the sequence

$$0 \to E_1 \to E_2 \to E_3 \to 0$$

is called an **(short) exact sequence of TVS** if all maps are continuous linear map and it is an exact sequence in the category of vector spaces.

Furthermore, it is called a **(short) topological exact sequence of TVS**, if all maps are homomorphisms of TVS. Equivalently this means $E_1 \to E_2$ and $E_2 \to E_3$ are homomorphisms of TVS, since the other two are automatically homomorphisms.

Lemma 2.7. *Any short exact sequence of Fréchet spaces (in the category of vector spaces) is automatically topological exact, i.e. all maps are homomorphisms.*

Proof. Let E_1, E_2, E_3 be three Fréchet spaces, and let

$$0 \to E_1 \to E_2 \to E_3 \to 0$$

be an exact sequence in the category of vector spaces. Then by Lemma 2.4, the $E_2 \to E_3$ is a homomorphism. Also since $\mathrm{Im}\{E_1 \to E_2\} = \mathrm{Ker}\{E_2 \to E_3\}$ which is a closed subspace of E_2, by Lemma 2.5, the $E_1 \to E_2$ is also a homomorphism. □

2.2.2 Properties of NF-Spaces and DNF-Spaces

The category of NF-spaces (or DNF-spaces) with homomorphisms between them form a good category, which is similar to the category of finite dimensional vector spaces. We are especially interested in the dual exact sequences, and topological tensor products.

Strong Dual and Transpose Maps

Lemma 2.8 ([18] p186 Lemma A.2). *Let* $(C^\bullet, d_\bullet)$ *be a complex of NF-spaces (resp. DNF-spaces), i.e. one has a complex*

$$\dots \xrightarrow{d_{n-1}} C^n \xrightarrow{d_n} C^{n+1} \xrightarrow{d_{n+1}} \dots$$

each space C^n *is a NF-space (resp. DNF-space), and each differential* d_n *is a homomorphism of TVS. Then the strong dual complex*

$$\dots \xrightarrow{d_{n+1}^t} (C^{n+1})' \xrightarrow{d_n^t} (C^n)' \xrightarrow{d'_{n-1}} (C^{n-1})' \to \dots$$

is a complex of DNF-spaces, i.e. all strong dual spaces $(C^n)'$ *are DNF-spaces, and all transpose maps* d_n^t *are homomorphisms of TVS. Moreover we have the isomorphism*

$$H^p(C'^\bullet) = H^{-p}(C^\bullet)'.$$

In partiular, we have

Lemma 2.9. *Let*

$$0 \to E_1 \xrightarrow{\phi} E_2 \xrightarrow{\psi} E_3 \to 0$$

be a topological exact sequence of NF-spaces (resp. DNF-spaces), then its strong dual

$$0 \to E_3' \xrightarrow{\psi^t} E_2' \xrightarrow{\phi^t} E_1' \to 0$$

is a topological exact sequence of DNF-spaces (resp. NF-spaces).

Topological Tensor Products

Let E be a nuclear TVS, and F be an arbitrary locally convex Hausdorff TVS. Then the inductive topology and the projective topology on $E \otimes F$ are the same:

$$E \otimes_\epsilon F = E \otimes_\pi F$$

hence they have the same completions:

$$E \mathbin{\widehat{\otimes}_\epsilon} F = E \mathbin{\widehat{\otimes}_\pi} F.$$

When one of the TVS in the tensor product is nuclear, we will not distinguish the inductive and projective tensor products, and simply denote the tensor product by $E \otimes F$, and its completion by $E \mathbin{\widehat{\otimes}} F$.

Lemma 2.10. *Let E, F be two locally convex Hausdorff TVS.*

- *If E, F are both nuclear, then so is $E \widehat{\otimes} F$. ([36] p514 Proposition 50.1)*
- *If E, F are both NF-spaces, then so is $E \widehat{\otimes} F$. ([26] Ch I, §1, no.3 Prop 5)*
- *If E, F are both DNF-spaces, then so is $E \widehat{\otimes} F$. ([26] Ch I, §1, no.3 Prop 5)([26] Ch I, §1, no.3 Prop 5)*

Lemma 2.11. *Let E, F be two Fréchet spaces. If E is nuclear, then*

- *the canonical map $E' \otimes F' \to (E \widehat{\otimes} F)'$ extends to an isomorphism ([36] p524 Proposition 50.7)*
$$E' \widehat{\otimes} F' \xrightarrow{\sim} (E \widehat{\otimes} F)' \tag{2.5}$$
- *the canonical map $E' \otimes F \to L(E, F)$ extends to an isomorphism ([36] p525 (50.17))*
$$E' \widehat{\otimes} F \xrightarrow{\sim} L(E, F) \tag{2.6}$$

Lemma 2.12 ([18] p187 Lemma A.3)**.** *Let F be an NF-space (resp. DNF-space), let*

$$0 \to E_1 \to E_2 \to E_3 \to 0$$

be a topological exact sequence of NF-spaces (resp. DNF-spaces). Then

$$0 \to E_1 \widehat{\otimes} F \to E_2 \widehat{\otimes} F \to E_3 \widehat{\otimes} F \to 0$$

is a topological exact sequence.

Direct Limits and Tensor Products

Let $\{F_i : i \in I\}$ be an inductive system of nuclear spaces, and let $F = \varinjlim_{i \in I} F_i$ be its inductive limit in the category of locally convex spaces. If I is countable and F is Hausdorff, then F is nuclear by [26] Ch II §2 no.2 Corollary 1. Moreover, if F_i are DNF-spaces, then so is $F = \varinjlim_{i \in I} F_i$.

Lemma 2.13 ([18] p187 Lemma A.4)**.** *Let E be a DNF-space, and $\{F_i : i \in I\}$ be a inductive system of DNF-spaces. Assume I is countable and $F = \varinjlim_{i \in I} F_i$ is Hausdorff. Then*

$$E \widehat{\otimes} F = \varinjlim_{i \in I} (E \widehat{\otimes} F_i). \tag{2.7}$$

2.3 Representations Studied in the Thesis

We introduce the category of representations studied in the thesis, i.e. the Harish-Chandra representations.

Throughout this section, let $G = \mathbf{G}(\mathbb{R})$ be the Lie group of real rational points of a linear algebraic group $\mathbf{G}$ defined over $\mathbb{R}$.

2.3.1 Basic Notions on Representations

In literatures, people add various of restrictions on the representation spaces. For example, unitary representations are on Hilbert spaces. We will study the intertwining operators and irreducibility, which is a very general problem. Thus it is natural to start from spaces with minimal confinement.

Continuous Representations

A TVS is

- **locally convex** ([36] p58 Definition 7.2), if its topology is defined by a family of seminorms.
- **Hausdorff**, if it is separated (T_2) in the ordinary sense , i.e. every two points are separated by two disjoint open subsets.
- **quasi-complete** ([14] III.6 Definition 6), if it is locally convex and every closed and bounded subset is complete.

This is the basic assumptions on TVS. *In this thesis, if not otherwise stated, all TVS are assumed to be locally convex Hausdorff and quasi-complete.*

Definition 2.14. A **continuous representation** (π, V) **of** G (V satisfying the above conditions), is a group representation on the TVS V such that the map

$$G \times V \to V, \quad (g, v) \mapsto \pi(g)v$$

is continuous (under the product topology).

Example 2.15. The left and right regular representation of G on the space $C^\infty(G, \mathbb{C})$ or $C_c^\infty(G, \mathbb{C})$.

Remark 2.16. The early works on Lie group representations usually have physics backgrounds, and the representations under concern are mostly unitary (Hilbert) representations, e.g. the work [5] of Bargmann. A lot of subsequent references define general representations on Hilbert spaces, for example the extensive works by D. Vogan including the standard reference [38], and "textbooks" like [41] and [30].

Harish-Chandra is one of the founders of algebraic study of Lie group representations. His early works in 1950's are on Banach representations, e.g. the "Representations of semisimple Lie groups I-VI". The groups in his early works are connected semisimple Lie groups. The work of representations on normed spaces traces back to [23] and [24]. In [27], Harish-Chandra started to work on representations on locally convex TVS (p5 §2) as in [15], and in [28], he generalized the groups under study to "groups of Harish-Chandra class".

By tracing back the reference chain, the French school seems to be the origin of studying representation theory on general TVS, by regarding Lie groups as locally compact topological groups (e.g.[15], [9] and [12]. The conditions "locally convex, Hausdorff and quasi-complete" on TVS originate from these references. The monograph [42] also adopt the general definition of representations on TVS, but in addition he assumed the representation space to be complete.

Smooth Vectors and K-Finite Vectors

Definition 2.17. Let (π, V) be a continuous representation of G. A vector $v \in V$ is called a **smooth vector**, if the map

$$G \to V, \quad g \mapsto \pi(g)v$$

is a smooth V-valued function on G. We denote the subspace of smooth vectors on V by

$$V^{\mathrm{sm}}.$$

A continuous representation (π, V) is called a **smooth representation** if $V = V^{\mathrm{sm}}$.

Let $K \subset G$ be a maximal compact subgroup, and (π, V) be a continuous representation of G.

Definition 2.18. A vector $v \in V$ is called a **K-finite vector**, if it is contained in a finite dimensional K-stable subspace of V. We denote by

$$V^K$$

the subspace of K-finite vectors on V.

All maximal compact subgroups of G are conjugate, hence the above notion of "K-finiteness" is independent of the choice of K.

Representations of Moderate Growth

We fix an embedding $\mathbf{G} \subset \mathbf{GL}_n$ defined over $\mathbb{R}$. Therefore the real point group $G = \mathbf{G}(\mathbb{R}) \subset \mathbf{GL}(n, \mathbb{R})$ is a group of real matrices, with the action on the vector space $\mathbb{R}^n$.

Definition 2.19 ([39] 1.6)**.** Let the matrix group $\mathbf{GL}(n, \mathbb{R})$ act on $\mathbb{R}^n \oplus \mathbb{R}^n$ by

$$g \cdot (x_1, x_2) := (g \cdot x_1, {}^t g^{-1} \cdot x_2),$$

and let $|g|$ be the operator norm of the above g as an operator on $\mathbb{R}^n \oplus \mathbb{R}^n$. We define the norm $|| \cdot ||$ on $G = \mathbf{G}(\mathbb{R}) \subset \mathbf{GL}(n, \mathbb{R})$ by

$$||g|| := |g|.$$

Definition 2.20 ([17] p391)**.** A continuous representation (π, V) of G is **of moderate growth**, if

(T1) the V is a Fréchet space, and assume its topology is defined by the family $\mathfrak{P}$ of seminorms;

(T2) for each seminorm $\rho \in \mathfrak{P}$, there is another seminorm $\rho' \in \mathfrak{P}$ and a positive integer N such that

$$\rho(\pi(g)v) \leq ||g||^N \rho'(v)$$

for all $g \in G, v \in V$ (the N doesn't depend on g and v).

Remark 2.21. If we choose a different embedding $\mathbf{G} \subset \mathbf{GL}_m$, we obtain another norm on G. The new norm is "equivalent" to the original one, in the sense that they are bounded by powers of each other. Hence the above notion "moderate growth" is independent of the choice of the embedding $\mathbf{G} \subset \mathbf{GL}_n$.

Remark 2.22. The above norm $||\cdot|| : \mathbf{G}(\mathbb{R}) \to \mathbb{R}$ is bounded by an algebraic functions. Hence the above notion of "moderate growth" is equivalent to the notion "croissance modérée" in Definition 1.4.1 on page 272 of [20].

Actually a wide class of representations (including all finite dimensional continuous representations) are of "moderate growth":

Lemma 2.23 ([39] p293 §2.2 Lemma)**.** *Every Banach representation is of moderate growth.*

2.3.2 Harish-Chandra Modules

The Notion of $(\mathfrak{g}, H)$-Module

Let $\mathfrak{g}$ be the complexified Lie algebra of G, and $H \subset G$ be a closed subgroup of G.

Definition 2.24 ([17] p393)**.** A vector space V is called a $(\mathfrak{g}, H)$**-module**, if one has an action of H on V and an $\mathfrak{g}$ on V, satisfying the following conditions:

(C1) Every vector is H-finite and H-continuous, i.e. it is contained in a finite dimensional subspace on which H acts continuously hence smoothly. (In particular, one can differentiate the H-action and obtain an $\mathfrak{h}$-action.)

(C2) The two actions of the complexified Lie algebra $\mathfrak{h}$ of H—one through subalgebra of $\mathfrak{g}$, the other one through differentiation of the H-action, are the same.

(C3) The H and $\mathfrak{g}$-actions are compatible in the following sense:

$$h \cdot (X \cdot v) = \mathrm{Ad}h(X) \cdot (h \cdot v)$$

for all $h \in H, X \in \mathfrak{g}, v \in V$.

If H is compact, then (C1) is equivalent to

(C1') As a H-representation, V is a direct sum of irreducible finite dimensional continuous H-representations.

The Underlying $(\mathfrak{g}, K)$-Module

We are particularly interested in the case when $H = K$ is a maximal compact subgroup. If (π, V) is a continuous representation, then the intersection $V^{\mathrm{sm}} \cap V^K$ of smooth vectors and K-finite vectors is a dense subspace of V, and stable under $\mathfrak{g}$ and K-actions, and this intersection satisfies the above conditions (C1)(C2)(C3).

Definition 2.25 ([17] p393)**.** The subspace $V^{\mathrm{sm}} \cap V^K$ of a continuous representation (π, V) is called the **underlying** $(\mathfrak{g}, K)$**-module of** (π, V). Conversely, let V be a $(\mathfrak{g}, K)$-module. A continuous representation (π_*, V_*) of G is called a G**-extension of** V if $V_*^{\mathrm{sm}} \cap V_*^K$ is isomorphic to V as $\mathfrak{g}$-module and K-representation.

Two representations are called **infinitesimally equivalent**, if their underlying $(\mathfrak{g}, K)$-modules are isomorphic as $(\mathfrak{g}, K)$-modules.

Definition 2.26 ([17] p393)**.** Let V be a $(\mathfrak{g}, K)$-module, it is called a **Harish-Chandra module**, if it satisfies the following conditions:

(H1) Any irreducible K-representation occurs in V with finite multiplicity. (Also called **admissible** in many literatures.)

(H2) V is annihilated by some ideal of finite codimension in $Z(\mathfrak{g})$.

(H3) V is finitely generated over $U(\mathfrak{g})$.

(H4) V has finite length.

(The (H2)(H3)(H4) are equivalent by [17] p408 Lemma 5.3.)

2.3.3 Harish-Chandra Representations

Definition 2.27 ([17] p394)**.** A continuous representation (π, V) of G is called a **Harish-Chandra representation**, if it satisfies the following conditions:

- (π, V) is a smooth representation, i.e. $V^{\mathrm{sm}} = V$.
- (π, V) is of moderate growth, in particular it is a Fréchet representation.
- Its $(\mathfrak{g}, K)$-module, i.e. $V^{\mathrm{sm}} \cap V^K$ is a Harish-Chandra module.

Remark 2.28. In [17], it is shown that every finitely generated Harish-Chandra module has a "globalization", i.e. a Harish-Chandra representation on a nuclear space which has its associated Harish-Chandra module is isomorphic to the given Harish-Chandra module. Therefore, up to infinitesimal equivalence, we will only study nuclear Harish-Chandra representations in this thesis.

We will see the following two families of representations are Harish-Chandra representations with nuclear representation space:

- Finite dimensional continuous representations;
- Smooth parabolic inductions of nuclear Harish-Chandra representations (see the next subsection).

2.3.4 Smooth Inductions

In this subsection, we introduce the main object–the smooth parabolic inductions of Harish-Chandra representations. Let G be a real point group of connected reductive linear algebraic group defined over $\mathbb{R}$, P be a standard real parabolic subgroup of G.

The Space $C^\infty\mathrm{Ind}_P^G\sigma$

Let (σ, V) be a smooth Fréchet representation of the subgroup P, we define the following space:

$$C^\infty\mathrm{Ind}_P^G\sigma := \{f \in C^\infty(G,V)|\ f(pg) = \sigma(p)f(g), \forall p \in P, g \in G\} \tag{2.8}$$

i.e. the space of smooth functions on G taking values in V, and satisfying the "σ-rule". The G acts on this space by the right regular action: $\forall g \in G, f \in C^\infty\mathrm{Ind}_P^G\sigma$

$$R_g(f)(g') = (g \cdot f)(g') := f(g'g), \quad \forall g' \in G. \tag{2.9}$$

The Topology on $C^\infty\mathrm{Ind}_P^G\sigma$

The elements in the real Lie algebra $\mathfrak{g}_0$ act as smooth left invariant vector fields on G. For each $f \in C^\infty(G, V)$ and $X \in \mathfrak{g}_0$, we define the $R_X f$ to be

$$(R_X f)(g) := \lim_{t\to 0} \frac{f(g\exp(tX)) - f(g)}{t} = \frac{d}{dt}|_{t=0} f(g\exp(tX)). \tag{2.10}$$

The $R_X f$ is still in $C^\infty(G, V)$. Moreover, if f is in the subspace $C^\infty\mathrm{Ind}_P^G\sigma$, then $R_X f$ also satisfies the σ-rule, hence $R_X f \in C^\infty\mathrm{Ind}_P^G\sigma$. By extension of scalar, we have the $\mathfrak{g}$-action hence the algebra action of $U(\mathfrak{g})$ on $C^\infty\mathrm{Ind}_P^G\sigma$, and we use the same notation R_X for all $X \in U(\mathfrak{g})$. In other words, the $C^\infty\mathrm{Ind}_P^G\sigma$ is a $\mathfrak{g}$-module.

Suppose the topology on V is given by a family of semi-norms $\{|\cdot|_\rho : \rho \in \mathfrak{P}\}$. For each element $X \in \mathfrak{g}$ and seminorm ρ on V, we define the following seminorm on $C^\infty\mathrm{Ind}_P^G\sigma$:

$$||f||_{X,\rho} := \sup_{k\in K} |R_X f(k)|_\rho.$$

These semi-norms give the $C^\infty\mathrm{Ind}_P^G\sigma$ a locally convex Hausdorff topology.

The Smooth Inductions

Under the above topology, the right regular G-action on $C^\infty(G, V, \sigma)$ is a smooth representation. And we have:

Lemma 2.29 ([17] p402 Proposition 4.1)**.** *If the (σ, V) is of moderate growth, then so is the $C^\infty\mathrm{Ind}_P^G\sigma$ (under right regular G-action). If (σ, V) is a Harish-Chandra representation, then so is $C^\infty\mathrm{Ind}_P^G\sigma$.*

Definition 2.30. The $C^\infty\mathrm{Ind}_P^G\sigma$ with right regular G-action is called the **smooth parabolic induction of (σ, V) from P to G.**

2.3.5 Frobenius Reciprocity

We formulate the Frobenius reciprocity (Lemma 2.31), in terms of the smooth inductions defined in the last subsection. Let G, P be as in the last subsection, and let (σ, V) be a Harish-Chandra representation of P and $C^\infty\mathrm{Ind}_P^G\sigma$ be the smooth parabolic induction.

Let $e \in G$ be the identity and let

$$\begin{aligned}\Omega_e : C^\infty\mathrm{Ind}_P^G\sigma &\to V\\ f &\mapsto f(e)\end{aligned}$$

be the delta function at e (evaluation at e). Then Ω_e is a continuous linear map from $C^\infty\mathrm{Ind}_P^G\sigma$ to V, and it is P-equivariant:

$$\Omega_e(R_p f) = \sigma(p)\Omega_e(f),$$

hence $\Omega_e \in \mathrm{Hom}_P(C^\infty\mathrm{Ind}_P^G\sigma, V)$.

Lemma 2.31 (**Frobenius reciprocity**, [17] Lemma 4.2)**.** *Let $C^\infty\mathrm{Ind}_P^G\sigma$ be the smooth induction of (σ, V) from P to G, let (τ, U) be a smooth representation of G. Then the following map is a linear isomorphism:*

$$\begin{aligned}\mathrm{Hom}_G(U, C^\infty\mathrm{Ind}_P^G\sigma) &\xrightarrow{\sim} \mathrm{Hom}_P(U, V)\\ T &\mapsto \Omega_e \circ T\end{aligned} \tag{2.11}$$

Remark 2.32. It is easy to write down the inverse map of (2.11). For a $\Phi \in \mathrm{Hom}_P(U, V)$, we let $T_\Phi : U \to C^\infty\mathrm{Ind}_P^G\sigma$ be the map defined as follows:

$$T_\Phi(u)(g) := \Phi(g \cdot u), \quad \forall u \in U, g \in G.$$

It is easy to verify that $T_\Phi(u) \in C^\infty\mathrm{Ind}_P^G\sigma$, $T_\Phi \in \mathrm{Hom}_G(U, C^\infty\mathrm{Ind}_P^G\sigma)$ and $\Phi \mapsto T_\Phi$ is the inverse of (2.11).

2.3.6 (Hilbert) Normalized Parabolic Inductions

In most references, people realized the parabolic inductions on a Hilbert space, called normalized parabolic inductions.

Let G, P be as above, and let (τ, V) be a representation of P on a Hilbert space V. Then let H_τ be the space of smooth functions $f \in C^\infty(G, V)$ satisfying

$$f(pg) = \tau(p)\delta_P^{1/2}(p)f(g), \forall p \in P, g \in G,$$

and we define a pre-Hilbert inner product on H_τ by

$$\langle f_1, f_2\rangle := \int_K \langle f_1(k), f_2(k)\rangle_\tau dk$$

where $\langle , \rangle_\tau$ is the Hilbert inner product on the Hilbert space V. Then let $\mathrm{Ind}_P^G\tau$ be the completion of H_τ with respect to the above pre-Hilbert inner product. Then right regular G-action on H_τ extends to the entire $\mathrm{Ind}_P^G\tau$ and makes it a Hilbert representation of G.

Definition 2.33. The $\mathrm{Ind}_P^G\tau$ with the right regular G-action, is called the **normalized (Hilbert) parabolic induction of τ (from P to G)**.

Remark 2.34. If (τ, V) is a unitary representation, the $\mathrm{Ind}_P^G\tau$ is also a unitary representation. This is the reason to add the factor $\delta_P^{1/2}$ in the definition.

There are different ways to define the normalized Hilbert parabolic inductions in various of references, and they are all infinitesimally equivalent to the above definition.

The relation between the normalized Hilbert parabolic induction and smooth parabolic induction is as follows:

Lemma 2.35. *Let τ be a Hilbert representation of P, and let $\sigma = \tau \otimes \delta_P^{1/2}$. Then the $C^\infty\mathrm{Ind}_P^G\sigma$ and $\mathrm{Ind}_P^G\tau$ are infinitesimally equivalent.*

Remark 2.36. The $C^\infty\mathrm{Ind}_P^G\sigma$ and $\mathrm{Ind}_P^G\tau$ are both admissible, hence their G-invariant closed subspaces are in one-to-one correspondence with the $(\mathfrak{g}, K)$-submodules of their (isomorphic) underlying $(\mathfrak{g}, K)$-modules. Therefore, *the $C^\infty\mathrm{Ind}_P^G\sigma$ and $\mathrm{Ind}_P^G\tau$ are irreducible/reducible simultaneously.*

Chapter 3

Preliminary

Summary of This Chapter

This assorted chapter lays the foundations for the following chapters, especially for the Chapter 4.

The first part consists of section 3.1, and 3.2, in which we introduce certain necessary geometric notions, for the definition of Schwartz functions in the next chapter. To define the Schwartz functions, one need a scale to measure if the functions decrease "rapidly". As for the classical Schwartz functions on $\mathbb{R}^n$, the rapidity of decreasing is measured by derivatives with polynomial coefficients. However on general smooth manifolds, there is no intrinsic "algebraic functions". If we look at the special class of manifolds– the Nash manifolds (or nonsingular real algebraic varieties), we do have the intrinsic notion of Nash functions (or polynomial functions), which gives us a scale to define the Schwartz functions on such manifolds.

Section 3.3 and 3.4 could be read independently.

- Section 3.1 provides a minimal set of knowledge of (affine) real algebraic variety. All terms and results are quoted from Chapter 3 of [6].
- In 3.2, we review the notion of affine Nash manifolds, following [1] and [2].
- In 3.3, we recall the double coset structures on the group $G = \mathbf{G}(\mathbb{R})$. Also pay attention to the term "anti-action", since we will mainly work with spaces with right translations.
- Section 3.4 is pure algebraic. We recall some basic notions on torsion submodules, and study the torsion subspace on tensor product modules.

3.1 Real Algebraic Varieties

We recall the notion of real algebraic varieties, following the Chapter 3 of [6]. The notions in this section are from the subject "real algebraic geometry", not from the ordinary algebraic geometry. "Real algebraic varieties" in this thesis have different meaning from "algebraic varieties defined over $\mathbb{R}$".

3.1.1 Algebraic Subsets of $\mathbb{R}^n$

Let $\mathbb{R}^n$ be the n-dimensional real affine space, and let $\mathbb{R}[X]$ (also denoted by $\mathbb{R}[X_1, \dots, X_n]$) be the (real coefficients) polynomial ring of n-variables. We denote a general point in $\mathbb{R}^n$ by x, or the coordinate form $x = (x_1, \dots, x_n)$ if necessary.

- An **algebraic subset of** $\mathbb{R}^n$ ([6] p23 Definition 2.1.1), is a subset of the form
$$Z(S) := \{x \in \mathbb{R}^n : f(x) = 0, \forall f \in S\}$$
where $S \subset \mathbb{R}[X]$ is a subset.

- For a subset $Y \subset \mathbb{R}^n$, we denote by
$$I(Y) := \{f \in \mathbb{R}[X] : f \equiv 0 \text{ on } Y\}$$
the set of polynomials vanishing on entire Y. This is an ideal of $\mathbb{R}[X]$, called the **vanishing ideal of** Y.

- The **Zariski topology on** $\mathbb{R}^n$ is the topology whose closed subsets are algebraic subsets of $\mathbb{R}^n$. On an algebraic subset $V \subset \mathbb{R}^n$, the induced topology from the Zariski topology on $\mathbb{R}^n$ is called the **Zariski topology on** V. The Euclidean topology on $\mathbb{R}^n$ and algebraic subset $V \subset \mathbb{R}^n$ are finer than the Zariski topologies on them. The Zariski topology is not Hausdorff, but is still Noetherian and in particular quasi-compact (every open cover has a finite subcover).

The following fact is one of the main differences between real algebraic subsets and general algebraic sets on arbitrary fields:

Lemma 3.1 ([6] p24 Proposition 2.1.3)**.** *Every real algebraic subset of* $\mathbb{R}^n$ *is of the form* $Z(f)$ *for a single polynomial* f*.*

3.1.2 Affine Real Algebraic Variety

Let $V \subset \mathbb{R}^n$ be an algebraic subset.

- The quotient ring $\mathcal{P}(V) = \mathbb{R}[X_1, \ldots, X_n]/I(V)$ is called the **ring of polynomial functions on** V, and elements in it are called **polynomial functions on** V. ([6] p62)
- Let $U \subset V$ be an Zariski open subset of V. A **regular function on** U, is a real valued function on U, which could be written as a quotient f/g, for some $f, g \in \mathcal{P}(V)$, such that g is nonzero at every point of U. The regular functions on U form a ring denoted by $\mathcal{R}(U)$. ([6] p62 Definition 3.2.1)
- The correspondence $\mathcal{R}_V : U \to \mathcal{R}(U)$ is a sheaf of rings over the Zariski topology on V, called the **sheaf of regular functions**. ([6] p62 Corollary 3.2.4)
- An **affine real algebraic variety** is a pair $(X, \mathcal{R}_X)$, consisting of a topological space X isomorphic to an algebraic subset V with its Zariski topology, and a sheaf of rings $\mathcal{R}_X$ isomorphic to the sheaf of regular functions $\mathcal{R}_V$. ([6] p63 Definition 3.2.9)

Zariski closed subsets of an affine real algebraic variety is still an affine real algebraic variety. By the Lemma 3.1, a Zariski open subset of an affine real algebraic variety is also an affine real algebraic variety:

Lemma 3.2 ([6] p63 Proposition 3.2.10)**.** *Let $(V, \mathcal{R}_V)$ be an affine real algebraic variety. Let $U \subset V$ be a Zariski open subset. Then the $(U, \mathcal{R}_V|_U)$ is an affine real algebraic variety.*

This Lemma tells us any Zariski locally closed subset of an affine real algebraic variety has a natural structure of affine real algebraic variety.

Remark 3.3. We can continue to define general **real algebraic varieties** by gluing affine real algebraic varieties ([6] p64 Definition 3.2.11). But we prefer to stop here to save space, since all real algebraic varieties studied in this thesis are locally closed hence affine real algebraic varieties.

The affine real algebraic varieties form a special class of affine Nash manifolds, which will be introduced in the next section.

3.2 Basic Notions on Nash Manifolds

This section is a short introduction to some basic notions in real algebraic geometry.

In the thesis (especially the next chapter), we will frequently quote results from [1] and [2], which are built on Nash manifolds. Thus to make the thesis self-contained but not too long, we write this short section.

The real algebraic geometry is not a branch of algebraic geometry, since the objects under study are not pure algebraic. However from this section the reader could see that the pattern to define terms in real algebraic geometry is similar to that of algebraic geometry.

3.2.1 Semi-algebraic Sets and Maps

For each positive integer n, let $\mathbb{R}^n$ be the n-dimensional real affine space, and we endow it with the Euclidean topology and canonical smooth structure. Let $x_1, \ldots, x_n$ be the coordinates on $\mathbb{R}^n$, and $\mathbb{R}[x_1, \ldots, x_n]$ be the ring of real polynomials on $\mathbb{R}^n$.

Definition 3.4. A subset $X \subset \mathbb{R}^n$ is called a **semi-algebraic subset of** $\mathbb{R}^n$, if it is a finite union of subsets of the form

$$\{x \in \mathbb{R}^n : f_i(x) > 0, g_j(x) = 0\},$$

for some polynomials $f_i, g_j \in \mathbb{R}[x_1, \ldots, x_n], i = 1, \ldots, r, j = 1, \ldots, s$.

Let $X \subset \mathbb{R}^m, Y \subset \mathbb{R}^n$ be two semi-algebraic subsets. A map $f : X \to Y$ is called a **semi-algebraic map** if its graph $\{(x, f(x)) : x \in X\} \subset \mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$ is a semi-algebraic subset in $\mathbb{R}^{m+n}$.

The *finite* unions, *finite* intersections and complements of semi-algebraic subsets are semi-algebraic. Images of semi-algebraic subsets under semi-algebraic maps are semi-algebraic. The Euclidean closures of semi-algebraic subsets are semi-algebraic.

Example 3.5. We give some examples of semi-algebraic subsets.

1. In $\mathbb{R}$, for all $-\infty \leq a < b \leq \infty$, the intervals $(a, b), [a, b], [a, b), (a, b]$ are semi-algebraic subsets in $\mathbb{R}$. Actually all semi-algebraic subsets of $\mathbb{R}$ are (finite) unions of such intervals.

2. An algebraic subset of $\mathbb{R}^n$ (see 3.1) is semi-algebraic. For example the "cross" in real plane:

$$\{(x, y) : x^2 - y^2 = 0\} \subset \mathbb{R}^2.$$

3. Here are some more examples in $\mathbb{R}^2$: the quadrant $\{(x, y) : x > 0, y > 0\}$; the unit cube $\{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$; a branch of hyperbola $\{(x, 1/x) : x > 0\}$.

4. The (algebraic) Lie group $\mathbf{GL}(n, \mathbb{R}) = \{g \in \mathbf{M}_n(\mathbb{R}) : \det g \neq 0\}$, and its connected component $\mathbf{GL}(n, \mathbb{R})^+ = \{g \in \mathbf{M}_n(\mathbb{R}) : \det g > 0\}$. In general, for an algebraic group $\mathbf{G}$ defined over $\mathbb{R}$, its real point group $\mathbf{G}(\mathbb{R})$ or a finite union of its Lie group components, are semi-algebraic.

Example 3.6. We give some examples of semi-algebraic maps.

1. Polynomials in $\mathbb{R}[x_1, \dots, x_n]$ are semi-algebraic maps from $\mathbb{R}^n$ to $\mathbb{R}$.

2. The absolute value function $|\cdot| : \mathbb{R} \to \mathbb{R}, x \mapsto |x|$ is semi-algebraic. The rational power function $\mathbb{R} \to \mathbb{R}, x \mapsto x^{m/n}$ is semi-algebraic (when it is well-defined).

3. Let $f_1, \dots, f_k$ be semi-algebraic maps from $\mathbb{R}^n$ to $\mathbb{R}$, then the functions $\max\{f_1, \dots, f_k\}$ and $\min\{f_1, \dots, f_k\}$ are semi-algebraic maps.

3.2.2 Nash Submanifolds of $\mathbb{R}^n$

Definition 3.7 ([2] Definition 2.3.3 and 2.3.5)**.** A semi-algebraic subset $X \subset \mathbb{R}^n$ is called a **Nash submanifold of** $\mathbb{R}^n$, if it is also a smooth regular submanifold of $\mathbb{R}^n$.

Let $X \subset \mathbb{R}^m$ be a Nash submanifold of $\mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be a Nash submanifold of $\mathbb{R}^n$. A map $f : X \to Y$ is called a **Nash map** if it is smooth and semi-algebraic.

Example 3.8. The semi-algebraic subset $\{(x, y) : x^2 - y^2 = 0\}$ is not a regular submanifold of $\mathbb{R}^2$ (the origin is a singular point), hence it is not a Nash submanifold of $\mathbb{R}^2$.

Another example is the closed quadrant $\{(x, y) : x \geq 0, y \geq 0\}$. It is a semi-algebraic subset of $\mathbb{R}^2$, but not a regular submanifold (the origin is a corner), hence not a Nash submanifold of $\mathbb{R}^2$.

Remark 3.9 ([2] Example 2.3.9)**.** Let $M \subset \mathbb{R}^m$ be a Nash submanifold of $\mathbb{R}^m$, then it has the induced topology from the Euclidean topology on $\mathbb{R}^m$. Let

$$\mathfrak{S}_M := \{U \subset M : U \text{ is open in } M \text{ and semi-algebraic in } \mathbb{R}^m\}$$

be the collection of subsets of M that is (Euclidean) open in M and semi-algebraic in $\mathbb{R}^m$. Then $\mathfrak{S}_M$ is a sub-family of the induced topology on M, and we have

- The $\emptyset$ and M are in $\mathfrak{S}_M$.
- $\mathfrak{S}_M$ is closed under *finite* union.
- $\mathfrak{S}_M$ is closed under *finite* intersection.

Remark 3.10 ([2] Definition 2.3.1 and Example 2.3.9)**.** Still let $M \subset \mathbb{R}^m$ be a Nash submaniofld of $\mathbb{R}^m$, and let $\mathfrak{S}_M$ be the collection in the above remark. Let $U \in \mathfrak{S}_M$, then U is open in M and semi-algebraic in $\mathbb{R}^m$, hence it is a regular submanifold of $\mathbb{R}^m$ and thus also a Nash submanifold of $\mathbb{R}^m$. A real valued function $f : U \to \mathbb{R}$ is called a **Nash function on** U, if it is a Nash map in the sense of Definition 3.7. Let

$$\mathcal{N}_M(U) = \{\text{Nash functions on } U\}$$

be the set of Nash functions on U. Then it is a $\mathbb{R}$-algebra. Conventionally we let $\mathcal{N}_M(\emptyset) = \{0\}$. For two subsets $U_1, U_2 \in \mathfrak{S}_M$ such that $U_1 \subset U_2$, one can restrict a Nash function from U_2 to U_1 and the restricted function is also a Nash function on U_1.

3.2.3 Restricted Topology, Sheaf and Cosheaf

We recall the notions of restricted topology, sheaves and cosheaves on them. These notions are introduced in [1] §3.2, appendix A.4 and [2] §2.2.

Definition 3.11 ([2] Definition 2.2.1)**.** Let M be a set. A family $\mathfrak{S}$ of subsets of M is called a **restricted topology on** M, if $\mathfrak{S}$ contains the empty set and M, and is closed under *finite* unions and *finite* intersections. A subset in $\mathfrak{S}$ is called a **restricted open** subset of M, and the complements of restricted open subsets are called **restricted closed subsets of** M.

A **restricted topological space** is a pair $(M, \mathfrak{S})$ consisting of a set M and a restricted topology $\mathfrak{S}$ on it. By abuse of terms, we also call M a restricted topological space if there is no ambiguity on the restricted topology.

A map between two restricted topological spaces is called **restricted continuous** if every restricted open subset has restricted open preimage.

A restricted topology is not a topology, since one cannot take infinite unions on it.

Example 3.12. Let $M \subset \mathbb{R}^m$ be a Nash submanifold of $\mathbb{R}^m$. Then the family $\mathfrak{S}_M$ in Remark 3.9 consisting of semi-algebraic and Euclidean open subsets of M, is a restricted topology.

Let $(M, \mathfrak{S})$ be a restricted topological space. The $\mathfrak{S}$ is a partially ordered set hence is also a category, one can define the sheaves and cosheaves on restricted topologies as for sheaves/cosheaves on ordinary topological spaces. The definition is very similar to that of the ordinary sheaves/cosheaves, except that one can only take *finite* (restricted) open covers.

Definition 3.13 ([2] Definition 2.2.4 and 2.2.5)**.** Let $\mathcal{C}$ be an abelian category (e.g. the category of abelian groups/vector spaces/TVS). A **presheaf on M (with value in $\mathcal{C}$)** is a contravariant functor from $\mathfrak{S}$ to $\mathcal{C}$.

More precisely, let $\mathcal{F}$ be a presheaf on M, then

- For each $U \in \mathfrak{S}$, one has an object $\mathcal{F}(U) \in \mathcal{C}$, and in particular $\mathcal{F}(\emptyset) = \{0\}$.
- For each pair of $U_1, U_2 \in \mathfrak{S}$ with $U_1 \subset U_2$, one has a **restriction map** $\mathrm{res}_{U_1}^{U_2} : \mathcal{F}(U_2) \to \mathcal{F}(U_1)$ which is a morphism in $\mathcal{C}$. And for three restricted open subsets $U_1, U_2, U_3 \in \mathfrak{S}$ with $U_1 \subset U_2 \subset U_3$ one has
$$\mathrm{res}_{U_1}^{U_3} = \mathrm{res}_{U_1}^{U_2} \circ \mathrm{res}_{U_2}^{U_3}.$$

An element $f \in \mathcal{F}(U)$ is called a **section on U**. A **morphism** between two presheaves on M is a natural transformation between these two contravariant functors.

A presheaf $\mathcal{F}$ on M (with value in $\mathcal{C}$) is called a **sheaf**, if it satisfies the following local conditions:

- Let $U \in \mathfrak{S}$ be a restricted open subset of M, and $\{U_i\}_{i=1}^k$ be a *finite* restricted open cover of U, i.e. $U_1, \ldots, U_k \in \mathfrak{S}$ and $U = \cup_{i=1}^k U_i$. Then a $f \in \mathcal{F}(U)$ is zero if and only if $\mathrm{res}_{U_i}^U(f) = 0$ for all $i = 1, \ldots, k$.
- Let $U \in \mathfrak{S}$ and $\{U_i\}_{i=1}^k$ be a *finite* restricted open cover of U. Let $f_i \in \mathcal{F}(U_i)$ be a section on U_i for each i. If $\mathrm{res}_{U_i \cap U_j}^{U_i}(f_i) = \mathrm{res}_{U_i \cap U_j}^{U_j}(f_j)$ for all $1 \leq i, j \leq k$, then there exists a $f \in \mathcal{F}(U)$ such that $f_i = \mathrm{res}_{U_i}^U(f)$.

Definition 3.14 ([2] Definition 2.2.6 and 2.2.7)**.** Let $\mathcal{C}$ be an abelian category (e.g. the category of abelian groups/vector spaces/TVS). A **precosheaf on M with value in $\mathcal{C}$** is a covariant functor from $\mathfrak{S}$ to $\mathcal{C}$.

More precisely, let $\mathcal{F}$ be a pre-cosheaf on M with value in $\mathcal{C}$, then

- For each $U \in \mathfrak{S}$, one has a object $\mathcal{F}(U) \in \mathcal{C}$. In particular $\mathcal{F}(\emptyset) = \{0\}$.

- For each pair of restricted open subsets $U_1, U_2 \in \mathfrak{S}$ with $U_1 \subset U_2$, one has an **extension map** $\mathrm{ex}_{U_1}^{U_2} : \mathcal{F}(U_1) \to \mathcal{F}(U_2)$ which is a morphism in the category $\mathcal{C}$. And for three restricted open subsets $U_1 \subset U_2 \subset U_3$, one has
$$\mathrm{ex}_{U_1}^{U_3} = \mathrm{ex}_{U_2}^{U_3} \circ \mathrm{ex}_{U_1}^{U_2}.$$

An element $f \in \mathcal{F}(U)$ is called a **section on** U. A **morphism** between two presheaves on M is a natural transformation between these two covariant functors.

A pre-cosheaf $\mathcal{F}$ on M (with value in $\mathcal{C}$) is called a **cosheaf**, if it satisfies the following local conditions:

- Let U be a restricted open subset of M and $\{U_i\}_{i=1}^k$ be a *finite* restricted open cover of U. Then for each $f \in \mathcal{F}(U)$, there exists a $f_i \in \mathcal{F}(U_i)$ such that $f = \sum_{i=1}^k \mathrm{ex}_{U_i}^U(f_i)$. In other words, every section on U is a sum of extensions of local sections.

- Let U be a restricted open subset and $\{U_i\}_{i=1}^k$ be a *finite* restricted open cover of U. For each $i = 1, \ldots, k$, let $f_i \in \mathcal{F}(U_i)$ be a local section. Then the $\sum_{i=1}^k \mathrm{ex}_{U_i}^U(f_i) = 0$ in $\mathcal{F}(U)$ if and only if there are $f_{ij} \in \mathcal{F}(U_i \cap U_j)$ for each $i, j \in \{1, \ldots, k\}$ such that $f_i = \sum_{j=1}^k [\mathrm{ex}_{U_i \cap U_j}^{U_i}(f_{ij}) - \mathrm{ex}_{U_j \cap U_i}^{U_i}(f_{ji})]$.

Example 3.15. Let $M \subset \mathbb{R}^m$ be a Nash submanifold and let $\mathfrak{S}_M$ be the restricted topology defined in Remark 3.9. For each $U \in \mathfrak{S}_M$, let $\mathcal{N}_M(U)$ be the ring of Nash functions defined in Remark 3.10, then it is easy to check the $\mathcal{N}_M : U \mapsto \mathcal{N}_M(U)$ is a sheaf on the restricted topology $\mathfrak{S}_M$.

Definition 3.16 ([2] Definition 2.2.9)**.** Let $(M, \mathfrak{S})$ be a restricted topological space and let $\mathcal{F}$ be a sheaf over $\mathfrak{S}$. Let $f \in \mathcal{F}(M)$ be a global section, and $Z \subset M$ be a restricted closed subset. Then f **is supported in** Z if $\mathrm{res}_{M-Z}^M(f) = 0$, i.e. f restricts to zero on the complement of Z.

Note that on restricted topology, one cannot take infinite intersections of restricted closed subsets. Hence there is no "closure" in restricted topology, and in general one cannot define "the support" of a section by taking closures as in ordinary topological spaces.

3.2.4 Affine Nash Manifolds and Abstract Nash Manifolds

Similar to the definition of affine algebraic varieties and general varieties, we need an intrinsic way to define Nash manifolds without embedding them into $\mathbb{R}^m$. We first have the following definition which is similar to the ringed spaces in algebraic geometry.

Definition 3.17 ([2] Definition 2.3.8)**.** An $\mathbb{R}$**-space** is a pair $(M, \mathcal{O}_M)$ consists of a restricted topological space M (with a restricted topology $\mathfrak{S}_M$) and a sheaf of $\mathbb{R}$-algebras over the restricted topology $\mathfrak{S}_M$, and the sheaf $\mathcal{O}_M$ is a subsheaf of the sheaf $\mathbb{R}_M$ of real valued functions on M.

A **morphism** between two $\mathbb{R}$-spaces $(M, \mathcal{O}_M)$ and $(N, \mathcal{O}_N)$ is a pair $(f, f^\sharp)$ consists of a restricted continuous map $f : M \to N$ and a sheaf morphism $f^\sharp : f^*\mathcal{R}_N \to \mathcal{R}_M$ of restricted sheaves, which maps the subsheaf $f^*\mathcal{O}_N$ to $\mathcal{O}_M$.

Example 3.18. Let $M \subset \mathbb{R}^m$ be a Nash submanifold of $\mathbb{R}^m$, and $\mathfrak{S}_M$ be the restricted topology defined in Remark 3.9, and $\mathcal{N}_M$ be the sheaf of Nash functions defined in Remark 3.10. Then the $(M, \mathcal{N}_M)$ is an $\mathbb{R}$-space.

Definition 3.19 ([2] Definition 2.3.10)**.** An **affine Nash manifold** is an $\mathbb{R}$-space which is isomorphic (as $\mathbb{R}$-space) to the $\mathbb{R}$-space of a *closed* Nash submanifold of some $\mathbb{R}^m$.

Example 3.20. Any real *nonsingular* $\mathbb{R}$-affine algebraic variety has a natural structure of affine Nash manifold.

Definition 3.21 ([2] Definition 2.3.16)**.** A **Nash manifold** is an $\mathbb{R}$-space $(M, \mathcal{O}_M)$ which has a finite cover $\{M_i\}_{i=1}^k$ by restricted open subsets of M, such that each $\mathbb{R}$-space $(M_i, \mathcal{O}_M|_{M_i})$ is isomorphic (as $\mathbb{R}$-space) to an affine Nash manifold. A **morphism** between two Nash manifolds are just morphisms of $\mathbb{R}$-spaces between them.

All Nash manifolds studied in the thesis are affine Nash manifolds, thus we stop the introduction here.

3.3 Bruhat Decompositions

3.3.1 Some Terms on Group Actions

Let G be an abstract group, E be a set. Let $\mathrm{Aut}E$ be the group of permutations on E.

- A G**-action on** E, is a group homomorphism $\sigma : G \to \mathrm{Aut}E$, i.e. $\forall g_1, g_2 \in G$, $\sigma(g_1 g_2) = \sigma(g_1) \circ \sigma(g_2)$.
- A G**-(anti)action on** E, is a group anti-homomorphism $\sigma : G \to \mathrm{Aut}E$, i.e. $\forall g_1, g_2 \in G$, $\sigma(g_1 g_2) = \sigma(g_2) \circ \sigma(g_1)$.

Let G be an abstract group, and H be a subgroup of G.

- The **left H-translation on** G is the H-action $h \mapsto l_h$, given by

$$l_h : g \mapsto hg, \quad \forall g \in G.$$

  The orbit space of the left H-translation on G is denoted by $H\backslash G$, it is exactly the set of right H-costs in G.

- The **right H-translation on** G is the H-action $h \mapsto r_h$, given by

$$r_h : g \mapsto gh, \quad \forall g \in G.$$

  The orbit space of the right H-translation on G is denoted by G/H, it is exactly the set of left H-cosets in G.

Let H_1, H_2 be two subgroup of G.

- The **left H_1-translation on** G/H_2, is given by

$$gH_2 \mapsto hgH_2, \quad \forall h \in H_1, gH_2 \in G/H_2.$$

  The orbit space of this action is denoted by $H_1\backslash(G/H_2)$.

- The **right H_2-translation on** $H_1\backslash G$, is given by

$$H_1g \mapsto H_1gh, \quad \forall h \in H_2, H_1g \in H_1\backslash G.$$

  The orbit space of this anti-action is denoted by $(H_1\backslash G)/H_2$.

- The (H_1, H_2)**-conjugation on** G, is the $H_1 \times H_2$-action on G given by

$$g \mapsto h_1gh_2^{-1}, \quad \forall g \in G, h_1 \in H_1, h_2 \in H_2.$$

  The orbit space of this conjugation on G is denoted by $H_1\backslash G/H_2$, it is exactly the space of (H_1, H_2)-double cosets in G.

The following easy fact is well-known:

Lemma 3.22. *The following three orbit spaces are in one-to-one correspondence:*

- *The orbit space $H_1\backslash(G/H_2)$ of left H_1-translation on G/H_2.*
- *The orbit space $(H_1\backslash G)/H_2$ of right H_2-translation on $H_1\backslash G$.*
- *The orbit space $H_1\backslash G/H_2$ of the (H_1, H_2)-conjugation on G.*

3.3.2 Bruhat Decomposition

Let

$$\begin{aligned}
\mathbf{G} &= \text{a connected reductive linear algebraic group defined over } \mathbb{R} \\
\mathbf{S} &= \text{a fixed maximal } \mathbb{R}\text{-split torus of } \mathbf{G} \\
\Sigma &= \text{the relative root system determined by } \mathbf{S} \\
W &= \text{the relative Weyl group of the root system } \Sigma \\
\mathbf{P}_\emptyset &= \text{a fixed minimal parabolic } \mathbb{R}\text{-subgroup containing } \mathbf{S} \\
\Delta &= \text{the base of } \Sigma \text{ determined by } \mathbf{P}_\emptyset \\
S &= \text{the set of simple reflections determined by } \Delta \\
\Sigma^+ &= \text{the positive system spanned by } \Delta
\end{aligned}$$

Let Θ, Ω be two subsets of Δ, and $\mathbf{P}_\Theta, \mathbf{P}_\Omega$ be the two standard parabolic $\mathbb{R}$-subgroups corresponding to them, and W_Θ, W_Ω be the two parabolic subgroup of W generated by simple reflections in Θ, Ω respectively. For each $w \in W$, one can choose and fix a representative in $\mathbf{N_G}(\mathbf{S})(\mathbb{R})$. *By abuse of notation, we denote this fixed representative by the same notation w.*

The Bruhat Decomposition

The $(\mathbf{G}(\mathbb{R}), \mathbf{P}_\emptyset(\mathbb{R}), \mathbf{N_G}(\mathbf{S})(\mathbb{R}), S)$ is a Tits system, with real point groups of parabolic $\mathbb{R}$-subgroups as its parabolic subgroups. By the Remark on page 22 of [13], we have

Lemma 3.23. *The $(\mathbf{P}_\Theta(\mathbb{R}), \mathbf{P}_\Omega(\mathbb{R}))$-double cosets on $\mathbf{G}(\mathbb{R})$ are in one-to-one correspondence with the double quotient $W_\Theta \backslash W / W_\Omega$. Explicitly, the correspondence is given by*

$$\begin{aligned}
\mathbf{P}_\Theta(\mathbb{R}) \backslash \mathbf{G}(\mathbb{R}) / \mathbf{P}_\Omega(\mathbb{R}) &\leftrightarrow W_\Theta \backslash W / W_\Omega \\
\mathbf{P}_\Theta(\mathbb{R}) w \mathbf{P}_\Omega(\mathbb{R}) &\leftrightarrow W_\Theta w W_\Omega
\end{aligned}$$

(Note that we have chosen and fixed a representative for each $w \in W$ from $\mathbf{N_G}(\mathbf{S})(\mathbb{R}) \subset \mathbf{G}(\mathbb{R})$, and by abuse of notation we denote this representative in $\mathbf{G}(\mathbb{R})$ by the same w.)

The Double Quotient $W_\Theta \backslash W / W_\Omega$

Among the elements in a double coset, one can choose a representative with minimal length:

Lemma 3.24 (Proposition 1.1.13 in [16])**.** *In each (W_Θ, W_Ω)-double coset of W, there is a unique element w characterized by the following equivalent conditions:*

- *w has minimal length in the double coset $W_\Theta w W_\Omega$.*
- *w has the minimal length in $W_\Theta w$ and wW_Ω.*
- *$w^{-1}\Theta \subset \Sigma^+$ and $w\Omega \subset \Sigma^+$.*

We call the element w satisfying the above conditions the ***minimal representative of the double coset*** *$W_\Theta w W_\Omega$. The set of minimal representatives in $W_\Theta \backslash W / W_\Omega$ is denoted by*

$$[W_\Theta \backslash W / W_\Omega] = \{w \in W : w^{-1}\Theta \subset \Sigma^+, w\Omega \subset \Sigma^+\}.$$

Remark 3.25. In general, for a Coexeter group W and two standard parabolic subgroup W_Θ, W_Ω. There is a unique minimal element and a unique maximal (under the Bruhat order of W) element in each double coset. The minimal element is exactly the element with minimal length as in the above Lemma. (See [19] Theorem 1.2)

There are two partial orders on the set $W_\Theta \backslash W / W_\Omega$ of double cosets: one by the Bruhat order on their maximal elements, and another one by the Bruhat order on their minimal elements. It is shown in [29] Theorem 1, that these two orders on $W_\Theta \backslash W / W_\Omega$ coincide.

Double Cosets on G

Let G be the Lie group $\mathbf{G}(\mathbb{R})$, and similarly $P_\Theta = \mathbf{P}_\Theta(\mathbb{R}), P_\Omega = \mathbf{P}_\Omega(\mathbb{R})$ be the corresponding Lie groups. The $\mathbf{P}_\Theta(\mathbb{R}) w \mathbf{P}_\Omega(\mathbb{R})$ is the same double coset as $P_\Theta w P_\Omega$ in G. We can summarize the above discussion as

Lemma 3.26. *The following four sets are in one-to-one correspondence*

- *The set of $(\mathbf{P}_\Theta(\mathbb{R}), \mathbf{P}_\Omega(\mathbb{R}))$-double cosets on $\mathbf{G}(\mathbb{R})$;*
- *The set of (P_Θ, P_Ω)-double coset on G;*
- *The double quotient $W_\Theta \backslash W / W_\Omega$;*
- *The set of minimal representatives $[W_\Theta \backslash W / W_\Omega]$.*

3.3.3 Closure Order on Double Cosets

Let

$$\mathbf{G} = \text{an algebraic group defined over } \mathbb{R}$$
$$\mathbf{P}, \mathbf{Q} = \text{two closed } \mathbb{R}\text{-subgroups of } \mathbf{G}$$

In the following part of this subsection, we assume $\mathbf{G}$ *has finitely many* $(\mathbf{P}, \mathbf{Q})$*-double cosets. In this thesis, we will only consider the case when* $\mathbf{P}, \mathbf{Q}$ *are* $\mathbb{R}$*-parabolic subgroups of a reductive linear algebraic group* $\mathbf{G}$ *defined over* $\mathbb{R}$*, hence it is safe to make this assumption.*

The left $\mathbf{P}$-translation and right $\mathbf{Q}$-translation on $\mathbf{G}$ are algebraic actions/antiactions defined over $\mathbb{R}$, so is the $(\mathbf{P}, \mathbf{Q})$-conjugation on $\mathbf{G}$. The $(\mathbf{P}, \mathbf{Q})$-double cosets on $\mathbf{G}$ are exactly the orbits under the such conjugations. By the Proposition in [10] (p53 §1.8), the double cosets are smooth $\mathbb{R}$-subvarieties of $\mathbf{G}$, and they are open in their closures. The boundary of each double coset is a disjoint union of double cosets of strictly lower dimension.

Since all the groups and actions are defined over $\mathbb{R}$, one has the similar results on the real points. Let G, P, Q be the Lie groups of real points of $\mathbf{G}, \mathbf{P}, \mathbf{Q}$ respectively. The real points of a $(\mathbf{P}, \mathbf{Q})$-double coset is exactly the corresponding (P, Q)-double coset on G. Under the Zariski topology on G, a (P, Q)-double coset on G is also a real smooth subvariety (in the classical sense), open in its Zariski closure, with its boundary a union of strictly lower dimensions. Hence if two double cosets have the same Zariski closure, then they are equal.

The Closure Order

Suppose the (P, Q)-double cosets in G are parameterized by a finite set I. Let G_i be the double coset corresponding to $i \in I$. Then we have the disjoint union

$$G = \coprod_{i \in I} G_i.$$

We define the relation $\leq$ on the set of double cosets by

$$G_i \leq G_j \text{ if and only if } G_i \subset \overline{G_j}$$

for two $i, j \in I$. Here $\overline{G_j}$ means the Zariski closure of G_j. The bijection $I \leftrightarrow \{G_i : i \in I\}$ thus induces a relation on I.

Lemma 3.27. *The relation "$\leq$" is a partial order on $\{G_i : i \in I\}$ (and also on I).*

Proof. First obviously $G_i \subset \overline{G_i}$, hence $i \leq i$. Second if $G_i \leq G_j$ and $G_j \leq G_i$, then $\overline{G_i} = \overline{G_j}$, hence $G_i = G_j$ and $i = j$. Third if $G_i \leq G_j \leq G_k$, for some $i, j, k \in I$, then $G_i \subset \overline{G_j} \subset \overline{G_k}$, hence $G_i \leq G_k$. $\square$

Definition 3.28 (Closure Order)**.** We call the above partial order $\leq$ the **closure order** on the set of (P, Q)-double cosets. We denote by $G_i < G_j$ (or $i < j$) if $G_i \leq G_j$ but $G_i \neq G_j$ ($i \leq j, i \neq j$).

Open Unions of Double Cosets

For an $i \in I$, let

$$G_{\geq i} := \coprod_{i \leq j} G_j \tag{3.1}$$

$$G_{>i} := \coprod_{i<j} G_j = G_{\geq i} - G_i$$

Lemma 3.29. *The $G_{\geq i}$ and $G_{>i}$ are open in G under the Zariski topology. The G_i is closed in $G_{\geq i}$.*

Proof. (1) We show $G_{\geq i}$ is open by showing the following claim:

Claim: For $i \in I$, one has

$$G_{\geq i} = (\cup_{k \not\geq i} \overline{G_k})^c.$$

("$\supset$") First note the complement

$$G_{\geq i}^c = \cup_{k \not\geq i} G_k \subset \cup_{k \not\geq i} \overline{G_k},$$

hence

$$G_{\geq i} \supset (\cup_{k \not\geq i} \overline{G_k})^c.$$

("$\subset$") Second if $k \not\geq i$, then $\overline{G_k} \cap G_{\geq i} = \emptyset$. Otherwise, we let $G_j \subset G_{\geq i} \cap \overline{G_k}$, then $k \geq j \geq i$ hence $k \geq i$ a contradiction. Hence $\overline{G_k}^c \supset G_{\geq i}$. By taking the intersection over all $k \not\geq i$, one has

$$G_{\geq i} \subset \cap_{k \not\geq i} (\overline{G_k})^c = (\cup_{k \not\geq i} \overline{G_k})^c.$$

By the claim, the $G_{\geq i}$ is a complement of a union of closed subsets, hence is open.

(2) We show G_i is closed in $G_{\geq i}$, thus its complement $G_{>i}$ in $G_{\geq i}$ is open in $G_{\geq i}$, and also open in G. We show the following claim:

Claim: For $i \in I$, one has

$$\overline{G_i} \cap G_{\geq i} = G_i \cap G_{\geq i} = G_i.$$

The second equality is trivial. The $\overline{G_i} \cap G_{\geq i} \supset G_i \cap G_{\geq i}$ is also trivial. We just need to show $\overline{G_i} \cap G_{\geq i} \subset G_i \cap G_{\geq i}$. Let $G_j \subset \overline{G_i} \cap G_{\geq i}$, then $j \geq i$ and $i \geq j$, hence $i = j$. Therefore the only double coset in $\overline{G_i} \cap G_{\geq i}$ is G_i itself, hence

$$\overline{G_i} \cap G_{\geq i} = G_i = G_i \cap G_{\geq i}.$$

□

Some Notations

Let $\mathbf{P} = \mathbf{P}_\Theta$, and $\mathbf{P}_\Omega$ be two standard parabolic $\mathbb{R}$-subgroups of $\mathbf{G}$. We are particularly interested in the case when

$$\Omega \subset \Theta.$$

Let $P = P_\Theta, P_\Omega$ be the corresponding groups of real points. In this case, the P_Ω is a subgroup of P_Θ. The (P_Θ, P_Ω)-double cosets on G are parameterized by the minimal representative set $[W_\Theta \backslash W / W_\Omega]$, and ordered by the closure order under Zariski topology.

For each $w \in [W_\Theta \backslash W / W_\Omega]$, we denote by

$$\begin{aligned}
G_w^\Omega &= \text{the double coset } PwP_\Omega \\
G_{\geq w}^\Omega &= \text{the Zariski open union of } G_x^\Omega \\
&\quad\ \text{such that } \overline{G_x^\Omega} \supset G_w^\Omega \\
&= \coprod_{x \geq w} G_x^\Omega \\
G_{>w}^\Omega &= \text{the Zariski open union of } G_x^\Omega \\
&\quad\ \text{such that } \overline{G_x^\Omega} \supset G_w^\Omega \text{ and } G_x^\Omega \neq G_w^\Omega \\
&= \coprod_{x \geq w, x \neq w} G_x^\Omega
\end{aligned}$$

For the particular case when $\Omega = \emptyset$ (empty set), we omit the superscript, and use the following simplified notations:

$$G_w = G_w^{\emptyset} = PwP_{\emptyset}$$
$$G_{\geq w} = G_{\geq w}^{\emptyset}$$
$$G_{>w} = G_{>w}^{\emptyset}$$

for all $w \in [W_\Theta \backslash W / W_\emptyset] = [W_\Theta \backslash W]$.

3.4 Algebraic Preliminary

This is a pure algebraic section. We study the torsion submodules on tensor product modules over Lie algebras. The main result is Lemma 3.52.

In this section, let $\mathfrak{h}$ be a complex Lie algebra and $U(\mathfrak{h})$ be its enveloping algebra. Let $(\mathfrak{h}^k)$ be the two-sided ideal of $U(\mathfrak{h})$ generated by k-products of elements in $\mathfrak{h}$.

- In 3.4.1, for a left $\mathfrak{h}$-module $\mathcal{M}$, we recall the definition of the left annihilators $\mathcal{M}^{[\mathfrak{h}^k]}$ of the ideal $(\mathfrak{h}^k)$ for each k, and the torsion submodule $\mathcal{M}^{[\mathfrak{h}^\bullet]}$. The annihilator sequence $\{\mathcal{M}^{[\mathfrak{h}^k]} : k \geq 0\}$ has no gap and is a strictly ascending sequence (see Lemma 3.38).

- In 3.4.2, we define the similar notions as in 3.4.1, for right $\mathfrak{h}$-modules.

- In 3.4.3, we show a product formula (3.2) on the tensor product module $\mathcal{M}_1 \otimes \mathcal{M}_2$ of two left $\mathfrak{h}$-modules $\mathcal{M}_1, \mathcal{M}_2$.

- In 3.4.4, for two left $\mathfrak{h}$-modules $\mathcal{M}_1, \mathcal{M}_2$, we show the tensor product of their torsion submodules is included in the torsion submodule of their tensor product (Lemma 3.43):

$$\mathcal{M}_1^{[\mathfrak{h}^\bullet]} \otimes \mathcal{M}_2^{[\mathfrak{h}^\bullet]} \subset (\mathcal{M}_1 \otimes \mathcal{M}_2)^{[\mathfrak{h}^\bullet]}.$$

  In general, the reversed inclusion is not true, see the example at the end of this subsection.

- In 3.4.5, we study a $\mathfrak{h}$-torsion module $\mathcal{M}$, i.e. $\mathcal{M} = \mathcal{M}^{[\mathfrak{h}^\bullet]}$. We define the depth function

$$\mathrm{dep} : \mathcal{M} \to \mathbb{Z}_{\geq 0}$$

  of elements in $\mathcal{M}$, and let $\mathrm{dep}(\mathcal{M})$ be its image. We show some basic properties of dep in Lemma 3.46, and show the $\mathrm{dep}(\mathcal{M})$ is either $\mathbb{Z}_{\geq 0}$ or a finite consecutive subset of the form $\{0, 1, \ldots, N\}$ in Lemma 3.47.

- In 3.4.6, we study a finite dimensional $\mathfrak{h}$-torsion module $\mathcal{M}$. We show $\mathcal{M}$ has a good basis (3.50), i.e. the annihilator $\mathcal{M}^{[\mathfrak{h}^k]}$ is spanned by vectors with depth $\leq k$ for all $k \in \operatorname{dep}(\mathcal{M})$.
- In 3.4.7, we assume $\mathcal{M}_1$ is a finite dimensional $\mathfrak{h}$-torsion module, and $\mathcal{M}_2$ is an arbitrary left $\mathfrak{h}$-module. We show the main theorem of this section (Lemma 3.52):

$$(\mathcal{M}_1 \otimes \mathcal{M}_2)^{[\mathfrak{h}^\bullet]} = \mathcal{M}_1^{[\mathfrak{h}^\bullet]} \otimes \mathcal{M}_2^{[\mathfrak{h}^\bullet]} = \mathcal{M}_1 \otimes \mathcal{M}_2^{[\mathfrak{h}^\bullet]}.$$

3.4.1 Left Torsion Submodule

In this subsection, let

$$\begin{aligned} \mathfrak{h} &= \text{a complex Lie algebra} \\ U(\mathfrak{h}) &= \text{the enveloping algebra of } \mathfrak{h} \\ \mathcal{M} &= \text{a left } \mathfrak{h}\text{-module} \\ &\quad\ \text{(equivalently a left } U(\mathfrak{h})\text{-module)} \end{aligned}$$

Definition 3.30. For $k \in \mathbb{Z}_{>0}$, let

$$\begin{aligned} (\mathfrak{h}^k) :&= \mathfrak{h}^k U(\mathfrak{h}) \\ &= \{X_1 \cdot \ldots \cdot X_k \cdot u : \forall X_1, \ldots, X_k \in \mathfrak{h}, \forall u \in U(\mathfrak{h})\} \end{aligned}$$

be the right ideal of $U(\mathfrak{h})$ generated by k-products of elements in $\mathfrak{h}$. Conventionally, we let $(\mathfrak{h}^0) = U(\mathfrak{h})$.

Lemma 3.31. *The $(\mathfrak{h}^k)$ is a two-sided ideal for each $k \in \mathbb{Z}_{>0}$, and one has the following inclusions:*

$$(\mathfrak{h}^0) \supset (\mathfrak{h}^1) \supset (\mathfrak{h}^2) \supset \ldots \supset (\mathfrak{h}^k) \supset (\mathfrak{h}^{k+1}) \supset \ldots.$$

Definition 3.32. For each $k > 0$, let

$$\begin{aligned} \mathcal{M}^{[\mathfrak{h}^k]} :&= \operatorname{Ann}_{\mathcal{M}}((\mathfrak{h}^k)) \\ &= \{m \in \mathcal{M} : (\mathfrak{h}^k) \cdot m = 0\} \\ &= \{m \in \mathcal{M} : X_1 \cdot \ldots \cdot X_k \cdot m = 0, \forall X_1, \ldots, X_k \in \mathfrak{h}\} \end{aligned}$$

be the **left annihilator of the ideal** $(\mathfrak{h}^k)$ **in** $\mathcal{M}$. By convention, we let $\mathcal{M}^{[\mathfrak{h}^0]} = \{0\}$.

Lemma 3.33. *The* $\mathcal{M}^{[\mathfrak{h}^k]}$ *is a left* $U(\mathfrak{h})$*-submodule of* $\mathcal{M}$ *and one has the inclusions:*

$$\{0\} = \mathcal{M}^{[\mathfrak{h}^0]} \subset \mathcal{M}^{[\mathfrak{h}^1]} \subset \mathcal{M}^{[\mathfrak{h}^2]} \subset \ldots \subset \mathcal{M}^{[\mathfrak{h}^k]} \subset \mathcal{M}^{[\mathfrak{h}^{k+1}]} \subset \ldots \mathcal{M}.$$

One also has the following easy fact:

$$(\mathfrak{h}^l) \cdot \mathcal{M}^{[\mathfrak{h}^k]} \subset \mathcal{M}^{[\mathfrak{h}^{k-l}]}$$

for all $k \geq l$*.*

Definition 3.34. Let

$$\mathcal{M}^{[\mathfrak{h}^\bullet]} := \bigcup_{k \in \mathbb{Z}_{\geq 0}} \mathcal{M}^{[\mathfrak{h}^k]}$$

be the union of all the left annihilators $\mathcal{M}^{[\mathfrak{h}^k]}$. We call this the **left $\mathfrak{h}$-torsion subspace of $\mathcal{M}$**. The module $\mathcal{M}$ is called **left $\mathfrak{h}$-torsion** if $\mathcal{M}^{[\mathfrak{h}^\bullet]} = \mathcal{M}$.

Lemma 3.35. *The* $\mathcal{M}^{[\mathfrak{h}^\bullet]}$ *is a left* $\mathfrak{h}$*-submodule of* $\mathcal{M}$ *since it is a direct limit of* $\mathfrak{h}$*-submodules. The annihilators* $\{\mathcal{M}^{[\mathfrak{h}^k]} : k \geq 0\}$ *form an exhaustive filtration of* $\mathcal{M}^{[\mathfrak{h}^\bullet]}$*.*

Lemma 3.36. *The correspondence* $\mathcal{M} \mapsto \mathcal{M}^{[\mathfrak{h}^\bullet]}$ *is a functor from the category of* $\mathfrak{h}$*-modules to itself.*

This functor is left exact, i.e. if $\mathcal{M}_1, \mathcal{M}_2$ *are two left* $\mathfrak{h}$*-modules, with an injective homomorphism of* $\mathfrak{h}$*-modules* $\mathcal{M}_1 \hookrightarrow \mathcal{M}_2$*, then this homomorphism restricts to a* $\mathfrak{h}$*-homomorphism*

$$\mathcal{M}_1^{[\mathfrak{h}^\bullet]} \hookrightarrow \mathcal{M}_2^{[\mathfrak{h}^\bullet]},$$

and we have

$$\mathcal{M}_1^{[\mathfrak{h}^\bullet]} = \mathcal{M}_1 \cap \mathcal{M}_2^{[\mathfrak{h}^\bullet]}$$

Remark 3.37. Note that $\mathcal{M}^{[\mathfrak{h}^1]}$ is just $H^0(\mathfrak{h}, \mathcal{M}) = \{m \in \mathcal{M} : X \cdot m = 0, \forall X \in \mathfrak{h}\}$.

What happens if $\mathcal{M}^{[\mathfrak{h}^1]} = \{0\}$? Assume this, and let $\forall m \in \mathcal{M}^{[\mathfrak{h}^2]}$, i.e. $X_1 X_2 \cdot m = 0$ for all $X_1, X_2 \in \mathfrak{h}$. We fix X_2 and let X_1 run through $\mathfrak{h}$, then we see $X_2 \cdot m \in \mathcal{M}^{[\mathfrak{h}^1]} = \{0\}$, i.e. $X_2 \cdot m = 0$. Now let X_2 run through $\mathfrak{h}$, we see $m \in \mathcal{M}^{[\mathfrak{h}^1]} = \{0\}$. In sum, $\mathcal{M}^{[\mathfrak{h}^2]} = \mathcal{M}^{[\mathfrak{h}^1]} = \mathcal{M}^{[\mathfrak{h}^0]} = \{0\}$. By iteration, we have all annihilators $\mathcal{M}^{[\mathfrak{h}^k]} = \{0\}$.

This works for all k, and we have the following lemma.

Lemma 3.38. *Let $\mathcal{M}$ be a left $\mathfrak{h}$-module, and let $\mathcal{M}^{[\mathfrak{h}^k]}$ be the left annihilator of $(\mathfrak{h}^k)$ for all $k \in \mathbb{Z}_{\geq 0}$. If there is a $k \geq 0$, such that $\mathcal{M}^{[\mathfrak{h}^k]} = \mathcal{M}^{[\mathfrak{h}^{k+1}]}$, then*

$$\mathcal{M}^{[\mathfrak{h}^k]} = \mathcal{M}^{[\mathfrak{h}^{k+1}]} = \mathcal{M}^{[\mathfrak{h}^{k+2}]} = \dots$$

i.e. the annihilator sequence stops ascending.

Proof. Suppose $\mathcal{M}^{[\mathfrak{h}^k]} = \mathcal{M}^{[\mathfrak{h}^{k+1}]}$ for some $k \geq 0$. Let $m \in \mathcal{M}^{[\mathfrak{h}^{k+2}]}$, then $\forall X_1, \dots, X_{k+1}, X_{k+2} \in \mathfrak{h}$, we have $X_1 \cdots X_{k+1} \cdot X_{k+2} \cdot m = 0$. Hence $X_{k+2} \cdot m \in \mathcal{M}^{[\mathfrak{h}^{k+1}]}$. By the assumption, we also have $X_{k+2} \cdot m \in \mathcal{M}^{[\mathfrak{h}^k]}$. Since X_{k+2} is arbitrary, we see $m \in \mathcal{M}^{[\mathfrak{h}^{k+1}]}$. Hence $\mathcal{M}^{[\mathfrak{h}^{k+2}]} = \mathcal{M}^{[\mathfrak{h}^{k+1}]} = \mathcal{M}^{[\mathfrak{h}^k]}$. By iteration, all the following annihilators equal to $\mathcal{M}^{[\mathfrak{h}^k]}$. $\square$

Remark 3.39. Let $\mathcal{M}$ be a $\mathfrak{h}$-torsion module, then there are only two possibilities:

1. The annihilator sequence is an *infinite strictly ascending sequence*:

$$\{0\} = \mathcal{M}^{[\mathfrak{h}^0]} \subsetneqq \mathcal{M}^{[\mathfrak{h}^1]} \subsetneqq \mathcal{M}^{[\mathfrak{h}^2]} \subsetneqq \dots \subsetneqq \mathcal{M}^{[\mathfrak{h}^k]} \subsetneqq \mathcal{M}^{[\mathfrak{h}^{k+1}]} \subsetneqq \dots,$$

   and all annihilators $\mathcal{M}^{[\mathfrak{h}^k]} \subsetneqq \mathcal{M}$ are proper subspaces.

2. The annihilator sequence is a *finite sequence "without jump"*, and there is an $N \geq 0$ such that

$$\{0\} = \mathcal{M}^{[\mathfrak{h}^0]} \subsetneqq \mathcal{M}^{[\mathfrak{h}^1]} \subsetneqq \dots \subsetneqq \mathcal{M}^{[\mathfrak{h}^N]} = \mathcal{M},$$

   and $\mathcal{M} = \mathcal{M}^{[\mathfrak{h}^N]} = \mathcal{M}^{[\mathfrak{h}^{N+1}]} = \mathcal{M}^{[\mathfrak{h}^{N+2}]} = \dots.$

3.4.2 Right Torsion Submodule

Similar to the left torsion submodule, we have the right torsion submodule of a right $\mathfrak{h}$-module. Let

$$\mathfrak{M} = \text{a right } \mathfrak{h}\text{-module}.$$

We still let $(\mathfrak{h}^k)$ be the ideal defined in Definition 3.30.

Definition 3.40. For each $k \in \mathbb{Z}_{>0}$, let

$$\begin{aligned}\mathfrak{M}_{[\mathfrak{h}^k]} &= \mathrm{Ann}_{\mathfrak{M}}((\mathfrak{h}^k)) \\ &= \{m \in \mathfrak{M} | m \cdot (\mathfrak{h}^k) = 0\} \\ &= \{m \in \mathfrak{M} | m \cdot X_1 \cdot \dots \cdot X_k = 0, \forall X_1, \dots, X_k \in \mathfrak{h}\}\end{aligned}$$

be the **right annihilator of the ideal** $(\mathfrak{h}^k)$ **in** $\mathfrak{M}$. Conventionally, we have $\mathfrak{M}_{[\mathfrak{h}^0]} = \{0\}$.

Let

$$\mathfrak{M}_{[\mathfrak{h}^\bullet]} = \bigcup_{k\in\mathbb{Z}_{\geq 0}} \mathfrak{M}_{[\mathfrak{h}^k]}$$

be the union of all right annihilators. We call this union the **right $\mathfrak{h}$-torsion subspace of** $\mathfrak{M}$. We say $\mathfrak{M}$ is **right $\mathfrak{h}$-torsion**, if $\mathfrak{M}_{[\mathfrak{h}^\bullet]} = \mathfrak{M}$.

Similar to the left picture, we have

Lemma 3.41. *Let $\mathfrak{M}$ be a right $\mathfrak{h}$-module.*

1. *We have the following inclusions:*

$$\{0\} = \mathfrak{M}_{[\mathfrak{h}^0]} \subset \mathfrak{M}_{[\mathfrak{h}^1]} \subset \ldots \mathfrak{M}_{[\mathfrak{h}^k]} \subset \mathfrak{M}_{[\mathfrak{h}^{k+1}]} \subset \ldots \subset \mathfrak{M},$$

   and each $\mathfrak{M}_{[\mathfrak{h}^k]}$ is a right $\mathfrak{h}$-submodule of $\mathfrak{M}$.

2. *For $k \geq l$, we have*

$$\mathfrak{M}_{[\mathfrak{h}^k]} \cdot (\mathfrak{h}^l) \subset \mathfrak{M}_{[\mathfrak{h}^{k-l}]}.$$

3. *The $\{\mathfrak{M}_{[\mathfrak{h}^k]} : k \geq 0\}$ form an exhaustive filtration on $\mathfrak{M}_{[\mathfrak{h}^\bullet]}$, and $\mathfrak{M}_{[\mathfrak{h}^\bullet]}$ is a right $\mathfrak{h}$-submodule of $\mathfrak{M}$.*

3.4.3 Tensor Product Module

Let $\mathcal{M}_1, \mathcal{M}_2$ be two left $\mathfrak{h}$-modules, and let $\mathcal{M}_1 \otimes \mathcal{M}_2$ be their tensor product module. In this subsection we show a multiplication formula (3.2) on the tensor product module $\mathcal{M}_1 \otimes \mathcal{M}_2$.

Let $m_1 \in \mathcal{M}_1, m_2 \in \mathcal{M}_2$ be arbitrary elements. Then an element $X \in \mathfrak{h}$ acts on $m_1 \otimes m_2 \in \mathcal{M}_1 \otimes \mathcal{M}_2$ as

$$X \cdot (m_1 \otimes m_2) = X \cdot m_1 \otimes m_2 + m_1 \otimes X \cdot m_2.$$

For a positive integer k, let

$$[1, k] = \{1, 2, \ldots, k\}$$

be the ordered set of the first kth positive integers. Let $X_1, \ldots, X_k$ be arbitrary k elements in $\mathfrak{h}$. For a subset $S \subset [1, k]$, we label the elements in S as

$$S = \{i_1, i_2, \ldots, i_s\} \text{ with } i_1 < i_2 < \ldots < i_s$$

And we use the following notation to denote the ordered product in $U(\mathfrak{h})$:

$$X_S = X_{i_1} \cdot X_{i_2} \cdots X_{i_s} \in U(\mathfrak{h}).$$

By convention, we let $X_\emptyset = 1$.

Lemma 3.42. *For arbitrary $m_1 \in \mathcal{M}_1, m_2 \in \mathcal{M}_2$ and elements $X_1, \ldots, X_k$ in $\mathfrak{h}$, we have the following multiplication formula*

$$X_1 \cdot X_2 \cdots X_k \cdot (m_1 \otimes m_2) = \sum_{S \subset [1,k]} (X_S \cdot m_1) \otimes (X_{S^c} \cdot m_2). \tag{3.2}$$

The $S^c \subset [1,k]$ is the complement of S in $[1,k]$. The right hand side is a sum of 2^k terms when S run through all subsets of $[1,k]$.

Proof. We proceed by induction on k. For $k = 1$, the formula is trivial. The $[1,1] = \{1\}$ has only two subsets $\emptyset$ and $[1,1]$ itself. The right hand side of (3.2) is

$$(X_\emptyset \cdot m_1) \otimes (X_{[1,1]} \cdot m_2) + (X_{[1,1]} \cdot m_1) \otimes (X_\emptyset \cdot m_2)$$

which is exactly $m_1 \otimes X_1 \cdot m_2 + X_1 \cdot m_1 \otimes m_2$.

Assume the formula is true for $k = n-1$. Then for $k = n$, we have

$$\begin{aligned}
&X_1 \cdot X_2 \cdots X_n (m_1 \otimes m_2) \\
&= X_1 \cdot (X_2 \cdots X_n \cdot (m_1 \otimes m_2)) \\
&= X_1 \cdot \Big(\sum_{S \subset [2,n]} (X_S \cdot m_1) \otimes (X_{S^c} \cdot m_2) \Big) \qquad \text{(induction hypothesis)} \\
&= \sum_{S \subset [2,n]} (X_1 \cdot X_S \cdot m_1) \otimes (X_{S^c} \cdot m_2) + \sum_{S \subset [2,n]} (X_S \cdot m_1) \otimes (X_1 \cdot X_{S^c} \cdot m_2) \\
&= \sum_{\substack{S \subset [1,n] \\ 1 \in S}} (X_S \cdot m_1) \otimes (X_{S^c} \cdot m_2) + \sum_{\substack{S \subset [1,n] \\ 1 \in S^c}} (X_S \cdot m_1) \otimes (X_{S^c} \cdot m_2) \\
&= \sum_{S \subset [1,n]} (X_S \cdot m_1) \otimes (X_{S^c} \cdot m_2)
\end{aligned}$$

□

3.4.4 Torsion Submodule of Tensor Products—I

Let $\mathcal{M}_1, \mathcal{M}_2$ be two left $\mathfrak{h}$-modules, and $\mathcal{M}_1 \otimes \mathcal{M}_2$ be their tensor product module. We discuss the relation between their left $\mathfrak{h}$-torsion subspaces.

Let $\mathcal{M}_1^{[\mathfrak{h}^\bullet]}, \mathcal{M}_2^{[\mathfrak{h}^\bullet]}$ and $(\mathcal{M}_1 \otimes \mathcal{M}_2)^{[\mathfrak{h}^\bullet]}$ be the left $\mathfrak{h}$-torsion subspaces of $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_1 \otimes \mathcal{M}_2$ respectively. We first have the following easy fact:

Lemma 3.43. *The tensor product of torsion subspaces is contained in the torsion subspace of tensor product:*

$$\mathcal{M}_1^{[\mathfrak{h}^\bullet]} \otimes \mathcal{M}_2^{[\mathfrak{h}^\bullet]} \subset (\mathcal{M}_1 \otimes \mathcal{M}_2)^{[\mathfrak{h}^\bullet]} \tag{3.3}$$

i.e. $\mathcal{M}_1^{[\mathfrak{h}^\bullet]} \otimes \mathcal{M}_2^{[\mathfrak{h}^\bullet]}$ *is a left* $\mathfrak{h}$*-submodule of* $(\mathcal{M}_1 \otimes \mathcal{M}_2)^{[\mathfrak{h}^\bullet]}$. *More precisely, we have*

$$\mathcal{M}_1^{[\mathfrak{h}^{k_1}]} \otimes \mathcal{M}_2^{[\mathfrak{h}^{k_2}]} \subset (\mathcal{M}_1 \otimes \mathcal{M}_2)^{[\mathfrak{h}^{k_1+k_2}]} \tag{3.4}$$

for all $k_1, k_2 \in \mathbb{Z}_{\geq 0}$.

Proof. The module structures are clear, and we just need to show the stronger inclusion (3.4). Let $m_1 \in \mathcal{M}_1^{[\mathfrak{h}^{k_1}]}$ and $m_2 \in \mathcal{M}_2^{[\mathfrak{h}^{k_2}]}$, i.e. m_1 is killed by k_1-products of elements in $\mathfrak{h}$ and m_2 is killed by k_2-products of elements in $\mathfrak{h}$.

Let $k = k_1 + k_2$, and let $X_1, \dots, X_k$ be arbitrary elements in $\mathfrak{h}$. By Lemma 3.42, we have the product formula

$$X_1 \cdot X_2 \cdot \dots \cdot X_k \cdot (m_1 \otimes m_2) = \sum_{S \subset [1,k]} (X_S \cdot m_1) \otimes (X_{S^c} \cdot m_2).$$

For each $S \subset [1,k]$ on the right hand side, either $|S| \geq k_1$ or $|S^c| \geq k_2$ is true. Otherwise we have $k = |S| + |S^c| < k_1 + k_2 = k$, a contradiction! Hence for each $S \subset [1,k]$, either $X_S \cdot m_1 = 0$ or $X_{S^c} \cdot m_2 = 0$. Hence the right hand side is always zero.

Since the $X_1, \dots, X_k$ are arbitrary, we see $m_1 \otimes m_2$ is in the annihilator $\mathrm{Ann}_{\mathcal{M}_1 \otimes \mathcal{M}_2}((\mathfrak{h}^k)) = (\mathcal{M}_1 \otimes \mathcal{M}_2)^{[\mathfrak{h}^k]} \subset (\mathcal{M}_1 \otimes \mathcal{M}_2)^{[\mathfrak{h}^\bullet]}$.

□

In general, the torsion subspace $(\mathcal{M}_1 \otimes \mathcal{M}_2)^{[\mathfrak{h}^\bullet]}$ is larger than the tensor product $\mathcal{M}_1^{[\mathfrak{h}^\bullet]} \otimes \mathcal{M}_2^{[\mathfrak{h}^\bullet]}$, see the following example.

Example 3.44. Let

$$\mathfrak{h} = \{x : x \in \mathbb{C}\}$$

be the one dimensional abelian Lie algebra. Let

$$V = \mathbb{C} = \{y : y \in \mathbb{C}\}$$

be a one dimensional $\mathfrak{h}$-module, with the $\mathfrak{h}$-action given by

$$x \cdot y = xy,$$

i.e. the scalar multiplication on $\mathbb{C}$. Then one can see

$$V^{[\mathfrak{h}^1]} = \{y \in \mathbb{C} : xy = 0, \forall x \in \mathbb{C}\} = \{0\}$$

and similarly

$$V^{[\mathfrak{h}^k]} = \{y \in \mathbb{C} : x_1 x_2 \dots x_k y = 0, \forall x_1, \dots, x_k \in \mathbb{C}\} = \{0\}.$$

Hence one has
$$V^{[\mathfrak{h}^\bullet]} = \{0\}.$$

Let
$$\hat{V} = \mathbb{C} = \{z : z \in \mathbb{C}\}$$
be the dual $\mathfrak{h}$-module of V, with the $\mathfrak{h}$-action given by

$$x \cdot z = -xz.$$

Then by the same argument one can see

$$\hat{V}^{[\mathfrak{h}^k]} = 0, \forall k \geq 0$$

and $\hat{V}^{[\mathfrak{h}^\bullet]} = \{0\}$.

The tensor product $V \otimes \hat{V}$ is a one dimensional trivial $\mathfrak{h}$-module, i.e. $\forall y, z \in \mathbb{C}$, we have

$$x \cdot (y \otimes z) = xy \otimes z + y \otimes (-xz) = 0.$$

Hence
$$(V \otimes \hat{V})^{[\mathfrak{h}^\bullet]} = (V \otimes \hat{V})^{[\mathfrak{h}^1]} = V \otimes \hat{V},$$
which is larger than $V^{[\mathfrak{h}^\bullet]} \otimes \hat{V}^{[\mathfrak{h}^\bullet]} = \{0\}$.

3.4.5 Depth and Torsion Modules

Let $\mathfrak{h}$ be a complex Lie algebra and $\mathcal{M}$ be a left $\mathfrak{h}$-module. In this subsection, we assume $\mathcal{M}$ is $\mathfrak{h}$-torsion, i.e.

$$\mathcal{M} = \mathcal{M}^{[\mathfrak{h}^\bullet]}.$$

The filtration by annihilators

$$\{0\} = \mathcal{M}^{[\mathfrak{h}^0]} \subset \mathcal{M}^{[\mathfrak{h}^1]} \subset \ldots \mathcal{M}^{[\mathfrak{h}^k]} \subset \mathcal{M}^{[\mathfrak{h}^{k+1}]} \subset \ldots$$

is an exhaustive filtration of $\mathcal{M}$, i.e. for all $m \in \mathcal{M}$, there exists a k such that $m \in \mathcal{M}^{[\mathfrak{h}^k]}$.

Definition 3.45. For a $\mathfrak{h}$-torsion left $\mathfrak{h}$-module $\mathcal{M}$, we define the **depth function on $\mathcal{M}$** by

$$\operatorname{dep}(m) = \min\{k : m \in \mathcal{M}^{[\mathfrak{h}^k]}\}, \quad \forall m \in \mathcal{M}. \tag{3.5}$$

Namely, the **depth of** $m \in \mathcal{M}$ is the minimal non-negative integer k such that m is annihilated by $(\mathfrak{h}^k)$. The $\mathrm{dep} : \mathcal{M} \to \mathbb{Z}_{\geq 0}$ is a function on $\mathcal{M}$ with values in non-negative integers, and we denote its image by

$$\mathrm{dep}(\mathcal{M}) = \{\mathrm{dep}(m) : m \in \mathcal{M}\}$$

i.e. the set of depth of elements in $\mathcal{M}$.

We have the following easy facts:

Lemma 3.46. *Let $m \in \mathcal{M}$, then*

(1) $\mathrm{dep}(m) = 0$ *if and only if* $m = 0$.

(2) $\mathrm{dep}(m) = 1$ *if and only if* $m \in H^0(\mathfrak{h}, \mathcal{M})$ *but* $m \neq 0$.

(3) $\mathcal{M}^{[\mathfrak{h}^k]} = \{m \in \mathcal{M} : \mathrm{dep}(m) \leq k\}$.

(4) For $k > 0$, the $\{m \in \mathcal{M} : \mathrm{dep}(m) = k\} = \mathcal{M}^{[\mathfrak{h}^k]} - \mathcal{M}^{[\mathfrak{h}^{k-1}]}$.

(5) For arbitrary $X \in \mathfrak{h}$, we have $\mathrm{dep}(X \cdot m) < \mathrm{dep}(m)$, *i.e. X strictly reduce the depth.*

(6) For $c \in \mathbb{C}$, we have

$$\mathrm{dep}(cm) = \begin{cases} \mathrm{dep}(m) & \text{if } c \neq 0 \\ 0 & \text{if } c = 0 \end{cases}$$

(7) Let $m_1, m_2 \in \mathcal{M}$, then

$$\mathrm{dep}(m_1 + m_2) \leq \max\{\mathrm{dep}(m_1), \mathrm{dep}(m_2)\}. \tag{3.6}$$

Proof. (1)(2)(3)(4)(6) are obvious by definition. We show (5) and (7).

(5): Let $k = \mathrm{dep}(m)$ and $X \in \mathfrak{h}$ be an arbitrary element. By definition we have $(\mathfrak{h}^k) \cdot m = \{0\}$. Then

$$(\mathfrak{h}^{k-1}) \cdot (X \cdot m) \subset (\mathfrak{h}^k) \cdot m = \{0\},$$

hence $X \cdot m \in \mathcal{M}^{[\mathfrak{h}^{k-1}]}$. Again by definition of dep, we have

$$\mathrm{dep}(X \cdot m) = \min\{i : X \cdot m \in \mathcal{M}^{[\mathfrak{h}^i]}\} \leq k - 1 < k,$$

i.e. $\mathrm{dep}(X \cdot m) < k = \mathrm{dep}(m)$.

(7): Let $k_1 = \mathrm{dep}(m_1), k_2 = \mathrm{dep}(m_2), k = \max\{k_1, k_2\}$. Then

$$m_1 \in \mathcal{M}^{[\mathfrak{h}^{k_1}]} \subset \mathcal{M}^{[\mathfrak{h}^k]}$$
$$m_2 \in \mathcal{M}^{[\mathfrak{h}^{k_2}]} \subset \mathcal{M}^{[\mathfrak{h}^k]}$$

Hence $m_1 + m_2 \in \mathcal{M}^{[\mathfrak{h}^k]}$, and by definition

$$\mathrm{dep}(m_1 + m_2) \leq k = \max\{k_1, k_2\} = \max\{\mathrm{dep}(m_1), \mathrm{dep}(m_2)\}.$$

$\square$

Lemma 3.47. *The depth function* $\mathrm{dep} : \mathcal{M} \to \mathbb{Z}_{\geq 0}$ *is either onto or with finite consecutive image. More precisely, the image* $\mathrm{dep}(\mathcal{M})$ *is either* $\mathbb{Z}_{\geq 0}$ *or of the form* $\{0, 1, \ldots, k\}$ *form some* $k \geq 0$*.*

Proof. This follows immediately from Remark 3.39. $\square$

In sum, the $\mathrm{dep}(\mathcal{M})$ is a well-ordered set, we can perform inductions on the depth of the module $\mathcal{M}$.

3.4.6 Finite Dimensional Torsion Modules

In this subsection, we look at the special case when $\mathcal{M}$ is a finite dimensional $\mathfrak{h}$-torsion module. First we have

Lemma 3.48. *Let* $\mathcal{M}$ *be a finite dimensional* $\mathfrak{h}$*-torsion module. Then the depth function* dep *on* $\mathcal{M}$ *is bounded, hence* $\mathrm{dep}(\mathcal{M})$ *is a finite consecutive subset of* $\mathbb{Z}_{\geq 0}$*.*

Proof. This follows immediately from Lemma 3.47. $\square$

Lemma 3.49. *Let* $\mathcal{M}$ *be a finite dimensional* $\mathfrak{h}$*-torsion module, and let*

$$\mathrm{dep}(\mathcal{M}) = \{0, 1, \ldots, N\},$$

then

(1) The annihilator sequence of $\mathcal{M}$ *is*

$$\{0\} = \mathcal{M}^{[\mathfrak{h}^0]} \subsetneqq \mathcal{M}^{[\mathfrak{h}^1]} \subsetneqq \ldots \subsetneqq \mathcal{M}^{[\mathfrak{h}^N]} = \mathcal{M}$$

and for all $k \geq N$*, the* $\mathcal{M}^{[\mathfrak{h}^k]} = \mathcal{M}$*.*

(2) *For any* $m \in \mathcal{M}$ *and* $k \in \operatorname{dep}(\mathcal{M})$*, we have* $\operatorname{dep}(m) = k$ *if and only if* $m \in \mathcal{M}^{[\mathfrak{h}^k]} - \mathcal{M}^{[\mathfrak{h}^{k-1}]}$*.*

Proof. Part (1) follows from the above lemma. Part (2) is just a reformulation of (4) of Lemma 3.46. $\square$

Let $\{m_1, \ldots, m_d\}$ be a basis of $\mathcal{M}$ ($d = \dim \mathcal{M}$). For each $k \in \operatorname{dep}(\mathcal{M})$, we consider the subspace

$$\operatorname{span}\{m_i : 1 \leq i \leq d, \operatorname{dep}(m_i) \leq k\}$$

spanned by all basis vectors with depth less than or equal to k. Obviously this space is contained in $\mathcal{M}^{[\mathfrak{h}^k]}$ by (6) (7) of Lemma 3.46.

Definition 3.50. Let $\mathcal{M}$ be a finite dimensional $\mathfrak{h}$-torsion module, and $\operatorname{dep}(\mathcal{M}) = \{0, 1, \ldots, N\}$. A basis $\{m_1, \ldots, m_d\}$ of $\mathcal{M}$ is called a **good basis**, if

$$\mathcal{M}^{[\mathfrak{h}^k]} = \operatorname{span}\{m_i : 1 \leq i \leq d, \operatorname{dep}(m_i) \leq k\}$$

for all $k \in \operatorname{dep}(\mathcal{M})$.

Lemma 3.51. *Let* $\mathcal{M}$ *be a finite dimensional* $\mathfrak{h}$*-torsion module. Then it has a good basis.*

Proof. We construct a good basis. Let $\operatorname{dep}(\mathcal{M}) = \{0, 1, \ldots, N\}$. Let

$$d_k = \dim \mathcal{M}^{[\mathfrak{h}^k]}$$

be the dimension of the kth annihilator, we have $0 = d_0 < d_1 < d_2 < \ldots < d_N = \dim \mathcal{M}$ (strictly increasing).

First let $\{m_1, \ldots, m_{d_1}\}$ be an *arbitrary* basis of $\mathcal{M}^{[\mathfrak{h}^1]}$. Since they are basis vectors, they are non-zero, hence $\operatorname{dep}(m_i) = 1$ for all $1 \leq i \leq d_1$. Obviously

$$\mathcal{M}^{[\mathfrak{h}^1]} = \operatorname{span}\{m_1, \ldots, m_{d_1}\}.$$

Since $\mathcal{M}^{[\mathfrak{h}^1]}$ is a finite dimensional (proper) subspace of $\mathcal{M}^{[\mathfrak{h}^2]}$ (also finite dimensional), one can extend the basis $\{m_1, \ldots, m_{d_1}\}$ of $\mathcal{M}^{[\mathfrak{h}^1]}$ to a basis $\{m_1, \ldots, m_{d_1}, m_{d_1+1}, \ldots, m_{d_2}\}$ of $\mathcal{M}^{[\mathfrak{h}^2]}$. Obviously, the $d_2 - d_1$ new basis vectors $\{m_{d_1+1}, \ldots, m_{d_2}\}$ are in the complement $\mathcal{M}^{[\mathfrak{h}^2]} - \mathcal{M}^{[\mathfrak{h}^1]}$, hence all their depth equal to 2 by (4) of Lemma 3.46. Obviously we have

$$\mathcal{M}^{[\mathfrak{h}^2]} = \operatorname{span}\{m_1, \ldots, m_{d_2}\}.$$

By iteration, for each $k < N$, assume we have constructed a basis $\{m_1, \ldots, m_{d_k}\}$ of $\mathcal{M}^{[\mathfrak{h}^k]}$, such that $\{m_1, \ldots, m_{d_i}\}$ is a basis of the subspace

$\mathcal{M}^{[\mathfrak{h}^i]}$, and $\text{dep}(m_{d_{i-1}+1}) = \ldots = \text{dep}(m_{d_i}) = i$ for all $i \leq k$. Then we can extend the basis $\{m_1, \ldots, m_{d_k}\}$ of $\mathcal{M}^{[\mathfrak{h}^k]}$ to a basis

$$\{m_1, \ldots, m_{d_k}, m_{d_k+1}, \ldots, m_{d_{k+1}}\}$$

of $\mathcal{M}^{[\mathfrak{h}^{k+1}]}$. The $d_{k+1} - d_k$ new vectors $\{m_{d_k+1}, \ldots, m_{d_{k+1}}\}$ are obviously in the complement $\mathcal{M}^{[\mathfrak{h}^{k+1}]} - \mathcal{M}^{[\mathfrak{h}^k]}$ and all their depth equal to $k+1$ by (4) of Lemma 3.46.

This iteration terminate when $k = N$, and we then obtain a basis $\{m_1, \ldots, m_{d_N}\}$ which is obviously a good basis. □

3.4.7 Torsion Submodule of Tensor Products—II

In this subsection, we show the reversed inclusion of Lemma 3.43. Let $\mathfrak{h}$ be a complex Lie algebra, $\mathcal{M}_1, \mathcal{M}_2$ be two left $\mathfrak{h}$-modules and $\mathcal{M}_1 \otimes \mathcal{M}_2$ be their tensor product module.

Moreover, we assume $\mathcal{M}_1$ is

- $\mathfrak{h}$-torsion, i.e. $\mathcal{M}_1 = \mathcal{M}_1^{[\mathfrak{h}^\bullet]}$;
- finite dimensional.

We have

Lemma 3.52. *Let $\mathcal{M}_1$ be a finite dimensional $\mathfrak{h}$-torsion module, and $\mathcal{M}_2$ be an arbitrary left $\mathfrak{h}$-module. Then*

$$(\mathcal{M}_1 \otimes \mathcal{M}_2)^{[\mathfrak{h}^\bullet]} = \mathcal{M}_1^{[\mathfrak{h}^\bullet]} \otimes \mathcal{M}_2^{[\mathfrak{h}^\bullet]} = \mathcal{M}_1 \otimes \mathcal{M}_2^{[\mathfrak{h}^\bullet]}.$$

We have seen one inclusion in Lemma 3.43, we just need to show the reversed inclusion

$$(\mathcal{M}_1 \otimes \mathcal{M}_2)^{[\mathfrak{h}^\bullet]} \subset \mathcal{M}_1^{[\mathfrak{h}^\bullet]} \otimes \mathcal{M}_2^{[\mathfrak{h}^\bullet]}.$$

Let $\text{dep}(\mathcal{M}_1) = \{0, 1, \ldots, N\}$. Then the $\{\mathcal{M}_1^{[\mathfrak{h}^k]} : 0 \leq k \leq N\}$ is a finite filtration which is strictly ascending. Let $d_k = \dim \mathcal{M}_1^{[\mathfrak{h}^k]}, \forall k \in \text{dep}(\mathcal{M}_1)$ and let $\{u_1, \ldots, u_{d_N}\}$ be a good basis of $\mathcal{M}_1$ (Definition 3.50), and we label them with non-decreasing depth:

$$\text{dep}(u_i) = k, \quad d_{k-1} + 1 \leq i \leq d_k$$

for all $k = 1, \ldots, N$.

The submodules $\{\mathcal{M}_1^{[\mathfrak{h}^k]} \otimes \mathcal{M}_2 : 0 \leq k \leq N\}$ form a finite exhaustive filtration of $\mathcal{M}_1 \otimes \mathcal{M}_2$, and the submodules $\{(\mathcal{M}_1^{[\mathfrak{h}^k]} \otimes \mathcal{M}_2)^{[\mathfrak{h}^\bullet]} : 0 \leq k \leq N\}$ form a finite exhaustive filtration of $(\mathcal{M}_1 \otimes \mathcal{M}_2)^{[\mathfrak{h}^\bullet]}$. We just need to show the inclusion

$$(\mathcal{M}_1^{[\mathfrak{h}^k]} \otimes \mathcal{M}_2)^{[\mathfrak{h}^\bullet]} \subset \mathcal{M}_1^{[\mathfrak{h}^\bullet]} \otimes \mathcal{M}_2^{[\mathfrak{h}^\bullet]},$$

or more precisely the inclusions

$$(\mathcal{M}_1^{[\mathfrak{h}^k]} \otimes \mathcal{M}_2)^{[\mathfrak{h}^l]} \subset \mathcal{M}_1^{[\mathfrak{h}]} \otimes \mathcal{M}_2^{[\mathfrak{h}^\bullet]},$$

for each $l \geq 0$, by (finite) induction (iteration) on k.

Proof. **For** $k = 0$, $\mathcal{M}_1^{[\mathfrak{h}^0]} = \{0\}$ and $(\mathcal{M}_1^{[\mathfrak{h}^0]} \otimes \mathcal{M}_2)^{[\mathfrak{h}^\bullet]} = \{0\}$ which is obviously contained in $\mathcal{M}_1^{[\mathfrak{h}^\bullet]} \otimes \mathcal{M}_2^{[\mathfrak{h}^\bullet]}$.

For $k = 1$, recall that the $\mathcal{M}_1^{[\mathfrak{h}^1]}$ has basis $\{u_1, \ldots, u_{d_1}\}$, and every element in $\mathcal{M}_1^{[\mathfrak{h}^1]} \otimes \mathcal{M}_2$ is uniquely written as

$$\sum_{i=1}^{d_1} u_i \otimes v_i$$

for some $v_i \in \mathcal{M}_2, 1 \leq i \leq d_1$. And it is zero if and only if all v_i are zero. If moreover the element $\sum_{i=1}^{d_1} u_i \otimes v_i$ is in

$$(\mathcal{M}_1^{[\mathfrak{h}^1]} \otimes \mathcal{M}_2)^{[\mathfrak{h}^l]}$$

then for any $X_1, \ldots, X_l \in \mathfrak{h}$, one has

$$X_1 \cdots X_l \cdot (\sum_{i=1}^{d_1} u_i \otimes v_i) = 0.$$

By Lemma 3.42, one has

$$\begin{aligned} X_1 \cdots X_l \cdot (\sum_{i=1}^{d_1} u_i \otimes v_i) &= \sum_{i=1}^{d_1} X_1 \cdots X_l \cdot (u_i \otimes v_i) \\ &= \sum_{i=1}^{d_1} \sum_{S \subset [1,l]} X_S u_i \otimes X_{S^c} v_i \end{aligned}$$

In the last term, if $S \neq \emptyset$, then $X_S u_i = 0$ since $u_i \in \mathcal{M}_1^{[\mathfrak{h}^1]}$. Hence only $S = \emptyset$ contributes to the last sum, and we have

$$\begin{aligned} X_1 \cdots X_l \cdot (\sum_{i=1}^{d_1} u_i \otimes v_i) &= \sum_{i=1}^{d_1} u_i \otimes X_{[1,l]} v_i \\ &= \sum_{i=1}^{d_1} u_i \otimes X_1 \cdots X_l \cdot v_i \end{aligned}$$

If this sum is zero, then $X_1 \cdots X_l \cdot v_i = 0$ for all $1 \leq i \leq d_1$. Thus $v_i \in \mathcal{M}_2^{[\mathfrak{h}^\bullet]}$ since the $X_1, \ldots, X_l$ are arbitrary. Hence the original element

$$\sum_{i=1}^{d_1} u_i \otimes v_i \in \mathcal{M}_1^{[\mathfrak{h}^1]} \otimes \mathcal{M}_2^{[\mathfrak{h}^l]}.$$

We thus have shown

$$(\mathcal{M}_1^{[\mathfrak{h}^1]} \otimes \mathcal{M}_2)^{[\mathfrak{h}^l]} \subset \mathcal{M}_1^{[\mathfrak{h}^1]} \otimes \mathcal{M}_2^{[\mathfrak{h}^l]} \subset \mathcal{M}_1^{[\mathfrak{h}^\bullet]} \otimes \mathcal{M}_2^{[\mathfrak{h}^\bullet]},$$

thus we have

$$(\mathcal{M}_1^{[\mathfrak{h}^1]} \otimes \mathcal{M}_2)^{[\mathfrak{h}^\bullet]} \subset \mathcal{M}_1^{[\mathfrak{h}^\bullet]} \otimes \mathcal{M}_2^{[\mathfrak{h}^\bullet]}.$$

(Induction Hypothesis) Assume

$$(\mathcal{M}_1^{[\mathfrak{h}^{k-1}]} \otimes \mathcal{M}_2)^{[\mathfrak{h}^\bullet]} \subset \mathcal{M}_1^{[\mathfrak{h}^\bullet]} \otimes \mathcal{M}_2^{[\mathfrak{h}^\bullet]}.$$

We need to show $(\mathcal{M}_1^{[\mathfrak{h}^k]} \otimes \mathcal{M}_2)^{[\mathfrak{h}^l]} \subset \mathcal{M}_1^{[\mathfrak{h}^\bullet]} \otimes \mathcal{M}_2^{[\mathfrak{h}^\bullet]}$, for all $l \geq 0$.

For general k, a generic element in $\mathcal{M}_1^{[\mathfrak{h}^k]} \otimes \mathcal{M}_2$ is uniquely written as

$$\sum_{i=1}^{d_k} u_i \otimes v_i$$

for some $v_i \in \mathcal{M}_2$. If moreover it is in $(\mathcal{M}_1^{[\mathfrak{h}^k]} \otimes \mathcal{M}_2)^{[\mathfrak{h}^l]}$, then

$$X_1 \cdots X_l \cdot (\sum_{i=1}^{d_k} u_i \otimes v_i) = 0$$

for all $X_1, \ldots, X_l \in \mathfrak{h}$. By Lemma 3.42, we have

$$\begin{aligned}
X_1 \cdots X_l \cdot (\sum_{i=1}^{d_k} u_i \otimes v_i) &= \sum_{i=1}^{d_k} X_1 \cdots X_l \cdot (u_i \otimes v_i) \\
&= \sum_{i=1}^{d_k} \sum_{S \subset [1,l]} X_S u_i \otimes X_{S^c} v_i \\
&= \sum_{S \subset [1,l]} \sum_{i=1}^{d_k} X_S u_i \otimes X_{S^c} v_i \\
&= \sum_{i=1}^{d_k} u_i \otimes X_{[1,l]} v_i + \sum_{\substack{S \subset [1,l] \\ S \neq \emptyset}} \sum_{i=1}^{d_k} X_S u_i \otimes X_{S^c} v_i \\
&= \sum_{i=d_{k-1}+1}^{d_k} u_i \otimes X_{[1,l]} v_i + \sum_{i=1}^{d_{k-1}} u_i \otimes X_{[1,l]} v_i \\
&\quad + \sum_{\substack{S \subset [1,l] \\ S \neq \emptyset}} \sum_{i=1}^{d_k} X_S u_i \otimes X_{S^c} v_i
\end{aligned}$$

The second term $\sum_{i=1}^{d_{k-1}} u_i \otimes X_{[1,l]} v_i$ is in $\mathcal{M}_1^{[\mathfrak{h}^{k-1}]} \otimes \mathcal{M}_2$ since $u_i \in \mathcal{M}_1^{[\mathfrak{h}^{k-1}]}$ for all $i \leq d_{k-1}$. The third term $\sum_{\substack{S \subset [1,l] \\ S \neq \emptyset}} \sum_{i=1}^{d_k} X_S u_i \otimes X_{S^c} v_i$ is also in $\mathcal{M}_1^{[\mathfrak{h}^{k-1}]} \otimes \mathcal{M}_2$, since all $X_S u_i$ are in $\mathcal{M}_1^{[\mathfrak{h}^{k-1}]}$. (Remember that $u_i \in \mathcal{M}_1^{[\mathfrak{h}^k]}$ and $\mathfrak{h}$ *strictly* reduce the depth, hence $X_S u_i \in \mathcal{M}_1^{[\mathfrak{h}^{k-1}]}$ when S is nonempty.)

In sum, the last two terms are all in $\mathcal{M}_1^{[\mathfrak{h}^{k-1}]} \otimes \mathcal{M}_2$, and the first term $\sum_{i=d_{k-1}+1}^{d_k} u_i \otimes X_{[1,l]} v_i$ is linear independent from the $\mathcal{M}_1^{[\mathfrak{h}^{k-1}]} \otimes \mathcal{M}_2$. Hence the $X_1 \cdots X_l \cdot (\sum_{i=1}^{d_k} u_i \otimes v_i)$ is zero if and only if both

$$\sum_{i=d_{k-1}+1}^{d_k} u_i \otimes X_{[1,l]} v_i = 0$$

and

$$\sum_{i=1}^{d_{k-1}} u_i \otimes X_{[1,l]} v_i + \sum_{\substack{S \subset [1,l] \\ S \neq \emptyset}} \sum_{i=1}^{d_k} X_S u_i \otimes X_{S^c} v_i = 0.$$

The $\sum_{i=d_{k-1}+1}^{d_k} u_i \otimes X_{[1,l]} v_i = 0$ implies $X_{[1,l]} v_i = X_1 \cdots X_l \cdot v_i = 0$ for all i such that $d_{k-1}+1 \leq i \leq d_k$. Hence the $v_i \in \mathcal{M}_2^{[\mathfrak{h}^\bullet]}$ for all $d_{k-1}+1 \leq i \leq d_k$, and

$$\sum_{i=d_{k-1}+1}^{d_k} u_i \otimes v_i \in \mathcal{M}_1^{[\mathfrak{h}^k]} \otimes \mathcal{M}_2^{[\mathfrak{h}^\bullet]}. \tag{3.7}$$

Then

$$\sum_{i=d_{k-1}+1}^{d_k} u_i \otimes v_i \in \mathcal{M}_1^{[\mathfrak{h}^k]} \otimes \mathcal{M}_2^{[\mathfrak{h}^\bullet]} \subset (\mathcal{M}_1^{[\mathfrak{h}^k]} \otimes \mathcal{M}_2)^{[\mathfrak{h}^\bullet]}$$

(replacing $\mathcal{M}_1$ by $\mathcal{M}_1^{[\mathfrak{h}^k]}$ in Lemma 3.43). Hence

$$\sum_{i=1}^{d_{k-1}} u_i \otimes v_i = \sum_{i=1}^{d_i} u_i \otimes v_i - \sum_{i=d_{k-1}+1}^{d_k} u_i \otimes v_i \in (\mathcal{M}_1^{[\mathfrak{h}^k]} \otimes \mathcal{M}_2)^{[\mathfrak{h}^\bullet]} \subset (\mathcal{M}_1 \otimes \mathcal{M}_2)^{[\mathfrak{h}^\bullet]}.$$

Also note that (good basis)

$$\sum_{i=1}^{d_{k-1}} u_i \otimes v_i \in \mathcal{M}_1^{[\mathfrak{h}^{k-1}]} \otimes \mathcal{M}_2.$$

Hence

$$\sum_{i=1}^{d_{k-1}} u_i \otimes v_i \in (\mathcal{M}_1^{[\mathfrak{h}^{k-1}]} \otimes \mathcal{M}_2) \cap (\mathcal{M}_1 \otimes \mathcal{M}_2)^{[\mathfrak{h}^\bullet]} = (\mathcal{M}_1^{[\mathfrak{h}^{k-1}]} \otimes \mathcal{M}_2)^{[\mathfrak{h}^\bullet]}.$$

By the induction hypothesis, we have

$$\sum_{i=1}^{d_{k-1}} u_i \otimes v_i \in \mathcal{M}_1^{[\mathfrak{h}^\bullet]} \otimes \mathcal{M}_2^{[\mathfrak{h}^\bullet]} \tag{3.8}$$

Combining (3.7) with (3.8), we have

$$\sum_{i=1}^{d_k} u_i \otimes v_i \in \mathcal{M}_1^{[\mathfrak{h}^\bullet]} \otimes \mathcal{M}_2^{[\mathfrak{h}^\bullet]}.$$

Hence

$$(\mathcal{M}_1^{[\mathfrak{h}^k]} \otimes \mathcal{M}_2)^{[\mathfrak{h}^l]} \subset \mathcal{M}_1^{[\mathfrak{h}^\bullet]} \otimes \mathcal{M}_2^{[\mathfrak{h}^\bullet]}$$

for all $l \geq 0$, and

$$(\mathcal{M}_1^{[\mathfrak{h}^k]} \otimes \mathcal{M}_2)^{[\mathfrak{h}^\bullet]} \subset \mathcal{M}_1^{[\mathfrak{h}^\bullet]} \otimes \mathcal{M}_2^{[\mathfrak{h}^\bullet]}.$$

□

Chapter 4

Schwartz Functions, Schwartz Inductions and Schwartz Distributions

Summary of This Chapter

In this chapter, we build the toolbox of Schwartz analysis for the study of intertwining operators (distributions). This chapter consists of three parts, the first part mainly follows the work [1] and [2], the second part is on some fundamental facts, and the third part is innovative and it combines the work of Aizenbud-Gourevitch with the notion of Schwartz inductions introduced in [20].

- The first part consists of section 4.1 (some notions), 4.2, 4.3 and 4.4, in which we recall the theory of Schwartz functions and distributions on affine Nash manifolds, developed in [1] and [2]. The key points of these four sections are
  - Schwartz $\mathbb{R}$-valued, $\mathbb{C}$-valued, E-valued functions, (the spaces $\mathcal{S}(M,\mathbb{R})$, $\mathcal{S}(M)$, $\mathcal{S}(M,E)$).
  - Crucial properties of Schwartz functions: Lemma 4.21, Proposition 4.30.
  - Schwartz distributions (Definition 4.31), restrictions of distributions (Definition 4.34) and crucial properties (Lemma 4.32, 4.35, 4.36).
  - The term "supported in", and the result: the space of Schwartz distributions supported in a closed Nash submanifold is independent of neighbourhoods (Lemma 4.38, Lemma 4.39).
- The second part is section 4.5, where we apply the notions and results in the first part to the nonsingular affine real algebraic varieties or

even real points of algebraic groups, which is a special class of affine Nash manifolds. The key points of this section are

- The Zariski topology on a nonsingular affine real algebraic variety is included in the restricted topology. Therefore by restricting to Zariski topology, one has the pseudo-cosheaf of Schwartz functions on Zariski topology and pseudo-sheaf of Schwartz distributions on Zariski topology.
- The Zariski topology is quasi-compact, hence a Schwartz distribution on a nonsingular affine real algebraic variety has a well-defined support, which is a Zariski closed subset.
- Let G be a real algebraic group, $Y \subset G$ be a subvariety stable under right translation of an algebraic subgroup H. Then the Schwartz function space $\mathcal{S}(Y, E)$ is preserved under right regular H-action and is a smooth H-representation.

• The third part is section 4.6, in which we combine the notion of Schwartz inductions in [20] with the above theory of Schwartz functions/distributions, to build a distribution theory on the local Schwartz inductions. The key points of this part are

- Schwartz inductions $\mathcal{S}\mathrm{Ind}_P^G\sigma$ and local Schwartz inductions (Definition 4.57, 4.64). The Schwartz induction $\mathcal{S}\mathrm{Ind}_P^G\sigma$ is exactly the smooth induction $C^\infty\mathrm{Ind}_P^G\sigma$ when the quotient manifold $P\backslash G$ is compact.
- Properties of local Schwartz inductions: NF-space (Lemma 4.65), open extensions (Lemma 4.67), closed restrictions (Lemma 4.69).
- Zariski P-topology, $\mathcal{S}\mathrm{Ind}_P^-\sigma$ is a pseudo-cosheaf on Zariski P-topology (Lemma 4.73).
- Distributions on Schwartz inductions (Definition 4.76), pseudo-sheaf property (Lemma 4.79), independency of neighbourhoods (Lemma 4.80).
- Right regular actions on Schwartz inductions.
- Schwartz inductions on fibrations: Lemma 4.91 and Lemma 4.92.

4.1 Some Notions

4.1.1 Nash Differential Operators

Let M be an affine Nash manifold, then it has a canonical structure of smooth manifold. Let $C^\infty(M, \mathbb{R})$ be the ring of smooth real valued functions on M.

Definition 4.1 ([1] Definition 3.5.1)**.** Let M be an affine Nash manifold, and let $\mathrm{Hom}_{\mathbb{R}}(C^\infty(M, \mathbb{R}), C^\infty(M, \mathbb{R}))$ be the ring of $\mathbb{R}$-linear homomorphisms on the $\mathbb{R}$-algebra $C^\infty(M, \mathbb{R})$. Let

$$\mathrm{Diff}^{Nash}(M)$$

be the subring ($\mathbb{R}$-subalgebra) of $\mathrm{Hom}_{\mathbb{R}}(C^\infty(M, \mathbb{R}), C^\infty(M, \mathbb{R}))$ generated by the ring $\mathcal{N}(M)$ of Nash functions and Nash vector fields. We call it the **ring of Nash differential operators on** M.

Remark 4.2. Each restricted open subset $U \subset M$ is also an affine Nash manifold, and one can define the ring $\mathrm{Diff}^{Nash}(U)$ of Nash differential operators on U. Since Nash functions and Nash vector fields form sheaves on the restricted topology, one can define the natural restriction of a Nash differential operator in $\mathrm{Diff}^{Nash}(M)$ to U, and obtain a Nash differential operator in $\mathrm{Diff}^{Nash}(U)$. This restriction makes the correspondence $U \mapsto \mathrm{Diff}^{Nash}(U)$ a sheaf on the restricted topology. We call it the **sheaf of Nash differential operators on** M, and denote it by $\mathrm{Diff}^{Nash} : U \mapsto \mathrm{Diff}^{Nash}(U)$.

4.1.2 Affine Real Algebraic Varieties as Affine Nash Manifolds

Let X be a nonsingular affine real algebraic variety ([6] p67 Definition 3.3.9). It has a canonical structure of affine Nash manifold ([1] Example 3.3.4). More precisely, we can embed X into an affine space $\mathbb{R}^n$ as a real algebraic set. Under the Euclidean topology on $\mathbb{R}^n$, the X has a canonical structure of real analytic space. Since the X is nonsingular as real algebraic variety, the real analytic space is actually a smooth manifold (and also a closed regular submanifold of $\mathbb{R}^n$). It is obviously semi-algebraic since it is algebraic. Hence X has a canonical structure of affine Nash manifold. The three structures of affine real algebraic variety, smooth manifold, and affine Nash manifold are all intrinsic and independent of the embedding $X \subset \mathbb{R}^n$.

Let

$$\begin{aligned}
\mathcal{T}^{Zar} &= \text{the Zariski topology on } X \\
\mathcal{T}^{Res} &= \text{the restricted topology on } X \\
\mathcal{T}^{Euc} &= \text{the Euclidean topology on } X
\end{aligned}$$

The $\mathcal{T}^{Zar}$ and $\mathcal{T}^{Euc}$ are topologies on X, while the $\mathcal{T}^{Res}$ is only a restricted topology on X (Definition 3.11). We will call subsets in these "topologies" **Zariski open, restricted open, Euclidean open** subsets respectively. It is obvious that

$$\mathcal{T}^{Zar} \subset \mathcal{T}^{Res} \subset \mathcal{T}^{Euc}.$$

Let

$$\begin{aligned}
\mathcal{R}_X &= \text{the sheaf of regular functions on } (X, \mathcal{T}^{Zar}) \\
\mathcal{N}_X &= \text{the (restricted) sheaf of Nash functions on } (X, \mathcal{T}^{Res}) \\
C^\infty(-, \mathbb{R}) &= \text{the sheaf of real-valued smooth functions on } (X, \mathcal{T}^{Euc})
\end{aligned}$$

These are sheaves on the Zariski topology, restricted topology and Euclidean topology respectively.

Let $U \in \mathcal{T}^{Zar}$ be a Zariski open subset of X, it is also a restricted open subset and Euclidean open subset. We have the inclusions of rings:

$$\mathcal{P}(U) \subset \mathcal{R}_X(U) \subset \mathcal{N}_X(U) \subset C^\infty(U, \mathbb{R}).$$

Here the four rings are rings of polynomial functions on U, regular functions on U, Nash functions on U and smooth functions on U. All these rings are rings of $\mathbb{R}$-valued functions.

Similar to the Nash differential operators in Definition 4.1, we have

Definition 4.3 ([1] Definition 3.5.1)**.** Let X be a nonsingular affine real algebraic variety, which is also regarded as a smooth manifold and affine Nash manifold. Let $C^\infty(X, \mathbb{R})$ be the ring of real-valued smooth functions on X. Let

$$\mathrm{Diff}^{Alg}(X)$$

be the subring of $\mathrm{Hom}_{\mathbb{R}}(C^\infty(M, \mathbb{R}), C^\infty(M, \mathbb{R}))$ generated by the ring $\mathcal{P}(X)$ of polynomial functions and algebraic vector fields. We call this ring the **ring of algebraic differential operators on** X.

4.2 Schwartz Functions on Affine Nash Manifolds

4.2.1 Definition of Schwartz Functions

In this subsection, we recall the notion Schwartz $\mathbb{R}$-valued functions defined in [1].

Schwartz Functions on Affine Nash Manifolds

Let M be an affine Nash manifold.

Definition 4.4 ([1] Definition 4.1.1)**.** A smooth function $f \in C^\infty(M, \mathbb{R})$ is called a **Schwartz function on** M, if for all Nash differential operator $D \in \mathrm{Diff}^{Nash}(M)$, the function Df is bounded on M, i.e.

$$\sup_{x \in M} |(Df)(x)| < \infty, \qquad \forall D \in \mathrm{Diff}^{Nash}(M).$$

We denote by

$$\mathcal{S}(M, \mathbb{R})$$

the space of Schwartz functions on M.

For each $D \in \mathrm{Diff}^{Nash}(M)$, we have a seminorm $|\cdot|_D$ on $\mathcal{S}(M, \mathbb{R})$ given by

$$|f|_D := \sup_{x \in M} |(Df)(x)|.$$

The seminorms $\{|\cdot|_D : D \in \mathrm{Diff}^{Nash}(M)\}$ define a locally convex Hausdorff topology on $\mathcal{S}(M, \mathbb{R})$ and make it into a topological vector space over $\mathbb{R}$.

Lemma 4.5 ([1] Corollary 4.1.2 and [2] Corollary 2.6.2)**.** *The space $\mathcal{S}(M, \mathbb{R})$ is a nuclear Fréchet space, under the topology defined by the seminorms $\{|\cdot|_D : D \in \mathrm{Diff}^{Nash}(M)\}$.*

Schwartz Functions on Nonsingular Affine Real Algebraic Varieties

Let X be a nonsingular affine real algebraic variety, then it has a natural structure of affine Nash manifold ([1] Example 3.3.4). We can define the $\mathcal{S}(M, \mathbb{R})$ of Schwartz functions by regarding it as an affine Nash manifold. However the Schwartz functions could be equivalently defined by algebraic differential operators.

Lemma 4.6 ([1] Lemma 3.5.5)**.** *Let X be a nonsingular affine real algebraic variety. The ring $\mathrm{Diff}^{Nash}(X)$ of Nash differential operators on X is generated by the subring $\mathrm{Diff}^{Alg}(X)$ of algebraic differential operators and the subring $\mathcal{N}(X)$ of Nash functions on X.*

Lemma 4.7 ([1] Proposition 4.1.1 and Corollary 4.1.3)**.** *A function $f \in C^\infty(X, \mathbb{R})$ is in $\mathcal{S}(M, \mathbb{R})$ if and only if Df is bounded on X for all algebraic differential operators $D \in \mathrm{Diff}^{Alg}(X)$. Hence Schwartz functions on X could be defined as smooth functions with all algebraic derivatives bounded on X.*

Remark 4.8. By the above Lemma, for the special case of affine real algebraic varieties, the notion of Schwartz functions in [1] is the same as the notion of Schwartz functions studied in [20].

4.2.2 Properties of Schwartz Functions

We summarize the crucial properties of Schwartz function spaces, for details see [1].

Open Extensions of Schwartz Functions

Let M be an affine Nash manifold, and let U be a restricted open subset (hence also an affine Nash manifold). Let $\mathcal{S}(M, \mathbb{R}), \mathcal{S}(U, \mathbb{R})$ be the corresponding spaces of Schwartz functions.

Let $f \in \mathcal{S}(U, \mathbb{R})$ be a Schwartz function on U, we define the naive extension $\mathrm{ex}_U^M f$ of f to M by

$$(\mathrm{ex}_U^M f)(x) := \begin{cases} f(x), & \text{if } x \in U \\ 0, & \text{if } x \in M - U \end{cases} \tag{4.1}$$

Lemma 4.9 ([1] Proposition 4.3.1, or [20] p265 Theorem 1.2.4(i))**.** *The naive extension $\mathrm{ex}_U^M f$ is in $\mathcal{S}(M, \mathbb{R})$. The map*

$$\begin{aligned} \mathcal{S}(U, \mathbb{R}) &\to \mathcal{S}(M, \mathbb{R}) \\ f &\mapsto \mathrm{ex}_U^M f \end{aligned} \tag{4.2}$$

*is an injective continuous linear map, called **extension by zero from** U **to** M. Its image is exactly the subspace*

$$\{f \in \mathcal{S}(M, \mathbb{R}) : Df \equiv 0 \text{ on } M - U, \forall D \in \mathrm{Diff}^{Nash}(M)\},$$

which is a closed subspace of $\mathcal{S}(M, \mathbb{R})$, hence the above extension map has closed image and is a homomorphism of TVS.

Partition of Unity

Lemma 4.10 ([1] Theorem 4.4.1, or [20] p268 Lemma 1.2.7)**.** *Let M be an affine Nash manifold, and let $\{U_i\}_{i=1}^k$ be a finite family of restricted open subsets of M, covering M. Then there exist smooth functions $\alpha_i \in C^\infty(M, \mathbb{R}), i = 1, \ldots, k$ such that* $\mathrm{supp}\alpha_i \subset U_i$ *and* $\sum_{i=1}^k \alpha_i \equiv 1$ *on M. Moreover, we can choose α_i such that for any $f \in \mathcal{S}(M, \mathbb{R})$, the $\alpha_i f \in \mathcal{S}(U_i, \mathbb{R})$.*

Remark 4.11. The α_i in this Lemma could be chosen to be tempered functions ([1] Definition 4.2.1), but we don't need this notion.

Cosheaf Property

Let U_1, U_2 be two restricted open subsets of M, and $\mathcal{S}(U_1, \mathbb{R}), \mathcal{S}(U_2, \mathbb{R})$ be the Schwartz function spaces over them. Assume $U_1 \subset U_2$, then the U_1 is a restricted open subset of the affine Nash manifold U_2, and we have the extension map $\mathrm{ex}_{U_1}^{U_2} : \mathcal{S}(U_1, \mathbb{R}) \to \mathcal{S}(U_2, \mathbb{R})$ as in Lemma 4.9. This makes the correspondence $\mathcal{S}(-, \mathbb{R}) : U \mapsto \mathcal{S}(U, \mathbb{R})$ into a pre-cosheaf on the restricted topology on M. Actually we have

Lemma 4.12 ([1] Proposition 4.4.4)**.** *The $\mathcal{S}(-, \mathbb{R}) : U \to \mathcal{S}(U, \mathbb{R})$ is a cosheaf on the restricted topology of M.*

Closed Restrictions of Schwartz Functions

In [1], the authors use the term "affine Nash submanifold $Z \subset M$" without defining it. We make the following definition, which is sufficient for our use.

Definition 4.13. Let $M \subset \mathbb{R}^n$ be a Nash submanifold of $\mathbb{R}^n$ which is closed under the Euclidean topology. A subset $Z \subset M$ is called a **closed submanifold of the Nash submanifold** $M \subset \mathbb{R}^n$, if

- $Z \subset \mathbb{R}^n$ is a Nash submanifold of $\mathbb{R}^n$, and is closed in M under the Euclidean topology.
- the embedding $Z \hookrightarrow M$ is a Nash map and regular embedding of manifolds.

With these two conditions, the inclusions $Z \hookrightarrow \mathbb{R}^n$ and $Z \hookrightarrow M$ are Nash maps, Z is closed in M and $\mathbb{R}^n$, and Z is indeed a regular submanifold of M in the sense of smooth manifolds.

Remark 4.14. In [1], the authors give a definition of general Nash manifold through ringed space as in algebraic geometry. In particular, the notion of "affine Nash manifold" (Definition 3.19 or [1] Definition 3.3.1) is defined as a ringed space, without embedding into any $\mathbb{R}^n$. However, we will only study affine Nash manifolds, and could find an embedding of them into Euclidean spaces.

Lemma 4.15 ([1] Theorem 4.6.1, or [20] p266 Theorem 1.2.4 (iii))**.** *Let $M \subset \mathbb{R}^n$ be a closed Nash submanifold of $\mathbb{R}^n$, and $Z \subset M$ is a closed Nash submanifold in the sense of Definition 4.13. For a Schwartz function $f \in \mathcal{S}(M, \mathbb{R})$, let $f|_Z$ be its restriction to Z. Then $f|_Z \in \mathcal{S}(Z, \mathbb{R})$, and the map*

$$\begin{aligned} \mathcal{S}(M, \mathbb{R}) &\to \mathcal{S}(Z, \mathbb{R}) \\ f &\mapsto f|_Z \end{aligned} \tag{4.3}$$

is a continuous linear map. It is surjective, hence is a homomorphism of TVS.

External Tensor Product

Let M_1, M_2 be two affine Nash manifolds, and their direct product $M_1 \times M_2$ has the natural structure of affine Nash manifold. We have

Lemma 4.16 ([2] Corollary 2.6.3, or [20] p268 Proposition 1.2.6(ii))**.** *The natural map*

$$\begin{aligned} \mathcal{S}(M_1, \mathbb{R}) \otimes \mathcal{S}(M_2, \mathbb{R}) &\to \mathcal{S}(M_1 \times M_2, \mathbb{R}) \\ f_1 \otimes f_2 &\mapsto \{f_1 \boxtimes f_2 : (x, y) \mapsto f_1(x) f_2(y)\} \end{aligned} \tag{4.4}$$

extends to an isomorphism of TVS:

$$\mathcal{S}(M_1, \mathbb{R}) \mathbin{\widehat{\otimes}} \mathcal{S}(M_2, \mathbb{R}) \xrightarrow{\sim} \mathcal{S}(M_1 \times M_2, \mathbb{R}) \tag{4.5}$$

4.2.3 Complex-Valued Schwartz Functions

Let M be an affine Nash manifold. We fix an $\mathbb{R}$-basis $\{1, i\}$ of $\mathbb{C}$. Each function $f \in C^\infty(M)$ is written as $f = (f_1, f_2)$ where f_1 is the real part and f_2 is the imaginary part. We have the canonical isomorphism

$$\begin{aligned} C^\infty(M, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} &\to C^\infty(M, \mathbb{C}) \\ f \otimes z &\mapsto \{x \mapsto f(x) z\} \end{aligned} \tag{4.6}$$

which is actually independent of the choice of the $\mathbb{R}$-basis. This isomorphism means a complex valued function is smooth if and only if its real and imaginary parts are smooth. The ring $\mathrm{Diff}^{Nash}(M)$ of Nash differential operators acts on $C^\infty(M,\mathbb{R})$, hence also on the $C^\infty(M,\mathbb{C})$.

Definition 4.17. A function $f \in C^\infty(M,\mathbb{C})$ is a **Schwartz ($\mathbb{C}$-valued) function on** M, if for all $D \in \mathrm{Diff}^{Nash}(M)$, the Df is bounded on M. We denote the space of Schwartz $\mathbb{C}$-valued functions by

$$\mathcal{S}(M).$$

We endow this space with the TVS structure induced by the semi-norms $|\cdot|_D$:

$$|f|_D := \sup_{x\in M} |(Df)(x)|$$

Let $\mathrm{Diff}^{Nash}(M) \otimes_{\mathbb{R}} \mathbb{C}$ be the (complexified) algebra of complex Nash differential operators. It acts on the $C^\infty(M,\mathbb{C}) = C^\infty(M,\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$ in the natural way. We have

Lemma 4.18. *A function $f \in C^\infty(M,\mathbb{C})$ is in $\mathcal{S}(M)$ if and only if Df is bounded on M for all $D \in \mathrm{Diff}^{Nash}(M) \otimes_{\mathbb{R}} \mathbb{C}$.*

Lemma 4.19. *For the space $\mathcal{S}(M)$ of $\mathbb{C}$-valued Schwartz functions, the natural map*

$$\begin{aligned} \mathcal{S}(M,\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} &\to \mathcal{S}(M) \\ f \otimes z &\mapsto \{x \mapsto f(x)z\} \end{aligned} \tag{4.7}$$

is an isomorphism between TVS. In particular, the $\mathcal{S}(M)$ is a (complex) nuclear Fréchet space.

Remark 4.20. By this crucial isomorphism, we can freely generalize the properties of $\mathcal{S}(M,\mathbb{R})$ to $\mathcal{S}(M)$. Actually the maps (4.2), (4.3) and (4.4) all extends to the spaces of $\mathbb{C}$-valued Schwartz functions, since the functor $-\otimes_{\mathbb{R}} \mathbb{C}$ is exact on NF-spaces.

We have $\mathbb{C}$-valued analogues of Lemma 4.9, 4.10, 4.12, 4.15 and 4.16. We summarize them as the following Lemma:

Lemma 4.21. *Let M be an affine Nash manifold, and for each restricted open subset $U \subset M$, let $\mathcal{S}(U)$ be the space of $\mathbb{C}$-valued Schwartz functions on the affine Nash manifold U.*

($\mathbb{C}-1$) *The space $\mathcal{S}(U)$ is a nuclear Fréchet space over $\mathbb{C}$ for each restricted open subset U.*

($\mathbb{C}-2$) *For each pair U_1, U_2 of restricted open subsets of M such that $U_1 \subset U_2$, the extension map (by zero)*

$$\mathrm{ex}_{U_1}^{U_2} : \mathcal{S}(U_1) \to \mathcal{S}(U_2)$$

is an injective homomorphism of TVS, with image equal to the closed subspace

$$\{f \in \mathcal{S}(U_2) : Df \equiv 0 \textit{ on } U_2 - U_1, \forall D \in \mathrm{Diff}^{Nash}(U_2)\}.$$

($\mathbb{C}-3$) *For each finite restricted open cover $\{U_i\}_{i=1}^k$ of an affine Nash manifold M, one can choose a smooth function $\alpha_i \in C^\infty(M, \mathbb{C})$ such that $\mathrm{supp}\alpha_i \subset U_i, \sum_{i=1}^k \alpha_i \equiv 1$ and $\alpha_i f \in \mathcal{S}(U_i)$ for all $f \in \mathcal{S}(M)$.*

($\mathbb{C}-4$) *The correspondence $U \mapsto \mathcal{S}(U)$ with the above extension maps form a cosheaf (of NF-spaces) on the restricted topology of M.*

($\mathbb{C}-5$) *For each closed affine Nash submanifold $Z \subset M$, the restriction map*

$$\mathcal{S}(M) \to \mathcal{S}(Z), \quad f \mapsto f|_Z$$

is a surjective homomorphism of TVS.

($\mathbb{C}-6$) *Let M_1, M_2 be two affine Nash manifolds, then the external tensor product induces an isomorphism of TVS:*

$$\mathcal{S}(M_1) \widehat{\otimes} \mathcal{S}(M_2) \xrightarrow{\sim} \mathcal{S}(M_1 \times M_2).$$

4.3 Vector Valued Schwartz Functions

4.3.1 Vector Valued Schwartz Functions on Affine Nash Manifolds

In this subsection, let

$$M = \text{an affine Nash manifold}$$
$$E = \text{a nuclear Fréchet TVS over } \mathbb{C}$$

We introduce the notion of E-valued Schwartz functions on M.

Definition of E-Valued Schwartz Functions

The E-valued Schwartz functions are defined in the similar way as $\mathbb{R}$-valued or $\mathbb{C}$-valued Schwartz functions.

Definition 4.22. A smooth function $f \in C^\infty(M, E)$ is called a **Schwartz E-valued function on** M, if for all $D \in \mathrm{Diff}^{Nash}(M)$, the derivative Df is bounded on M. More precisely, let $\{|\cdot|_\rho : \rho \in \mathfrak{P}\}$ be the family of semi-norms defining the Fréchet structure of E, then a function $f \in C^\infty(M, E)$ is a Schwartz E-valued function if for all semi-norm $|\cdot|_\rho$, and $D \in \mathrm{Diff}^{Nash}(M)$, one has

$$\sup_{x \in M} |(Df)(x)|_\rho < \infty.$$

We denote by

$$\mathcal{S}(M, E)$$

the space of Schwartz E-valued functions on M, and endow it with the topology defined by seminorms $\{q_{D,\rho} : D \in \mathrm{Diff}^{Nash}(M), \rho \in \mathfrak{P}\}$ where

$$q_{D,\rho}(f) := \sup_{x \in M} |(Df)(x)|_\rho.$$

Lemma 4.23. *The $\mathcal{S}(M, E)$ is a nuclear Fréchet space, under the topology defined by seminorms $\{q_{D,\rho} : D \in \mathrm{Diff}^{Nash}(M), \rho \in \mathfrak{P}\}$.*

Lemma 4.24 ([20] p268 Proposition 1.2.6)**.** *The natrual map*

$$\begin{aligned} \mathcal{S}(M) \otimes E &\to \mathcal{S}(M, E) \\ f \otimes v &\mapsto \{x \mapsto f(x)v\} \end{aligned} \tag{4.8}$$

is a continuous linear map which extends to an isomorphism of TVS

$$\mathcal{S}(M) \mathbin{\widehat{\otimes}} E \xrightarrow{\sim} \mathcal{S}(M, E) \tag{4.9}$$

Open Extensions of Schwartz E-Valued Functions

Let M be an affine Nash manifold, and U be a restricted open subset of M. Then U is also an affine Nash manifold, and we have the spaces $\mathcal{S}(U)$ and $\mathcal{S}(U, E)$ of Schwartz functions on U.

Given a $f \in \mathcal{S}(U, E)$, let $\mathrm{ex}_U^X f$ be the naive extension of f to M by zero:

$$\mathrm{ex}_U^M f(x) := \begin{cases} f(x), & \text{if } x \in U \\ 0, & \text{if } x \in M - U \end{cases}$$

Lemma 4.25. *The* $\mathrm{ex}_U^X f$ *is in* $\mathcal{S}(X,E)$*. And the map*

$$\begin{aligned} \mathrm{ex}_U^X : \mathcal{S}(U,E) &\to \mathcal{S}(X,E) \\ f &\mapsto \mathrm{ex}_U^X f \end{aligned} \tag{4.10}$$

is an injective continuous linear map between TVS. Its image is the subspace

$$\{f \in \mathcal{S}(M,E) | (Df)|_{M-U} = 0, \forall D \in \mathrm{Diff}^{Nash}(M)\}$$

which is a closed subspace. Hence ex_U^M *is a homomorphism of TVS.*

By Lemma 4.24, the map ex_U^M maps the subspace $\mathcal{S}(U) \otimes E$ of $\mathcal{S}(U,E)$ continuously to the subspace $\mathcal{S}(M) \otimes E$, hence extends to a continuous linear map on the completions $\mathcal{S}(U,E) \to \mathcal{S}(M,E)$. Obviously this extension is exactly the ex_U^M.

Partition of Unity

Lemma 4.26. *Let* M *be an affine Nash manifold, and* $\{U_i\}_{i=1}^k$ *be a finite restricted open cover of* M*. There exists a partition of unity, i.e. a smooth function* $\alpha_i \in C^\infty(M,\mathbb{R})$*,* $\mathrm{supp}\alpha_i \subset U_i$ *and* $\sum_{i=1}^k \alpha_i \equiv 1$ *on* M*. Moreover, for each* $f \in \mathcal{S}(M,E)$*, we have* $\alpha_i f \in \mathcal{S}(U_i,E)$*.*

We can use the same partition of unity as in Lemma 4.10.

Cosheaf Property of $\mathcal{S}(-,E)$

Lemma 4.27. *The correspondence* $\mathcal{S}(-,E) : U \mapsto \mathcal{S}(U,E)$ *is a cosheaf of NF-spaces on the restricted topology of* M*.*

Given a finite restricted open cover $U = \cup_{i=1}^n U_i$ of a restricted open subset U, the sequence

$$\prod_{i=1}^{n-1} \prod_{j=i+1}^{n} \mathcal{S}(U_i \cap U_j) \to \prod_{i=1}^{n} \mathcal{S}(U_i) \to \mathcal{S}(U) \to 0$$

is exact since $\mathcal{S}(-)$ is a cosheaf on the restricted topology. Since E is nuclear Fréchet, the functor $- \widehat{\otimes} E$ is exact. We obtain the exact sequence

$$\prod_{i=1}^{n-1} \prod_{j=i+1}^{n} \mathcal{S}(U_i \cap U_j, E) \to \prod_{i=1}^{n} \mathcal{S}(U_i, E) \to \mathcal{S}(U,E) \to 0$$

by Lemma 4.24.

Closed Restrictions of Schwartz E-Valued Fnctions

Lemma 4.28. *Let M be an affine Nash manifold, $Z \subset M$ be a closed Nash submanifold. The restriction map*

$$\begin{aligned} \mathcal{S}(M,E) &\to \mathcal{S}(Z,E) \\ f &\mapsto f|_Z \end{aligned} \tag{4.11}$$

is a surjective homomorphism of TVS.

The restriction $f|_Z$ is obviously Schwartz. The above map is obtained by tensoring the map $\mathcal{S}(M,\mathbb{C}) \to \mathcal{S}(Z,\mathbb{C})$ in Lemma 4.21, by the E. Since E is nuclear, the $-\widehat{\otimes}\, E$ is exact, and the above map is still surjective.

Externel Tensor Products

Lemma 4.29. *Let M_1, M_2 be two affine Nash manifolds, E_1, E_2 be two nuclear Fréchet spaces. The natural map*

$$\begin{aligned} \mathcal{S}(M_1,E_1) \otimes \mathcal{S}(M_2,E_2) &\to \mathcal{S}(M_1 \times M_2, E_1 \widehat{\otimes} E_2) \\ f_1 \otimes f_2 &\mapsto \{f_1 \boxtimes f_2 : (x_1,x_2) \mapsto f_1(x_1) \otimes f_2(x_2)\} \end{aligned} \tag{4.12}$$

is continuous linear and extends to an isomorphism

$$\mathcal{S}(M_1,E_1) \widehat{\otimes} \mathcal{S}(M_2,E_2) \xrightarrow{\sim} \mathcal{S}(M_1 \times M_2, E_1 \widehat{\otimes} E_2). \tag{4.13}$$

By Lemma 4.24, we have the natural map

$$\mathcal{S}(M_1) \otimes \mathcal{S}(M_2) \otimes E_1 \otimes E_2 \to \mathcal{S}(M_1 \times M_2, E_1 \widehat{\otimes} E_2)$$

which extends to an isomorphism on the completion.

4.3.2 Summary of Properties of Schwartz Functions

We summarize the crucial properties about Schwartz E-valued functions in Lemma 4.23, 4.24, 4.25, 4.26, 4.27, 4.28 and 4.29, as the following proposition. The Lemma 4.21 is thus the special case of the following proposition when $E = \mathbb{C}$.

Proposition 4.30. *Let M be an affine Nash manifold, E be a nuclear Fréchet space, U be an arbitrary restricted open subset of M. Let $\mathcal{S}(M)$ (resp. $\mathcal{S}(U)$) be the space of $\mathbb{C}$-valued Schwartz functions on M (resp. U), and $\mathcal{S}(M,E)$ (resp. $\mathcal{S}(U,E)$) be the space of E-valued Schwartz functions on M (resp. U).*

(E − 1) *For each restricted open U, the $\mathcal{S}(U,E)$ is a nuclear Fréchet TVS, and the natural map $\mathcal{S}(U)\otimes E\to\mathcal{S}(U,E)$ extends to an isomorphism of TVS:*

$$\mathcal{S}(U)\,\widehat{\otimes}\,E \xrightarrow{\sim} \mathcal{S}(U,E).$$

(E − 2) *For a restricted open subset $U\subset M$, the extension map (by zero)*

$$\mathcal{S}(U,E)\hookrightarrow\mathcal{S}(X,E) \tag{4.14}$$

is an injective homomorphism of TVS.

(E − 3) *Let $\{U_i\}_{i=1}^k$ be a restricted open cover of M, then there exists a partition of unity subordinate to the cover, i.e. there exist smooth functions α_i on M such that $\mathrm{supp}\alpha_i\subset U_i, \sum_{i=1}^k\alpha_i\equiv 1$ and $\alpha_i f\in\mathcal{S}(U_i,E)$ for all $f\in\mathcal{S}(M,E)$.*

(E − 4) *The correspondence $U\mapsto\mathcal{S}(U,E)$ is a cosheaf of NF-spaces on the restricted topology of M.*

(E − 5) *Let $Z\subset M$ be a closed Nash submanifold of M and $\mathcal{S}(Z,E)$ be the space of E-valued Schwartz functions on Z. The restriction map*

$$\mathcal{S}(X,E)\twoheadrightarrow\mathcal{S}(Z,E) \tag{4.15}$$

is a surjective homomorphism of TVS.

(E − 6) *Let M_1,M_2 be two smooth real algebraic manifolds, let E_1,E_2 be two nuclear Fréchet spaces, and $\mathcal{S}(X_i,E_i), i=1,2$ be the spaces of E_i-valued Schwartz functions on M_i. The natural map*

$$\mathcal{S}(M_1,E_1)\,\widehat{\otimes}\,\mathcal{S}(M_2,E_2)\xrightarrow{\sim}\mathcal{S}(M_1\times M_2,E_1\,\widehat{\otimes}\,E_2). \tag{4.16}$$

is an isomorphism of TVS.

4.4 Schwartz Distributions

In this section, let

$$\begin{aligned} M &= \text{an affine Nash manifold}\\ E,F &= \text{two nuclear Fréchet spaces}\\ \mathcal{S}(-,E) &= \text{the cosheaf of Schwartz } E\text{-valued functions}\\ &\quad\ \text{on the restricted topology of } M \end{aligned}$$

Definition 4.31. A continuous linear map from $\mathcal{S}(M, E)$ to F is called a **Schwartz F-valued E-distribution on** M, and we denote by

$$L(\mathcal{S}(M, E), F)$$

the **space of Schwartz F-valued E-distributions on** M. It is exactly the space of continuous linear maps from $\mathcal{S}(M, E)$ to F.

If $F = \mathbb{C}$, we simply call an element $\Phi \in L(\mathcal{S}(M, E), \mathbb{C})$ a **Schwartz E-distribution on** M, and the space of Schwartz E-distributions on M is also denoted by

$$\mathcal{S}(M, E)' = L(\mathcal{S}(M, E), \mathbb{C})$$

i.e. it is exactly the strong dual of $\mathcal{S}(M, E)$ with strong topology.

If further $E = F = \mathbb{C}$, we simply call an element $\Phi \in L(\mathcal{S}(M), \mathbb{C})$ a **Schwartz distribution on** M, and the space of Schwartz distributions on M is exactly strong dual space $\mathcal{S}(M)'$.

By abuse of terms, we simply call them **distributions**, when there is no ambiguity on the spaces E and F.

Since $\mathcal{S}(M, E)$ and F are both nuclear Fréchet, by (50.18) on page 525 of [36], we have

Lemma 4.32. *The natural map $\mathcal{S}(M, E)' \otimes F \to L(\mathcal{S}(M, E), F)$ extends to an isomorphism of TVS:*

$$\mathcal{S}(M, E)' \widehat{\otimes} F \xrightarrow{\sim} L(\mathcal{S}(M, E), F) \tag{4.17}$$

In particular, the space $L(\mathcal{S}(M, E), F)$ is a tensor product of an NF-space F with a DNF-space $\mathcal{S}(M, E)'$, hence is a nuclear space.

Remark 4.33. Note that in general $L(\mathcal{S}(M, E), F)$ is not a Fréchet space. For example, when $F = \mathbb{C}$, the distribution space is $\mathcal{S}(M, E)'$–a dual of a Fréchet space, which is almost never Fréchet unless it is finite dimensional (e.g. M is a point, and E is finite dimensional, then the $\mathcal{S}(M, E) = E$ and $\mathcal{S}(M, E)' = E'$.)

4.4.1 Sheaf of Schwartz Distributions

We keep the M, E, F as the beginning of this section.

Definition 4.34. Let U_1, U_2 be two restricted open subsets of M such that $U_1 \subset U_2$. We have the open extension map as in Lemma 4.25

$$\mathrm{ex}_{U_1}^{U_2} : \mathcal{S}(U_1, E) \hookrightarrow \mathcal{S}(U_2, E).$$

The F-transpose of this extension map gives the following continuous linear map

$$\begin{aligned}\operatorname{res}_{U_1}^{U_2} : L(\mathcal{S}(U_2, E), F) &\to L(\mathcal{S}(U_1, E), F) \\ D &\mapsto D \circ \operatorname{ex}_{U_1}^{U_2}\end{aligned} \tag{4.18}$$

called the **restriction map of Schwartz distributions from U_2 to U_1**.

The restriction maps are homomorphisms of TVS, and they are all surjective homomorphisms, i.e. all local Schwartz distributions are restrictions of global Schwartz distributions.

The correspondence $U \mapsto L(\mathcal{S}(U, E), F)$ is a presheaf (of nuclear TVS) on the restricted topology of M. Actually we have

Lemma 4.35. *The correspondence $U \mapsto L(\mathcal{S}(U, E), F)$ is sheaf on the restricted topology on M.*

Proof. By Lemma 4.32, we just need to show the case $F = \mathbb{C}$. The "sheaf-sequence" is exact because it is the dual sequence of the "cosheaf-sequence", which is exact. □

4.4.2 Extension of Schwartz Distributions from Closed Nash Submanifolds

Let M, E, F be as the beginning of this section. Let $Z \subset M$ be a closed Nash submanifold. By Proposition 4.30, we have the map

$$\mathcal{S}(M, E) \twoheadrightarrow \mathcal{S}(Z, E), f \mapsto f|_Z$$

which is a surjective homomorphism of TVS. Its F-transpose gives a homomorphism on the space of F-valued Schwartz E-distributions:

Lemma 4.36. *The F-transpose map of the homomorphism $\mathcal{S}(M, E) \twoheadrightarrow \mathcal{S}(Z, E)$ (surjective) is an injective homomorphism of TVS:*

$$L(\mathcal{S}(Z, E), F) \hookrightarrow L(\mathcal{S}(M, E), F) \tag{4.19}$$

called the ***extension of Schwartz distributions from Z to M****.*

4.4.3 Distributions Supported in Closed Subsets

Definition 4.37. A subset Z of a affine Nash manifold M is called a **restricted closed subset**, if its complement $M - Z$ is a restricted open subset.

Let $U \subset M$ be a restricted open subset, $Z = M - U$ be its complement, (a restricted closed subset of M) and $D \in L(\mathcal{S}(M, E), F)$ be a Schwartz F-valued E-distribution. We say D **is supported in** Z, or D **vanishes on** U, if $\mathrm{res}^M_U(D) = 0$, i.e. the restriction of D to U is zero. The space of distributions supported in Z (vanishing on U) is exactly the kernel $\mathrm{Ker}(\mathrm{res}^M_U)$, which is a closed subspace of $L(\mathcal{S}(M, E), F)$.

Lemma 4.38. *Let U_1, U_2 be two restricted open subsets of M such that $U_1 \subset U_2$, let Z be a restricted closed subset of U_1 (the $U_1 - Z$ and $U_2 - Z$ are restricted open subsets of M). Then the restriction map*

$$\mathrm{res}^{U_2}_{U_1} : L(\mathcal{S}(U_2, E), F) \to L(\mathcal{S}(U_1, E), F)$$

sends $\mathrm{Ker}(\mathrm{res}^{U_2}_{U_2 - Z})$ *isomorphically to* $\mathrm{Ker}(\mathrm{res}^{U_1}_{U_1 - Z})$.

The open embeddings

$$\begin{tikzcd} U_1 \arrow[r, hook] & U_2 \\ U_1 - Z \arrow[u, hook] \arrow[r, hook] & U_2 - Z \arrow[u, hook] \end{tikzcd}$$

induce open extension maps

$$\begin{tikzcd} \mathcal{S}(U_1, E) \arrow[r, hook] & \mathcal{S}(U_2, E) \\ \mathcal{S}(U_1 - Z, E) \arrow[u, hook] \arrow[r, hook] & \mathcal{S}(U_2 - Z, E) \arrow[u, hook] \end{tikzcd}$$

The F-transpose of this diagram thus gives the following diagram of restriction maps of distributions:

$$
\begin{tikzcd}
\mathrm{Ker}(\mathrm{res}^{U_1}_{U_1-Z}) \arrow[d, hook] & \mathrm{Ker}(\mathrm{res}^{U_2}_{U_2-Z}) \arrow[l] \arrow[d, hook] \\
L(\mathcal{S}(U_1,E),F) \arrow[d, "\mathrm{res}^{U_1}_{U_1-Z}"'] & L(\mathcal{S}(U_2,E),F) \arrow[l, "\mathrm{res}^{U_2}_{U_1}"'] \arrow[d, "\mathrm{res}^{U_2}_{U_2-Z}"] \\
L(\mathcal{S}(U_1-Z,E),F) & L(\mathcal{S}(U_2-Z,E),F) \arrow[l, "\mathrm{res}^{U_2-Z}_{U_1-Z}"]
\end{tikzcd}
$$

Figure 4.1: Independent of Neighbourhoods

This Lemma says: the top horizontal arrow is an isomorphism of TVS.

Proof. **(Injectivity):** Let $D \in \mathrm{Ker}(\mathrm{res}^{U_2}_{U_2-Z})$ be a Schwartz distribution on U_2 vanishing on $U_2 - Z$. If $\mathrm{res}^{U_2}_{U_1}(D) = 0$, then D satisfies

$$\mathrm{res}^{U_2}_{U_1}(D) = 0, \quad \mathrm{res}^{U_2}_{U_2-Z}(D) = 0.$$

Note that $U_1 \cup (U_2 - Z) = U_2$, by the sheaf property of $L(\mathcal{S}(-,E),F)$, we see $D = 0$ as a distribution on U_2. Hence $\mathrm{res}^{U_2}_{U_1}$ maps $\mathrm{Ker}(\mathrm{res}^{U_2}_{U_2-Z})$ injectively to $\mathrm{Ker}(\mathrm{res}^{U_1}_{U_1-Z})$.

(Surjectivity): Let $D \in \mathrm{Ker}(\mathrm{res}^{U_1}_{U_1-Z}) \subset L(\mathcal{S}(U_1,E),F)$ be a Schwartz distribution on U_1 vanishing on $U_1 - Z$. Consider the zero distribution 0_{U_2-Z} on the restricted open subset $U_2 - Z$. Obviously we have

$$\mathrm{res}^{U_1}_{U_1-Z}(D) = 0 = \mathrm{res}^{U_2-Z}_{U_1-Z}(0_{U_2-Z})$$

i.e. the D and 0_{U_2-Z} agree on the restricted open subset $U_1 - Z = U_1 \cap (U_2 - Z)$. By the sheaf property, they glue up to a distribution $\widetilde{D}$ on $U_1 \cup (U_2 - Z) = U_2$. Obviously $\mathrm{res}^{U_2}_{U_1}(\widetilde{D}) = D$, and $\mathrm{res}^{U_2}_{U_2-Z}(\widetilde{D}) = 0$, hence we find a pre-image $\widetilde{D}$ of D in $\mathrm{Ker}(\mathrm{res}^{U_2}_{U_2-Z})$. $\square$

As a corollary to the above Lemma, we have

Lemma 4.39. *Let U_1, U_2 be two restricted open subsets of M, and let $Z \subset M$ be a subset of U_1 and U_2 which is restricted open in both of them. Then the two kernels $\mathrm{Ker}(\mathrm{res}^{U_1}_{U_1-Z})$ and $\mathrm{Ker}(\mathrm{res}^{U_2}_{U_2-Z})$ are isomorphic.*

Proof. We just need to apply the above Lemma to U_1, U_2 and $U_1 \cap U_2$, and the following three kernels are canonically isomorphic:

$$\mathrm{Ker}(\mathrm{res}^{U_1}_{U_1-Z}) \simeq \mathrm{Ker}(\mathrm{res}^{U_1\cap U_2}_{U_1\cap U_2-Z}) \simeq \mathrm{Ker}(\mathrm{res}^{U_2}_{U_2-Z}).$$

□

4.5 Application to Nonsingular Affine Real Algebraic Varieties

We will apply the above theory of Schwartz functions and distributions, to nonsingular affine real algebraic varieties, or more specifically the $G = \mathbf{G}(\mathbb{R})$.

In [20], Fokko du Cloux has defined the Schwartz functions on "semi-algebraic varieties". For the particular case of affine algebraic varieties, the definition of Schwartz functions in [1] coincide with the definition in [20] ([1] p6 Remark 1.6.3). Hence we can identify the two notions of $\mathbb{R}$-valued Schwartz functions in [1] and [20].

Remark 4.40. After the Chapter 4 is written, we found out that B. Elazar and A. Shaviv have generalized the work of Aizenbud-Gourevitch to (non-affine) real algebraic varieties (arXiv 1701.07334[math.AG]).

4.5.1 Pseudo-Sheaf and Pseudo-Cosheaf

In this subsection, we introduce two terms: pseudo-sheaves and pseudo-cosheaves, on topological spaces. In a word, they are just sheaves/cosheaves with the open covers in the sheaf/cosheaf axioms replaced by *finite* open covers.

Let

$$(X, \mathcal{T}) = \text{a topological space},$$

i.e. X is a set and $\mathcal{T}$ is a topology on X.

The topology $\mathcal{T}$ is a partial orderd set which could be regarded as a category, with open subsets as objects, open inclusions as morphisms, the empty set $\emptyset$ as initial object and X as the final object.

Let $\mathcal{C}$ be an abelian category. One can define the notion of presheaves and precosheaves on $(X, \mathcal{T})$ in the ordinary sense:

Definition 4.41. A **presheaf on** $(X, \mathcal{T})$ **with value in** $\mathcal{C}$ is a contravariant functor from the category $\mathcal{T}$ to $\mathcal{C}$. When there is no ambiguity on the topology and category $\mathcal{C}$, we simply call them **presheaves**. More precisely, let $\mathcal{F}$ be a presheaf, then

- Each open subset $U \in \mathcal{T}$ is associated with an object $\mathcal{F}(U) \in \mathcal{C}$, and in particular $\mathcal{F}(\emptyset) = \{0\}$.
- For each pair of open subset $U_1 \subset U_2$, one has a morphism

  $$\mathrm{res}_{U_1}^{U_2} : \mathcal{F}(U_2) \to \mathcal{F}(U_1)$$

  in $\mathcal{C}$ called the **restriction from U_2 to U_1**. And for three open subsets $U_1 \subset U_2 \subset U_3$, the restriction morphisms satisfy

  $$\mathrm{res}_{U_1}^{U_2} \circ \mathrm{res}_{U_2}^{U_3} = \mathrm{res}_{U_1}^{U_3}.$$

A **precosheaf on $(X, \mathcal{T})$ with value in $\mathcal{C}$** is a covariant functor from the category $\mathcal{T}$ to $\mathcal{C}$. When there is no ambiguity on the topology and $\mathcal{C}$, we simply call them **precosheaves**. More precisely, let $\mathcal{E}$ be a precosheaf, then

- Each open subset $U \in \mathcal{T}$ is associated with an object $\mathcal{E}(U) \in \mathcal{C}$, and in particular $\mathcal{E}(\emptyset) = \{0\}$.
- For each pair of open subset $U_1 \subset U_2$, one has a morphism

  $$\mathrm{ex}_{U_1}^{U_2} : \mathcal{E}(U_1) \to \mathcal{E}(U_2)$$

  in $\mathcal{C}$ called the **extension from U_1 to U_2**. And for three open subsets $U_1 \subset U_2 \subset U_3$, the restriction morphisms satisfy

  $$\mathrm{ex}_{U_3}^{U_2} \circ \mathrm{ex}_{U_2}^{U_1} = \mathrm{ex}_{U_3}^{U_1}.$$

Definition 4.42. Let $\mathcal{F}$ be a presheaf on $(X, \mathcal{T})$ with values in $\mathcal{C}$. We say $\mathcal{F}$ is a **pseudo-sheaf**, if it satisfies the following two axioms:

- Let U be an open subset and $\{U_i\}_{i=1}^n$ be a *finite* open cover of U: $U = \cup_{i=1}^n U_i$. Let $f \in \mathcal{F}(U)$ be a section on U. If $\mathrm{res}_{U_i}^U f = 0$ for all $i = 1, \ldots, n$, then $f = 0$.
- Let U and $\{U_i\}_{i=1}^n$ be as above (i.e. *finite cover*). For each i, let $f_i \in \mathcal{F}(U_i)$ be a section on U_i. If $\mathrm{res}_{U_i \cap U_j}^{U_i} f_i = \mathrm{res}_{U_i \cap U_j}^{U_j} f_j$ for all $1 \le i, j \le n$, then there is a $f \in \mathcal{F}(U)$ such that $f_i = \mathrm{res}_{U_i}^U f$.

Equivalently, a presheaf $\mathcal{F}$ is a pseudo-sheaf, if for each open U and *finite* open cover $\{U_i\}_{i=1}^n$ of U, the following sequence is exact in $\mathcal{C}$:

$$0 \to \mathcal{F}(U) \xrightarrow{\Phi_1} \prod_{i=1}^{n} \mathcal{F}(U_i) \xrightarrow{\Phi_2} \prod_{i=1}^{n-1} \prod_{j=i+1}^{n} \mathcal{F}(U_i \cap U_j). \tag{4.20}$$

Here the map Φ_1 is

$$\begin{aligned} \Phi_1 : \mathcal{F}(U) &\to \prod_{i=1}^{n} \mathcal{F}(U_i) \\ f &\mapsto (\mathrm{res}_{U_i}^{U} f)_{i=1}^{n} \end{aligned} \tag{4.21}$$

and the map Φ_2 is

$$\begin{aligned} \Phi_2 : \prod_{i=1}^{n} \mathcal{F}(U_i) &\to \prod_{i=1}^{n-1} \prod_{j=i+1}^{n} \mathcal{F}(U_i \cap U_j) \\ (f_i)_{i=1}^{n} &\mapsto ((\mathrm{res}_{U_i \cap U_j}^{U_i} f_i - \mathrm{res}_{U_i \cap U_j}^{U_j} f_j)_{j=i+1}^{n})_{i=1}^{n-1} \end{aligned} \tag{4.22}$$

Definition 4.43. Let $\mathcal{E}$ be a precosheaf on $(X, \mathcal{T})$ with values in $\mathcal{C}$. We say $\mathcal{E}$ is a **pseudo-cosheaf**, if it satisfies the following two axioms:

- Let U be an open subset and $\{U_i\}_{i=1}^n$ be a *finite* open cover of U: $U = \cup_{i=1}^n U_i$. Let $f \in \mathcal{E}(U)$ be a section on U. There exists a $f_i \in \mathcal{E}(U_i)$ for each i, such that
$$f = \sum_{i=1}^{n} \mathrm{ex}_{U_i}^{U} f_i.$$
- Let U and $\{U_i\}_{i=1}^n$ be as above (i.e. *finite cover*). For each i, let $f_i \in \mathcal{E}(U_i)$ be a section on U_i. If $\sum_{i=1}^n \mathrm{ex}_{U_i}^{U} f_i = 0$, then there exists a $f_{ij} \in \mathcal{E}(U_i \cap U_j)$ for all $1 \leq i < j \leq n$, such that
$$f_i = -\sum_{1 \leq k < i} \mathrm{ex}_{U_k \cap U_i}^{U_i} f_{ki} + \sum_{i < k \leq n} \mathrm{ex}_{U_i \cap U_k}^{U_i} f_{ik}$$
for each $1 \leq i \leq n$.

Equivalently, a precosheaf $\mathcal{E}$ is a pseudo-sheaf, if for each open U and *finite* open cover $\{U_i\}_{i=1}^n$ of U, the following sequence is exact in $\mathcal{C}$:

$$\bigoplus_{i=1}^{n-1} \bigoplus_{j=i+1}^{n} \mathcal{E}(U_i \cap U_j) \xrightarrow{\Psi_2} \bigoplus_{k=1}^{n} \mathcal{E}(U_k) \xrightarrow{\Psi_1} \mathcal{E}(U) \to 0. \tag{4.23}$$

Here the map Ψ_1 is

$$\Psi_1 : \bigoplus_{k=1}^{n} \mathcal{E}(U_k) \to \mathcal{E}(U) \tag{4.24}$$
$$(f_k)_{k=1}^n \mapsto \sum_{k=1}^{n} \mathrm{ex}_{U_k}^{U} f_k$$

and the map Ψ_2 is

$$\Psi_2 : \bigoplus_{i=1}^{n-1} \bigoplus_{j=i+1}^{n} \mathcal{E}(U_i \cap U_j) \to \bigoplus_{k=1}^{n} \mathcal{E}(U_k) \tag{4.25}$$
$$((f_{ij})_{j=i+1}^n)_{i=1}^{n-1} \mapsto (-\sum_{1\leq i<k} \mathrm{ex}_{U_i\cap U_k}^{U_k} f_{ik} + \sum_{k<j\leq n} \mathrm{ex}_{U_k\cap U_j}^{U_k} f_{kj})_{k=1}^n$$

Remark 4.44.

- Note that even for a presheaf $\mathcal{F}$, the sequence (4.20) is a complex, and for a precosheaf $\mathcal{E}$, the sequence (4.23) is a complex.
- We call them "pseudo", since they only satisfies the axioms of sheaves and cosheaves on *finite* covers.

4.5.2 Schwartz Functions and Distributions on Nonsingular Affine Real Algebraic Varieties

Let X be a nonsingular affine real algebraic variety. As in 4.1.2, we have seen that it has a natural structure of affine Nash manifold, and all Zariski open subsets of X are restricted open, i.e. the Zariski topology is included in the restricted topology:

$$\mathcal{T}_X^{Zar} \subset \mathcal{T}_X^{Res}.$$

Note that the restricted topology $\mathcal{T}_X^{Res}$ is not a topology since infinite unions of restricted open subsets need not be restricted open. However, the Zariski topology $\mathcal{T}_X^{Zar}$ is indeed a topology.

For two nuclear Fréchet spaces E, F, we have

$$\mathcal{S}(-,E) = \text{the cosheaf of Schwartz } E\text{-valued functions on } X$$
$$L(\mathcal{S}(-,E),F) = \text{the sheaf of Schwartz } F\text{-valued } E\text{-distributions on } X$$

These cosheaf and sheaf are on the restricted topology $\mathcal{T}_X^{Res}$ of X, and we can restrict them to the Zariski topology $\mathcal{T}_X^{Zar}$. With the terms "pseudo-cosheaf" and "pseudo-sheaf" defined in 4.5.1, we have

Lemma 4.45. • *The cosheaf $\mathcal{S}(-, E)$ on $\mathcal{T}_X^{Res}$ restricted to the Zariski topology $\mathcal{T}_X^{Zar}$, is a pseudo-cosheaf on $\mathcal{T}_X^{Zar}$.*

- *The sheaf $L(\mathcal{S}(-, E), F)$ on $\mathcal{T}_X^{Res}$ restricted to the Zariski topology $\mathcal{T}_X^{Zar}$, is a pseudo-sheaf on $\mathcal{T}_X^{Zar}$.*

Remark 4.46. In this thesis, we will only apply the Schwartz analysis to nonsingular affine real algebraic varieties, i.e. the $G = \mathbf{G}(\mathbb{R})$ and its Zariski open/closed subvarieties. We will restrict the cosheaf of Schwartz functions and sheaf of Schwartz distributions to the Zariski topologies, and regard them as pseudo-cosheaves/pseudo-sheaves.

4.5.3 Supports and Maximal Vanishing Subsets of Distributions

As in 4.5.2, let X be a nonsingular affine real algebraic variety, and we define the cosheaves of Schwartz functions and sheaves of Schwartz distributions on it, by regarding it as an affine Nash manifold. As in Lemma 4.45, we restrict the cosheaves of Schwartz functions and sheaves of Schwartz distributions to the Zariski topology of X and obtain the pseudo-cosheaves of Schwartz functions and pseudo-sheaves of Schwartz distributions.

An advantage of restricting the sheaf of Schwartz distributions to the Zariski topology is—such distributions do have supports on the Zariski topology.

Remark 4.47. In classical distribution analysis, the support of a distribution $D \in C_c^\infty(M)'$ on a smooth manifold M, is defined to be the complement of the maximal vanishing open subset of D (see [36] p255 Corollary 2 and Definition 24.2). More precisely, let $U \subset M$ be the maximal (Euclidean) open subset of M such that $\langle D, f \rangle = 0$ for all $f \in C_c^\infty(U)$. Then the support of D denoted by supp$D$ is defined to be the complement of U. Here the maximal vanishing open subset of D is obtained by taking union of all open subsets on which D vanishes. And by Corollary 2 on p255 of [36], the distribution D vanishes on this union.

Note that on restricted topology, an infinite union of restricted open subsets may not be a restricted open subset. Hence one cannot define "the support" of a Schwartz distribution as in classical distribution analysis. And we only have the notion **"support in"** as in Definition 4.37.

However, if we restrict the sheaf of Schwartz distributions to the Zariski topology, we do have the notion of support of a Schwartz distribution, since the Zariski topology is indeed a topology on which one can take infinite unions.

Definition 4.48. Let X be a nonsingular affine real algebraic variety, and E, F be two nuclear Fréchet spaces. We regard $\mathcal{S}(-,E)$ as a pseudo-cosheaf on X and $L(\mathcal{S}(-,E),F)$ as a pseudo-sheaf on X.

Let $D \in L(\mathcal{S}(X,E),F)$ be a Schwartz F-valued E-distribution on X. Let

$$\begin{aligned} \mathrm{van}D &= \text{the union of all Zariski open subsets } U \subset X \\ &\quad\ \text{such that } \mathrm{res}_U^X(D) = 0 \\ \mathrm{supp}D &= (\mathrm{van}D)^c \\ &= \text{the complement of } \mathrm{van}D \end{aligned}$$

We call $\mathrm{van}D$ the **maximal vanishing subset of** D and $\mathrm{supp}D$ the **support of** D.

Lemma 4.49. *For $D \in L(\mathcal{S}(X,E),F)$, the* $\mathrm{van}D$ *is a Zariski open subset of X, and* $\mathrm{supp}D$ *is a Zariski closed subset of X. The* $\mathrm{van}D$ *is the largest Zariski open subset of X on which D vanishes, i.e. the name "maximal vanishing subset" is reasonable.*

Proof. The first part is trivial. We just need to show D indeed vanishes on $\mathrm{van}D$.

By definition, $\mathrm{van}D$ is the union of all Zariski open U such that $D|_U = 0$. Hence $\mathrm{van}D$ is covered by $\{U \in \mathcal{T}_X^{Zar} : D|_U = 0\}$. The Zariski topology is quasi-compact, hence we can choose a finite subcover of $\mathrm{van}D$ from the family $\{U \in \mathcal{T}_X^{Zar} : D|_U = 0\}$, say $\{U_i\}_{i=1}^m$. Hence $D|_{U_i} = 0$ for $i = 1, \ldots, m$ and $\mathrm{van}D = \cup_{i=1}^m U_i$. By the partition of unity property, we can find α_i smooth functions on U_i, such that $\mathrm{supp}\alpha_i \subset U_i$ and $\sum_{i=1}^m \alpha_i \equiv 1$. Then for an arbitrary $f \in \mathcal{S}(\mathrm{van}D, E)$, we have

$$\langle D, f\rangle = \langle D, \sum_{i=1}^m \alpha_i f\rangle = \sum_{i=1}^m \langle \mathrm{res}_{U_i}^X(D), \alpha_i f\rangle = 0.$$

Hence $D|_{\mathrm{van}D} = 0$, i.e. D vanishes on $\mathrm{van}D$. □

4.5.4 Geometry on $G = \mathbf{G}(\mathbb{R})$

We list some well-known facts about $G = \mathbf{G}(\mathbb{R})$, and its subspaces (subvarieties/submanifolds). Note that $G = \mathbf{G}(\mathbb{R})$ is a

- Real affine algebraic variety;
- Smooth manifold (Lie group);
- Affine Nash manifold.

G as a Real Algebraic Variety

The algebraic group $\mathbf{G}$ is defined over $\mathbb{R}$, hence is $\mathbb{R}$-closed. Since $\mathbf{G}$ is affine, this means we can embed $\mathbf{G}$ into a general linear group $\mathbf{GL}_n \subset \mathbb{A}^{n^2}$ as an closed subvariety, and the defining ideal of $\mathbf{G}$ is generated by polynomials with real coefficients.

Lemma 4.50. *For the group $G = \mathbf{G}(\mathbb{R})$, we have*

- *G is an affine real algebraic variety.*
- *For any $\mathbb{R}$-closed subgroup $\mathbf{H}$ of $\mathbf{G}$, the $H = \mathbf{H}(\mathbb{R})$ is an affine real algebraic variety, and Zariski closed subset in G.*
- *Let $\mathbf{P}, \mathbf{Q}$ be two $\mathbb{R}$-closed subgroups of $\mathbf{G}$, and $P = \mathbf{P}(\mathbb{R}), Q = \mathbf{Q}(\mathbb{R})$. Then every (P, Q)-double coset on G is locally closed under Zariski topology, hence is an affine real algebraic variety.*
- *Let $\mathbf{P}, \mathbf{Q}$ be as above, and assume there are finitely many (P, Q)-double cosets on G parameterized by a finite set I. For each $i \in I$, the Zariski open subsets $G_{\geq i}, G_{>i}$ are affine real algebraic varieties.*

G as a Smooth Manifold

We regard $G = \mathbf{G}(\mathbb{R})$ as a Lie group. Let $\mathbf{G} \subset \mathbf{GL}_n$ be an embedding of the variety $\mathbf{G}$ into general linear group (define over $\mathbb{R}$). This induces the embedding on real points $G = \mathbf{G}(\mathbb{R}) \subset \mathbf{GL}(n, \mathbb{R}) \subset \mathbb{R}^{n^2}$, which gives the canonical smooth structure on G.

Lemma 4.51. *For the group $G = \mathbf{G}(\mathbb{R})$ with the canonical structure of smooth manifold, we have*

- *$G = \mathbf{G}(\mathbb{R})$ is a regular submanifold of $\mathbf{GL}(n, \mathbb{R})$ (and $\mathbb{R}^{n^2}$).*
- *For $\mathbb{R}$-closed subgroup $\mathbf{H}$ of $\mathbf{G}$, the $H = \mathbf{H}(\mathbb{R})$ is a regular closed submanifold of G (also of $\mathbf{GL}(n, \mathbb{R})$).*

- *For* $\mathbf{P}, \mathbf{Q}$ *be two* $\mathbb{R}$*-closed subgroup of* $\mathbf{G}$*, and* $P = \mathbf{P}(\mathbb{R}), Q = \mathbf{Q}(\mathbb{R})$*. The* (P, Q)*-double cosets on* G *are locally closed regular submanifolds of* G*.*
- *Let* $\mathbf{P}, \mathbf{Q}$ *be as above, and assume there are finitely many* (P, Q)*-double cosets on* G *parameterized by* I*. Then for each* $i \in I$*, the Zariski open subsets* $G_{\geq i}, G_{>i}$ *are open under the manifold (Euclidean) topology and they are open regular submanifolds of* G*.*

G as an Affine Nash Manifold

Every nonsingular affine real algebraic variety has a canonical structure of affine Nash manifold. Combining the above two subsections, we can study $G = \mathbf{G}(\mathbb{R})$ as an affine Nash manifold.

Lemma 4.52. *For* $G = \mathbf{G}(\mathbb{R})$*, we have*

- *The* $G \subset \mathbb{R}^{n^2}$ *is a closed Nash submanifold, hence an affine Nash manifold.*
- *For* $\mathbf{H} \subset \mathbf{G}$ *a* $\mathbb{R}$*-closed subgroup, the* $H = \mathbf{H}(\mathbb{R})$ *is a closed affine Nash submanifold of* G*.*
- *For* $\mathbf{P}, \mathbf{Q}$ *two* $\mathbb{R}$*-closed subgroups of* $\mathbf{G}$*, and* $P = \mathbf{P}(\mathbb{R}), Q = \mathbf{Q}(\mathbb{R})$*, each* (P, Q)*-double coset is a affine Nash manifold.*
- *For* $\mathbf{P}, \mathbf{Q}$ *as above, assume there are finitely many* (P, Q)*-double cosets on* G*, parameterized by the finite set* I*. Then for each* $i \in I$*, the* $G_{\geq i}, G_{>i}$ *are affine Nash manifolds.*

4.5.5 Right Regular Actions on Schwartz Function Spaces

In this subsection, we let $G = \mathbf{G}(\mathbb{R})$ be the real point group of a connected reductive linear algebraic group $\mathbf{G}$ defined over $\mathbb{R}$. It is a nonsingular affine real algebraic variety and an affine Nash manifold. Let

$$
\begin{aligned}
E &= \text{a nuclear Fréchet space} \\
\mathbf{H} &= \text{a } \mathbb{R}\text{-closed subgroup of } \mathbf{G} \\
\mathbf{Y} &= \text{a nonsingular } \mathbb{R}\text{-subvariety of } \mathbf{G} \\
&\quad\ \text{stable under the right } \mathbf{H}\text{-translation}
\end{aligned}
$$

Then $H = \mathbf{H}(\mathbb{R})$ is a Lie subgroup of G, $Y = \mathbf{Y}(\mathbb{R})$ is a real subvariety of G stable under the right H-translation. And the H acts algebraically on Y by the right translation.

The Y is nonsingular affine real algebraic variety, and it has the space $\mathcal{S}(Y, E)$ of Schwartz E-valued functions on it.

Definition 4.53 (**Right regular actions on Schwartz spaces**)**.** The **right regular H-action on the space** $\mathcal{S}(Y, E)$ of Schwartz E-valued functions on Y, is defined to be

$$[R_h f](y) := f(yh), \qquad \forall h \in H, y \in Y, f \in \mathcal{S}(Y, E).$$

Lemma 4.54. *For the right regular H-action on $\mathcal{S}(Y, E)$, we have:*

1. *The $\mathcal{S}(Y, E)$ is a smooth H-representation. It is isomorphic to the tensor product representation $\mathcal{S}(Y) \widehat{\otimes} E$, where $\mathcal{S}(Y)$ is the space of $\mathbb{C}$-valued Schwartz functions on Y with the right regular H-action, E is considered as a trivial H-representation.*

2. *Let U_1, U_2 be two right H-stable Zariski open subsets of G, and $U_1 \subset U_2$. The inclusion map*

$$\mathcal{S}(U_1, E) \hookrightarrow \mathcal{S}(U_2, E)$$

   is H-equivariant, i.e. an H-intertwining operator.

3. *Let U be a right H-stable Zariski open subset of G, O be a right H-stable nonsingular Zariski closed subset of U. The restriction map*

$$\mathcal{S}(U, E) \to \mathcal{S}(O, E)$$

   is H-equivariant, i.e. an H-intertwining operator.

Example 4.55. For $H = P_\emptyset$, $Y = G$ or $G_{\geq w}, G_{>w}, \forall w \in [W_\Theta \backslash W]$, the spaces $\mathcal{S}(G, E), \mathcal{S}(G_{\geq w}, E)$ and $\mathcal{S}(G_{>w}, E)$ have the above right regular $P_\emptyset$-actions, and they are smooth $P_\emptyset$-representations. The following inclusion maps are all $P_\emptyset$-equivariant (intertwining operators):

$$\mathcal{S}(G_{>w}, E) \hookrightarrow \mathcal{S}(G_{\geq w}, E) \hookrightarrow \mathcal{S}(G, E).$$

Example 4.56. For $H = P$, $Y = G$ or P, the $\mathcal{S}(G, E)$ and $\mathcal{S}(P, E)$ are smooth P-representations. The surjective restriction map

$$\mathcal{S}(G, E) \to \mathcal{S}(P, E)$$

is P-equivariant.

For $H = P_\emptyset$, $Y = G_{\geq w}$ or $PwP_\emptyset$, the $\mathcal{S}(G_{\geq w}, E)$ and $\mathcal{S}(PwP_\emptyset, E)$ are smooth $P_\emptyset$-representations, and the surjective restriction map

$$\mathcal{S}(G_{\geq w}, E) \to \mathcal{S}(PwP_\emptyset, E)$$

is $P_\emptyset$-equivariant.

4.6 Schwartz Inductions

In this section, we recall the notion of Schwartz inductions, and show some properties on such spaces and their strong dual.

The Schwartz induction is first introduced in section 2 of [20]. We have seen that notion of Schwartz functions defined in [20] is the same as the Schwartz functions defined in [1], in case the manifolds under consideration are nonsingular affine real algebraic varieties. Hence we could combine the works in [20] and [1], to prove parallel a set of properties on Schwartz inductions, as summarized in Proposition 4.30.

4.6.1 Schwartz Induction $\mathcal{S}\mathrm{Ind}_P^G\sigma$

Let

$$\begin{aligned}
\mathbf{G} &= \text{a linear algebraic group defined over } \mathbb{R} \\
\mathbf{P} &= \text{a } \mathbb{R}\text{-(closed) subgroup of } \mathbf{G} \\
G = \mathbf{G}(\mathbb{R}) &= \text{the Lie group of real points of } \mathbf{G} \\
P = \mathbf{P}(\mathbb{R}) &= \text{the Lie group of real points of } \mathbf{P} \\
dp &= \text{a fixed right invariant measure on } P \\
(\sigma, V) &= \text{a Harish-Chandra representation of } P \\
&\quad\ \text{(In particular } V \text{ is Fréchet and of moderate growth)} \\
\mathcal{S}(G, V) &= \text{the space of } V\text{-valued Schwartz functions on } G
\end{aligned}$$

The σ-Mean Value and the Schwartz Induction

For a function $f \in \mathcal{S}(G, V)$, we can define its σ**-mean value** by

$$f^\sigma(g) := \int_P \sigma(p)^{-1} f(pg)dp, \quad \forall g \in G. \tag{4.26}$$

This integration converges, since the f is Schwartz and σ is of moderate growth.

The f^σ is a smooth V-valued function on G, satisfying the following **"σ-rule"**:

$$f^\sigma(pg) = \sigma(p) f^\sigma(g), \quad \forall p \in P, \forall g \in G.$$

In other words, the f^σ is in the smooth induction space $C^\infty \mathrm{Ind}_P^G \sigma$.

The map $\mathcal{S}(G, V) \to C^\infty \mathrm{Ind}_P^G \sigma, f \mapsto f^\sigma$ is continuous linear and G-equivariant when $\mathcal{S}(G, V)$ and $C^\infty \mathrm{Ind}_P^G \sigma$ are both endowed with right regular G-actions.

Definition 4.57 ([20] p273 Definition 2.1.2)**.** We denote the image of the map $f \mapsto f^\sigma$ in $C^\infty \mathrm{Ind}_P^G \sigma$ by

$$\mathcal{S}\mathrm{Ind}_P^G(\sigma, V)$$

or simply $\mathcal{S}\mathrm{Ind}_P^G \sigma$ if there is no ambiguity. We endow the space $\mathcal{S}\mathrm{Ind}_P^G(\sigma, V)$ with the quotient topology from $\mathcal{S}(G, V)$. The $\mathcal{S}\mathrm{Ind}_P^G(\sigma, V)$ is called the **Schwartz induction space of** (σ, V) **(from** P **to** G**)**.

Since a quotient of a NF-space is still a NF-space, we have:

Lemma 4.58. *The $\mathcal{S}\mathrm{Ind}_P^G(\sigma, V)$ is a nuclear Fréchet space, and it is a smooth representation of G.*

Schwartz Inductions vs. Smooth Inductions

By definition, the Schwartz induction space $\mathcal{S}\mathrm{Ind}_P^G \sigma$ is a subspace of the smooth induction space $C^\infty \mathrm{Ind}_P^G \sigma$. By a partition of unity argument, we see the space

$$C_c^\infty \mathrm{Ind}_P^G \sigma = \text{smooth induction with compact support modulo } P$$

is contained in $\mathcal{S}\mathrm{Ind}_P^G \sigma$, hence one has the following inclusions:

$$C_c^\infty \mathrm{Ind}_P^G \sigma \subset \mathcal{S}\mathrm{Ind}_P^G \sigma \subset C^\infty \mathrm{Ind}_P^G \sigma.$$

Actually, if the quotient manifold $P \backslash G$ is compact, the above inclusions are equalities:

Lemma 4.59 ([20] p273 Remark 2.1.4)**.** *If the quotient manifold $P \backslash G$ is compact, then the three spaces of inductions coincide:*

$$C_c^\infty \mathrm{Ind}_P^G \sigma = \mathcal{S}\mathrm{Ind}_P^G \sigma = C^\infty \mathrm{Ind}_P^G \sigma.$$

In particular, the $\mathcal{S}\mathrm{Ind}_P^G(\sigma, V)$ is a Harish-Chandra representation.

We will mainly apply this Lemma to the case of parabolic inductions. If $\mathbf{P}$ is a parabolic $\mathbb{R}$-subgroup of $\mathbf{G}$, then the $\mathbf{P} \backslash \mathbf{G}$ is complete, and the manifold of real points $P \backslash G$ is compact by [11] p146 Proposition 14.2.

Schwartz Inductions are NOT Schwartz on G

The functions in the Schwartz inductions are not necessarily rapidly decreasing on G. We give some examples.

Example 4.60. Let $G = (\mathbb{R}, +)$ be the additive group of real numbers, and let $P = G$. The irreducible representations of $P = \mathbb{R}$ are characters of the form

$$\begin{aligned} \chi_\lambda : \mathbb{R} &\to \mathbb{C}^\times \\ r &\mapsto e^{\lambda r} \end{aligned}$$

for some $\lambda \in \mathbb{C}$. It is of moderate growth if and only if χ_λ is unitary or equivalently $\lambda \in i\mathbb{R}$.

Suppose $\lambda \in i\mathbb{R}$, we consider the Schwartz induction $\mathcal{S}\mathrm{Ind}_P^G \chi_\lambda$. A function ϕ in it is a smooth function satisfying

$$\phi(r \cdot x) = \phi(x + r) = \chi_\lambda(r)\phi(x) = e^{\lambda r}\phi(x), \quad \forall r \in P = \mathbb{R}, x \in G = \mathbb{R}.$$

In particular, for arbitrary $r \in \mathbb{R}$, we have $\phi(r) = e^{\lambda r}\phi(0)$. We see

$$\begin{aligned} \lim_{r \to +\infty} \phi(r) &= \phi(0) \lim_{r \to +\infty} e^{\lambda r} \\ \lim_{r \to -\infty} \phi(r) &= \phi(0) \lim_{r \to -\infty} e^{\lambda r} \end{aligned}$$

If the ϕ is Schwartz, then both limits should be zero. However if $\lambda \neq 0$, the limits on the right-hand-side do not exist (periodic function with image on unit circle), if $\lambda = 0$, the limits equal to $\phi(0)$ which may not be zero.

Example 4.61. Let $G = (\mathbb{R}^2, +)$ be the additive group of real plane (with coordinates (x, y)), and let $P = \mathbb{R}_y$ be the y-axis. The P is a closed subgroup of G, with embedding $y \mapsto (0, y)$. For a $\lambda \in i\mathbb{R}$, let χ_λ be a character of P as the above example.

Let $\phi \in \mathcal{S}\mathrm{Ind}_P^G \chi_\lambda$ be an arbitrary function in the Schwartz induction. Then it is a smooth function on $G = \mathbb{R}^2$ satisfying

$$\phi(r \cdot (x, y)) = \phi(x, y + r) = e^{\lambda r}\phi(x, y).$$

In particular, $\phi(0, r) = e^{\lambda r}\phi(0, 0)$, and we see the restriction of ϕ to $\mathbb{R}_y$ is not a Schwartz function. However, for every fixed y, the $\phi(x, y)$ as a function of x, is still rapidly decreasing.

Example 4.62. In general, let G be the real point group of a connected reductive linear algebraic group defined over $\mathbb{R}$, P be the real point group of a parabolic subgroup. Let σ be the one dimensional trivial character of P. Then $\mathcal{S}\mathrm{Ind}_P^G\sigma$ is not contained in $\mathcal{S}(G,\mathbb{C})$. Suppose not, and assume a $\phi \in \mathcal{S}\mathrm{Ind}_P^G\sigma$ is a Schwartz function on G, then its restriction to P is still a Schwartz function (by (E-5) of Proposition 4.30). However,

$$\phi(p) = \sigma(p)\phi(e) = \phi(e), \forall p \in P.$$

Hence ϕ restricted to P is a constant function, which is not Schwartz on P.

Remark 4.63. Although functions in $\mathcal{S}\mathrm{Ind}_P^G\sigma$ are not rapidly decreasing on the entire G, we will see later that they are rapidly decreasing along the "orthogonal direction" of P.

4.6.2 Local Schwartz Inductions

We generalize the above construction and define the local Schwartz inductions. This notion is defined in 2.2.4 of [20]. We keep the setting as in last subsection, and let

$$\begin{aligned} Y = &\text{ a locally closed nonsingular subvariety of } G \\ &\text{ which is stable under left } P\text{-translation.} \end{aligned}$$

Since Y is a nonsingular locally closed subvariety of G, one has the space $\mathcal{S}(Y,V)$ of Schwartz V-valued functions on Y, and one can define the σ-mean value function as in (4.26), i.e. for a $f \in \mathcal{S}(Y,V)$, and $y \in Y$ let

$$f^\sigma(y) := \int_P \sigma(p^{-1}) f(py) dp \tag{4.27}$$

Then f^σ is a smooth V-valued function on Y, satisfying the σ-rule:

$$f^\sigma(py) = \sigma(p) f^\sigma(y), \quad \forall p \in P, y \in Y.$$

Let

$$C^\infty(Y,V,\sigma) = \{f \in C^\infty(G,V) : f(py) = \sigma(p)f(y), \forall p \in P, y \in Y\}$$

be the space of smooth V-valued functions satisfying the σ-rule. The correspondence $f \mapsto f^\sigma$ is a continuous linear map from $\mathcal{S}(Y,V)$ to $C^\infty(Y,V,\sigma)$.

Definition 4.64. The image of the map $\mathcal{S}(Y,V) \to C^\infty(Y,V,\sigma), f \mapsto f^\sigma$ is denoted by

$$\mathcal{S}\mathrm{Ind}_P^Y(\sigma, V)$$

or simply $\mathcal{S}\mathrm{Ind}_P^Y\sigma$ if there is no ambiguity, and we call it the **local Schwartz induction space of** σ **from** P **to** Y. We endow this space with the quotient topology from $\mathcal{S}(Y,V)$, and one has a surjective homomorphism of TVS:

$$\begin{aligned} \mathcal{S}(Y,V) &\twoheadrightarrow \mathcal{S}\mathrm{Ind}_P^Y\sigma \\ f &\mapsto f^\sigma \end{aligned} \tag{4.28}$$

Lemma 4.65. *The* $\mathcal{S}\mathrm{Ind}_P^Y\sigma$ *is a nuclear Fréchet TVS.*

4.6.3 Open Extensions and Closed Restrictions of Schwartz Inductions

Similar to the Schwartz V-valued functions, we have the extension from open subsets and restriction to closed subsets.

Open Extensions of Schwartz Inductions

Let U_1, U_2 be two Zariski open subvarieties of G, which are stable under left P-translation, and assume $U_1 \subset U_2$.

The open embedding induces the injective homomorphism of TVS (extension of the cosheaf):

$$\mathcal{S}(U_1, V) \hookrightarrow \mathcal{S}(U_2, V), \quad f \mapsto \mathrm{ex}_{U_1}^{U_2} f.$$

Let $\phi \in \mathcal{S}\mathrm{Ind}_P^{U_1}\sigma$, and let $f \in \mathcal{S}(U_1, V)$ be a Schwartz function such that $\phi = f^\sigma$. Then the above embedding gives the function $\mathrm{ex}_{U_1}^{U_2} f \in \mathcal{S}(U_2, V)$ (cosheaf extension). Then its σ-mean value function $(\mathrm{ex}_{U_1}^{U_2} f)^\sigma$ is in $\mathcal{S}\mathrm{Ind}_P^{U_2}\sigma$. We have

Lemma 4.66. *The* $(\mathrm{ex}_{U_1}^{U_2} f)^\sigma$ *is independent of the choice of* $f \in \mathcal{S}(U_1, V)$*, and it is exactly the extension of* ϕ *by zero. More precisely, let*

$$\mathrm{Ex}_{U_1}^{U_2}\phi(x) = \begin{cases} \phi(x), & \text{if } x \in U_1 \\ 0, & \text{if } x \in U_2 - U_1 \end{cases}$$

Then $(\mathrm{ex}_{U_1}^{U_2} f)^\sigma = \mathrm{Ex}_{U_1}^{U_2}\phi$ *for all* $f \in \mathcal{S}(U_1, V)$ *such that* $f^\sigma = \phi$*. Hence the* $\mathrm{Ex}_{U_1}^{U_2}\phi$ *is in* $\mathcal{S}\mathrm{Ind}_P^{U_2}\sigma$*.*

Proof. (1) We first verify that the $(\mathrm{ex}_{U_1}^{U_2} f)^\sigma$ is independent of the choice of f. Let $f_0 \in \mathcal{S}(U_1, V)$ satisfy $(f_0)^\sigma = 0$ in $\mathcal{S}\mathrm{Ind}_P^{U_1}\sigma$. Let $\mathrm{ex}_{U_1}^{U_2} f_0 \in \mathcal{S}(U_2, V)$ be its extension to U_2, and let $(\mathrm{ex}_{U_1}^{U_2} f_0)^\sigma \in \mathcal{S}\mathrm{Ind}_P^{U_2}\sigma$ be its σ-mean value function.

By definition, for all $x \in U_2$

$$(\mathrm{ex}_{U_1}^{U_2} f_0)^\sigma(x) = \int_P \sigma(p^{-1}) \mathrm{ex}_{U_1}^{U_2} f_0(px) dp.$$

If $x \in U_1$, since U_1 is stable under left P-translation, the $px \in U_1$ for all $p \in P$, and $\mathrm{ex}_{U_1}^{U_2} f_0(px) = f_0(px)$. Hence

$$(\mathrm{ex}_{U_1}^{U_2} f_0)^\sigma(x) = \int_P \sigma(p^{-1}) f_0(px) dp = (f_0)^\sigma(x) = 0.$$

If $x \in U_2 - U_1$, then $px \in U_2 - U_1$ (the complement $U_2 - U_1$ is also left P-stable), and $\mathrm{ex}_{U_1}^{U_2} f_0(px) = 0$. Hence

$$(\mathrm{ex}_{U_1}^{U_2} f_0)^\sigma(x) = \int_P \sigma(p^{-1}) 0 dp = 0.$$

In sum, if $(f_0)^\sigma = 0$, then $(\mathrm{ex}_{U_1}^{U_2} f_0)^\sigma = 0$. Hence the $(\mathrm{ex}_{U_1}^{U_2} f)^\sigma$ is independent of the choice of f.

(2) Second we check $(\mathrm{ex}_{U_1}^{U_2} f)^\sigma = \mathrm{Ex}_{U_1}^{U_2} \phi$. Let $f \in \mathcal{S}(U_1, V)$ be a Schwartz function satisfying $f^\sigma = \phi$. Still for all $x \in U_2$, one has

$$(\mathrm{ex}_{U_1}^{U_2} f)^\sigma(x) = \int_P \sigma(p^{-1}) \mathrm{ex}_{U_1}^{U_2} f(px) dp.$$

If $x \in U_1$, then $px \in U_1, \forall p \in P$, and $\mathrm{ex}_{U_1}^{U_2} f(px) = f(px)$ and

$$(\mathrm{ex}_{U_1}^{U_2} f)^\sigma(x) = \int_P \sigma(p^{-1}) f(px) dp = f^\sigma(x) = \phi(x) = \mathrm{Ex}_{U_1}^{U_2} \phi(x).$$

If $x \in U_2 - U_1$, then $px \in U_2 - U_1, \forall p \in P$, and $\mathrm{ex}_{U_1}^{U_2} f(px) = 0$ and

$$(\mathrm{ex}_{U_1}^{U_2} f)^\sigma(x) = \int_P \sigma(p^{-1}) 0 dp = 0 = \mathrm{Ex}_{U_1}^{U_2} \phi(x).$$

□

Lemma 4.67. *The map*

$$\begin{aligned}\mathcal{S}\mathrm{Ind}_P^{U_1}\sigma &\to \mathcal{S}\mathrm{Ind}_P^{U_2}\sigma \\ \phi &\mapsto \mathrm{Ex}_{U_1}^{U_2}\phi\end{aligned} \tag{4.29}$$

is a continuous linear map and an injective homomorphism of TVS. This homomorphism makes the following diagram commute:

$$\begin{tikzcd} \mathcal{S}(U_1,V) \arrow[r, hook] \arrow[d, two heads] & \mathcal{S}(U_2,V) \arrow[d, two heads] \\ \mathcal{S}\mathrm{Ind}_P^{U_1}\sigma \arrow[r, hook] & \mathcal{S}\mathrm{Ind}_P^{U_2}\sigma \end{tikzcd}$$

Proof. The map is obviously linear, and the above diagram commute. It is continuous because the topology on $\mathcal{S}\mathrm{Ind}_P^{U_1}\sigma$ is the quotient topology, and the composition map

$$\mathcal{S}(U_1,V) \to \mathcal{S}\mathrm{Ind}_P^{U_1}\sigma \to \mathcal{S}\mathrm{Ind}_P^{U_2}\sigma$$

is continuous since the composition other way is continuous. It is injective since the map is extension by zero. It is a homomorphism of TVS since it has closed image and all spaces are Fréchet. □

Closed Restriction of Schwartz Inductions

Let U be a Zariski open subvariety of G, and O be a Zariski closed subvariety of U (locally closed in G). Assume U and O are all stable under left P-translation. Let $\mathcal{S}\mathrm{Ind}_P^U\sigma, \mathcal{S}\mathrm{Ind}_P^O\sigma$ be their Schwartz induction spaces.

One has the surjective homomorphism of TVS

$$\mathcal{S}(U,V) \twoheadrightarrow \mathcal{S}(O,V), \quad f \mapsto f|_O$$

given by the restriction of Schwartz functions to O.

Let $\phi \in \mathcal{S}\mathrm{Ind}_P^U\sigma$ be an arbitrary element. Let $f \in \mathcal{S}(U,V)$ be a Schwartz function such that $f^\sigma = \phi$, and let $f|_O \in \mathcal{S}(O,V)$ be its restriction to O. Its σ-mean value function $(f|_O)^\sigma$ is in $\mathcal{S}\mathrm{Ind}_P^O\sigma$. We have

Lemma 4.68. *The function $(f|_O)^\sigma \in \mathcal{S}\mathrm{Ind}_P^O\sigma$ is independent of the choice of $f \in \mathcal{S}(U,V)$. More precisely, let*

$$\phi|_O = \textit{the restriction of } \phi \textit{ to } O.$$

Then $\phi|_O = (f|_O)^\sigma$ for all $f \in \mathcal{S}(U,V)$ such that $f^\sigma = \phi$. Hence the $\phi|_O$ is in the Schwartz induction space $\mathcal{S}\mathrm{Ind}_P^O\sigma$.

Proof. We just need to check the $(f|_O)^\sigma = \phi|_O$. Let $f \in \mathcal{S}(U, V)$ be a Schwartz function such that $f^\sigma = \phi$ in $\mathcal{S}\mathrm{Ind}_P^U\sigma$. For $x \in O$, the

$$(f|_O)^\sigma(x) = \int_P \sigma(p^{-1})[f|_O](px)dp.$$

Note that $px \in O, \forall p \in P$, hence $[f|_O](px) = f(px)$. The above integration is exactly $f^\sigma(x)$ for all $x \in O$. Hence as functions on O, the $(f|_O)^\sigma$ is exactly the restriction $\phi|_O$. ☐

Lemma 4.69. *The restriction map*

$$\begin{aligned} \mathcal{S}\mathrm{Ind}_P^U\sigma &\to \mathcal{S}\mathrm{Ind}_P^O\sigma \\ \phi &\mapsto \phi|_O \end{aligned} \tag{4.30}$$

is a continuous linear map and also a surjective homomorphism of TVS. It makes the following diagram commute

$$\begin{tikzcd} \mathcal{S}(U,V) \arrow[r, two heads] \arrow[d, two heads] & \mathcal{S}(O,V) \arrow[d, two heads] \\ \mathcal{S}\mathrm{Ind}_P^U\sigma \arrow[r, two heads] & \mathcal{S}\mathrm{Ind}_P^O\sigma \end{tikzcd}$$

Proof. Obviously the map is linear and the diagram commutes. The map (4.30) is continuous since its composition with $\mathcal{S}(U, V) \to \mathcal{S}\mathrm{Ind}_P^U\sigma$ is continuous. It is a homomorphism of TVS since it is a surjective map to a Fréchet space hence always has closed image. ☐

4.6.4 Pseudo-Cosheaf Property of Schwartz Inductions

Similar to the Schwartz function spaces, the Schwartz induction spaces also form a pseudo-cosheaf on certain open subsets of G.

The Zariski P-Topology on G

Let G, P be as the beginning of this section. The flag variety

$$P\backslash G = \mathbf{P}(\mathbb{R})\backslash\mathbf{G}(\mathbb{R}) \simeq (\mathbf{P}\backslash\mathbf{G})(\mathbb{R})$$

is a real projective algebraic variety, and the quotient morphism

$$\pi : G \to P\backslash G$$

is continuous under Zariski topologies of G and $P\backslash G$. We can pull back the Zariski topology on $P\backslash G$ to have a subtopology of the Zariski topology on G.

Definition 4.70. A subset of the form $\pi^{-1}(U)$ where $U \subset P\backslash G$ is a Zariski open subset of $P\backslash G$, is called a **Zariski P-open subset of** G. Let $\mathcal{T}_G^P$ be the family of all Zariski P-open subsets of G, and we call it the **Zariski P-topology on** G.

We have the following easy facts:

Lemma 4.71. *The $\mathcal{T}_G^P$ is a topology on G. In particular, the Zariski P-topology is quasi-compact.*

A Zariski open subset of G is Zariski P-open if and only if it is stable under left P-translation.

Example 4.72. Let $P_\Omega = \mathbf{P}_\Omega(\mathbb{R})$ be the real point group of a standard parabolic $\mathbb{R}$-subgroup $\mathbf{P}_\Omega$ of $\mathbf{G}$. For each $w \in [W_\Theta\backslash W/W_\Omega]$, the open subsets $G^\Omega_{\geq w}, G^\Omega_{>w}$ are Zariski P-open subsets of G (see 3.3.3 for notations).

Let $\overline{N}_P$ be the opposite unipotent radical of P, then the $P\overline{N}_P$ and its right translations are Zariski P-open subsets of G.

Pseudo-Cosheaf Property of Schwartz Induction

Let $U \in \mathcal{T}_G^P$ be a Zariski P-open subset. We have the space of $\mathcal{S}(U, V)$ of Schwartz V-valued functions on U, and the local Schwartz induction space $\mathcal{S}\mathrm{Ind}_P^U\sigma$ on U. We also have the σ-mean value map

$$\mathcal{S}(U, V) \to \mathcal{S}\mathrm{Ind}_P^U\sigma, \quad f \mapsto f^\sigma$$

which is a surjective homomorphism between NF-spaces.

Let $U_1 \subset U_2$ be two Zariski P-open subsets, we have discussed the open extension map in Lemma 4.67:

$$\mathrm{Ex}_{U_1}^{U_2} : \mathcal{S}\mathrm{Ind}_P^{U_1}\sigma \hookrightarrow \mathcal{S}\mathrm{Ind}_P^{U_2}\sigma$$

and seen it is an injective homomorphism of NF-spaces.

With the local Schwartz induction spaces and extension maps defined above, the correspondence $U \mapsto \mathcal{S}\mathrm{Ind}_P^U\sigma$ is a pre-cosheaf on the Zariski P-topology of G. Moreover, by diagram-chasing, we see the coshear sequences are exact, hence we have

Lemma 4.73. *The correspondence $\mathcal{S}\mathrm{Ind}_P^-\sigma : U \mapsto \mathcal{S}\mathrm{Ind}_P^U\sigma$ is a pseudo-cosheaf of NF-spaces, on the Zariski P-topology of G.*

Remark 4.74. If the representation σ is a Nash-representation, the associated vector bundle $\sigma \times_P G$ is a Nash bundle. The above cosheaf is exactly the pull-back cosheaf of the pseudo-cosheaf of Schwartz sections of the associated vector bundle.

Lemma 4.75. *For each Zariski P-open subset U, the surjective homomorphism*

$$\mathcal{S}(U,V) \twoheadrightarrow \mathcal{S}\mathrm{Ind}_P^U\sigma$$

is functorial, hence we have a morphism

$$\mathcal{S}(-,V) \to \mathcal{S}\mathrm{Ind}_P^-\sigma \tag{4.31}$$

of cosheaves of NF-spaces on the Zariski P-topology, which is a surjective homomorphism on each Zariski P-open subset.

4.6.5 Distributions on Schwartz Induction Spaces

We keep the setting as last subsection, and let F be an NF-space. Similar to the pseudo-sheaf of distributions on Schwartz functions, we define the F-valued distributions on the Schwartz induction spaces, and show they form a pseudo-sheaf on the Zariski P-topology on G.

Definition of Distributions on Schwartz Inductions

Definition 4.76. Let $L(\mathcal{S}\mathrm{Ind}_P^U\sigma, F)$ be the space of continuous linear maps from $\mathcal{S}\mathrm{Ind}_P^U\sigma$ to F. The elements in it are called **F-valued distributions on $\mathcal{S}\mathrm{Ind}_P^U\sigma$.**

Since $\mathcal{S}\mathrm{Ind}_P^U\sigma$ is a NF-space, we have

Lemma 4.77. *The canonical map $(\mathcal{S}\mathrm{Ind}_P^U\sigma)' \otimes F \to L(\mathcal{S}\mathrm{Ind}_P^U\sigma, F)$ extends to an isomorphism of TVS:*

$$(\mathcal{S}\mathrm{Ind}_P^U\sigma)' \mathbin{\widehat{\otimes}} F \xrightarrow{\sim} L(\mathcal{S}\mathrm{Ind}_P^U\sigma, F). \tag{4.32}$$

Pseudo-Sheaf Property of Distributions on Schwartz Inductions

Let $U_1, U_2 \in \mathcal{T}_G^P$ be two open subsets in the Zariski P-topology of G, and $U_1 \subset U_2$. We have the extension map $\mathrm{Ex}_{U_1}^{U_2} : \mathcal{S}\mathrm{Ind}_P^{U_1}\sigma \to \mathcal{S}\mathrm{Ind}_P^{U_2}\sigma$, and let

$$\begin{aligned}\mathrm{Res}_{U_1}^{U_2} : L(\mathcal{S}\mathrm{Ind}_P^{U_2}\sigma, F) &\to L(\mathcal{S}\mathrm{Ind}_P^{U_1}\sigma, F)\\ D &\mapsto D \circ \mathrm{Ex}_{U_1}^{U_2}\end{aligned}$$

be its F-transpose map. This map is a homomorphism of TVS, called the **restriction map** of distributions.

Remark 4.78. Keep in mind that we use the uppercase notation $\mathrm{Ex}, \mathrm{Res}$ to denote the extensions of Schwartz inductions and restrictions of distributions on Schwartz inductions, and lowercase notation $\mathrm{ex}, \mathrm{res}$ to denote the extensions of Schwartz functions and restrictions of Schwartz distributions.

The correspondence

$$L(\mathcal{S}\mathrm{Ind}_P^{-}\sigma, F) : U \mapsto L(\mathcal{S}\mathrm{Ind}_P^{U}\sigma, F)$$

(with the restriction maps Res) is a presheaf on the Zariski P-topology of G. Moreover, we have

Lemma 4.79. *The correspondence $L(\mathcal{S}\mathrm{Ind}_P^{-}\sigma, F) : U \mapsto L(\mathcal{S}\mathrm{Ind}_P^{U}\sigma, F)$ is a pseudo-sheaf on the Zariski P-topology on G.*

Proof. By Lemma 4.77, we just need to show the case $F = \mathbb{C}$. The pseudo-sheaf sequence is exact because it is the strong dual sequence of the pseudo-cosheaf sequence, which is exact by Lemma 4.73. $\square$

Distributions Supported in Zariski P-Closed Subsets

By the pseudo-sheaf property of $L(\mathcal{S}\mathrm{Ind}_P^{-}\sigma, F)$, we have the following results similar to the Lemma 4.38 and Lemma 4.39.

Lemma 4.80. *Let U_1, U_2 be two Zariski P-open subsets of G, such that $U_1 \subset U_2$. Let Z be a nonsingular closed subvariety of U_1 (hence also nonsingular subvariety of G). Assume Z is stable under left P-translation. Then $U_1 - Z, U_2 - Z$ are Zariski P-open subsets of G, and one has the restriction maps*

$$\mathrm{Res}^{U_1}_{U_1-Z} : L(\mathcal{S}\mathrm{Ind}_P^{U_1}\sigma, F) \to L(\mathcal{S}\mathrm{Ind}_P^{U_1-Z}\sigma, F)$$
$$\mathrm{Res}^{U_2}_{U_2-Z} : L(\mathcal{S}\mathrm{Ind}_P^{U_2}\sigma, F) \to L(\mathcal{S}\mathrm{Ind}_P^{U_2-Z}\sigma, F)$$

and the restriction map $\mathrm{Res}^{U_2}_{U_1}$ sends the kernel$\mathrm{Ker}(\mathrm{Res}^{U_2}_{U_2-Z})$ isomorphically to the kernel $\mathrm{Ker}(\mathrm{Res}^{U_1}_{U_1-Z})$.

Lemma 4.81. *Let U_1, U_2 be two Zariski P-open subsets, and $Z \subset G$ be a nonsingular closed subvariety of both U_1 and U_2, and assume Z is stable under left P-translation. Then the two kernel spaces $\mathrm{Ker}(\mathrm{Res}^{U_1}_{U_1-Z})$ and $\mathrm{Ker}(\mathrm{Res}^{U_2}_{U_2-Z})$ are canonically isomorphic.*

The proof of the above two Lemmas are exactly the same as Lemma 4.38 and Lemma 4.39.

Extensions of Distributions from Closed Subvarieties

Let U be a open subvariety of G, and $O \subset U$ be a nonsingular closed subvariety of U, assume both U and O are left P-stable. We have the surjective homomorphism $\mathcal{S}\mathrm{Ind}_P^U\sigma \to \mathcal{S}\mathrm{Ind}_P^O\sigma$ by Lemma 4.69. The F-transpose of this map gives a homomorphism between the distribution spaces on Schwartz inductions, and we have

Lemma 4.82. *The F-transpose of the map $\mathcal{S}\mathrm{Ind}_P^U\sigma \to \mathcal{S}\mathrm{Ind}_P^O\sigma$ is an injective homomorphism of TVS:*

$$L(\mathcal{S}\mathrm{Ind}_P^O\sigma, F) \hookrightarrow L(\mathcal{S}\mathrm{Ind}_P^U\sigma, F) \tag{4.33}$$

Definition 4.83. We call the above map (4.33) the **extension of distributions from $L(\mathcal{S}\mathrm{Ind}_P^O\sigma, F)$ to $L(\mathcal{S}\mathrm{Ind}_P^U\sigma, F)$.**

4.6.6 Group Actions on Local Schwartz Inductions

Suppose H is a closed algebraic subgroup of G, and suppose the Y is left P-stable and is an H-subvariety of G, i.e. it is a smooth subvariety of G, and the embedding $Y \to G$ is H-equivariant under the right H-translations on Y and G. Let $\mathcal{S}\mathrm{Ind}_P^Y\sigma$ be the local Schwartz induction of σ from P to Y.

Definition 4.84 (Right regular action on local Schwartz inductions). Let $\phi \in \mathcal{S}\mathrm{Ind}_P^Y\sigma$ be an arbitrary element in the Schwartz induction. Let $h \in H$, and we define $R_h\phi$ by

$$[R_h\phi](y) := \phi(yh), \quad \forall y \in Y.$$

This is a group action of H on the vector space $\mathcal{S}\mathrm{Ind}_P^Y\sigma$, called the **right regular H-action on $\mathcal{S}\mathrm{Ind}_P^Y\sigma$.**

We have

Lemma 4.85. *Under the right regular H-action, the $\mathcal{S}\mathrm{Ind}_P^Y\sigma$ is a smooth H-representation. The map (4.28)*

$$\mathcal{S}(Y, V) \twoheadrightarrow \mathcal{S}\mathrm{Ind}_P^Y\sigma$$

is H-equivariant hence an intertwining operator between H-representations.

Group Actions are Compatible with Open Extensions

Let U_1, U_2 be two Zariski P-open H-subvarieties of G, i.e. they are open subvarieties stable under right H-translation and the embeddings are H-equivariant. Assume U_1, U_2, then as in Lemma 4.67, we have the extension map from $\mathcal{S}\mathrm{Ind}_P^{U_1}\sigma$ to $\mathcal{S}\mathrm{Ind}_P^{U_2}\sigma$.

Lemma 4.86. *Under the right regular H-actions on $\mathcal{S}\mathrm{Ind}_P^{U_1}\sigma$ and $\mathcal{S}\mathrm{Ind}_P^{U_2}\sigma$, the extension map (4.29)*

$$\mathcal{S}\mathrm{Ind}_P^{U_1}\sigma \hookrightarrow \mathcal{S}\mathrm{Ind}_P^{U_2}\sigma$$

is an H-intertwining operator. Hence the following diagram

$$\begin{tikzcd} \mathcal{S}(U_1, V) \arrow[r, hook] \arrow[d, two heads] & \mathcal{S}(U_2, V) \arrow[d, two heads] \\ \mathcal{S}\mathrm{Ind}_P^{U_1}\sigma \arrow[r, hook] & \mathcal{S}\mathrm{Ind}_P^{U_2}\sigma \end{tikzcd}$$

is a diagram in the category of H-representations, with all four spaces endowed with the right regular H-actions.

Group Actions are Compatible with Closed Restrictions

Let U be a Zariski P-open subvariety of G, O be a Zariski P-closed subvariety of U. Assume they are all H-subvarieties of G. Let $\mathcal{S}\mathrm{Ind}_P^U\sigma, \mathcal{S}\mathrm{Ind}_P^O\sigma$ be the Schwartz inductions on them respectively, and they have the right regular H-actions, and the restriction map between them is H-equivariant.

Lemma 4.87. *Under the right regular H-actions, the restriction map (4.30)*

$$\mathcal{S}\mathrm{Ind}_P^U\sigma \twoheadrightarrow \mathcal{S}\mathrm{Ind}_P^O\sigma$$

is an H-intertwining operator. Then following diagram

$$\begin{tikzcd} \mathcal{S}(U, V) \arrow[r, two heads] \arrow[d, two heads] & \mathcal{S}(O, V) \arrow[d, two heads] \\ \mathcal{S}\mathrm{Ind}_P^U\sigma \arrow[r, two heads] & \mathcal{S}\mathrm{Ind}_P^O\sigma \end{tikzcd}$$

is in the category of H-representations, with all four spaces endowed with the right regular H-actions.

4.6.7 Local Schwartz Inductions on Fibrations

We study the Schwartz induction spaces on subvarieties of G, which are isomorphic to direct products of P and subvarieties of G.

In this subsection, we let P be the real point group of a parabolic $\mathbb{R}$-subgroup, and let $\overline{N}_P$ be the unipotent radical of its opposite parabolic subgroup.

Local Trivialization of $\pi : G \to P\backslash G$

Let $\pi : G \to P\backslash G$ be the algebraic quotient map. The $(G, P\backslash G, \pi)$ is an algebraic fibre bundle, and it is locally trivial on each Zariski open subset

$$Y_w := \pi(P\overline{N}_P w)$$

of $P\backslash G$. The inverse image

$$Z_w := \pi^{-1}(Y_w) = P\overline{N}_P w$$

is isomorphic to a direct product of varieties, i.e. the following map is an isomorphism of variety:

$$\begin{aligned} P \times w^{-1}\overline{N}_P w &\xrightarrow{\sim} Z_w = P\overline{N}_P w \\ (p, n) &\mapsto pwn \end{aligned}$$

Direct Product Decomposition

Let $Y \subset Y_w$ be a nonsingular subvariety of $Y_w \subset P\backslash G$, and let $Z = \pi^{-1}(Y)$. We assume there is a subvariety O of G, and the multiplication map on G induces an isomorphism of varieties:

$$\begin{aligned} P \times O &\xrightarrow{\sim} Z = \pi^{-1}(Y) \\ (p, x) &\mapsto px \end{aligned}$$

Then this isomorphism is a P-equivariant isomorphism, with $P \times O$ and Z endowed with the left P-translations. The O is isomorphic to the subvariety $\{e\} \times O$ of Z.

Example 4.88. We will only consider the following two concrete examples in this thesis:

- Let $Y = Y_w$ (and $Z = Z_w$ as above), the O is exactly the $w^{-1}\overline{N}_P w$.

- Let $Y = \pi(PwP_\emptyset)$, it is the $P_\emptyset$-orbit on the $P\backslash G$ through the point $x_w = \pi(Pw)$. Then $Z = PwP_\emptyset$, and the O is the subgroup $w^{-1}\overline{N}_P w \cap N_\emptyset$.

The isomorphism $P \times O \xrightarrow{\sim} Z$ induces the following isomorphisms of TVS:

$$\mathcal{S}(P,V)\,\widehat{\otimes}\,\mathcal{S}(O,\mathbb{C}) \xrightarrow{\sim} \mathcal{S}(Z,V)$$
$$\mathcal{S}(P,\mathbb{C})\,\widehat{\otimes}\,\mathcal{S}(O,V) \xrightarrow{\sim} \mathcal{S}(Z,V)$$

(This follows from E-6 of Proposition 4.30.) We know the algebraic tensor products $\mathcal{S}(P,V)\otimes\mathcal{S}(O,\mathbb{C})$ is dense in $\mathcal{S}(P,V)\widehat{\otimes}\mathcal{S}(O,\mathbb{C})$, and the $\mathcal{S}(P,\mathbb{C})\otimes\mathcal{S}(O,V)$ is dense in $\mathcal{S}(P,\mathbb{C})\widehat{\otimes}\mathcal{S}(O,V)$. We first study the images of functions in these algebraic tensor products, under the σ-mean value map.

Lemma 4.89. *Let $\phi \in \mathcal{S}(P,V)$ and $\psi \in \mathcal{S}(O,\mathbb{C})$, and let*

$$\begin{aligned} \phi\otimes\psi : Z \simeq P\times O &\to V \\ (p,x) &\mapsto \psi(x)\phi(p) \end{aligned}$$

Then $\phi\otimes\psi \in \mathcal{S}(Z,V)$, and its image under the map $\mathcal{S}(Z,V)\to\mathcal{S}\mathrm{Ind}_P^Z\sigma$ is

$$(\phi\otimes\psi)^\sigma(px) = \psi(x)\phi^\sigma(p) \tag{4.34}$$

Proof. This is easy to verify:

$$\begin{aligned} (\phi\otimes\psi)^\sigma(px) &= \int_P \sigma(q^{-1})(\phi\otimes\psi)(q\cdot px)dq \\ &= \int_P \sigma(q^{-1})(\phi\otimes\psi)(qpx)dq \\ &= \int_P \sigma(q^{-1})[\psi(x)\phi(qp)]dq \\ &= \psi(x)\int_P \sigma(q^{-1})\phi(qp)dq \\ &= \psi(x)\phi^\sigma(p) \end{aligned}$$

□

Lemma 4.90. *Let $\phi \in \mathcal{S}(P,\mathbb{C})$ and $\psi \in \mathcal{S}(O,V)$, and let*

$$\begin{aligned} \phi\otimes\psi : Z = P\times O &\to V \\ (p,x) &\mapsto \phi(p)\psi(x) \end{aligned}$$

Then $\phi \otimes \psi \in \mathcal{S}(Z, V)$, and its image under map $\mathcal{S}(Z, V) \to \mathcal{S}\mathrm{Ind}_P^Z\sigma$ is

$$(\phi \otimes \psi)^\sigma(px) = \sigma(p) \int_P \phi(q)\sigma(q^{-1})\psi(x)dq \tag{4.35}$$

Proof. Actually

$$\begin{aligned}
(\phi \otimes \psi)^\sigma(px) &= \int_P \sigma(q^{-1})(\phi \otimes \psi)(qpx)dq \\
&= \int_P \phi(qp)\sigma(q^{-1})\psi(x)dq \\
&= \int_P \phi(qp)\sigma(p)\sigma(qp)^{-1}\psi(x)dq \\
&= \sigma(p) \int_P \phi(qp)\sigma(qp)^{-1}\psi(x)dq \\
&= \sigma(p) \int_P \phi(qp)\sigma(qp)^{-1}\psi(x)dqp \\
&= \sigma(p) \int_P \phi(q)\sigma(q)^{-1}\psi(x)dq
\end{aligned}$$

□

The First Isomorphism

By Lemma 4.89, we have the following map

$$\begin{aligned}
\mathcal{S}\mathrm{Ind}_P^P\sigma \otimes \mathcal{S}(O, \mathbb{C}) &\to \mathcal{S}\mathrm{Ind}_P^Z\sigma \\
\phi^\sigma \otimes \psi &\mapsto (\phi \otimes \psi)^\sigma
\end{aligned}$$

It is easy to see this map is well-defined, and is independent of the choice of $\phi \in \mathcal{S}(P, V)$. Actually let $\phi_0 \in \mathrm{Ker}\{\mathcal{S}(P, V) \to \mathcal{S}\mathrm{Ind}_P^P\sigma\}$, then for any $\psi \in \mathcal{S}(O, \mathbb{C})$, the $\phi_0 \otimes \psi$ is in the kernel of map $\mathcal{S}(Z, V) \to \mathcal{S}\mathrm{Ind}_P^Z\sigma$ by Lemma 4.89. Moreover, we have

Lemma 4.91. *The above map $\phi^\sigma \otimes \psi \mapsto (\phi \otimes \psi)^\sigma$ extends to an isomorphism on the completion:*

$$\mathcal{S}\mathrm{Ind}_P^P\sigma \,\widehat{\otimes}\, \mathcal{S}(O, \mathbb{C}) \xrightarrow{\sim} \mathcal{S}\mathrm{Ind}_P^Z\sigma \tag{4.36}$$

and the following diagram commutes:

$$\begin{tikzcd}
\mathcal{S}(P,V)\,\widehat{\otimes}\,\mathcal{S}(O,\mathbb{C}) \arrow[r, "\simeq"] \arrow[d, two heads] & \mathcal{S}(Z,V) \arrow[d, two heads] \\
\mathcal{S}\mathrm{Ind}_P^P\sigma\,\widehat{\otimes}\,\mathcal{S}(O,\mathbb{C}) \arrow[r, "(4.36)"] & \mathcal{S}\mathrm{Ind}_P^Z\sigma
\end{tikzcd}$$

Proof. By Lemma 4.89, the diagram commutes, hence the (4.36) is surjective. If $(\phi\otimes\psi)^\sigma = 0$, then by Lemma 4.89 again, we see $\psi(x)\phi^\sigma(p) = 0$ for all $p \in P, x \in O$. This is means either $\phi^\sigma = 0$ or $\psi = 0$, otherwise there exist $p \in P, x \in O$ such that $\phi^\sigma(p) \neq 0, \psi(x) \neq 0$ thus $\psi(x)\phi^\sigma(p) \neq 0$. Hence the map $\phi^\sigma \otimes \psi \mapsto (\phi\otimes\psi)^\sigma$ is injective, so is the map (4.36). $\square$

The Second Isomorphism

For a $F \in \mathcal{S}\mathrm{Ind}_P^Z\sigma$, it is a smooth function on Z. Let $F|_O$ be its restriction to the submanifold O (remember O is embedded into Z as $\{e\} \times O$), then $F|_O$ is a smooth V-valued function on O.

Lemma 4.92. *We have*

(1) The $F|_O$ is a Schwartz function on O, i.e. $F|_O \in \mathcal{S}(O,V)$.

(2) The restriction map

$$\begin{aligned} \mathcal{S}\mathrm{Ind}_P^Z\sigma &\to \mathcal{S}(O,V) \\ F &\mapsto F|_O \end{aligned} \tag{4.37}$$

is a homomorphism of TVS.

(3) The restriction map (4.37) is an isomorphism between TVS.

Proof. (1) We first consider F of the form $(\phi\otimes\psi)^\sigma$ where $\phi \in \mathcal{S}(P,V)$ and $\psi \in \mathcal{S}(O,\mathbb{C})$. By Lemma 4.89, we have

$$F(p,x) = \psi(x)\phi^\sigma(p), \quad \forall p \in P, x \in O.$$

Hence $F|_O(x) = F(e,x) = \psi(x)\phi^\sigma(e)$. This is a scalar Schwartz function, since $\phi^\sigma(e)$ is a fixed vector in V, and ψ is a scalar valued Schwartz function. Hence for $F = (\phi\otimes\psi)^\sigma$, we have $F|_O \in \mathcal{S}(O,V)$.

Since the $\mathcal{S}(O,V)$ is complete, and functions of the form $(\phi\otimes\psi)^\sigma$ are dense in $\mathcal{S}\mathrm{Ind}_P^Z\sigma$, we see the map $F \mapsto F|_O$ has its image in $\mathcal{S}(O,V)$.

(2) All spaces are nuclear, hence we just need to show (4.37) is continuous. Actually it is continuous since its composition with $\mathcal{S}(Z,V) \to \mathcal{S}\mathrm{Ind}_P^Z\sigma$ is continuous on the dense subspace $\mathcal{S}(P,V)\otimes\mathcal{S}(O,\mathbb{C})$.

(3) We show the (4.37) is an isomorphism. Obviously it is injective, since a function in $\mathcal{S}\mathrm{Ind}_P^Z\sigma$ is uniquely determined by its values on $O \simeq \{e\} \times O$. We just need to show it is surjective.

For any $\Psi \in \mathcal{S}(O, V)$, let $\gamma \in C_c^\infty(P, \mathbb{C})$ be a bump function satisfying

$$\int_P \gamma(p)dp = \gamma(e) = 1.$$

We construct a function F on $Z = P \times O$ by

$$F(p, x) := \gamma(p)\sigma(p)\Psi(x), \quad \forall p \in P, x \in O.$$

This F is in $\mathcal{S}(Z, V)$ since it is smooth with compact support. It is easy to verify

$$\begin{aligned}
F^\sigma(p, x) &= \int_P \sigma(q^{-1})F(qp, x)dq \\
&= \int_P \sigma(q^{-1})\gamma(qp)\sigma(qp)\Psi(x)dq \\
&= \int_P \gamma(qp)\sigma(p)\Psi(x)dq \\
&= [\int_P \gamma(qp)dq] \cdot \sigma(p)\Psi(x) \\
&= [\int_P \gamma(q)dq] \cdot \sigma(p)\Psi(x) \\
&= \sigma(p)\Psi(x)
\end{aligned}$$

Then $F^\sigma|_O = \Psi$. Hence (4.37) is surjective. □

The Trivial Case $\mathcal{S}\mathrm{Ind}_P^P\sigma$

If $P = G$, a function in $\mathcal{S}\mathrm{Ind}_P^P\sigma$ is uniquely determined by its value at identity e:

Lemma 4.93. *The delta function*

$$\begin{aligned}
\Omega_e : \mathcal{S}\mathrm{Ind}_P^P\sigma &\to V \\
\phi &\mapsto \phi(e)
\end{aligned}$$

is an isomorphism of TVS.

Proof. This is the special case of the above Lemma. Since $P = G$, the quotient variety $P\backslash G = \{*\}$ is a singleton. Then $Y = \{*\}$, $Z = G$, and $O = \{e\}$. The $\mathcal{S}(O, V) = V$, and the map $\mathcal{S}\mathrm{Ind}_P^P\sigma \to \mathcal{S}(O, V), F \mapsto F|_O$ is exactly the delta function Ω_e. □

Chapter 5

Intertwining Distributions

Summary of This Chapter

This is a conceptual chapter, in which we apply the tools of Schwartz analysis developed in the last chapter, to study intertwining operators. We embed the space $\mathrm{Hom}_G(I, J)$ of intertwining operators between two smooth parabolic inductions I, J, to the space of equivariant Schwartz distributions on the Schwartz induction spaces, and study such distributions by looking at their restrictions to various Zariski open subsets (unions of double cosets) of G.

In the first three sections, we will work on more general situation, since our long term goal is to apply the theory to study all intertwining operators between two arbitrary parabolic inductions. The main theme of the thesis is the irreducibility of a single parabolic induction, and in the last two sections we will look at the space of self-intertwining operators.

Part I—General Intertwining Distributions

In section 5.1, 5.2, 5.3, we work in general situation, and let

$$\begin{aligned}
\mathbf{G} &= \text{a connected reductive linear algebraic group defined over } \mathbb{R}\\
\mathbf{P}, \mathbf{Q} &= \text{two parabolic } \mathbb{R}\text{-subgroups of } \mathbf{G}\\
G, P, Q &= \text{the corresponding Lie groups of real points of } \mathbf{G}, \mathbf{P}, \mathbf{Q}\\
(\sigma_1, V_1) &= \text{a nuclear Harish-Chandra representation of } P\\
(\sigma_2, V_2) &= \text{a nuclear Harish-Chandra representation of } Q\\
I &= C^\infty \mathrm{Ind}_P^G \sigma_1 = \mathcal{S}\mathrm{Ind}_P^G \sigma_1\\
J &= C^\infty \mathrm{Ind}_Q^G \sigma_2 = \mathcal{S}\mathrm{Ind}_Q^G \sigma_2
\end{aligned}$$

For two TVS E_1, E_2, let

$$\begin{aligned}
L(E_1, E_2) &= \mathrm{Hom}_{cont}(E_1, E_2)\\
&= \text{the space of continuous linear maps from } E_1 \text{ to } E_2.
\end{aligned}$$

We study the space $\mathrm{Hom}_G(I, J)$ of intertwining operators between the two smooth inductions (Schwartz inductions) I and J.

In 5.1, we embed the space $\mathrm{Hom}_G(I, J)$ of intertwining operators, into the space $L(I, V_2) \subset L(\mathcal{S}(G, V_1), V_2)$ of V_2-valued Schwartz distributions on $\mathcal{S}\mathrm{Ind}_P^G \sigma_1$ (or $\mathcal{S}(G, V_1)$), and starting from this section we call elements in $\mathrm{Hom}_G(I, J)$ **intertwining distributions**.

In 5.2, we show the intertwining distributions in $\mathrm{Hom}_G(I, J)$ have supports equal to closed unions of (P, Q)-double cosets. Since there are finitely many double cosets on G, we find the first discreteness of intertwining operators: they are sorted into finitely many families with different supports. (One can also replace the Q by its algebraic subgroups H such that G has finitely many (P, H)-double cosets.)

In 5.3, we first introduce the **"maximal double coset"** in the support. If an intertwining distribution D has "a" maximal double coset G_w in its support, then the restriction of D to the open subset $G_{\geq w}$ is a nonzero element which vanishes on the smaller open subset $G_{>w}$, i.e. this restriction is a nonzero element in the kernel of the restriction map from $G_{\geq w}$ to $G_{>w}$.

Part II—Self-Intertwining Distributions and Irreducibilities

In section 5.4, 5.5, we work on the self-intertwining distributions, and let

- $\mathbf{G}$ (resp. $\mathbf{S}, \mathbf{P}_\emptyset$) be a connected reductive linear algebraic group defined over $\mathbb{R}$ (resp. a maximal $\mathbb{R}$-split torus, a minimal parabolic $\mathbb{R}$-subgroup containing $\mathbf{S}$).

- $\mathbf{P} = \mathbf{P}_\Theta$ be a standard parabolic $\mathbb{R}$-subgroup containing $\mathbf{P}_\emptyset$ (corresponding to a subset Θ of the base). In this chapter we will not use the particular set Θ and we drop the subscript Θ since there is no ambiguity.

- $\mathbf{M_P} = \mathbf{M}_\Theta, \mathbf{N_P} = \mathbf{N}_\Theta$ be the standard $\mathbb{R}$-Levi factor and unipotent radical of $\mathbf{P}$, $\mathbf{M}_\emptyset, \mathbf{N}_\emptyset$ be the Levi $\mathbb{R}$-factor and unipotent radical of $\mathbf{P}_\emptyset$;

- $G, P_\emptyset, M_\emptyset, N_\emptyset, P, M_P, N_P$ etc. be the Lie groups of $\mathbb{R}$-rational points corresponding to the above algebraic groups denoted by boldface letters.

- (σ, V) be a nuclear Harish-Chandra representation of P.

- $I = C^\infty\mathrm{Ind}_P^G \sigma = \mathcal{S}\mathrm{Ind}_P^G \sigma$ be the smooth (Schwartz) induction.

We apply the notions and results in Part I to the case when $P = Q, \sigma_1 = \sigma_2, I = J$, and study the space $\mathrm{Hom}_G(I, I)$ and the irreducibility of the smooth induction I.

Section 5.4 now consists of some notations and basic settings. Also we recall the fact that the intertwining distributions corresponding to scalar intertwining operators are supported in $\mathrm{supp}D$ (see Lemma 5.21).

In 5.5, we give the local descriptions of intertwining distributions based on the maximal double cosets in their supports. We summarize two main steps to show the $\mathrm{Hom}_G(I, I) = \mathbb{C}$.

5.1 Intertwining Distributions

In this and the following two sections, we let $\mathbf{G}$ and $G = \mathbf{G}(\mathbb{R})$ be the same as the beginning of this chapter. Let $\mathbf{P}, \mathbf{Q}$ be two parabolic $\mathbb{R}$-subgroups of $\mathbf{G}$, and let $P = \mathbf{P}(\mathbb{R}), Q = \mathbf{Q}(\mathbb{R})$ be their corresponding Lie groups of real points. Let

$$(\sigma_1, V_1) = \text{a Harish-Chandra representation of } P$$
$$(\sigma_2, V_2) = \text{a Harish-Chandra representation of } Q$$

In particular, the V_1, V_2 are nuclear Fréchet spaces and smooth representations of P, Q respectively.

Let $\mathcal{S}\mathrm{Ind}_P^G\sigma_1, \mathcal{S}\mathrm{Ind}_Q^G\sigma_2$ be the Schwartz inductions of σ_1, σ_2 respectively. Since the quotient manifolds $P\backslash G, Q\backslash G$ are compact, the Schwartz inductions agree with the smooth inductions. For simplicity, we use the following notations for the Schwartz (smooth) inductions:

$$I = C^\infty\mathrm{Ind}_P^G\sigma_1 = \mathcal{S}\mathrm{Ind}_P^G\sigma_1$$
$$J = C^\infty\mathrm{Ind}_Q^G\sigma_2 = \mathcal{S}\mathrm{Ind}_Q^G\sigma_2$$

5.1.1 Intertwining Distributions

As in 4.6.1, for a Schwartz V_1-valued function $f \in \mathcal{S}(G, V_1)$, its σ_1-mean value function

$$f^{\sigma_1}(g) = \int_P \sigma_1(p^{-1})f(pg)dp, \quad g \in G$$

is in the Schwartz induction space $\mathcal{S}\mathrm{Ind}_P^G\sigma_1$, and the map $\mathcal{S}(G, V_1) \to \mathcal{S}\mathrm{Ind}_P^G\sigma_1, f \mapsto f^{\sigma_1}$ is a surjective homomorphism of TVS. And Frobenius reciprocity (Lemma 2.31) tells us the map $T \mapsto \Omega_e \circ T$ is an isomorphism between (finite dimensional) vector spaces.

Given an intertwining operator $T \in \mathrm{Hom}_G(I, J)$, we consider the composition map $D_T : I \to V_2$ of the three maps: (1) the mean value map $\mathcal{S}(G, V_1) \to I, f \mapsto f^{\sigma_1}$; (2) the operator $T : I \to J$; (3) the delta function $\Omega_e : J \to V_2$:

$$\begin{tikzcd}
\mathcal{S}(G,V_1) \arrow[r, dashed, "D_T"] \arrow[d, two heads] & V_2 \\
I \arrow[r, "T"] & J \arrow[u, "\Omega_e"']
\end{tikzcd}$$

The D_T is a V_2-valued Schwartz V_1-distribution on G, and it is obviously Q-equivariant, since Ω_e is Q-equivariant, T and $f \mapsto f^{\sigma_1}$ are G-equivariant.

Since the σ_1-mean value map $\mathcal{S}(G, V_1) \to I$ is surjective, its adjoint map is injective, and the correspondence $T \mapsto \Omega_e \circ T$ is bijective, hence we have:

Lemma 5.1. *The following linear map*

$$\begin{aligned} \mathrm{Hom}_G(I, J) &\to \mathrm{Hom}_Q(\mathcal{S}(G, V_1), V_2) \subset L(\mathcal{S}(G, V_1), V_2) \\ T &\mapsto D_T \end{aligned} \tag{5.1}$$

is one-to-one.

Thus we have embedded the finite dimensional space $\mathrm{Hom}_G(I, J)$, as a subspace of $L(\mathcal{S}(G, V_1), V_2)$. From now on, we identify the space $\mathrm{Hom}_G(I, J)$ with its image in the $L(\mathcal{S}(G, V_1), V_2)$, and identify an intertwining operator T with its corresponding distribution D_T.

Definition 5.2. Let $D \in \mathrm{Hom}_G(I, J)$.

- We call the $D_T \in L(\mathcal{S}(G, V_1), V_2)$ a **V_2-valued intertwining distribution on $\mathcal{S}(G, V_1)$**.
- We call the $\Omega_e \circ T \in L(I, V_2)$ a **V_2-valued intertwining distribution on I**.

Remark 5.3. The above term "intertwining distribution" is an emphasis of the fact that a distribution come from an intertwining operator.

- When the spaces under discussion are clear without ambiguity, we will simply use the term **intertwining distributions**, without mentioning the spaces $\mathcal{S}(G, V_1), I, V_2$ etc.
- In application we will also study the restrictions of intertwining distributions on $\mathcal{S}(G, V_1)$ or I to their subspaces, and we will also call these restricted distributions intertwining distributions (abuse of terms).

- By abuse of terms, we will also call elements in $\mathrm{Hom}_G(I, J)$ intertwining distributions, instead of intertwining operators. The space of intertwining distributions is exactly $\mathrm{Hom}_G(I, J)$.

The finiteness of $\mathrm{Hom}_G(I, J)$ indicates that there should not be many intertwining distributions, hence there should be a lot of restrictions on such distributions. In the next section, we will see the first good property of intertwining distributions on $\mathcal{S}(G, V_1)$, i.e. their supports are good subsets of G.

5.2 Supports of Intertwining Distributions

We keep the setting as in 5.1, and let $D \in \mathrm{Hom}_G(I, J) \subset L(\mathcal{S}(G, V_1), V_2)$ be an intertwining distribution. As we have seen in section 4.4, the D has a well-defined support under the Zariski topology of G. Let $\mathrm{supp}D$ and $\mathrm{van}D$ be the support and maximal vanishing subset as in Definition 4.48. In this section, we show

Lemma 5.4. *The support* $\mathrm{supp}D$ *and maximal vanishing subset* $\mathrm{van}D$ *of* D *are stable under left* P*-translation and right* Q*-translation. In particular, the* $\mathrm{supp}D$ *is a (Zariski closed) union of* (P, Q)*-double cosets in* G*, the* $\mathrm{van}D$ *is a (Zariski open) union of* (P, Q)*-double cosets in* G*.*

Remark 5.5. Let $\mathbf{P}' \subset \mathbf{P}$ and $\mathbf{Q}' \subset \mathbf{Q}$ be two parabolic $\mathbb{R}$-subgroups contained in $\mathbf{P}, \mathbf{Q}$ respectively, and let P', Q' be the corresponding Lie subgroups of $\mathbb{R}$-rational points. Then the $\mathrm{supp}D$ and $\mathrm{van}D$ are also unions of (P', Q')-double cosets in G.

Before studying the supports, we have the following trivial Lemma:

Lemma 5.6. *Let* D *be an intertwining distribution. Then the following statements are equivalent:*

- $D = 0$ *or equivalently the corresponding intertwining operator is zero.*
- $\mathrm{supp}D = \emptyset$.

5.2.1 Some Topological Facts

Let $H \subset G$ be an algebraic subgroup, i.e. there is an $\mathbb{R}$-closed subgroup $\mathbf{H}$ of $\mathbf{G}$ such that $H = \mathbf{H}(\mathbb{R})$.

Definition 5.7. For a Zariski open subset $U \subset G$, and an element $g \in G$, let Ug be the right translation of U by g, and gU be the left translation of U by g. We call the

$$UH := \bigcup_{h \in H} Uh$$

the **right H-augmentation of U**, and similarly call the

$$HU := \bigcup_{h \in H} hU$$

the **left H-augmentation of U**.

We have the following basic facts:

Lemma 5.8. *Let $U \subset G$ be a Zariski open subset.*

1. *For a $g \in G$, the Ug and gU are Zariski open in G.*

2. *For an algebraic subgroup $H \subset G$, the UH and HU are Zariski open subsets of G, since they are union of translations of U. Actually, the UH is a finite union of Zariski open subsets of the form Uh for some $h \in H$, and HU is a finite union of open subsets of the form hU for some $h \in H$.*

3. *A subset S of G is **left H-stable** (i.e. $hS = S$ for all $h \in H$), if and only if $HS = S$; the S is **right H-stable** (i.e. $Sh = S$ for all $h \in H$), if and only if $SH = S$.*

Proof. Part 1 and 2 are trivial. We show part 3 for the "left part", the "right part" is similar.

Suppose S is left H-stable, i.e. $\forall h \in H$, we have $hS = S$. Then $hS \subset S$ for all $h \in H$, hence their union HS is contained in S, and $S \subset HS$ is obvious. Hence $HS = S$.

Suppose $HS = S$, then for all $h \in H$, $hS \subset HS \subset S$. By the same argument $h^{-1}S \subset S$, and translate both sides by h we have $S \subset hS$ for all $h \in H$. Hence for all h, we have $hS = S$. □

5.2.2 The supp$D$ and van$D$ are Right Q-Stable

Let D be an intertwining distribution. We show the van$D$ is stable under right Q-translation, which implies supp$D$ is also right Q-stable since it is the complement of van$D$. More precisely, by part 3 of the above Lemma, we need to show

$$(\mathrm{van}D)Q = \mathrm{van}D.$$

The inclusion $\mathrm{van}D \subset (\mathrm{van}D)Q$ is obvious, hence we just need to show $(\mathrm{van}D)Q \subset \mathrm{van}D$, or equivalently $(\mathrm{van}D)q \subset \mathrm{van}D$ for all $q \in Q$. By the maximality of $\mathrm{van}D$, we just need to show D vanishes on $(\mathrm{van}D)q$. By Lemma 4.49, we know D vanishes on $\mathrm{van}D$, and we just need to apply the following Lemma to $U = \mathrm{van}D$:

Lemma 5.9. *Let U be a Zariski open subset of G, and $D|_U = 0$. Then $D|_{Uq} = 0$ for all $q \in Q$.*

Proof. Suppose the intertwining distribution $D \in \mathrm{Hom}_Q(\mathcal{S}(G, V_1), V_2)$ vanishes on U, namely $\langle D, \phi \rangle = 0$ for all $\phi \in \mathcal{S}(U, V_1) \subset \mathcal{S}(G, V_1)$. We show $\langle D, f \rangle = 0$ for all $f \in \mathcal{S}(Uq, V_1) \subset \mathcal{S}(G, V_1)$.

Actually

$$\begin{aligned}
&f \text{ is in the subspace } \mathcal{S}(Uq, V_1) \subset \mathcal{S}(G, V_1)\\
\Leftrightarrow &f \text{ vanishes with all derivatives on } G - Uq\\
\Leftrightarrow &R_q f \text{ vanishes with all derivatives on } G - U\\
\Leftrightarrow &R_q f \in \mathcal{S}(U, V_1) \subset \mathcal{S}(G, V_1)
\end{aligned}$$

Hence if $f \in \mathcal{S}(Uq, V_1)$, we know the right translation $R_q f$ is in $\mathcal{S}(U, V_1)$ and by the assumption on D, we have $\langle D, R_q f \rangle = 0$. Now we have

$$\begin{aligned}
\langle D, f \rangle &= \langle D, R_{q^{-1}} R_q f \rangle\\
&= \sigma_2(q^{-1}) \langle D, R_q f \rangle \qquad (D \text{ is } Q\text{-equivariant})\\
&= \sigma_2(q^{-1}) 0
\end{aligned}$$

Thus we have shown $\langle D, f \rangle = 0$ for all $f \in \mathcal{S}(Uq, V_1)$, hence D vanishes on Uq. □

5.2.3 The $\mathrm{supp}D$ and $\mathrm{van}D$ are Left P-Stable

Let D be an intertwining distribution as above. We show the $\mathrm{van}D$ is left P-stable, which implies $\mathrm{supp}D$ is also left P-stable. Similar to the above discussion, we just need to show $P(\mathrm{van}D) \subset \mathrm{van}D$, or equivalently $p(\mathrm{van}D) \subset \mathrm{van}D$ for all $p \in P$.

Lemma 5.10. *Let $f \in \mathcal{S}(G, V_1)$ be and arbitrary Schwartz V_1-valued function, and $p \in P$ be an arbitrary element. Let*

$$(Lpf)(g) := f(p^{-1}g), \forall g \in G$$

be the left translation of f by p. Let $\sigma_1(p)f$ be the following composition function

$$[\sigma_1(p)f](g) := \sigma_1(p)f(g),$$

i.e. $\sigma_1(p)$ acts on the vector $f(g) \in V_1$. Then we have

(1) The $L_p f$ is in $\mathcal{S}(G, V_1)$.

(2) The $\sigma_1(p)f$ is in $\mathcal{S}(G, V_1)$.

(3) The σ_1-mean value function of $L_p f$ (the image of $L_p f$ in $\mathcal{S}\mathrm{Ind}_P^G \sigma_1$) is

$$(L_p f)^{\sigma_1} = \delta_P(p)[\sigma_1(p^{-1})f]^{\sigma_1},$$

i.e. it is the σ_1-mean value function of $\sigma_1(p^{-1})f$ multiplied by a constant $\delta_P(p)$.

Proof. Part (1) is true because the left translation by p is an algebraic isomorphism on G, and it induces an isomorphism on Schwartz function spaces:

$$L_p : \mathcal{S}(G, V_1) \xrightarrow{\sim} \mathcal{S}(G, V_1).$$

Part (2) is true because the function $\sigma_1(p)f$ is simply a composition of the Schwartz function f with a single linear operator $\sigma_1(p)$.

We verify part (3), let $f \in \mathcal{S}(G, V_1)$ and $p \in P$ be arbitrary elements. Then

$$\begin{aligned}
(L_p f)^{\sigma_1}(g) &= \int_P \sigma_1(q^{-1})[L_p f](qg)dq \\
&= \int_P \sigma_1(q^{-1})f(p^{-1}qg)dq \\
&= \int_P \sigma_1(q^{-1})\sigma_1(p)\sigma_1(p^{-1})f(p^{-1}qg)dq \\
&= \int_P \sigma_1(p^{-1}q)^{-1}[\sigma_1(p^{-1})f](p^{-1}qg)dq \\
&= \int_P \sigma_1(p')^{-1}[\sigma_1(p^{-1})f](p'g)d(pp') \qquad (p' = p^{-1}q) \\
&= \delta_P(p) \int_P \sigma_1(p')^{-1}[\sigma_1(p^{-1})f](p'g)dp' \\
&= \delta_P(p)[\sigma_1(p^{-1})f]^{\sigma_1}(g)
\end{aligned}$$

hence $(L_p f)^{\sigma_1} = \delta_P(p)[\sigma_1(p^{-1})f]^{\sigma_1}$. $\square$

Similar to the proof of right Q-stability of van$D$, we show the following Lemma which implies van$D$ is left P-stable:

Lemma 5.11. *Suppose U is a Zariski open subset on which D vanishes ($D|_U = 0$), then for an arbitrary $p \in P$, we have $D|_{pU} = 0$, namely D also vanishes on the left translation pU.*

Proof. We just need to show $\langle D, f\rangle = 0$ for all $f \in \mathcal{S}(pU, V_1)$. Suppose the intertwining distribution D come from an intertwining operator $T \in \mathrm{Hom}_G(I, J)$.

First the isomorphism

$$L_{p^{-1}} : \mathcal{S}(G, V_1) \xrightarrow{\sim} \mathcal{S}(G, V_1)$$

maps the subspace $\mathcal{S}(pU, V_1)$ of $\mathcal{S}(G, V_1)$ isomorphically to the subspace $\mathcal{S}(U, V_1)$, i.e. $f \in \mathcal{S}(pU, V_1)$ is equivalent to $L_{p^{-1}} f \in \mathcal{S}(U, V_1)$. Then for all $f \in \mathcal{S}(pU, V_1)$

$$\begin{aligned}
\langle D, f\rangle &= \langle D, L_p L_{p^{-1}} f\rangle \\
&= \langle \Omega_e \circ T, [L_p L_{p^{-1}} f]^{\sigma_1}\rangle \\
&= \langle \Omega_e \circ T, \delta_P(p)[\sigma_1(p^{-1}) L_{p^{-1}} f]^{\sigma_1}\rangle \\
&= \delta_P(p)\langle \Omega_e \circ T, [\sigma_1(p^{-1}) L_{p^{-1}} f]^{\sigma_1}\rangle \\
&= \delta_P(p)\langle D, \sigma_1(p^{-1}) L_{p^{-1}} f\rangle \\
&= \delta_P(p) \cdot 0
\end{aligned}$$

The last equality holds because $L_{p^{-1}} f$ is in $\mathcal{S}(U, V_1)$ and the composition function $\sigma_1(p^{-1}) L_{p^{-1}} f$ is also in $\mathcal{S}(U, V_1)$, since it vanishes with all derivatives on $G - U$. By the assumption $D|_U = 0$, we know $\langle D, \sigma_1(p^{-1}) L_{p^{-1}} f\rangle = 0$.

Therefore we have shown $\langle D, f\rangle = 0$ for all $f \in \mathcal{S}(pU, V_1)$, hence D vanishes on pU. □

Remark 5.12. We summarize the above results we have proved for the support of intertwining distributions. We first note the following sequence of inclusions:

$$\mathrm{Hom}_G(I, J) = \mathrm{Hom}_Q(I, V_2) \subset \mathrm{Hom}_Q(\mathcal{S}(G, V_1), V_2) \subset L(\mathcal{S}(G, V_1), V_2).$$

A distribution D in the largest space $L(\mathcal{S}(G, V_1), V_2)$ has a well-defined support as in Chapter 4. If D is in the subspace $\mathrm{Hom}_Q(\mathcal{S}(G, V_1), V_2)$, then its support is right Q-stable. If further D is in the $\mathrm{Hom}_G(I, J) = \mathrm{Hom}_Q(I, V_2)$, i.e. it factor through the $\mathcal{S}(G, V_1) \to I$, then its support is also left P-stable.

5.3 Some Notions of Distribution Analysis on G

Let $G, P, Q, (\sigma_1, V_1), (\sigma_2, V_2), I, J$ be the same as in 5.1. In this section, we introduce the notions of maximal double coset(s) in the supports, and the diagonal actions on Schwartz distribution spaces. Each subsection could be read independently.

5.3.1 Maximal Double Cosets in Supports

Let $D \in \mathrm{Hom}_G(I, J)$ be an intertwining distribution G, then the supp$D$ is a closed union of (P, Q)-double cosets. Let $H \subset Q$ be an algebraic subgroup, i.e. assume there is a $\mathbb{R}$-closed subgroup $\mathbf{H} \subset \mathbf{Q}$ such that $H = \mathbf{H}(\mathbb{R})$.

In this subsection, we assume there are finitely many (P, H)-double coset on G, and they are parameterized by a finite set $\mathcal{W}$.

Since there are finitely many (P, H)-double cosets, as in 3.3.3, the double cosets form a partial ordered set, ordered by their Zariski closures. And we can find a maximal (P, H)-double coset from the supp$D$ if supp$D \neq \emptyset$ (or equivalently $D \neq 0$).

Remark 5.13. Note that the closure order on double cosets is only a partial order. If a double coset is maximal in supp$D$, it doesn't mean it is "greater" than all other double cosets in supp$D$, but means it is "not smaller" than any other double cosets in supp$D$.

We call it "a", but not "the" maximal double coset, because there might be more than one maximal double cosets in supp$D$ since the order is only a partial order.

For a $w \in \mathcal{W}$, we denote the corresponding (P, H)-double coset by G_w. We can define the Zariski open subsets $G_{\geq w}, G_{>w}$ as in 3.3.3, and $G_w = G_{\geq w} - G_{>w}$ is closed in $G_{\geq w}$. We denote by

$$\mathrm{res}^G_{G_{\geq w}} = \text{the restriction map } L(\mathcal{S}(G, V_1), V_2) \to L(\mathcal{S}(G_{\geq w}, V_1), V_2)$$
$$\mathrm{res}^G_{G_{>w}} = \text{the restriction map } L(\mathcal{S}(G, V_1), V_2) \to L(\mathcal{S}(G_{>w}, V_1), V_2)$$

The following easy lemma gives an *algebraic description* of "a" maximal double coset in supp$D$:

Lemma 5.14. *Let $D \in \mathrm{Hom}_Q(\mathcal{S}(G, V_1), V_2)$ be an intertwining distribution and* assume $D \neq 0$ *to make sure it has nonempty support, then the following 3 statements are equivalent:*

1. *The G_w is "a" maximal double coset in* supp$D$.

2. *The restriction* $D|_{G_{\geq w}} = \mathrm{res}^G_{G_{\geq w}}(D) \neq 0$ *and the restriction* $D|_{G_{>w}} = \mathrm{res}^G_{G_{>w}}(D) = 0$.

3. *The* $D \in \mathrm{Ker}(\mathrm{res}^G_{G_{>w}})$ *but* $D \notin \mathrm{Ker}(\mathrm{res}^G_{G_{\geq w}})$.

Proof. We show 1. $\Leftrightarrow$ 2. since 2. $\Leftrightarrow$ 3. is trivial.

(1. $\Rightarrow$ 2.) Suppose G_w is a maximal double coset in supp$D$.

First we have $G_w \subset \mathrm{supp}D$, hence $G_{\geq w} \cap \mathrm{supp}D \supset G_w \neq \emptyset$. Hence $G_{\geq w} \nsubseteq \mathrm{van}D$ and $D|_{G_{\geq w}} \neq 0$.

Second if $D|_{G_{>w}} \neq 0$, then $G_{>w} \nsubseteq \mathrm{van}D$ hence $G_{>w} \cap \mathrm{supp}D \neq \emptyset$. There exists a $G_x \subset G_{>w} \cap \mathrm{supp}D$ (thus $G_x > G_w$). Hence G_w is not maximal, contradiction!

(2. $\Rightarrow$ 1.) Suppose $D|_{G_{\geq w}} \neq 0$ and $D|_{G_{>w}} = 0$. First we have $G_{>w} \subset \mathrm{van}D$.

Second we show $G_w \subset \mathrm{supp}D$. Suppose not, then $G_w \subset \mathrm{van}D$, then $G_{\geq w} = G_w \cup G_{>w} \subset \mathrm{van}D$, and $D|_{G_{\geq w}} = 0$, contradiction!

Now we see the G_w is a double coset in supp$D$, and for all $x > w$, the $G_x \subset G_{>w} \subset \mathrm{van}D$, i.e. all orbits "greater" than G_w are not contained in supp$D$. Hence G_w is "a" maximal orbit in supp$D$.

□

Remark 5.15. For $D \neq 0$, since there is at least one maximal double coset in supp$D$, the subsets $\mathrm{Ker}(\mathrm{res}^G_{G_{>w}}) - \mathrm{Ker}(\mathrm{res}^G_{G_{\geq w}})$ "cover" the subset $\mathrm{Hom}_G(I, J) - \{0\}$ of $L(\mathcal{S}(G, V_1), V_2)$. However this is not a partition, since maximal double coset(s) may not be unique.

5.3.2 Diagonal Actions on Distribution Spaces—I

Let $P, Q, (\sigma_1, V_1), (\sigma_2, V_2)$ be as the beginning of this section. As in 4.5.5, let

H = a closed algebraic subgroup of Q

Y = a real subvariety of G which is closed under right H-translation

One has the right regular H-action on the Schwartz function space $\mathcal{S}(Y, V_1)$, and the H-action on V_2 through the representation σ_2. (Note that $H \subset Q$ is a subgroup).

Definition 5.16 (**Diagonal H-action on distribution space**)**.** We define the following H-action $L(\mathcal{S}(Y, V_1), V_2)$: for all $\Phi \in L(\mathcal{S}(Y, V_1), V_2), h \in H$ and $f \in \mathcal{S}(Y, V_1)$, the $h \cdot \Phi$ is given by

$$\langle h \cdot \Phi, f \rangle := \sigma_2(h) \langle \Phi, R_{h^{-1}} f \rangle. \tag{5.2}$$

Here the

$$\langle , \rangle : L(\mathcal{S}(Y, V_1), V_2) \times \mathcal{S}(Y, V_1) \to V_2$$

is the pairing between the Schwartz distributions and Schwartz functions, $R_{h^{-1}}f$ is the right regular action of h^{-1} on f, the $\langle \Phi, R_{h^{-1}}f \rangle$ is a vector in V_2, and h acts on it through the representation σ_2.

We call this action the **diagonal H-action on the distribution space** $L(\mathcal{S}(Y, V_1), V_2)$.

Lemma 5.17. *For the above diagonal H-action on $L(\mathcal{S}(Y, V_1), V_2)$, we have:*

1. *The $L(\mathcal{S}(Y, V_1), V_2)$ is a smooth H-representation. It is isomorphic to the tensor product representation*

$$L(\mathcal{S}(Y, V_1), V_2) \simeq \mathcal{S}(Y, V_1)' \,\widehat{\otimes}\, V_2 \tag{5.3}$$

of H, where $\mathcal{S}(Y, V_1)'$ is endowed with the contragredient H-action of the above right regular H-action, and V_2 is endowed with the restricted representation $\sigma_2|_H$ of H.

2. *The space $\mathrm{Hom}_H(\mathcal{S}(Y, V_1), V_2)$ of H-equivariant V_2-valued Schwartz V_1-distributions on Y is exactly the space of H-invariants on the space $L(\mathcal{S}(Y, V_1), V_2)$:*

$$\mathrm{Hom}_H(\mathcal{S}(Y, V_1), V_2) = H^0(H, L(\mathcal{S}(Y, V_1), V_2)) \tag{5.4}$$

3. *Let U_1, U_2 be two right H-stable Zariski open subvarieties of G and $U_1 \subset U_2$, the restriction map of distributions*

$$\mathrm{res}_{U_1}^{U_2} : L(\mathcal{S}(U_2, V_1), V_2) \to L(\mathcal{S}(U_1, V_1), V_2)$$

is an H-intertwining operator. Its kernel is a H-subrepresentation of $L(\mathcal{S}(U_2, V_1), V_2)$ under the diagonal H-action.

4. *Let U be a right H-stable Zariski open subvariety of G, $O \subset U$ be a right H-stable Zariski closed subvariety of U. Then the inclusion map (extension of distributions from a closed subset)*

$$L(\mathcal{S}(O, V_1), V_2) \hookrightarrow L(\mathcal{S}(U, V_1), V_2)$$

is an H-intertwining operator.

5.3.3 Diagonal Actions on Distribution Spaces—II

Let $P, Q, (\sigma_1, V_1), (\sigma_2, V_2)$ be as above. Similar to the diagonal action on Schwartz distributions on Schwartz function spaces, we can define the diagonal action on Schwartz distributions on Schwartz induction spaces.

Let

$$\begin{aligned} H &= \text{a closed algebraic subgroup of } Q \\ Y &= \text{a real subvariety of } G \text{ which is stable under left } P\text{-translation} \\ &\quad\ \text{and right } H\text{-translation} \end{aligned}$$

Since Y is left a P-stable subvariety of G, one can define the local Schwartz induction space

$$\mathcal{S}\mathrm{Ind}_P^Y \sigma_1$$

as in 4.6. This space is a smooth H-representation, under the right regular H-action on it. The H also acts on the V_2 through σ_2 since it is a subgroup of Q.

Definition 5.18 (**Diagonal H-action on Schwartz induction spaces**). Let $L(\mathcal{S}\mathrm{Ind}_P^Y \sigma_1, V_2)$ be the space of V_2-valued Schwartz distributions on $\mathcal{S}\mathrm{Ind}_P^Y \sigma_1$. We define the following H-action on it: for each $h \in H, \Phi \in L(\mathcal{S}\mathrm{Ind}_P^Y \sigma_1, V_2)$, $\phi \in \mathcal{S}\mathrm{Ind}_P^Y \sigma_1$, let $h \cdot \Phi$ be the distribution

$$\langle h \cdot \Phi, \phi \rangle := \sigma(h) \langle \Phi, R_{h^{-1}} \phi \rangle \tag{5.5}$$

Here $R_{h^{-1}}\phi$ is the right regular action of h^{-1} on $\phi \in \mathcal{S}\mathrm{Ind}_P^Y \sigma_1$, the

$$\langle , \rangle : L(\mathcal{S}\mathrm{Ind}_P^Y \sigma_1, V_2) \times \mathcal{S}\mathrm{Ind}_P^Y \sigma_1 \to V_2$$

is the pairing between Schwartz induction space $\mathcal{S}\mathrm{Ind}_P^Y \sigma_1$ and Schwartz distribution space on it, and $\langle \Phi, R_{h^{-1}} \phi \rangle$ is a vector in V_2 and $\sigma_2(h)$ acts on it.

We call this H-action the **diagonal H-action on the distribution space** $L(\mathcal{S}\mathrm{Ind}_P^Y \sigma_1, V_2)$.

Lemma 5.19. *For the above diagonal H-action, we have*

1. *The $L(\mathcal{S}\mathrm{Ind}_P^Y \sigma_1, V_2)$ is a smooth H-representation under the diagonal H-action. It is isomorphic to the tensor product H-representation*

$$L(\mathcal{S}\mathrm{Ind}_P^Y \sigma_1, V_2) \simeq (\mathcal{S}\mathrm{Ind}_P^Y \sigma_1)' \,\widehat{\otimes}\, V_2 \tag{5.6}$$

where the $(\mathcal{S}\mathrm{Ind}_P^Y \sigma_1)'$ is the contragredient H-representation of the right regular representation, V_2 is regarded as the representation space of restricted representation $\sigma_2|_H$.

2. *The space* $\mathrm{Hom}_H(\mathcal{S}\mathrm{Ind}_P^Y\sigma_1, V_2)$ *of* H*-intertwining operators between* $\mathcal{S}\mathrm{Ind}_P^Y\sigma_1$ *and* $(\sigma_2|_H, V_2)$ *is exactly the space of* H*-invariant of the space* $L(\mathcal{S}\mathrm{Ind}_P^Y\sigma_1, V_2)$*:*
$$\mathrm{Hom}_H(\mathcal{S}\mathrm{Ind}_P^Y\sigma_1, V_2) = H^0(H, L(\mathcal{S}\mathrm{Ind}_P^Y\sigma_1, V_2)).$$

3. *Let* U_1, U_2 *be two Zariski open subvarieties of* G*, stable under left* P*-translation and right* H*-translation, and assume* $U_1 \subset U_2$*. Then the restriction map of distribution*
$$\mathrm{Res}_{U_1}^{U_2} : L(\mathcal{S}\mathrm{Ind}_P^{U_2}\sigma_1, V_2) \to L(\mathcal{S}\mathrm{Ind}_P^{U_1}\sigma_1, V_2)$$
is an H*-intertwining operator under the diagonal* H*-actions. Its kernel is an* H*-subrepresentation of* $L(\mathcal{S}\mathrm{Ind}_P^{U_2}\sigma_1, V_2)$*.*

4. *Let* U *be a Zariski open subvariety of* G*,* O *be a Zariski closed subvariety of* U *(thus also a locally closed subvariety of* G*), and assume both* U *and* O *are stable under left* P*-translation and right* H*-translation. Then the inclusion map of distributions*
$$L(\mathcal{S}\mathrm{Ind}_P^O\sigma_1, V_2) \hookrightarrow L(\mathcal{S}\mathrm{Ind}_P^U\sigma_1, V_2)$$
is an H*-intertwining operator.*

5.4 Self-Intertwining Distributions

Starting from this section through the entire chapter, we consider the self-intertwining distributions (operators) on a single smooth parabolic induction. We let
$$\begin{aligned}
\mathbf{P} = \mathbf{P}_\Theta &= \text{a standard parabolic } \mathbb{R}\text{-subgroup} \\
&\quad\ \text{corresponding to a subset } \Theta \text{ of the base } \Delta \\
P = P_\Theta &= \mathbf{P}(\mathbb{R}) \\
(\sigma, V) &= \text{a Harish-Chandra representation of } P \\
I &= \mathcal{S}\mathrm{Ind}_P^G\sigma = C^\infty\mathrm{Ind}_P^G\sigma
\end{aligned}$$
We apply the notions and results in the previous three sections, to the case when
$$\begin{gathered}
\mathbf{P} = \mathbf{Q} = \mathbf{P}_\Theta \\
P = Q = P_\Theta \\
(\sigma_1, V_1) = (\sigma_2, V_2) = (\sigma, V) \\
I = J = \mathcal{S}\mathrm{Ind}_P^G\sigma
\end{gathered}$$

i.e. P, Q are the same parabolic subgroup P_Θ, and $(\sigma_1, V_1), (\sigma_2, V_2)$ are the same representation (σ, V), I, J are the same Schwartz induction. And we study the space $\mathrm{Hom}_G(I, I)$ of **self-intertwining distributions (operators)**.

Let $D \in \mathrm{Hom}_G(I, I) = \mathrm{Hom}_P(I, V) \subset \mathrm{Hom}_P(\mathcal{S}(G, V), V)$ be an intertwining distribution. As we have seen in 5.2, its support $\mathrm{supp} D$ is a union of (P, P)-double cosets of G. Actually for any algebraic subgroup $H \subset P$, the support $\mathrm{supp} D$ is also a union of (P, H)-double cosets. We are particularly interested in the case when

$$H = P_\Omega,$$

where Ω is a subset of Θ, and P_Ω is the standard real parabolic subgroup corresponds to Ω.

5.4.1 (P, P_Ω)-Stable Subsets of G and Local Schwartz Inductions

As above, let

$$\begin{aligned} \Omega &= \text{a subset of } \Theta \\ \mathbf{P}_\Omega &= \text{the standard parabolic } \mathbb{R}\text{-subgroup corresponding to } \Omega \\ P_\Omega &= \mathbf{P}_\Omega(\mathbb{R}) = \text{the Lie group of real points of } \mathbf{P}_\Omega \end{aligned}$$

Then by $\emptyset \subset \Omega \subset \Theta$, we have $P_\emptyset \subset P_\Omega \subset P_\Theta$.

Notations on Double Cosets

As in 3.3, the (P, P_Ω)-double cosets in G are parameterized by the set

$$[W_\Theta \backslash W / W_\Omega] = \{w \in W : w^{-1}\Theta > 0, w\Omega > 0\}$$

of minimal representatives of the double quotient $W_\Theta \backslash W / W_\Omega$. The double cosets form a partial ordered set by the closure order on them, which also give the parameter set $[W_\Theta \backslash W / W_\Omega]$ a partial order. We adopt the same notations as in 3.3:

$$\begin{aligned}
G_w^\Omega &= PwP_\Omega = \text{the double coset corresponding to } w\\
G_{\geq w}^\Omega &= \coprod_{x\geq w} G_x^\Omega\\
&= \text{the open union of double cosets } PxP_\Omega \text{ s.t. } PwP_\Omega \subset \overline{PxP_\Omega}\\
G_{>w}^\Omega &= \coprod_{x\geq w, x\neq w} G_x^\Omega\\
&= \text{the open complement of } G_w^\Omega \text{ in } G_{\geq w}^\Omega
\end{aligned}$$

In particular, if $\Omega = \emptyset$ (the empty set), we omit the superscript Ω, i.e. for $w \in [W_\Theta\backslash W] = [W_\Theta\backslash W/W_\emptyset]$, we use the following simplified notations:

$$\begin{aligned}
G_w &= G_w^\emptyset = PwP_\emptyset\\
G_{\geq w} &= G_{\geq w}^\emptyset\\
G_{>w} &= G_{>w}^\emptyset
\end{aligned}$$

Remark 5.20. Note that for different $\Omega \subset \Theta$, the double cosets are parameterized by different sets of minimal representatives $[W_\Theta\backslash W/W_\Omega]$. However *all of them contain the minimal element e (identity of W). And we always have*

$$\begin{aligned}
G_e^\Omega &= P\\
G_{\geq e}^\Omega &= G\\
G_{>e}^\Omega &= G - P
\end{aligned}$$

no matter what Ω is, and they are stable under left and right P-translations. In this sense, *the minimal double coset $G_e^\Omega = P$ will be singled out and studied separately.*

Some Notations on Local Schwartz Inductions

Let $Y = G_w^\Omega, G_{\geq w}^\Omega$, or $G_{>w}^\Omega$. Obviously they are left P-stable and right P_Ω-stable, and the local Schwartz inductions $\mathcal{S}\mathrm{Ind}_P^Y\sigma$ are smooth nuclear representations of P_Ω under the right regular P_Ω-actions. We use the following simplified notations to denote the local Schwartz inductions on them:

$$\begin{aligned}
I_w^\Omega &= \mathcal{S}\mathrm{Ind}_P^{G_w^\Omega}\sigma\\
I_{\geq w}^\Omega &= \mathcal{S}\mathrm{Ind}_P^{G_{\geq w}^\Omega}\sigma\\
I_{>w}^\Omega &= \mathcal{S}\mathrm{Ind}_P^{G_{>w}^\Omega}\sigma
\end{aligned}$$

They are nuclear Fréchet spaces and smooth representations of P_Ω under the right regular P_Ω-actions.

As before, *when* $\Omega = \emptyset$ *(empty set), we omit the superscript* Ω, and for each $w \in [W_\Theta \backslash W]$, we let

$$\begin{aligned} I_w &= I_w^\emptyset \\ I_{\geq w} &= I_{\geq w}^\emptyset \\ I_{>w} &= I_{>w}^\emptyset \end{aligned}$$

5.4.2 Scalar Intertwining Operators

Let G, P, σ, I be as above. The space $\mathrm{Hom}_G(I, I)$ always contain the 1-dimensional subspace of scalar intertwining operators.

Given a $\lambda \in \mathbb{C}$, let $\lambda \mathrm{id} \in \mathrm{Hom}_G(I, I)$ be the scalar intertwining operator. Then the corresponding intertwining distribution (by Frobenius reciprocity) is obviously the

$$\lambda\Omega_e : I \to V.$$

We call this $\lambda\Omega_e \in \mathrm{Hom}_P(I, I)$ a **scalar intertwining distribution**. For such distributions, we obviously have

Lemma 5.21. *A scalar intertwining distribution* $\lambda\Omega_e \in \mathrm{Hom}_P(I, V)$ *have its support contained in* P*. More precisely, we have*

$$\mathrm{supp}(\lambda\Omega_e) = \begin{cases} P, & \text{if } \lambda \neq 0 \\ \emptyset, & \text{otherwise.} \end{cases} \tag{5.7}$$

5.4.3 Analysis of Schwartz Distributions

Let Y be a subvariety of G, which is stable under left P-translation and right P_Ω-translation. We have the Schwartz function space $\mathcal{S}(Y, V)$ and the Schwartz induction space $\mathcal{S}\mathrm{Ind}_P^Y \sigma$, and a surjective homomorphism of TVS

$$\mathcal{S}(Y, V) \twoheadrightarrow \mathcal{S}\mathrm{Ind}_P^Y \sigma, \quad f \mapsto f^\sigma,$$

where f^σ is the σ-mean value function of f. These two spaces are smooth P_Ω-representations, and the above surjective homomorphism is a homomorphism of NF-spaces and P_Ω-representations.

The $L(\mathcal{S}(Y, V), V), L(\mathcal{S}\mathrm{Ind}_P^Y \sigma, V)$ are the Schwartz distribution spaces, and they have the diagonal P_Ω-actions on them which make them into smooth P_Ω-representations. The V-transpose of the above surjective homomorphism is an injective homomorphism of TVS and P_Ω-representations:

$$L(\mathcal{S}\mathrm{Ind}_P^Y \sigma, V) \hookrightarrow L(\mathcal{S}(Y, V), V).$$

Restriction of Distributions to Open Subsets

Let $Y = G^{\Omega}_{\geq w}, G^{\Omega}_{>w}$ or G, they are all open in G. We have the two commutative diagrams of TVS and P_{Ω}-representations in Figure 5.1. (Remember that restriction maps of the pseudo-sheaf $L(\mathcal{S}(-,V),V)$ are denoted by res, while restriction maps of the pseudo-sheaf $L(\mathcal{S}\mathrm{Ind}_P^{-}\sigma, V)$ are denoted by Res.)

On the left diagram (a), the four inclusion maps (vertical arrows) are induced by the inclusions $G^{\Omega}_{>w} \subset G^{\Omega}_{\geq w} \subset G$. The three horizontal maps are surjective homomorphisms of TVS by the definition of Schwartz inductions. All six spaces in the left diagram (a) have the right regular P_{Ω}-actions, and all maps in the left diagram (a) are P_{Ω}-equivariant, hence they are P_{Ω}-intertwining operators between smooth P_{Ω}-representations.

The right diagram (b) is the V-transpose of the left diagram (a). All four vertical arrows are restrictions of distributions, and three horizontal arrows are injective homomorphisms of TVS. All six distribution spaces have the diagonal P_{Ω}-actions, and all maps in diagram (b) are intertwining operators of P_{Ω}-representations.

$$\begin{tikzcd}
I^{\Omega}_{>w} \arrow[d, hook] & \mathcal{S}(G^{\Omega}_{>w}, V) \arrow[l, two heads] \arrow[d, hook] \\
I^{\Omega}_{\geq w} \arrow[d, hook] & \mathcal{S}(G^{\Omega}_{\geq w}, V) \arrow[l, two heads] \arrow[d, hook] \\
I & \mathcal{S}(G, V) \arrow[l, two heads]
\end{tikzcd}$$

(a) Schwartz spaces

$$\begin{tikzcd}
L(I^{\Omega}_{>w}, V) \arrow[r, hook] & L(\mathcal{S}(G^{\Omega}_{>w}, V), V) \\
L(I^{\Omega}_{\geq w}, V) \arrow[r, hook] \arrow[u, "\mathrm{Res}^{G^{\Omega}_{\geq w}}_{G^{\Omega}_{>w}}"'] & L(\mathcal{S}(G^{\Omega}_{\geq w}, V), V) \arrow[u, "\mathrm{res}^{G^{\Omega}_{\geq w}}_{G^{\Omega}_{>w}}"'] \\
L(I, V) \arrow[r, hook] \arrow[u, "\mathrm{Res}^{G}_{G^{\Omega}_{\geq w}}"'] & L(\mathcal{S}(G, V), V) \arrow[u, "\mathrm{res}^{G}_{G^{\Omega}_{\geq w}}"']
\end{tikzcd}$$

(b) Distribution spaces

Figure 5.1: Schwartz and Distribution-Open

Extension of Distributions from Closed Subsets

Let $Y = G^{\Omega}_{w}, G^{\Omega}_{\geq w}$ or $G^{\Omega}_{>w}$. The $G^{\Omega}_{\geq w}, G^{\Omega}_{>w}$ are open in G, while G^{Ω}_{w} is closed in $G^{\Omega}_{\geq w}$. We have the two commutative diagrams of TVS and P_{Ω}-representations in Figure 5.2.

On the left diagram (a), the above two vertical arrows are inclusions of TVS induced by the inclusion $G^{\Omega}_{>w} \subset G^{\Omega}_{\geq w}$. The two vertical arrows below are surjective homomorphisms of TVS, given by restriction map to closed subvariety G^{Ω}_{w}. The three horizontal arrows are surjective homomorphisms of TVS by the definition of Schwartz inductions. The composition of two consecutive vertical arrows are zero, i.e. each vertical sequence in diagram (a) is a complex. All six spaces in diagram (a) have the right regular P_{Ω}-actions, and all maps in the diagram (a) are P_{Ω}-equivariant hence intertwining operators between smooth P_{Ω}-representations.

The right diagram (b) is the V-transpose of the diagram (a). In diagram (b): The two vertical arrows above are restriction maps of distributions. The two bottom vertical arrows are "extension of distributions from closed subsets" and they are injective homomorphisms of TVS. The three horizontal maps are inclusions of distributions spaces. All six distribution spaces in diagram (b) have the diagonal P_{Ω}-actions and they are smooth P_{Ω}-representations. All maps in diagram (b) are P_{Ω}-intertwining operators. The compositions of two consecutive vertical arrows are zero.

$$\begin{tikzcd}
I^{\Omega}_{>w} \arrow[d, hook] & \mathcal{S}(G^{\Omega}_{>w}, V) \arrow[l, two heads] \arrow[d, hook] \\
I^{\Omega}_{\geq w} \arrow[d, two heads] & \mathcal{S}(G^{\Omega}_{\geq w}, V) \arrow[l, two heads] \arrow[d, two heads] \\
I^{\Omega}_{w} & \mathcal{S}(G^{\Omega}_{w}, V) \arrow[l, two heads]
\end{tikzcd}$$

(a) Schwartz spaces

$$\begin{tikzcd}
L(I^{\Omega}_{>w}, V) \arrow[r, hook] & L(\mathcal{S}(G^{\Omega}_{>w}, V), V) \\
L(I^{\Omega}_{\geq w}, V) \arrow[r, hook] \arrow[u, "\mathrm{Res}^{G^{\Omega}_{\geq w}}_{G^{\Omega}_{>w}}"'] & L(\mathcal{S}(G^{\Omega}_{\geq w}, V), V) \arrow[u, "\mathrm{res}^{G^{\Omega}_{\geq w}}_{G^{\Omega}_{>w}}"'] \\
L(I^{\Omega}_{w}, V) \arrow[r, hook] \arrow[u, hook] & L(\mathcal{S}(G^{\Omega}_{w}, V), V) \arrow[u, hook]
\end{tikzcd}$$

(b) Distribution spaces

Figure 5.2: Schwartz and Distribution-Closed

5.5 The Irreducibility of Unitary Parabolic Inductions

In this section, we keep the notations as in the last section, and let (σ, V) be a Harish-Chandra representation, and $I = C^{\infty}\mathrm{Ind}_{P}^{G}\sigma = \mathcal{S}\mathrm{Ind}_{P}^{G}\sigma$ be the

Schwartz induction (also the smooth induction). We want to study when the following equality holds:

$$\mathrm{Hom}_G(I, I) = \mathbb{C},$$

i.e. the space of self-intertwining distributions is one dimensional.

Remark 5.22. When the σ is of the form $\sigma = \tau \otimes \delta_P^{1/2}$ for a unitary (Hilbert) representation (τ, V), the smooth induction $I = C^\infty \mathrm{Ind}_P^G \sigma$ is infinitesimally equivalent to the normalized unitary induction $\mathrm{Ind}_P^G \tau$. If two representations are infinitesimally equivalent, then they are irreducible/reducible simultaneously. Moreover, we have $\mathrm{Hom}_G(I, I) = \mathrm{Hom}_G(\mathrm{Ind}_P^G \tau, \mathrm{Ind}_P^G \tau)$ and $\mathrm{Ind}_P^G \tau$ is irreducible if and only if $\mathrm{Hom}_G(\mathrm{Ind}_P^G \tau, \mathrm{Ind}_P^G \tau) = \mathbb{C}$. Therefore, if $\sigma = \tau \otimes \delta_P^{1/2}$ for a unitary representation τ, the following statements are equivalent:

- The $I = C^\infty \mathrm{Ind}_P^G \sigma = \mathcal{S}\mathrm{Ind}_P^G \sigma$ is irreducible;
- The $\mathrm{Ind}_P^G \tau$ is irreducible;
- The $\mathrm{Hom}_G(\mathrm{Ind}_P^G \tau, \mathrm{Ind}_P^G \tau) = \mathbb{C}$;
- $\mathrm{Hom}_G(I, I) = \mathbb{C}$.

When the $\sigma = \tau \otimes \delta_P^{1/2}$, to show the I is irreducible, it is sufficient to show $\mathrm{Hom}_G(I, I) = \mathbb{C}$.

In this section, we find sufficient conditions for $\mathrm{Hom}_G(I, I)$ to be 1-dimensional, and in the Chapter 9, we will apply the results in this section to study the irreducibility of unitary inductions.

Let $D \in \mathrm{Hom}_P(I, V)$ be an intertwining distribution. For each fixed subset $\Omega \subset \Theta$, we have seen its support $\mathrm{supp} D$ is a Zariski closed union of (P, P_Ω)-double cosets. Then one has two possibilities of its support:

- **the extreme case:** the support is contained in the identity double coset P;
- **the general case:** the support is not contained in the identity double coset P.

We will discuss these two cases separately in following subsections.

5.5.1 General Case $\mathrm{supp} D \nsubseteq P$

Suppose the support $\mathrm{supp}D$ is not contained in P, then for each fixed subset $\Omega \subset \Theta$, there exists a maximal (P, P_Ω)-double coset contained in $\mathrm{supp}D$, say G_w^Ω for some $w \in [W_\Theta \backslash W/W_\Omega], w \neq e$.

For this particular Ω and w, let $I_{\geq w}^\Omega, I_{>w}^\Omega$ be the two local Schwartz inductions on $G_{\geq w}^\Omega$ and $G_{>w}^\Omega$ respectively, and one has the inclusions $I_{>w}^\Omega \subset I_{\geq w}^\Omega \subset I$. By Lemma 5.14, we see the restriction of D to the subspace $I_{>w}^\Omega$ is zero, and the restriction of D to the subspace $I_{\geq w}^\Omega$ is nonzero. We denote the restriction of D to $I_{\geq w}^\Omega$ by $D_{\geq w}^\Omega$, then it is in the up-left space in the following diagram

$$\begin{tikzcd}
\mathrm{Hom}_{P_\Omega}(I_{\geq w}^\Omega, V) \arrow[r, hook] \arrow[d, "restriction"'] & \mathrm{Hom}_{P_\Omega}(\mathcal{S}(G_{\geq w}^\Omega, V), V) \arrow[d, "restriction"'] \\
\mathrm{Hom}_{P_\Omega}(I_{>w}^\Omega, V) \arrow[r, hook] & \mathrm{Hom}_{P_\Omega}(\mathcal{S}(G_{>w}^\Omega, V), V)
\end{tikzcd}$$

Moreover the $D_{\geq w}^\Omega \in \mathrm{Hom}_{P_\Omega}(I_{\geq w}^\Omega, V)$ is a *nonzero element* in the kernel of the restriction map

$$\mathrm{Res}_w^\Omega : \mathrm{Hom}_{P_\Omega}(I_{\geq w}^\Omega, V) \to \mathrm{Hom}_{P_\Omega}(I_{>w}^\Omega, V)$$

Since $D_{\geq w}^\Omega$ vanishes on the subspace $I_{>w}^\Omega$ of $I_{\geq w}^\Omega$, it factor through a P_Ω-equivariant continuous linear map on the quotient:

$$\overline{D}_{\geq w}^\Omega : I_{\geq w}^\Omega / I_{>w}^\Omega \to V$$

which makes the following diagram commute

$$\begin{tikzcd}
I_{\geq w}^\Omega \arrow[r, "D_{\geq w}^\Omega"] \arrow[d, "quotient"'] & V \\
I_{\geq w}^\Omega / I_{>w}^\Omega \arrow[ur, "\overline{D}_{\geq w}^\Omega"'] &
\end{tikzcd}$$

To summarize the above, we have the following lemma which gives a *local description of intertwining distributions which have G_w^Ω as a maximal (P, P_Ω)-double coset in their supports*:

Lemma 5.23. *Let $D \in \mathrm{Hom}_P(I, V)$ be an intertwining distribution with support* $\mathrm{supp}D$.

(1) *If* G_w^Ω *is a maximal double coset in* $\mathrm{supp} D$*, then the element* $\overline{D}_{\geq w}^\Omega$ *defined above is a* nonzero element in $\mathrm{Ker}(\mathrm{Res}_w^\Omega)$*.*

(2) *The correspondence*

$$\mathrm{Ker}(\mathrm{Res}_w^\Omega) \to \mathrm{Hom}_{P_\Omega}(I_{\geq w}^\Omega / I_{>w}^\Omega, V)$$

is a linear isomorphism. Moreover, since $V, I_{\geq w}^\Omega, I_{>w}^\Omega$ *are nuclear hence reflexive, we have*

$$\mathrm{Hom}_{P_\Omega}(I_{\geq w}^\Omega / I_{>w}^\Omega, V) \simeq \mathrm{Hom}_{P_\Omega}(V', (I_{\geq w}^\Omega / I_{>w}^\Omega)')$$

where $V', (I_{\geq w}^\Omega / I_{>w}^\Omega)'$ *are the dual representation of* V *and* $I_{\geq w}^\Omega / I_{>w}^\Omega$ *respectively.*

5.5.2 Extreme Case $\mathrm{supp} D \subset P$

In this case, it does not matter what Ω is, since for all Ω, one has $G_e^\Omega = P, G_{\geq e}^\Omega = G, G_{>e}^\Omega = G - P$, and

$$I_{\geq e}^\Omega = I_{\geq e} = I$$
$$I_{>e}^\Omega = I_{>e} = \mathcal{S}\mathrm{Ind}_P^{G-P}\sigma$$

Thus we simply drop the superscript Ω.

Since $\mathrm{supp} D \subset P$, we have $\mathrm{van} D \supset G - P$, equivalently the $D \in \mathrm{Hom}_P(I, V)$ vanishes on the subspace $I_{>e}$ (which is also a subrepresentation of I). Therefore the D factor through the quotient representation $I/I_{>e}$, and gives a P-equivariant map

$$\overline{D} : I/I_{>e} \to V$$

which makes the following diagram commute

$$\begin{tikzcd} I \arrow[r, "D"] \arrow[d, "quotient"'] & V \\ I/I_{>e} \arrow[ur, "\overline{D}"'] & \end{tikzcd}$$

Similar to the general case, we have the following local description of intertwining distributions supported in P:

Lemma 5.24. *Let $D \in \mathrm{Hom}_P(I, V)$ be an intertwining distribution with support* $\mathrm{supp}D$*. The correspondence*

$$\begin{aligned}\{D \in \mathrm{Hom}_P(I, V) : \mathrm{supp}D \subset P\} &\to \mathrm{Hom}_P(I/I_{>e}, V)\\ D &\mapsto \overline{D}\end{aligned}$$

is a one-to-one linear map. Moreover, since the $V, I, I_{>e}$ are all nuclear hence reflexive, we have

$$\mathrm{Hom}_P(I/I_{>e}, V) \simeq \mathrm{Hom}_P(V', (I/I_{>e})'),$$

where the V' and $(I/I_{>e})'$ means the contragredient representation of V and $I/I_{>e}$ respectively.

In sum, the intertwining distributions with supports contained in P, are in one-to-one correspondence with the space $\mathrm{Hom}_P(V', (I/I_{>e})')$*.*

5.5.3 Irreducibility of I

We summarize the above two lemma about local descriptions of intertwining distributions:

Theorem 5.25. *Let $D \in \mathrm{Hom}_G(I, I) = \mathrm{Hom}_P(I, V)$ be an intertwining distribution. Then either* $\mathrm{supp}D \subset P$ *or* $\mathrm{supp}D \not\subseteq P$*.*

(1) If $\mathrm{supp}D \subset P$*, then the adjoint map of $\overline{D}$ is in the space*

$$\mathrm{Hom}_P(V', (I/I_{>e})').$$

(2) If $\mathrm{supp}D \not\subseteq P$*, then* for each $\Omega \subset \Theta$*, there exists a $w \in [W_\Theta \backslash W / W_\Omega]$, where $w \neq e$, such that the adjoint map of (restricted distribution) $\overline{D}^{\Omega}_{\geq w}$ is a* nonzero element *in*

$$\mathrm{Hom}_{P_\Omega}(V', (I^{\Omega}_{\geq w}/I^{\Omega}_{>w})')$$

Remark 5.26. Combining with Lemma 5.21, to show $\mathrm{Hom}_G(I, I) = \mathbb{C}$, (equivalently I is irreducible when $\sigma = \tau \otimes \delta_P^{1/2}$ for a unitary τ), we just need to:

(1) Show
$$\mathrm{Hom}_P(V', (I/I_{>e})') = \mathbb{C}.$$

(2) Find a subset $\Omega \subset \Theta$, and show for all $w \in [W_\Theta \backslash W / W_\Omega], w \neq e$, one has

$$\mathrm{Hom}_{P_\Omega}(V', (I^\Omega_{\geq w}/I^\Omega_{>w})') = \{0\}.$$

(Therefore it contains no nonzero elements.)

The (2) guarantees that there is no intertwining distribution with supports not contained in P. Otherwise suppose D has its support not contained in P, then for *every* Ω, one can find a maximal double coset G^Ω_w in supp$D$, and a nonzero element in some $\mathrm{Hom}_{P_\Omega}(V', (I^\Omega_{\geq w}/I^\Omega_{>w})')$, a contradiction!

Then the (1) guarantees that the only possible intertwining distributions are scalar intertwining distributions, therefore $\mathrm{Hom}_G(I, I) = \mathrm{Hom}_P(I, V) = \mathbb{C}$.

Chapter 6

Schwartz Distributions Supported in Double Cosets

Summary of This Chapter

In the last chapter, we are required to study the spaces

$$(I^{\Omega}_{\geq w}/I^{\Omega}_{>w})'$$

for various $\Omega \subset \Theta$ and $w \in [W_\Theta \backslash W / W_\Omega]$. Recall that we have the following exact sequence of NF-spaces

$$0 \to I^{\Omega}_{>w} \to I^{\Omega}_{\geq w} \to I^{\Omega}_{\geq w}/I^{\Omega}_{>w} \to 0$$

Since all spaces are NF-spaces, its dual sequence is still exact:

$$0 \to (I^{\Omega}_{\geq w}/I^{\Omega}_{>w})' \to (I^{\Omega}_{\geq w})' \to (I^{\Omega}_{>w})' \to 0$$

The space $(I^{\Omega}_{\geq w}/I^{\Omega}_{>w})'$ is exactly the kernel of the restriction map $(I^{\Omega}_{\geq w})' \to (I^{\Omega}_{>w})'$. We will show the kernel of the restriction map $(I^{\Omega}_{\geq w})' \to (I^{\Omega}_{>w})'$ consists of transverse derivatives of distributions on G^{Ω}_{w}. In this chapter, we will study the case $\Omega = \emptyset$ and the general cases of Ω will be studied in the last chapter.

Notations and Settings

Let $G, P = P_\Theta, P_\emptyset, W, W_\Theta, [W_\Theta \backslash W], (\sigma, V)$ be the same as in Chapter 5. The $(P, P_\emptyset)$-double cosets in G are parameterized by the set $[W_\Theta \backslash W]$ of minimal representatives, and they are ordered by the closure order (see Definition 3.28).

For a $w \in [W_\Theta \backslash W]$, let $G_w, G_{\geq w}, G_{>w}$ and $I_w, I_{\geq w}, I_{>w}$ be the same as in 5.4. Remember that $G_{\geq w}, G_{>w}$ are Zariski open in G and G_w is the complement of $G_{>w}$ in $G_{\geq w}$ hence G_w is closed in $G_{\geq w}$. The local Schwartz induction $I_{>w}$ is a subspace of $I_{\geq w}$, and the restriction to G_w gives a surjective homomorphism $I_{\geq w} \to I_w$.

We will work on a more general setting. Let

$$F = \text{a nuclear Fréchet space}$$

and let

$$\begin{aligned} \mathcal{S}(-, V) &= \text{the pseudo-cosheaf of } V\text{-valued Schwartz functions on } G \\ L(\mathcal{S}(-, V), F) &= \text{the pseudo-sheaf of } F\text{-valued Schwartz distributions on } G \end{aligned}$$

We use the following notations to denote the extension maps (by zero) of Schwartz V-valued functions and Schwartz inductions:

$$\begin{aligned} \mathrm{ex}_{G_{>w}}^{G_{\geq w}} &: \mathcal{S}(G_{>w}, V) \hookrightarrow \mathcal{S}(G_{\geq w}, V) \\ \mathrm{Ex}_{G_{>w}}^{G_{\geq w}} &: I_{>w} \hookrightarrow I_{\geq w} \end{aligned}$$

and the following notations to denote the restriction maps of Schwartz F-valued distributions:

$$\begin{aligned} \mathrm{res}_{G_{>w}}^{G_{\geq w}} &: L(\mathcal{S}(G_{\geq w}, V), F) \to L(\mathcal{S}(G_{>w}, V), F) \\ \mathrm{Res}_{G_{>w}}^{G_{\geq w}} &: L(I_{\geq w}, F) \to L(I_{>w}, F) \end{aligned}$$

and for the special case $F = \mathbb{C}$:

$$\begin{aligned} \mathrm{res}_w &: \mathcal{S}(G_{\geq w}, V)' \to \mathcal{S}(G_{>w}, V)' \\ \mathrm{Res}_w &: I'_{\geq w} \to I'_{>w} \end{aligned}$$

They fit into the following diagram

$$\begin{tikzcd}
\mathrm{Ker}(\mathrm{res}_{G_{>w}}^{G_{\geq w}}) \arrow[r] & L(\mathcal{S}(G_{\geq w}, V), F) \arrow[r, "\mathrm{res}_{G_{>w}}^{G_{\geq w}}"] & L(\mathcal{S}(G_{>w}, V), F) \\
\mathrm{Ker}(\mathrm{Res}_{G_{>w}}^{G_{\geq w}}) \arrow[r] \arrow[u, hook] & L(I_{\geq w}, F) \arrow[r, "\mathrm{Res}_{G_{>w}}^{G_{\geq w}}"] \arrow[u, hook] & L(I_{>w}, F) \arrow[u, hook]
\end{tikzcd}$$

Figure 6.1: Restriction maps and kernels

The two leftmost kernels (for the case $F = \mathbb{C}$) are the spaces we need to study in this chapter.

New Notations in This Chapter

Let $\overline{N}_P$ be the opposite unipotent radical of P. We introduce the following new notations:

$$\begin{aligned} N_w &= \dot{w}^{-1}\overline{N}_P\dot{w} \\ N_w^+ &= \dot{w}^{-1}\overline{N}_P\dot{w} \cap N_\emptyset \\ &= N_w \cap N_\emptyset \\ N_w^- &= \dot{w}^{-1}\overline{N}_P\dot{w} \cap \overline{N}_\emptyset \\ &= N_w \cap \overline{N}_\emptyset \end{aligned}$$

The N_w^+, N_w^- are closed subgroups of N_w and $N_w^+ \cap N_w^- = \{e\}$, and the multiplication map $N_w^+ \times N_w^- \to N_w$ is a smooth diffeomorphism and an isomorphism of real algebraic varieties (not a group isomorphism). Let $\mathfrak{n}_w, \mathfrak{n}_w^+, \mathfrak{n}_w^-$ be their complexified Lie algebras respectively, and one has $\mathfrak{n}_w = \mathfrak{n}_w^+ + \mathfrak{n}_w^-$.

Main Theorem of This Chapter

The main theorem in this chapter is (see 6.1.1 for more details):

Theorem (Theorem 6.1)**.** *The right multiplication of $U(\mathfrak{n}_w^-)$ (as derivatives), on the distribution spaces* $\mathrm{Ker}(\mathrm{res}^{G_{\geq w}}_{G_{>w}})$ *and* $\mathrm{Ker}(\mathrm{Res}^{G_{\geq w}}_{G_{>w}})$*, gives the following isomorphisms (of TVS):*

$$L(\mathcal{S}(G_w, V), F) \otimes U(\mathfrak{n}_w^-) \xrightarrow{\sim} \mathrm{Ker}(\mathrm{res}^{G_{\geq w}}_{G_{>w}})$$
$$L(I_w, F) \otimes U(\mathfrak{n}_w^-) \xrightarrow{\sim} \mathrm{Ker}(\mathrm{Res}^{G_{\geq w}}_{G_{>w}})$$

In this thesis, we will only need the case $F = \mathbb{C}$, but we prefer to include the general result since there is no essential difficulty to prove the general case.

This theorem means, Schwartz distributions supported in G_w, are exactly the transverse derivatives of Schwartz distributions on G_w. We will explain the term "transverse" in 7.1.5 of the next chapter.

6.1 Preparation

- In 6.1.1, we first formulate the main theorem in this chapter, namely explain the meaning of the maps (6.1) and (6.2) in the main theorem.
- In 6.1.2, we construct a Zariski open tubular neighbourhood Z_w of the G_w.

- In 6.1.3, we change the neighbourhood from $G_{\geq w}$ to Z_w. Since the Schwartz distributions have the pseudo-sheaf property, the space of Schwartz distributions supported in G_w is independent of the choice of neighbourhoods (Lemma 4.38 and Lemma 4.80).

6.1.1 Formulating the Main Theorem

We explain the main theorem at the beginning of this chapter. Remember we want to study the algebraic structure of the following two kernels of restriction maps of distributions:

$$\operatorname{Ker}(\operatorname{res}^{G_{\geq w}}_{G_{>w}}) = \operatorname{Ker}\{L(\mathcal{S}(G_{\geq w}, V), F) \to L(\mathcal{S}(G_{>w}, V), F)\}$$
$$\operatorname{Ker}(\operatorname{Res}^{G_{\geq w}}_{G_{>w}}) = \operatorname{Ker}\{L(I_{\geq w}, F) \to L(I_{>w}, F)\}$$

where F is an arbitrary NF-space. (Here we abuse the notation, and use the same $\operatorname{Res}, \operatorname{res}$ to denote restrictions with different target spaces F. This will not create ambiguity since we will only work on a single target space F at each time.)

Finally we will see for different F, they all have similar structure and we only need to study the case $F = \mathbb{C}$.

The Kernel $\operatorname{Ker}\{L(\mathcal{S}(G_{\geq w}, V), F) \to L(\mathcal{S}(G_{>w}, V), F)\}$

The $G_{\geq w}$ and $G_{>w}$ are Zariski open subset of G, hence the $U(\mathfrak{g})$ acts on them as algebraic differential operators, and makes the Schwartz function spaces $\mathcal{S}(G_{\geq w}, V)$ and $\mathcal{S}(G_{>w}, V)$ into left $U(\mathfrak{g})$-modules. The inclusion map

$$\mathcal{S}(G_{>w}, V) \hookrightarrow \mathcal{S}(G_{\geq w}, V)$$

is a left $U(\mathfrak{g})$-homomorphism.

The transpose action of differential operators makes the distribution spaces $L(\mathcal{S}(G_{\geq w}, V), F)$ and $L(\mathcal{S}(G_{>w}, V), F)$ into right $U(\mathfrak{g})$-modules. And the restriction map

$$\operatorname{res}^{G_{\geq w}}_{G_{>w}} : L(\mathcal{S}(G_{\geq w}, V), F) \to L(\mathcal{S}(G_{>w}, V), F)$$

is a right $U(\mathfrak{g})$-homomorphism. Hence the kernel

$$\operatorname{Ker}(\operatorname{res}^{G_{\geq w}}_{G_{>w}}) = \operatorname{Ker}\{L(\mathcal{S}(G_{\geq w}, V), F) \to L(\mathcal{S}(G_{>w}, V), F)\}$$

is also a right $U(\mathfrak{g})$-module (submodule of $L(\mathcal{S}(G_{\geq w}, V), F)$.)

Since G_w is closed in $G_{\geq w}$, by Lemma 4.36, we have the following inclusion of TVS:

$$L(\mathcal{S}(G_w, V), F) \hookrightarrow L(\mathcal{S}(G_{\geq w}, V), F).$$

We regard the $L(\mathcal{S}(G_w, V), F)$ as a subspace of $L(\mathcal{S}(G_{\geq w}, V), F)$. This subspace is contained in the kernel $\mathrm{Ker}(\mathrm{res}^{G_{\geq w}}_{G_{>w}})$, since the composition

$$\mathcal{S}(G_{>w}, V) \hookrightarrow \mathcal{S}(G_{\geq w}, V) \twoheadrightarrow \mathcal{S}(G_w, V)$$

is zero.

The $U(\mathfrak{n}_w^-)$ is a subalgebra of $U(\mathfrak{g})$, and the right multiplication of (algebraic differential operators) on distribution space gives the following map

$$\begin{gathered} L(\mathcal{S}(G_w, V), F) \otimes U(\mathfrak{n}_w^-) \to \mathrm{Ker}(\mathrm{res}^{G_{\geq w}}_{G_{>w}}) \\ \Phi \otimes u \mapsto \Phi \cdot u \end{gathered} \tag{6.1}$$

where the distribution $\Phi \cdot u$ is given by

$$\langle \Phi \cdot u, f \rangle := \langle \Phi, (R_u \cdot f)|_{G_w} \rangle.$$

The Kernel $\mathrm{Ker}\{L(I_{\geq w}, F) \to L(I_{>w}, F)\}$

The situation for distributions on Schwartz inductions is similar. The $G_{\geq w}, G_{>w}$ are in the Zariski P-topology of G, hence we have the inclusion of left $U(\mathfrak{g})$-modules

$$I_{>w} \hookrightarrow I_{\geq w}.$$

And its F-transpose gives the restriction map

$$\mathrm{Res}^{G_{\geq w}}_{G_{>w}} : L(I_{\geq w}, F) \to L(I_{>w}, F)$$

which is a right $U(\mathfrak{g})$ homomorphism, with its kernel $\mathrm{Ker}(\mathrm{Res}^{G_{\geq w}}_{G_{>w}})$ a right $U(\mathfrak{g})$-module (submodule).

Since G_w is closed in $G_{\geq w}$ and both are left P-stable, by Lemma 4.82, we have the injective homomorphism

$$L(I_w, F) \hookrightarrow L(I_{\geq w}, F)$$

with its image contained in the kernel $\mathrm{Ker}(\mathrm{Res}^{G_{\geq w}}_{G_{>w}})$, since the composition $I_{>w} \hookrightarrow I_{\geq w} \twoheadrightarrow I_w$ is zero.

The right multiplication of the subalgebra $U(\mathfrak{n}_w^-)$ of $U(\mathfrak{g})$ on the subspace $L(I_w, F)$ gives the following map

$$\begin{aligned} L(I_w, F) \otimes U(\mathfrak{n}_w^-) &\to L(I_{\geq w}, F) \\ \Phi \otimes u &\mapsto \Phi \cdot u \end{aligned} \tag{6.2}$$

where the $\Phi \cdot u$ is given by

$$\langle \Phi \cdot u, \phi \rangle := \langle \Phi, (R_u \phi)|_{G_w} \rangle, \quad \forall \phi \in I_{\geq w} = \mathcal{S}\mathrm{Ind}_P^{G_{\geq w}} \sigma.$$

The Main Theorem

The main theorem of this chapter is;

Theorem 6.1. *The map (6.1) and (6.2) are isomorphisms of TVS.*

Remark 6.2. The $U(\mathfrak{n}_w^-)$ is endowed with the inductive limit topology of $U_n(\mathfrak{n}_w^-)$ and is a LF-space and nuclear space. Here $U_n(\mathfrak{n}_w^-)$ means the finite dimensional subspace spanned by n products of elements in $\mathfrak{n}_w^-$.

6.1.2 G_w and its Tubular Neighbourhood Z_w

The $G_{\geq w}$ is an open neighbourhood of G_w, but its structure is hard to describe. We introduce another open neighbourhood Z_w of G_w which is a tubular neighbourhood.

The Neighbourhood Z_w

For each $w \in [W_\Theta \backslash W]$, remember we abuse the notation and denote the fixed representative of it in G by the same w. Let $N_w = w^{-1}\overline{N}_P w$ be as the beginning of this chapter. We let

$$Z_w = PwN_w = P\overline{N}_P w,$$

i.e. it is the (P, N_w)-double coset through the base point w, and it is exactly the right translation of $P\overline{N}_P$ by w.

Lemma 6.3. *The Z_w is a Zariski open subset of G. And the map*

$$\begin{aligned} P \times N_w &\to Z_w \\ (p, n) &\mapsto pwn \end{aligned} \tag{6.3}$$

is an isomorphism of varieties and manifolds.

The Structure of G_w

Recall that $G_w = PwP_\emptyset$ is the $(P, P_\emptyset)$-double coset through the base point w. It is easy to see G_w is also the $(P, N_\emptyset)$-double coset through w:

$$G_w = PwN_\emptyset,$$

since $wM_\emptyset w^{-1} = M_\emptyset$ for all $w \in W$.

The $N_\emptyset$ is the product of two subgroups, i.e. the multiplication map

$$\begin{aligned}(N_\emptyset \cap w^{-1}Pw) \times (N_\emptyset \cap w^{-1}\overline{N}_P w) &\to N_\emptyset \\ (n_1, n_2) &\mapsto n_1 n_2\end{aligned} \tag{6.4}$$

is an isomorphism of real algebraic variety and smooth manifolds (*NOT a group isomorphism*). Since the w-conjugation of first part is contained in P, we see

$$G_w = Pw(N_\emptyset \cap w^{-1}\overline{N}_P w) = PwN_w^+$$

where $N_w^+ = N_\emptyset \cap w^{-1}\overline{N}_P w$ as the beginning of this chapter.

Lemma 6.4. *The map*

$$\begin{aligned}P \times N_w^+ &\to G_w \\ (p, n) &\mapsto (pwn)\end{aligned} \tag{6.5}$$

is an isomorphism of varieties and manifolds. Hence the G_w is a nonsingular closed subvariety of Z_w and closed regular submanifold of Z_w. The following diagram commutes in the category of real algebraic varieties and smooth manifolds

$$\begin{array}{ccc} P \times N_w & \xrightarrow{(6.3)} & Z_w \\ \uparrow & & \uparrow \\ P \times N_w^+ & \xrightarrow{(6.5)} & G_w \end{array}$$

Here the two vertical arrows are closed embeddings and the two horizontal arrows are isomorphisms.

6.1.3 Change of Neighbourhoods

We now have two Zariski P-open neighbourhoods of G_w: $G_{\geq w}$ and Z_w, hence the intersection $G_{\geq w} \cap Z_w$ is also a Zariski P-open neighbourhood of

G_w. The complements of G_w in them are $G_{>w}$, Z_w-G_w and $G_{\geq w}\cap Z_w-G_w$ respectively (all three complements are Zariski P-open in G).

We then have the following two commutative diagrams, where all horizontal arrows are restriction maps of distributions, and all vertical sequences are exact:

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
\mathrm{Ker}(\mathrm{res}^{Z_w}_{Z_w-G_w}) \arrow[r, "(a)"] \arrow[d] & \mathrm{Ker}(\mathrm{res}^{Z_w\cap G_{\geq w}}_{Z_w\cap G_{\geq w}-G_w}) \arrow[d] & \mathrm{Ker}(\mathrm{res}^{G_{\geq w}}_{G_{>w}}) \arrow[l, "(b)"'] \arrow[d] \\
L(\mathcal{S}(Z_w,V),F) \arrow[r, "\mathrm{res}"] \arrow[d, "\mathrm{res}^{Z_w}_{Z_w-G_w}"] & L(\mathcal{S}(Z_w\cap G_{\geq w},V),F) \arrow[d, "\mathrm{res}^{Z_w\cap G_{\geq w}}_{Z_w\cap G_{\geq w}-G_w}"] & L(\mathcal{S}(G_{\geq w},V),F) \arrow[l, "\mathrm{res}"'] \arrow[d, "\mathrm{res}^{G_{\geq w}}_{G_{>w}}"] \\
L(\mathcal{S}(Z_w-G_w,V),F) \arrow[r, "\mathrm{res}"] \arrow[d] & L(\mathcal{S}(Z_w\cap G_{\geq w}-G_w,V),F) \arrow[d] & L(\mathcal{S}(G_{>w},V),F) \arrow[l, "\mathrm{res}"'] \arrow[d] \\
0 & 0 & 0
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
\mathrm{Ker}(\mathrm{Res}^{Z_w}_{Z_w-G_w}) \arrow[r, "(c)"] \arrow[d] & \mathrm{Ker}(\mathrm{Res}^{Z_w\cap G_{\geq w}}_{Z_w\cap G_{\geq w}-G_w}) \arrow[d] & \mathrm{Ker}(\mathrm{Res}^{G_{\geq w}}_{G_{>w}}) \arrow[l, "(d)"'] \arrow[d] \\
L(\mathcal{S}\mathrm{Ind}_P^{Z_w}\sigma,F) \arrow[r, "\mathrm{Res}"] \arrow[d, "\mathrm{Res}^{Z_w}_{Z_w-G_w}"] & L(\mathcal{S}\mathrm{Ind}_P^{Z_w\cap G_{\geq w}}\sigma,F) \arrow[d, "\mathrm{Res}^{Z_w\cap G_{\geq w}}_{Z_w\cap G_{\geq w}-G_w}"] & L(I_{\geq w},F) \arrow[l, "\mathrm{Res}"'] \arrow[d, "\mathrm{Res}^{G_{\geq w}}_{G_{>w}}"] \\
L(\mathcal{S}\mathrm{Ind}_P^{Z_w-G_w}\sigma,F) \arrow[r, "\mathrm{Res}"] \arrow[d] & L(\mathcal{S}\mathrm{Ind}_P^{Z_w\cap G_{\geq w}-G_w}\sigma,F) \arrow[d] & L(I_{>w},F) \arrow[l, "\mathrm{Res}"'] \arrow[d] \\
0 & 0 & 0
\end{tikzcd}$$

By Lemma 4.38, the maps (a)(b) are isomorphisms of TVS, and by Lemma 4.80, the maps (c)(d) are isomorphisms of TVS. Hence as in Lemma 4.39 and Lemma 4.81, we have the following isomorphisms of kernels of restriction maps:

$$\operatorname{Ker}(\operatorname{res}^{G_{\geq w}}_{G_{>w}}) \simeq \operatorname{Ker}(\operatorname{res}^{Z_w}_{Z_w - G_w}) \tag{6.6}$$
$$\operatorname{Ker}(\operatorname{Res}^{G_{\geq w}}_{G_{>w}}) \simeq \operatorname{Ker}(\operatorname{Res}^{Z_w}_{Z_w - G_w})$$

Remark 6.5. As we have seen in the last subsection, the Z_w is a tubular neighbourhood of G_w and we have good product decomposition of both G_w and Z_w in Lemma 6.3 and Lemma 6.4. This makes the analysis of the kernels $\operatorname{Ker}(\operatorname{res}^{Z_w}_{Z_w - G_w})$ and $\operatorname{Ker}(\operatorname{Res}^{Z_w}_{Z_w - G_w})$ much easier than the original kernels $\operatorname{Ker}(\operatorname{res}^{G_{\geq w}}_{G_{>w}})$ and $\operatorname{Ker}(\operatorname{Res}^{G_{\geq w}}_{G_{>w}})$.

6.2 Schwartz Distributions with Point Supports

In this section, let

$$\begin{aligned}
\mathbf{N} &= \text{a unipotent algebraic group defined over } \mathbb{R} \\
N &= \mathbf{N}(\mathbb{R}) \\
e &= \text{the identity of } N \\
N^* &= N - \{e\} \\
\mathfrak{n}_0 &= \text{the real Lie algebra of } N \\
\mathfrak{n} &= \text{the complexification of } \mathfrak{n}_0
\end{aligned}$$

The N is a nonsingular affine real algebraic variety and affine Nash manifold, we have the pseudo-cosheaf $\mathcal{S}(-,\mathbb{C})$ of $\mathbb{C}$-valued Schwartz functions on N, and the pseudo-sheaf $\mathcal{S}(-,\mathbb{C})'$ of Schwartz distributions on N. The $\{e\}$ is a nonsingular closed subvariety of N and a closed Nash submanifold of N, the N^* is Zariski open and restricted open in N.

Since N^* is Zariski open in N, we have the inclusion map

$$\mathcal{S}(N^*,\mathbb{C}) \hookrightarrow \mathcal{S}(N,\mathbb{C})$$

under which the $\mathcal{S}(N^*,\mathbb{C})$ is identified with the subspace

$$\{f \in \mathcal{S}(N,\mathbb{C}) : Df(e) = 0, \forall D \in \mathcal{D}(N)\}$$

of Schwartz functions vanishing with all derivatives at e.

Remark 6.6. Here the $\mathcal{D}$ is the sheaf of (complexified) algebraic differential operators, and the $\mathcal{D}(N)$ is the (complexified) ring of algebraic differential operators on N. Since N is affine real algebraic variety, the $\mathcal{D}(N)$ generates the ring of (complexified) Nash differential operators over the ring of complex valued Nash functions. Hence a Schwartz function vanishes with all Nash derivatives at e, is equivalent to say it vanishes with all algebraic derivatives at e. In this section, all geometric spaces are affine real algebraic varieties, hence it is safe to work only on algebraic derivatives.

Also if not otherwise stated, all differential operators are complexified, since we work on complex valued smooth (Schwartz) functions.

The transpose of the above inclusion map is the restriction map of distributions, denoted by

$$\mathrm{res} : \mathcal{S}(N, \mathbb{C})' \to \mathcal{S}(N^*, \mathbb{C})'.$$

In this section, we study the kernel of this restriction map. It is a closed subspace of $\mathcal{S}(N, \mathbb{C})'$, and fits into the exact sequence

$$0 \to \mathrm{Ker}(\mathrm{res}) \to \mathcal{S}(N, \mathbb{C})' \xrightarrow{\mathrm{res}} \mathcal{S}(N^*, \mathbb{C})' \to 0 \tag{6.7}$$

The main result in this section is

Lemma 6.7. *The kernel* $\mathrm{Ker}(\mathrm{res})$ *is isomorphic to* $U(\mathfrak{n})$*. More precisely, the following map is an isomorphism of TVS:*

$$\begin{aligned} U(\mathfrak{n}) &\mapsto \mathrm{Ker}(\mathrm{res}) \\ u &\mapsto \Omega_e \cdot u \end{aligned} \tag{6.8}$$

where $\Omega_e \cdot u$ *is given by*

$$\langle \Omega_e \cdot u, f \rangle := \langle \Omega_e, R_u f \rangle = (R_u f)(e)$$

for all $f \in \mathcal{S}(N, \mathbb{C})$*. And the* $U(\mathfrak{n})$ *is endowed with the direct limit topology from finite dimensional subspaces* $U_n(\mathfrak{n})$*.*

We omit the proof since it is exactly the same as the proof of Theorem XXXVI on page 101 of [35].

6.3 Proof of the Main Theorem

We first prove the theorem for the case $F = \mathbb{C}$ and we use the following simplified notations:

$$\begin{aligned} \mathrm{res}^w &: \mathcal{S}(Z_w, V)' \to \mathcal{S}(Z_w - G_w, V)' \\ \mathrm{Res}^w &: (\mathcal{S}\mathrm{Ind}_P^{Z_w}\sigma)' \to (\mathcal{S}\mathrm{Ind}_P^{Z_w - G_w}\sigma)' \end{aligned}$$

i.e. they are the restriction maps of $\mathbb{C}$-valued distributions on Schwartz functions and Schwartz inductions respectively. In this section, we show the following special case ($F = \mathbb{C}$) of the main theorem:

Lemma 6.8. *The following maps, given by multiplication of derivatives in $U(\mathfrak{n}_w^-)$, are isomorphisms of TVS:*

$$\mathcal{S}(G_w, V)' \otimes U(\mathfrak{n}_w^-) \to \mathrm{Ker}(\mathrm{res}^w), \quad \Phi \otimes u \mapsto \Phi \cdot u \tag{6.9}$$

$$(\mathcal{S}\mathrm{Ind}_P^{G_w} \sigma)' \otimes U(\mathfrak{n}_w^-) \to \mathrm{Ker}(\mathrm{Res}^w), \quad \Phi \otimes u \mapsto \Phi \cdot u \tag{6.10}$$

More precisely, let $\Phi \in \mathcal{S}(G_w, V)'$ (resp. $(\mathcal{S}\mathrm{Ind}_P^{G_w} \sigma)'$) and $u \in U(\mathfrak{n}_w^-)$, the $\Phi \cdot u$ is

$$\langle \Phi \cdot u, f \rangle = \langle \Phi, (R_u f)|_{G_w} \rangle$$

for all $f \in \mathcal{S}(Z_w, V)$ (resp. $\mathcal{S}\mathrm{Ind}_P^{Z_w} \sigma$).

Later we will use this Lemma to show the main theorem for general NF-spaces F.

6.3.1 The Kernel $\mathrm{Ker}(\mathrm{res}^w)$

Let

$$\mathrm{res}^w : \mathcal{S}(Z_w, V)' \to \mathcal{S}(Z_w - G_w, V)'$$

be the restriction map of $\mathbb{C}$-valued Schwartz V-distributions on Z_w, as in the beginning of this section.

Isomorphisms between Varieties (Affine Nash Manifolds)

Combining the Lemma 6.3 and Lemma 6.4 and the isomorphism $N_w^+ \times N_w^- \xrightarrow{\sim} N_w$, we have the following isomorphisms:

$$\begin{gathered} P \times N_w^+ \times N_w^- \xrightarrow{\sim} Z_w \\ P \times N_w^+ \times (N_w^- - \{e\}) \xrightarrow{\sim} Z_w - G_w \\ P \times N_w^+ \times \{e\} \xrightarrow{\sim} G_w \end{gathered}$$

Then by (E-6) of Proposition 4.30, we have the following isomorphisms of Schwartz V-valued function spaces:

$$\begin{gathered} \mathcal{S}(P \times N_w^+, V) \widehat{\otimes} \mathcal{S}(N_w^-, \mathbb{C}) \xrightarrow{\sim} \mathcal{S}(Z_w, V) \\ \mathcal{S}(P \times N_w^+, V) \widehat{\otimes} \mathcal{S}(N_w^- - \{e\}, \mathbb{C}) \xrightarrow{\sim} \mathcal{S}(Z_w - G_w, V) \\ \mathcal{S}(P \times N_w^+, V) \widehat{\otimes} \mathcal{S}(\{e\}, \mathbb{C}) \xrightarrow{\sim} \mathcal{S}(G_w, V) \end{gathered}$$

Isomorphisms between Distribution Spaces

Since all the above Schwartz spaces are nuclear, we have the corresponding isomorphisms on their strong dual:

$$\begin{aligned}\mathcal{S}(Z_w,V)' &\xrightarrow{\sim} \mathcal{S}(P\times N_w^+,V)'\,\widehat{\otimes}\,\mathcal{S}(N_w^-,\mathbb{C})'\\ \mathcal{S}(Z_w-G_w,V)' &\xrightarrow{\sim} \mathcal{S}(P\times N_w^+,V)'\,\widehat{\otimes}\,\mathcal{S}(N_w^- -\{e\},\mathbb{C})'\\ \mathcal{S}(G_w,V)' &\xrightarrow{\sim} \mathcal{S}(P\times N_w^+,V)'\,\widehat{\otimes}\,\mathcal{S}(\{e\},\mathbb{C})'\end{aligned}$$

Note that $\mathcal{S}(\{e\},\mathbb{C})=\mathbb{C}$, i.e. Schwartz functions at a point are exactly "constant", and $\mathcal{S}(\{e\},\mathbb{C})'=\mathbb{C}\Omega_e^{\{e\}}$ is the one dimensional space spanned by the "delta function" on $\{e\}$.

It is easy to verify that the following diagrams commutes:

$$\begin{tikzcd}\mathcal{S}(Z_w,V)' \arrow[r, "\simeq"] \arrow[d, "\mathrm{res}^w"] & \mathcal{S}(P\times N_w^+,V)'\,\widehat{\otimes}\,\mathcal{S}(N_w^-,\mathbb{C})' \arrow[d, "\mathrm{id}\otimes r_w"] \\ \mathcal{S}(Z_w-G_w,V)' \arrow[r, "\simeq"] & \mathcal{S}(P\times N_w^+,V)'\,\widehat{\otimes}\,\mathcal{S}(N_w^- -\{e\},\mathbb{C})'\end{tikzcd}$$

Here the rightmost vertical arrow is the tensor product of the identity map on $\mathcal{S}(P\times N_w^+,V)'$ with the restriction map

$$r_w:\mathcal{S}(N_w^-,\mathbb{C})'\to\mathcal{S}(N_w^- -\{e\},\mathbb{C})'.$$

Therefore, we have the isomorphism between the kernels of the above two vertical maps:

$$\mathrm{Ker}(\mathrm{res}^w)\simeq\mathcal{S}(P\times N_w^+,V)'\,\widehat{\otimes}\,\mathrm{Ker}(r_w).$$

The Tensor Product $\mathcal{S}(P\times N_w^+,V)'\otimes\mathrm{Ker}(r_w)$

In section 6.2, we have seen the following isomorphism between TVS

$$U(\mathfrak{n}_w^-)\xrightarrow{\sim}\mathrm{Ker}(r_w),$$

where $U(\mathfrak{n}_w^-)$ is given the limit topology from the finite dimensional subspaces $U_n(\mathfrak{n}_w^-)$, and is a LF-space.

By Lemma 2.13, we have

$$\begin{aligned}
\mathrm{Ker}(\mathrm{res}_w) &\simeq \mathcal{S}(P \times N_w^+, V)' \mathbin{\widehat{\otimes}} U(\mathfrak{n}_w^-) \\
&= \mathcal{S}(P \times N_w^+, V)' \mathbin{\widehat{\otimes}} \varinjlim_n U_n(\mathfrak{n}_w^-) \\
&= \varinjlim_n [\mathcal{S}(P \times N_w^+, V)' \mathbin{\widehat{\otimes}} U_n(\mathfrak{n}_w^-)] \\
&= \varinjlim_n [\mathcal{S}(P \times N_w^+, V)' \otimes U_n(\mathfrak{n}_w^-)] \\
&= \mathcal{S}(P \times N_w^+, V)' \otimes U(\mathfrak{n}_w^-)
\end{aligned}$$

In other word, the algebraic tensor product $\mathcal{S}(P\times N_w^+, V)'\otimes U(\mathfrak{n}_w^-)$ is already complete, and we don't need the completion on the $\otimes$.

In sum, we have an isomorphism of TVS

$$\mathrm{Ker}(\mathrm{res}^w) \simeq \mathcal{S}(P \times N_w^+, V)' \otimes U(\mathfrak{n}_w^-) \tag{6.11}$$

6.3.2 The Isomorphism (6.9)

In the last subsection, we have shown the isomorphism (6.11):

$$\mathrm{Ker}(\mathrm{res}^w) \simeq \mathcal{S}(P \times N_w^+, V)' \otimes U(\mathfrak{n}_w^-)$$

which is further isomorphic to $\mathcal{S}(G_w, V)' \otimes U(\mathfrak{n}_w^-)$ since $\mathcal{S}(G_w, V)' \simeq \mathcal{S}(P \times N_w^+, V)'$. Hence To show the map (6.9) is an isomorphism, we just need to verify it is the same as the isomorphism (6.11).

More precisely, we just need to verify that the following diagram commutes

$$\begin{tikzcd}
& \mathcal{S}(P \times N_w^+, V)' \otimes U(\mathfrak{n}_w^-) \arrow[dl] \arrow[dr] & \\
\mathcal{S}(P \times N_w^+, V)' \mathbin{\widehat{\otimes}} \mathrm{Ker}(r_w) \arrow[d, hook] & & \mathrm{Ker}(\mathrm{res}^w) \arrow[d, hook] \\
\mathcal{S}(P \times N_w^+, V)' \otimes \mathcal{S}(N_w^-, \mathbb{C})' \arrow[dr] & & \mathcal{S}(Z_w, V)' \arrow[dl] \\
& (\mathcal{S}(P \times N_w^+, V) \otimes \mathcal{S}(N_w^-, \mathbb{C}))' &
\end{tikzcd}$$

This is routine and we omit the detail. One just need to pick a function of the form $\phi \otimes \psi$ for $\phi \in \mathcal{S}(P \times N_w^+, V)$ and $\psi \in \mathcal{S}(N_w^-, \mathbb{C})$, and check the value $\langle \Phi \cdot u, \phi \otimes \psi \rangle$ through two ways, and see they are equal. The crucial reason is the Lie algebra $\mathfrak{n}_w^-$ acts on Schwartz functions through left invariant vector fields, i.e. by differentiation of the right regular N_w-action, and it will not affect the P-component on the left.

6.3.3 The Kernel $\mathrm{Ker}(\mathrm{Res}^w)$ and the Isomorphism (6.10)

To show the map (6.10) is an isomorphism, we just need to combine the isomorphism $U(\mathfrak{n}_w^-) \simeq \mathrm{Ker}\{\mathcal{S}(N_w^-, \mathbb{C}) \to \mathcal{S}(N_w^- - \{e\}, \mathbb{C})\}$ with the isomorphism in Lemma 4.91.

By Lemma 4.91, we have the following isomorphisms

$$\mathcal{S}\mathrm{Ind}_P^{Z_w}\sigma \simeq \mathcal{S}\mathrm{Ind}_P^P\sigma \,\widehat{\otimes}\, \mathcal{S}(N_w, \mathbb{C})$$
$$\mathcal{S}\mathrm{Ind}_P^{Z_w - G_w}\sigma \simeq \mathcal{S}\mathrm{Ind}_P^P\sigma \,\widehat{\otimes}\, \mathcal{S}(N_w - N_w^+, \mathbb{C})$$

and the isomorphisms on their dual

$$(\mathcal{S}\mathrm{Ind}_P^{Z_w}\sigma)' \simeq (\mathcal{S}\mathrm{Ind}_P^P\sigma)' \,\widehat{\otimes}\, \mathcal{S}(N_w, \mathbb{C})'$$
$$(\mathcal{S}\mathrm{Ind}_P^{Z_w - G_w}\sigma)' \simeq (\mathcal{S}\mathrm{Ind}_P^P\sigma)' \,\widehat{\otimes}\, \mathcal{S}(N_w - N_w^+, \mathbb{C})'$$

Thus the following diagram commutes:

$$\begin{tikzcd}
(\mathcal{S}\mathrm{Ind}_P^{Z_w}\sigma)' \arrow[r, "\mathrm{Res}^w"] \arrow[d, "\simeq"] & (\mathcal{S}\mathrm{Ind}_P^{Z_w - G_w}\sigma)' \arrow[d, "\simeq"] \\
(\mathcal{S}\mathrm{Ind}_P^P\sigma)' \,\widehat{\otimes}\, \mathcal{S}(N_w, \mathbb{C})' \arrow[r] & (\mathcal{S}\mathrm{Ind}_P^P\sigma)' \,\widehat{\otimes}\, \mathcal{S}(N_w - N_w^+, \mathbb{C})'
\end{tikzcd}$$

where the bottom arrow is the tensor product map $\mathrm{id} \otimes \mathrm{res}^{N_w}_{N_w - N_w^+}$. Hence we have the isomorphism on the kernels

$$\mathrm{Ker}(\mathrm{Res}^w) \simeq (\mathcal{S}\mathrm{Ind}_P^P\sigma)' \,\widehat{\otimes}\, \mathrm{Ker}(\mathrm{res}^{N_w}_{N_w - N_w^+}).$$

Therefore

$$\begin{aligned}
\mathrm{Ker}(\mathrm{Res}^w) &\simeq (\mathcal{S}\mathrm{Ind}_P^P\sigma)' \,\widehat{\otimes}\, \mathrm{Ker}(\mathrm{res}^{N_w}_{N_w - N_w^+}) \\
&\simeq (\mathcal{S}\mathrm{Ind}_P^P\sigma)' \,\widehat{\otimes}\, \mathcal{S}(N_w^+, \mathbb{C})' \\
&\quad \widehat{\otimes}\, \mathrm{Ker}\{\mathcal{S}(N_w^-, \mathbb{C})' \to \mathcal{S}(N_w^- - \{e\}, \mathbb{C})'\} \\
&\simeq (\mathcal{S}\mathrm{Ind}_P^P\sigma)' \,\widehat{\otimes}\, \mathcal{S}(N_w^+, \mathbb{C})' \,\widehat{\otimes}\, U(\mathfrak{n}_w^-) \\
&\simeq (\mathcal{S}\mathrm{Ind}_P^{G_w}\sigma)' \,\widehat{\otimes}\, U(\mathfrak{n}_w^-)
\end{aligned}$$

And by the same argument as in the last subsection, we can verify this isomorphism is exactly the map (6.10), hence (6.10) is an isomorphism of TVS. The proof of Lemma 6.8 is completed.

6.3.4 General Case of F

We prove the general case for arbitrary nuclear Fréchet space F, although we won't use this result in the thesis. We only show the isomorphism

$$L(I_w, F) \otimes U(\mathfrak{n}_w^-) \simeq \mathrm{Ker}\{\mathrm{Res}_{G_{>w}}^{G_{\geq w}} : L(I_{\geq w}, F) \to L(I_{>w}, F)\}$$

since the other isomorphism is similar.

First we have the isomorphism:

$$\mathrm{Ker}\{\mathrm{Res}_{G_{>w}}^{G_{\geq w}} : L(I_{\geq w}, F) \to L(I_{>w}, F)\} \simeq \mathrm{Ker}\{I'_{\geq w} \to I'_{>w}\} \mathbin{\widehat{\otimes}} F$$

Second we have the natural maps

$$\begin{aligned}
(I'_w \otimes U(\mathfrak{n}_w^-)) \otimes F &\xrightarrow{\sim} (I'_w \otimes F) \otimes U(\mathfrak{n}_w^-) \\
&\hookrightarrow (I'_w \mathbin{\widehat{\otimes}} F) \otimes U(\mathfrak{n}_w^-) \\
&\xrightarrow{\sim} L(I_w, F) \otimes U(\mathfrak{n}_w^-)
\end{aligned}$$

and their composition extends to a continuous linear map

$$(I'_w \otimes U(\mathfrak{n}_w^-)) \mathbin{\widehat{\otimes}} F \to L(I_w, F) \otimes U(\mathfrak{n}_w^-),$$

which makes the following diagram commute

$$\begin{tikzcd}
(I'_w \otimes U(\mathfrak{n}_w^-)) \mathbin{\widehat{\otimes}} F \arrow[r, "\sim"] \arrow[d] & \mathrm{Ker}\{I'_{\geq w} \to I'_{>w}\} \mathbin{\widehat{\otimes}} F \arrow[d, "\sim"] \\
L(I_w, F) \otimes U(\mathfrak{n}_w^-) \arrow[r] & \mathrm{Ker}\{L(I_{\geq w}, F) \to L(I_{>w}, F)\}
\end{tikzcd}$$

To show the bottom arrow is an isomorphism, we just need to show the left vertical arrow is an isomorphism. Actually

$$\begin{aligned}
(I'_w \otimes U(\mathfrak{n}_w^-)) \mathbin{\widehat{\otimes}} F &= \varinjlim_n [I'_w \otimes U_n(\mathfrak{n}_w^-)] \mathbin{\widehat{\otimes}} F \\
&\simeq \varinjlim_n [(I'_w \otimes U_n(\mathfrak{n}_w^-)) \mathbin{\widehat{\otimes}} F] \\
&\simeq \varinjlim_n [(I'_w \mathbin{\widehat{\otimes}} F) \otimes U_n(\mathfrak{n}_w^-)] \\
&= \varinjlim_n [L(I_w, F) \otimes U_n(\mathfrak{n}_w^-)] \\
&\simeq L(I_w, F) \otimes U(\mathfrak{n}_w^-)
\end{aligned}$$

(Here we use Lemma 2.13 twice in the first and last steps.)

Chapter 7

Torsions of Distributions

Summary of This Chapter

In the last chapter, we have shown the spaces $(I_{\geq w}/I_{>w})'$ are isomorphic to

$$I'_w \otimes U(\mathfrak{n}_w^-)$$

for all $w \in [W_\Theta \backslash W]$. In this chapter, we continue to study the $\mathfrak{n}_\emptyset$-torsion subspaces of them, and show their $\mathfrak{n}_\emptyset$-torsion subspaces are given by the torsion subspace on the first component I'_w.

Notations in This Chapter

As in Chapter 6, we use the following notations to denote the restriction maps of (scalar valued) distributions:

$$\begin{aligned} \mathrm{res}_w &: \mathcal{S}(G_{\geq w}, V)' \to \mathcal{S}(G_{>w}, V)' \\ \mathrm{Res}_w &: I'_{\geq w} \to I'_{>w} \end{aligned}$$

These two restriction maps are $P_\emptyset$-homomorphisms of $P_\emptyset$-representations, when the four dual spaces are endowed with the contragredient actions of $P_\emptyset$. We have seen that

$$\begin{aligned} \mathrm{Ker}(\mathrm{res}_w) &= (\mathcal{S}(G_{\geq w}, V)/\mathcal{S}(G_{>w}, V))' \\ \mathrm{Ker}(\mathrm{Res}_w) &= (I_{\geq w}/I_{>w})' \end{aligned}$$

and these two spaces are the main objects studied in the last chapter.

For each $n \in \mathbb{Z}_{\geq 0}$, let $U_n(\mathfrak{n}_w^-)$ be the (finite dimensional) subspace of $U(\mathfrak{n}_w^-)$ spanned by products of k-elements in $\mathfrak{n}_w^-$ for all $k \leq n$. By convention, let $U_0(\mathfrak{n}_w^-) = \mathbb{C}$. The $\{U_n(\mathfrak{n}_w^-), n \geq 0\}$ is an exhaustive filtration on $U(\mathfrak{n}_w^-)$.

The Main Theorem of This Chapter

The main theorem of this chapter is (see 7.1.1 for details):

Theorem (Theorem 7.1). *The $\mathfrak{n}_\emptyset$-torsion subspaces of the two kernels are given by:*

$$(\mathrm{Ker}(\mathrm{res}_w))^{[\mathfrak{n}_\emptyset{}^\bullet]} = [\mathcal{S}(G_w,V)' \otimes U(\mathfrak{n}_w^-)]^{[\mathfrak{n}_\emptyset{}^\bullet]} = [\mathcal{S}(G_w,V)']^{[\mathfrak{n}_\emptyset{}^\bullet]} \otimes U(\mathfrak{n}_w^-)$$
$$(\mathrm{Ker}(\mathrm{Res}_w))^{[\mathfrak{n}_\emptyset{}^\bullet]} = [I'_w \otimes U(\mathfrak{n}_w^-)]^{[\mathfrak{n}_\emptyset{}^\bullet]} = [I'_w]^{[\mathfrak{n}_\emptyset{}^\bullet]} \otimes U(\mathfrak{n}_w^-)$$

Moreover, these two equalities are $M_\emptyset$-equivariant when both sides are endowed with the obvious $M_\emptyset$-actions.

Reading Guide

- Section 7.1 consists of preliminary works. We explain the main theorem with more details, show a product formula in enveloping algebra which will be frequently used, and discuss the transverse decomposition of tangent vector fields on the submanifolds G_w. In particular, we show the first equality in the above main theorem implies the second equality, and we just need to show the first one, namely

$$[\mathcal{S}(G_w,V)' \otimes U(\mathfrak{n}_w^-)]^{[\mathfrak{n}_\emptyset{}^\bullet]} = [\mathcal{S}(G_w,V)']^{[\mathfrak{n}_\emptyset{}^\bullet]} \otimes U(\mathfrak{n}_w^-).$$

- In 7.2, we show the inclusion

$$[\mathcal{S}(G_w,V)' \otimes U(\mathfrak{n}_w^-)]^{[\mathfrak{n}_\emptyset{}^\bullet]} \supset [\mathcal{S}(G_w,V)']^{[\mathfrak{n}_\emptyset{}^\bullet]} \otimes U(\mathfrak{n}_w^-).$$

  This is the easy part of the main theorem, and is proved by pure algebraic tricks.

- In 7.3, we show the reversed inclusion

$$[\mathcal{S}(G_w,V)' \otimes U(\mathfrak{n}_w^-)]^{[\mathfrak{n}_\emptyset{}^\bullet]} \subset [\mathcal{S}(G_w,V)']^{[\mathfrak{n}_\emptyset{}^\bullet]} \otimes U(\mathfrak{n}_w^-).$$

  This is the most subtle part of the entire thesis. In a word, we choose a "good" linear order on the PBW-basis of the enveloping algebra $U(\mathfrak{n}_w^-)$, filter the left-hand-side by this order, and prove each filtered part is included in the right-hand-side by induction on this linear order.

7.1 Preparation

This section is the preparation work for the proof of the main theorem (Theorem 7.1). Each subsection could be read independently.

- In 7.1.1, we discuss the main theorem in more details. The entire chapter is reduced to proving the first equality in the main theorem:
$$(\mathcal{S}(G_w, V)' \otimes U(\mathfrak{n}_w^-))^{[\mathfrak{n}_\emptyset{}^\bullet]} = [\mathcal{S}(G_w, V)']^{[\mathfrak{n}_\emptyset{}^\bullet]} \otimes U(\mathfrak{n}_w^-).$$
The second equality in the main theorem is implied by this equality.

- In 7.1.2, we recall the notion of transpose modules over an enveloping algebra. The $U(\mathfrak{g})$-actions on the distribution spaces are essentially the transpose $U(\mathfrak{g})$-module of the Schwartz function spaces. It is more natural and convenient to consider the distribution spaces as right $U(\mathfrak{g})$-modules instead of left $U(\mathfrak{g})$-modules. *In the following sections of this chapter, we will regard all distribution spaces as right modules.*

- In 7.1.3, we introduce some simplified notations which are convenient for our proof.

- In 7.1.4, we show a formula in the enveloping algebra, which will be used repeatedly in the proof.

- In 7.1.5, we recall the necessary geometric notions: left and right invariant vector fields, transverse tangent fields, and show that the elements in $\mathfrak{n}_w^-$ are indeed transverse to the submanifold G_w.

7.1.1 Main Theorem in this Chapter

We have seen the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{S}(G_{>w}, V) \arrow[r] \arrow[d, two heads] & \mathcal{S}(G_{\geq w}, V) \arrow[r] \arrow[d, two heads] & \frac{\mathcal{S}(G_{\geq w}, V)}{\mathcal{S}(G_{>w}, V)} \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & I_{>w} \arrow[r] & I_{\geq w} \arrow[r] & \frac{I_{\geq w}}{I_{>w}} \arrow[r] & 0
\end{tikzcd}$$

where the two rows are exact sequence of NF-spaces, the three verticall arrows are surjective and all arrows are $P_\emptyset$-equivariant. By taking the strong dual, we have the following commutative diagram of DNF-spaces (and dual $P_\emptyset$-representations), in which all two rows are exact sequences and three vertical arrows are inclusions:

$$\begin{tikzcd}
0 \arrow[r] & (\frac{\mathcal{S}(G_{\geq w}, V)}{\mathcal{S}(G_{>w}, V)})' \arrow[r] & \mathcal{S}(G_{\geq w}, V)' \arrow[r, "\mathrm{res}_w"] & \mathcal{S}(G_{>w}, V)' \arrow[r] & 0 \\
0 \arrow[r] & (\frac{I_{\geq w}}{I_{>w}})' \arrow[r] \arrow[u, hook] & I'_{\geq w} \arrow[r, "\mathrm{Res}_w"] \arrow[u, hook] & I'_{>w} \arrow[r] \arrow[u, hook] & 0
\end{tikzcd}$$

The two leftmost spaces, namely

$$\begin{aligned}\mathrm{Ker}(\mathrm{res}_w) &= (\mathcal{S}(G_{\geq w}, V)/\mathcal{S}(G_{>w}, V))' \\ \mathrm{Ker}(\mathrm{Res}_w) &= (I_{\geq w}/I_{>w})'\end{aligned}$$

are the main object to study. The two kernels are $P_\emptyset$-subrepresentations of $\mathcal{S}(G_{\geq w}, V)'$ and $I'_{\geq w}$ respectively, and in particular they are also $\mathfrak{n}_\emptyset$-modules (actually even $\mathfrak{g}$-modules since $G_{\geq w}$ is open), with the Lie algebras acting on them as algebraic derivatives. In this chapter we will compute the $\mathfrak{n}_\emptyset$-torsion subspaces on the two $\mathfrak{n}_\emptyset$-modules $\mathrm{Ker}(\mathrm{Res}_w)$ and $\mathrm{Ker}(\mathrm{res}_w)$.

The Main Theorem in this Chapter

In Chapter 6, we have shown the $\mathrm{Ker}(\mathrm{res}_w)$ (resp. $\mathrm{Ker}(\mathrm{Res}_w)$) is given by $U(\mathfrak{n}_w^-)$-transverse derivatives of distributions in $\mathcal{S}(G_w, V)'$ (resp. I'_w), i.e. we have the two horizontal linear isomorphisms in the following diagram

$$\begin{tikzcd} \mathcal{S}(G_w, V)' \otimes U(\mathfrak{n}_w^-) \arrow[r, equal] & \mathrm{Ker}(\mathrm{res}_w) \\ I'_w \otimes U(\mathfrak{n}_w^-) \arrow[u, hook] \arrow[r, equal] & \mathrm{Ker}(\mathrm{Res}_w) \arrow[u, hook] \end{tikzcd}$$

Figure 7.1: $\mathrm{Ker}(\mathrm{res}_w)$ and $\mathrm{Ker}(\mathrm{Res}_w)$ as transverse derivatives

The right side inclusion $\mathrm{Ker}(\mathrm{Res}_w) \hookrightarrow \mathrm{Ker}(\mathrm{res}_w)$ is $P_\emptyset$-equivariant (under the contragredient $P_\emptyset$-actions on $\mathcal{S}(G_{\geq w}, V)'$ and $I'_{\geq w}$ respectively), in particular it is a homomorphism of $\mathfrak{n}_\emptyset$-modules.

The two horizontal isomorphisms transport the $\mathfrak{n}_\emptyset$-actions on the right two kernel spaces to the left two tensor products. The $\mathfrak{n}_\emptyset$ acts on them as derivatives, while the $U(\mathfrak{n}_w^-)$ also acts as (transverse) derivatives, hence one regard $U(\mathfrak{n}_\emptyset)$ and $U(\mathfrak{n}_w^-)$ as subrings of the $U(\mathfrak{g})$, which also acts as derivatives on distribution spaces. In particular, one can multiply elements in $U(\mathfrak{n}_\emptyset)$ and $U(\mathfrak{n}_w^-)$ inside $U(\mathfrak{g})$.

These derivative actions are complicated, and there is no easy algebraic descriptions. In other words, the tensor products $\mathcal{S}(G_w, V)' \otimes U(\mathfrak{n}_w^-)$ and $I'_w \otimes U(\mathfrak{n}_w^-)$ are not tensor products of $\mathfrak{n}_\emptyset$-modules, as one can see the $U(\mathfrak{n}_w^-)$ is not stable under $\mathfrak{n}_\emptyset$-multiplication.

However, their $\mathfrak{n}_\emptyset$-torsion subspaces, behave like tensor product modules:

Theorem 7.1. *Under the two horizontal isomorphisms in Figure 7.1, the $\mathfrak{n}_\emptyset$-torsion subspaces on* $\mathrm{Ker}(\mathrm{res}_w)$ *and* $\mathrm{Ker}(\mathrm{Res}_w)$ *are given by*

$$[\mathrm{Ker}(\mathrm{res}_w)]^{[\mathfrak{n}_\emptyset{}^\bullet]} = (\mathcal{S}(G_w, V)' \otimes U(\mathfrak{n}_w^-))^{[\mathfrak{n}_\emptyset{}^\bullet]} = (\mathcal{S}(G_w, V)')^{[\mathfrak{n}_\emptyset{}^\bullet]} \otimes U(\mathfrak{n}_w^-) \tag{7.1}$$

$$[\mathrm{Ker}(\mathrm{Res}_w)]^{[\mathfrak{n}_\emptyset{}^\bullet]} = (I_w' \otimes U(\mathfrak{n}_w^-))^{[\mathfrak{n}_\emptyset{}^\bullet]} = (I_w')^{[\mathfrak{n}_\emptyset{}^\bullet]} \otimes U(\mathfrak{n}_w^-)$$

Since the inclusion $\mathrm{Ker}(\mathrm{Res}_w) \hookrightarrow \mathrm{Ker}(\mathrm{res}_w)$ is an $\mathfrak{n}_\emptyset$-homomorphism, by Lemma 3.36, we see

$$[\mathrm{Ker}(\mathrm{Res}_w)]^{[\mathfrak{n}_\emptyset{}^\bullet]} = \mathrm{Ker}(\mathrm{Res}_w) \cap [\mathrm{Ker}(\mathrm{res}_w)]^{[\mathfrak{n}_\emptyset{}^\bullet]}.$$

If we can show the first equality in the above theorem, then we have

$$\begin{aligned}
[\mathrm{Ker}(\mathrm{Res}_w)]^{[\mathfrak{n}_\emptyset{}^\bullet]} &= \mathrm{Ker}(\mathrm{Res}_w) \cap [\mathrm{Ker}(\mathrm{res}_w)]^{[\mathfrak{n}_\emptyset{}^\bullet]} \\
&= [I_w' \otimes U(\mathfrak{n}_w^-)] \cap [\mathcal{S}(G_w, V)' \otimes U(\mathfrak{n}_w^-)]^{[\mathfrak{n}_\emptyset{}^\bullet]} \\
&= [I_w' \otimes U(\mathfrak{n}_w^-)] \cap \{[\mathcal{S}(G_w, V)']^{[\mathfrak{n}_\emptyset{}^\bullet]} \otimes U(\mathfrak{n}_w^-)\} \\
&= \{I_w' \cap [\mathcal{S}(G_w, V)']^{[\mathfrak{n}_\emptyset{}^\bullet]}\} \otimes U(\mathfrak{n}_w^-) \\
&= [I_w']^{[\mathfrak{n}_\emptyset{}^\bullet]} \otimes U(\mathfrak{n}_w^-)
\end{aligned}$$

Hence the second equality in the above theorem is implied by the first one, we just need to show the first equality. The rest of this chapter is devoted to this object.

Remark 7.2. Showing the second equality directly, is much more difficult than imagined. The Schwartz induction spaces $I_{>w}, I_{\geq w}$ are included in smooth function spaces on corresponding algebraic manifolds, and one can take derivatives of functions in them, however the resulting functions need not satisfy the σ-rule. Although one can take the derivatives in Lie algebra, we will see in the following sections that derivatives in Lie algebra are not sufficient for the discussion.

7.1.2 Left vs. Right

On Schwartz distribution spaces, the left action of Lie algebra is the transpose action of the Lie algebra on the corresponding Schwartz function spaces.

For example, the $\mathcal{S}(G_{\geq w}, V)$ is a left $U(\mathfrak{g})$-module, since $U(\mathfrak{g})$ acts as algebraic differential operators on G and $G_{\geq w}$ is open in G. On the corresponding distribution space $\mathcal{S}(G_{\geq w}, V)'$, the left $U(\mathfrak{g})$-module structure is given by the transpose $U(\mathfrak{g})$-action:

$$\langle X \cdot \Phi, f\rangle := \langle \Phi, -X \cdot f\rangle$$

for all $\Phi \in \mathcal{S}(G_{\geq w}, V)', f \in \mathcal{S}(G_{\geq w}, V), X \in \mathfrak{g}$. Here the $-X \cdot f = R_{-X} f$ is the differentiation of the right regular action. More generaly, for $X_1, \ldots, X_k \in \mathfrak{g}$, one has

$$\langle X_1 \cdots X_k \cdot \Phi, f\rangle = \langle \Phi, (-1)^k X_k \cdots X_1 \cdot f\rangle.$$

To get rid of the annoying $(-1)^k$ and the reversed order, it is more natural to regard a distribution space as a right module over an enveloping algebra.

Remark 7.3. On distribution spaces, it is more natural to let differential operators act on the right, since the differential operators actually act on functions rather than distributions.

On Schwartz function spaces, the right regular action is an action, and the Lie algebra action makes them into left modules. There is no ambiguity about left or right in discussion.

Transpose Right Module of a Left Module

Definition 7.4 (**Involution on Enveloping Algebra**)**.** Let $\mathfrak{h}$ be an arbitrary complex Lie algebra, and $U(\mathfrak{h})$ be its enveloping algebra. The map $-1 : \mathfrak{h} \to \mathfrak{h}, X \mapsto -X$ induces an involution on the $U(\mathfrak{h})$

$$\begin{aligned} U(\mathfrak{h}) &\to U(\mathfrak{h}) \\ u &\mapsto {}^t u \end{aligned}$$

More precisely, let $X_1, \ldots, X_k \in \mathfrak{h}$ be arbitrary elements, then ${}^t(X_1 \cdots X_k) = (-1)^k (X_k \cdots X_1)$.

Definition 7.5 (**Transpose Module of Left $U(\mathfrak{h})$-Module**)**.** Let $\mathcal{M}$ be a left $U(\mathfrak{h})$-module. Then it has a natural structure of right $U(\mathfrak{h})$-module, given by

$$m \cdot u := {}^t u \cdot m$$

for all $u \in U(\mathfrak{h}), m \in \mathcal{M}$. We call $\mathcal{M}$ with this right $U(\mathfrak{h})$-module structure the **transpose (right) $U(\mathfrak{h})$-module of the left $U(\mathfrak{h})$-module $\mathcal{M}$.**

We keep the notation as in 3.4. Keep in mind that we use superscript for torsion subspaces of left module, and subscript for torsions of right module. The right transpose $U(\mathfrak{h})$-module of a left $U(\mathfrak{h})$-module $\mathcal{M}$ share the same torsion submodule with $\mathcal{M}$, i.e. we have the following easy fact:

Lemma 7.6. *Let $\mathcal{M}$ be a left $U(\mathfrak{h})$-module, and by abuse of notation we denote its transpose right $U(\mathfrak{h})$-module by the same $\mathcal{M}$. For $k \in \mathbb{Z}_{\geq 0}$, let*

$$\mathcal{M}^{[\mathfrak{h}^k]} = \textit{the annihilator of } (\mathfrak{h}^k) \textit{ in the left } U(\mathfrak{h})\textit{-module } \mathcal{M}$$
$$\mathcal{M}_{[\mathfrak{h}^k]} = \textit{the annihilator of } (\mathfrak{h}^k) \textit{ in the transpose right } U(\mathfrak{h})\textit{-module } \mathcal{M}$$
$$\mathcal{M}^{[\mathfrak{h}^\bullet]} = \textit{the } \mathfrak{h}\textit{-torsion submodule of the left } U(\mathfrak{h})\textit{-module } \mathcal{M}$$
$$\mathcal{M}_{[\mathfrak{h}^\bullet]} = \textit{the } \mathfrak{h}\textit{-torsion submodule of the right transpose } U(\mathfrak{h})\textit{-module } \mathcal{M}$$

Then

$$\mathcal{M}^{[\mathfrak{h}^k]} = \mathcal{M}_{[\mathfrak{h}^k]} \qquad \forall k \geq 0$$
$$\mathcal{M}^{[\mathfrak{h}^\bullet]} = \mathcal{M}_{[\mathfrak{h}^\bullet]}$$

Left Module vs. Right Module

For a Schwartz distribution space with appropriate left module structure, e.g. the left $U(\mathfrak{p}_\emptyset)$-module $\mathcal{S}(G_w, V)'$, or left $U(\mathfrak{g})$-module $\mathcal{S}(G_{\geq w}, V)'$ and $\mathrm{Ker}(\mathrm{res}_w)$, we will consider their transpose right module, and torsion subspaces.

For example, on $\mathcal{S}(G_{\geq w}, V)'$ or its submodule $\mathrm{Ker}(\mathrm{res}_w)$, the transpose right $U(\mathfrak{g})$-action is

$$\begin{aligned}\langle \Phi \cdot u, f \rangle &= \langle {}^t u \cdot \Phi, f \rangle \\ &= \langle \Phi, {}^t({}^t u) \cdot f \rangle \\ &= \langle \Phi, u \cdot f \rangle\end{aligned}$$

for all $\Phi \in \mathcal{S}(G_{\geq w}, V)', f \in \mathcal{S}(G_{\geq w}, V), u \in U(\mathfrak{g})$. In other words, the right transpose action is exactly the right multiplication of derivatives.

Similarly on $\mathcal{S}(G_w, V)'$ we have the transpose right $U(\mathfrak{p}_\emptyset)$-module structure, given by the right multiplication of derivatives. The inclusion map $\mathcal{S}(G_w, V)' \hookrightarrow \mathcal{S}(G_{\geq w}, V)'$ is a right $U(\mathfrak{p}_\emptyset)$-homomorphism.

Remark 7.7. *Starting from this subsection, through the entire chapter, we will consider the* $\mathrm{Ker}(\mathrm{res}_w)$ *and $\mathcal{S}(G_{\geq w}, V)'$ as a right $U(\mathfrak{g})$-module, and similarly $\mathcal{S}(G_w, V)'$ as a right $U(\mathfrak{p}_\emptyset)$-module, with the transpose right module structures from the original left module structures on them.*

We will study the right $\mathfrak{n}_\emptyset$-torsion subspace

$$[\mathrm{Ker}(\mathrm{res}_w)]_{[\mathfrak{n}_\emptyset{}^\bullet]}$$

which is exactly the left torsion subspace $[\mathrm{Ker}(\mathrm{res}_w)]^{[\mathfrak{n}_\emptyset{}^\bullet]}$ by Lemma 7.6.

7.1.3 Setting and Notation for the Proof

Remember that in 7.1.2, we have regarded all modules as right modules. Let

$$\begin{aligned}
\mathcal{U} &= \mathcal{S}(G_w, V)' \\
\mathcal{K} &= \mathrm{Ker}(\mathrm{res}_w) \\
&= \mathrm{Ker}\{\mathcal{S}(G_{\geq w}, V)' \to \mathcal{S}(G_{>w}, V)'\} \\
&= \mathcal{S}(G_w, V)' \otimes U(\mathfrak{n}_w^-) \\
&= \mathcal{U} \otimes U(\mathfrak{n}_w^-) \\
\mathcal{U}_{[\mathfrak{n}_\emptyset \bullet]} &= \text{the } \mathfrak{n}_\emptyset\text{-torsion submodule of } \mathcal{S}(G_w, V)' \\
\mathcal{M} &= \mathcal{U}_{[\mathfrak{n}_\emptyset \bullet]} \otimes U(\mathfrak{n}_w^-)
\end{aligned}$$

The $\mathcal{K}$ is the space of (scalar valued) distributions on $\mathcal{S}(G_{\geq w}, V)$ vanishing on the subspace $\mathcal{S}(G_{>w}, V)$, and $\mathcal{M}$ is the space of $U(\mathfrak{n}_w^-)$-transverse derivatives of (scalar valued) $\mathfrak{n}_\emptyset$-torsion distributions on $\mathcal{S}(G_w, V)$. The $\mathcal{M}$ is a subspace of $\mathcal{K}$, and we need to show it is exactly the $\mathfrak{n}_\emptyset$-torsion submodule of $\mathcal{K}$:

$$\mathcal{K}_{[\mathfrak{n}_\emptyset \bullet]} = \mathcal{M}, \tag{7.2}$$

or equivalently $(\mathcal{U} \otimes U(\mathfrak{n}_w^-))_{[\mathfrak{n}_\emptyset \bullet]} = \mathcal{U}_{[\mathfrak{n}_{\emptyset_\bullet}]} \otimes U(\mathfrak{n}_w^-)$.

For each $n \in \mathbb{Z}_{\geq 0}$, let

$$U_n(\mathfrak{n}_w^-) = \mathrm{span}_{\mathbb{C}}\{X_1 \cdot X_2 \cdots X_k : k \leq n, X_1, \ldots, X_k \in \mathfrak{n}_w^-\}$$

be the finite dimensional subspace of $U(\mathfrak{n}_w^-)$, spanned by $\leq n$ products of elements in $\mathfrak{n}_w^-$. Then $\{U_n(\mathfrak{n}_w^-)\}_{n\geq 0}$ form an exhaustive filtration of $U(\mathfrak{n}_w^-)$:

$$U(\mathfrak{n}_w^-) = \bigcup_{n \geq 0} U_n(\mathfrak{n}_w^-).$$

Let

$$\begin{aligned}
\mathcal{K}^n &= \mathcal{U} \otimes U_n(\mathfrak{n}_w^-) \qquad (7.3)\\
\mathcal{K}^n_{[\mathfrak{n}_\emptyset \bullet]} &= \mathcal{K}_{[\mathfrak{n}_\emptyset \bullet]} \cap \mathcal{K}^n \\
\mathcal{M}^n &= \mathcal{U}_{[\mathfrak{n}_\emptyset \bullet]} \otimes U_n(\mathfrak{n}_w^-)
\end{aligned}$$

Obviously we have $\mathcal{M}^n \subset \mathcal{K}^n$ for all $n \geq 0$, hence $\mathcal{M} \subset \mathcal{K}$. The $\{\mathcal{K}^n\}_{n\geq 0}$ (respectively $\{\mathcal{K}^n_{[\mathfrak{n}_\emptyset \bullet]}\}_{n\geq 0}, \{\mathcal{M}^n\}_{n\geq 0}$) form an exhaustive filtration of $\mathcal{K}$ (respectively $\mathcal{K}_{[\mathfrak{n}_\emptyset \bullet]}, \mathcal{M}$).

To show $\mathcal{K}_{[\mathfrak{n}_\emptyset \bullet]} = \mathcal{M}$, we just need to show the two filtration $\{\mathcal{K}^n_{[\mathfrak{n}_\emptyset \bullet]}\}_{n\geq 0}$ and $\{\mathcal{M}^n\}_{n\geq 0}$ are cofinal.

7.1.4 A Formula

In this subsection, we show a multiplication formula in the enveloping algebra. Namely, let $\mathfrak{g}$ be an arbitrary complex Lie algebra, and $U(\mathfrak{g})$ be its enveloping algebra, let $Y_1, \ldots, Y_k$ be k arbitrary elements in $\mathfrak{g}$ and $X \in \mathfrak{g}$ be another one. We want to compute the product $(Y_1 \cdot Y_2 \cdot \ldots \cdot Y_k) \cdot X$, and move the $Y_1, \ldots, Y_k$ to the right as much as we can.

Some Notations

For an element $X \in \mathfrak{g}$, let

$$\begin{aligned} R_X &: U(\mathfrak{g}) \to U(\mathfrak{g}), u \mapsto u \cdot X \\ L_X &: U(\mathfrak{g}) \to U(\mathfrak{g}), u \mapsto X \cdot u \\ \mathrm{ad}X &: U(\mathfrak{g}) \to U(\mathfrak{g}), u \mapsto [X, u] = X \cdot u - u \cdot X \end{aligned}$$

be the right multiplication, left multiplication, and adjoint action of X on $U(\mathfrak{g})$. In the algebra $\mathrm{Hom}_{\mathbb{C}}(U(\mathfrak{g}), U(\mathfrak{g}))$, these three elements satisfy

$$\mathrm{ad}X = L_X - R_X. \tag{7.4}$$

The $U(\mathfrak{g})$ is an associative algebra, hence for $X, Y \in \mathfrak{g}$, we have

$$L_X \circ R_Y = R_Y \circ L_X \tag{7.5}$$

i.e. the L_X commutes with R_Y in $\mathrm{Hom}_{\mathbb{C}}(U(\mathfrak{g}), U(\mathfrak{g}))$ for all $X, Y \in \mathfrak{g}$, although the $\mathrm{Hom}_{\mathbb{C}}(U(\mathfrak{g}), U(\mathfrak{g}))$ is not a commutative algebra.

For a positive integer k, we denote by

$$[1, k] = \{1, 2, \ldots, k-1, k\}$$

the set of the first k positive integers. For a subset $S \subset [1, k]$, let $s = |S|$ be its size, and $S^c = [1, k] - S$ be its complement in $[1, k]$. For the subsets S and S^c, we always arrange their elements in the increasing order.

Let $Y_1, Y_2, \ldots, Y_k \in \mathfrak{g}$ be k elements labeled by $[1, k]$. Let $S = \{i_1, \ldots, i_s\}$ be a subset of $[1, k]$ and let $S^c = \{j_1, \ldots, j_{k-s}\}$ be its complement. We introduce the following notations:

$$\begin{aligned} \langle Y_S, - \rangle &= \mathrm{ad}Y_{i_1} \circ \mathrm{ad}Y_{i_2} \circ \cdots \circ \mathrm{ad}Y_{i_s} \\ Y_S &= Y_{i_1} \cdot Y_{i_2} \cdots Y_{i_s} \\ R_{Y_S} &= R_{Y_{i_s}} \circ \cdots \circ R_{Y_{i_1}} \end{aligned}$$

i.e. the $\langle Y_S, -\rangle$ is the following element in the $\mathrm{Hom}_{\mathbb{C}}(U(\mathfrak{g}), U(\mathfrak{g}))$

$$\begin{aligned}\langle Y_S, -\rangle : U(\mathfrak{g}) &\to U(\mathfrak{g})\\ u &\mapsto [Y_{i_1}, [Y_{i_2}, [\cdots, [Y_{i_s}, u]]]],\end{aligned}$$

the Y_S is the ordered product $Y_1 \cdots Y_s$ in $U(\mathfrak{g})$, and the R_{Y_S} is an element in $\mathrm{Hom}_{\mathbb{C}}(U(\mathfrak{g}), U(\mathfrak{g}))$ which multiplies every element in $U(\mathfrak{g})$ by the product Y_S on the right.

By convention, if $S = \emptyset$, we let $Y_S = 1 \in U(\mathfrak{g})$, and $\langle Y_S, -\rangle = \mathrm{id}$ and $R_{Y_S} = \mathrm{id}$ in $\mathrm{Hom}_{\mathbb{C}}(U(\mathfrak{g}), U(\mathfrak{g}))$.

The Attempts for Small k's

Let $Y_1, \ldots, Y_k$ be k arbitrary elements in $\mathfrak{g}$ indexed by $[1, k]$ and $X \in \mathfrak{g}$ be an arbitrary element. We want to write the product

$$Y_1 \cdots Y_k \cdot X = L_{Y_1} \circ \ldots \circ L_{Y_k}(X)$$

in the form that Y_i's are on the right of X. We first look at the examples for $k = 1, 2, 3$, which inspire the general formula for all k. These are routine, but they help to understand the general formula.

For $k = 1$, we see $Y_1 \cdot X = X \cdot Y_1 + [Y_1, X]$. In other word, we have

$$L_{Y_1}(X) = R_{Y_1}(X) + \mathrm{ad}Y_1(X) \tag{7.6}$$

as in (7.4).

For $k = 2$, we have

$$\begin{aligned}(Y_1Y_2) \cdot X &= L_{Y_1} \circ L_{Y_2}(X)\\ &= L_{Y_1} \circ (R_{Y_2} + \mathrm{ad}Y_2)(X) \qquad \text{(by (7.6))}\\ &= L_{Y_1} \circ R_{Y_2}(X) + L_{Y_1} \circ \mathrm{ad}Y_2(X)\\ &= R_{Y_2} \circ L_{Y_1}(X) + L_{Y_1} \circ \mathrm{ad}Y_2(X) \qquad \text{(by (7.5))}\\ &= R_{Y_2}[R_{Y_1}(X) + \mathrm{ad}Y_1(X)]\\ &\quad + R_{Y_1}[\mathrm{ad}Y_2(X)] + \mathrm{ad}Y_1[\mathrm{ad}Y_2(X)] \qquad \text{(by (7.6))}\end{aligned}$$

i.e. in $\mathrm{Hom}_{\mathbb{C}}(U(\mathfrak{g}), U(\mathfrak{g}))$ we have

$$L_{Y_1}L_{Y_2} = R_{Y_2}R_{Y_1} + R_{Y_1}\mathrm{ad}Y_2 + R_{Y_2}\mathrm{ad}Y_1 + \mathrm{ad}Y_1\mathrm{ad}Y_2. \tag{7.7}$$

For $k = 3$, we have

$$
\begin{aligned}
(Y_1Y_2Y_3) \cdot X &= L_{Y_1}L_{Y_2}L_{Y_3}(X) \\
&= L_{Y_1}L_{Y_2}[R_{Y_3}(X) + \mathrm{ad}Y_3(X)] \qquad \text{(by (7.6))} \\
&= L_{Y_1}L_{Y_2}R_{Y_3}(X) + L_{Y_1}L_{Y_2}\mathrm{ad}Y_3(X) \\
&= R_{Y_3}L_{Y_1}L_{Y_2}(X) + L_{Y_1}L_{Y_2}\mathrm{ad}Y_3(X) \qquad \text{(by (7.5))} \\
&= R_{Y_3}R_{Y_2}R_{Y_1}(X) + R_{Y_3}R_{Y_1}\mathrm{ad}Y_2(X) \\
&\quad + R_{Y_3}R_{Y_2}\mathrm{ad}Y_1(X) + R_{Y_3}\mathrm{ad}Y_1\mathrm{ad}Y_2(X) \\
&\quad + R_{Y_2}R_{Y_1}\mathrm{ad}Y_3(X) + R_{Y_1}\mathrm{ad}Y_2\mathrm{ad}Y_3(X) \\
&\quad + R_{Y_2}\mathrm{ad}Y_1\mathrm{ad}Y_3(X) + \mathrm{ad}Y_1\mathrm{ad}Y_2\mathrm{ad}Y_3(X) \qquad \text{(by (7.7))}
\end{aligned}
$$

i.e. in $\mathrm{Hom}_{\mathbb{C}}(U(\mathfrak{g}), U(\mathfrak{g}))$, we have

$$
\begin{aligned}
L_{Y_1}L_{Y_2}L_{Y_3} &= R_{Y_3}R_{Y_2}R_{Y_1} + R_{Y_3}R_{Y_1}\mathrm{ad}Y_2 \\
&\quad + R_{Y_3}R_{Y_2}\mathrm{ad}Y_1 + R_{Y_3}\mathrm{ad}Y_1\mathrm{ad}Y_2 \\
&\quad + R_{Y_2}R_{Y_1}\mathrm{ad}Y_3 + R_{Y_1}\mathrm{ad}Y_2\mathrm{ad}Y_3 \\
&\quad + R_{Y_2}\mathrm{ad}Y_1\mathrm{ad}Y_3 + \mathrm{ad}Y_1\mathrm{ad}Y_2\mathrm{ad}Y_3 \qquad (7.8)
\end{aligned}
$$

The General Formula

We now show the general formula:

Lemma 7.8. *For a positive integer k, and elements $Y_1, \ldots, Y_k, X \in \mathfrak{g}$, we have*

$$(Y_1Y_2 \ldots Y_k) \cdot X = \sum_{S \subset [1,k]} \langle Y_S, X\rangle \cdot Y_{S^c} \qquad (7.9)$$

Equivalently in the algebra $\mathrm{Hom}_{\mathbb{C}}(U(\mathfrak{g}), U(\mathfrak{g}))$ *we can write*

$$
\begin{aligned}
L_{Y_1} \circ \ldots \circ L_{Y_k}(-) &= \sum_{S \subset [1,k]} (R_{Y_{S^c}} \circ \langle Y_S, -\rangle) \\
&= \sum_{S \subset [1,k]} \langle Y_S, -\rangle \cdot Y_{S^c}
\end{aligned}
$$

Here the sum on the right hand side has 2^k terms when S runs through all subsets of $[1, k]$.

Proof. As suggested by the cases of $k = 1, 2, 3$, we prove the (7.9) by induction on k. The case $k = 1, 2, 3$ are shown in the above subsubsection. Assume the (7.9) is true for $k - 1$ elements $Y_1, \ldots, Y_{k-1}$.

For $(Y_1 Y_2 \dots Y_k) \cdot X = L_{Y_1} L_{Y_2} \dots L_{Y_k}(X)$, we have

$$\begin{aligned} L_{Y_1} L_{Y_2} \dots L_{Y_k}(X) &= L_{Y_1} L_{Y_2} \dots L_{Y_{k-1}}[L_{Y_k}(X)] \\ &= L_{Y_1} L_{Y_2} \dots L_{Y_{k-1}}[R_{Y_k}(X) + \mathrm{ad}Y_k(X)] \\ &= L_{Y_1} L_{Y_2} \dots L_{Y_{k-1}} R_{Y_k}(X) + L_{Y_1} L_{Y_2} \dots L_{Y_{k-1}} \mathrm{ad}Y_k(X) \end{aligned}$$

For the first term, note that $L_{Y_1} \dots L_{Y_{k-1}} R_{Y_k} = R_{Y_k} L_{Y_1} \dots L_{Y_{k-1}}$ by (7.5), we have

$$\begin{aligned} L_{Y_1} L_{Y_2} \dots L_{Y_{k-1}} R_{Y_k}(X) &= R_{Y_k} L_{Y_1} L_{Y_2} \dots L_{Y_{k-1}}(X) \\ &= R_{Y_k} \sum_{S \subset [1,k-1]} \langle Y_S, X \rangle \cdot Y_{S^c} \\ &= \sum_{S \subset [1,k-1]} (\langle Y_S, X \rangle \cdot Y_{S^c}) \cdot Y_k \\ &= \sum_{\substack{S \subset [1,k] \\ k \in S^c}} \langle Y_S, X \rangle \cdot Y_{S^c} \end{aligned}$$

For the second term, we have

$$\begin{aligned} L_{Y_1} L_{Y_2} \dots L_{Y_{k-1}} \mathrm{ad}Y_k(X) &= \sum_{S \subset [1,k-1]} \langle Y_S, \mathrm{ad}Y_k(X) \rangle \cdot Y_{S^c} \\ &= \sum_{S \subset [1,k-1]} \langle Y_{S \cup \{k\}}, X \rangle \cdot Y_{S^c} \\ &= \sum_{\substack{S \subset [1,k] \\ k \in S}} \langle Y_S, X \rangle \cdot Y_{S^c} \end{aligned}$$

Hence the sum of the above two terms is

$$\sum_{\substack{S \subset [1,k] \\ k \in S^c}} \langle Y_S, X \rangle \cdot Y_{S^c} + \sum_{\substack{S \subset [1,k] \\ k \in S}} \langle Y_S, X \rangle \cdot Y_{S^c} = \sum_{S \subset [1,k]} \langle Y_S, X \rangle \cdot Y_{S^c}$$

i.e.

$$L_{Y_1} L_{Y_2} \dots L_{Y_k}(X) = \sum_{S \subset [1,k]} \langle Y_S, X \rangle \cdot Y_{S^c}.$$

□

By the same argument, or simply move the leading term in the sum to the left, we have

Lemma 7.9. *Let $Y_1, \ldots, Y_k, X \in \mathfrak{g}$ be arbitrary elements. Then*

$$X \cdot (Y_1 \cdots Y_k) = \sum_{S \subset [1,k]} Y_S \cdot \langle X, Y_{S^c} \rangle \tag{7.10}$$

Here the last notation means

$$\langle X, Y_{S^c} \rangle = [\cdots [X, Y_{j_1}] \cdots Y_{j_{k-s}}]$$

if $S^c = \{j_1, \ldots, j_{k-s}\}$.

7.1.5 Geometric Preparation

We study the tangent spaces on the regular submanifold G_w of Z_w. Recall that $Z_w = P\overline{N}_P w$ is a Zariski open subset of G, and $G_w = PwP_\emptyset = PwN_\emptyset = PwN_w^+$ is a Zariski closed subset of Z_w. As manifolds, the Z_w is an open submanifold of G and G_w is a regular closed submanifold of Z_w.

Let $\mathfrak{g}_0$ be the (real) Lie algebra of G, and similarly for a subgroup of G denoted by an uppercase letter, we use the same fraktur letter with a subscript 0 to denote its (real) Lie algebra.

We regard the $\mathfrak{g}_0$ as the abstract Lie algebra of G, and for an element $X \in \mathfrak{g}_0$, we denote by X^L the corresponding left invariant vector field, and $X_g^L \in T_g G$ be the tangent vector of the vector field X^L at $g \in G$. The X^L and X_g^L are given by

$$(X^L f)(g) = X_g^L f = \frac{d}{dt}|_{t=0} f(ge^{tX}), \qquad \forall f \in C^\infty(G).$$

Similarly, we denote by X^R the corresponding right invariant vector field on G and $X_g^R \in T_g G$ its tangent vector at g. The X^R and X_g^R are given by

$$(X^R f)(g) = X_g^R f = \frac{d}{dt}|_{t=0} f(e^{tX} g), \qquad \forall f \in C^\infty(G).$$

Remark 7.10. In most textbooks, the Lie algebra is defined to be the space $\{X^L : X \in \mathfrak{g}_0\}$ of left invariant vector fields, and is canonically identified with $T_e G = \{X_e^L : X \in \mathfrak{g}_0\}$. In this chapter, we will regard Lie algebra elements as vector fields (differential operators of degree 1), hence it is convenient to distinguish vector fields from tangent vectors.

We have the following easy facts in Lie theory:

Lemma 7.11. *For the two vector fields X^L, X^R corresponding to an element $X \in \mathfrak{g}_0$, they satisfy*

- $X_e^L = X_e^R$, *i.e. the two vector fields have the same tangent vector at the identity.*
- $X_g^R = (g^{-1}Xg)_g^L$ *and* $X_g^L = (gXg^{-1})_g^R$, *for all* $g \in G$. *(Here* gXg^{-1} *means the adjoint action* $\mathrm{Ad}g(X)$*.)*

The Tangent Space $T_w G_w$

We consider the tangent space $T_w G_w$ of the submanifold G_w at w, where w is a fixed representative in G (abuse of notation) of the Weyl group element w. The $T_w G_w$ is a subspace of $T_w G = T_w Z_w$, since G_w is a embedded submanifold of Z_w.

At the point $w \in G_w \subset G$, the tangent space $T_w G = T_w Z_w$ is

$$T_w G = T_w Z_w = \mathrm{span}_{\mathbb{R}}\{X_w^L : X \in \mathfrak{g}_0\} = \mathrm{span}_{\mathbb{R}}\{X_w^R : X \in \mathfrak{g}_0\}.$$

Recall that $N_w = w^{-1}\overline{N}_P w, N_w^+ = N_w \cap N_\emptyset, N_w^- = N_w \cap \overline{N}_\emptyset$, and $\mathfrak{n}_{w0}, \mathfrak{n}_{w0}^+, \mathfrak{n}_{w0}^-$ are their real Lie algebras respectively. Also the $\mathfrak{p}_0, \overline{\mathfrak{n}}_{P0}$ are the real Lie algebras of $P, \overline{N}_P$ respectively. Let

$$\begin{aligned} R_w^{\mathfrak{p}_0} &:= \mathrm{span}_{\mathbb{R}}\{X_w^R : X \in \mathfrak{p}_0\} \\ L_w^{\mathfrak{n}_{w0}} &:= \mathrm{span}_{\mathbb{R}}\{X_w^L : X \in \mathfrak{n}_{w0}\} \\ L_w^{\mathfrak{n}_{w0}^+} &:= \mathrm{span}_{\mathbb{R}}\{X_w^L : X \in \mathfrak{n}_{w0}^+\} \\ L_w^{\mathfrak{n}_{w0}^-} &:= \mathrm{span}_{\mathbb{R}}\{X_w^L : X \in \mathfrak{n}_{w0}^-\} \end{aligned}$$

e.g. $R_w^{\mathfrak{p}_0} \subset T_w G$ is the subspace of the $T_w G$, spanned by tangent vectors X_w^R for all $X \in \mathfrak{p}_0$, etc. Then we have:

Lemma 7.12. *The above four spaces are subspaces of* $T_w G = T_w Z_w$*, and*

(1) $L_w^{\mathfrak{n}_{w0}} = L_w^{\mathfrak{n}_{w0}^+} \oplus L_w^{\mathfrak{n}_{w0}^-}$.

(2) $T_w G = T_w Z_w = R_w^{\mathfrak{p}_0} \oplus L_w^{\mathfrak{n}_{w0}}$.

(3) $T_w G_w = R_w^{\mathfrak{p}_0} \oplus L_w^{\mathfrak{n}_{w0}^+}$.

($\oplus$ means direct sum of vector spaces)

Proof. The (1) is trivial, since $\mathfrak{n}_{w0} = \mathfrak{n}_{w0}^+ + \mathfrak{n}_{w0}^-$.

For the (2), first note that $R_w^{\mathfrak{p}_0} \subset T_w Z_w$ since $Z_w = PwN_w$ is left P-stable, and $L_w^{\mathfrak{n}_{w0}} \subset T_w Z_w$ since $Z_w = PwN_w$ is right N_w-stable (note $N_w = w^{-1}\overline{N}_P w$). Hence

$$T_w G = T_w Z_w \supset R_w^{\mathfrak{p}_0} + L_w^{\mathfrak{n}_{w0}}.$$

By counting the dimension, we just need to verify the two subspaces $R_w^{\mathfrak{p}_0}$ and $L_w^{\mathfrak{n}_{w0}}$ are linear independent. Actually, let $X \in \mathfrak{n}_{w0}$, then $X_w^L = (wXw^{-1})_w^R$, and $wXw^{-1} \in w(w^{-1}\overline{\mathfrak{n}}_{P0}w)w^{-1} = \overline{\mathfrak{n}}_{P0}$ which is linear independent from $\mathfrak{p}_0$. Hence the tangent vector $X_w^L = (wXw^{-1})_w^R$ is linear independent with the subspace $R_w^{\mathfrak{p}_0}$.

For the (3), we also have $T_wG_w \supset R_w^{\mathfrak{p}_0} + L_w^{\mathfrak{n}_{w0}^+}$ since $G_w = PwN_w^+$ is left P-stable and right N_w^+-stable. Also by counting dimension, we just need to show the two subspaces $R_w^{\mathfrak{p}_0}, L_w^{\mathfrak{n}_{w0}^+}$ are linear independent. This is obvious from (2) since $L_w^{\mathfrak{n}_{w0}^+}$ is a subspace of $L_w^{\mathfrak{n}_{w0}}$. □

The Tangent Spaces $T_{pwn}G_w$

Let pwn ($p \in P, n \in N_w^+$) be an arbitrary point in G_w (every element in G_w is uniquely written as pwn for a pair of $p \in P, n \in N_w^+$). We study the tangent space $T_{pwn}G_w$ at $pwn \in G_w$.

Similar to the tangent space at w, we have the following subspaces of $T_{pwn}G = T_{pwn}Z_w$

$$\begin{aligned}
R_{pwn}^{\mathfrak{p}_0} &:= \mathrm{span}_{\mathbb{R}}\{X_{pwn}^R : X \in \mathfrak{p}_0\} \\
L_{pwn}^{\mathfrak{n}_{w0}} &:= \mathrm{span}_{\mathbb{R}}\{X_{pwn}^L : X \in \mathfrak{n}_{w0}\} \\
L_{pwn}^{\mathfrak{n}_{w0}^+} &:= \mathrm{span}_{\mathbb{R}}\{X_{pwn}^L : X \in \mathfrak{n}_{w0}^+\} \\
L_{pwn}^{\mathfrak{n}_{w0}^-} &:= \mathrm{span}_{\mathbb{R}}\{X_{pwn}^L : X \in \mathfrak{n}_{w0}^-\}
\end{aligned}$$

i.e. $R_{pwn}^{\mathfrak{p}_0}$ is the subspace of $T_{pwn}G = T_{pwn}Z_w$ spanned by tangent vectors at pwn of right invariant vector fields in $\mathfrak{p}_0$ etc, and all these four are subspaces of $T_{pwn}G = T_{pwn}Z_w$.

As a generalization of the previous Lemma, we have

Lemma 7.13. *For arbitrary $pwn \in G_w \subset Z_w \subset G$, the above four vector spaces satisfy*

(1) $L_{pwn}^{\mathfrak{n}_{w0}} = L_{pwn}^{\mathfrak{n}_{w0}^+} \oplus L_{pwn}^{\mathfrak{n}_{w0}^-}$.

(2) $T_{pwn}G = T_{pwn}Z_w = R_{pwn}^{\mathfrak{p}_0} \oplus L_{pwn}^{\mathfrak{n}_{w0}}$.

(3) $T_{pwn}G_w = R_{pwn}^{\mathfrak{p}_0} \oplus L_{pwn}^{\mathfrak{n}_{w0}^+}$.

(The Lemma 7.12 is the special case when $p = n = e$.)

Proof. (1)(3) are exactly the same as the previous lemma. The (2) is similar. Let $X \in \mathfrak{n}_{w0}$, then

$$X^L_{pwn} = (pwnXn^{-1}w^{-1}p^{-1})^R_{pwn}.$$

Since $n \in N_w^+ \subset N_w$, one has $nXn^{-1} \in \mathfrak{n}_{w0}$ and $wnXn^{-1}w^{-1} \in \overline{\mathfrak{n}}_{P0}$. Then since $\overline{\mathfrak{n}}_{P0}$ is linear independent with $\mathfrak{p}_0$, we see $pwnXn^{-1}w^{-1}p^{-1} \in p(\overline{\mathfrak{n}}_{P0})p^{-1}$ is linear independent with $p(\mathfrak{p}_0)p^{-1} = \mathfrak{p}_0$. Hence the subspace $L^{\mathfrak{n}_{w0}}_{pwn}$ is linear independent with the subspace $R^{\mathfrak{p}_0}_{pwn}$. □

The $\mathfrak{n}_{w0}^-$ is Transverse to G_w

In the last chapter, we call elements in $U(\mathfrak{n}_w^-)$ "transverse derivatives". We now explain this term.

Definition 7.14. Let V be a set of vector fields on a manifold M, and $N \subset M$ is a regular submanifold. Then we say V **is transverse to the submanifold** N, if

$$T_xM = T_xN \oplus \mathrm{span}_{\mathbb{R}}\{X_x : X \in V\},$$

for every $x \in N$, i.e. the vector fields in V give a linear complement of T_xN in T_xM at every point x of N.

The G_w is a submanifold of Z_w, with a smooth embedding $i : G_w \hookrightarrow Z_w$. Let TZ_w (resp. TG_w) be the tangent bundle on Z_w (resp. G_w), and let i^*TZ_w be the pull-back bundle over G_w. Then TG_w is a subbundle of i^*TZ_w, and at every point $pwn \in G_w$, the fibre (tangent) space $(TG_w)_{pwn} = T_{pwn}G_w$ is a subspace of $(i^*TZ_w)_{pwn} = T_{pwn}Z_w$.

By the Lemma 7.13, we have

$$T_{pwn}Z_w = T_{pwn}G_w \oplus L^{\mathfrak{n}_{w0}^-}_{pwn}$$

(direct sum of vector spaces), i.e. the left invariant vector fields in $\mathfrak{n}_{w0}^-$ give a linear complement space of $T_{pwn}G_w$ in $T_{pwn}Z_w$ at every point pwn in G_w. Hence the Lemma 7.13 tells us

Lemma 7.15. *The subalgebra $\mathfrak{n}_{w0}^-$ (regarded as the space of left invariant vector fields) is transverse to the double coset G_w.*

The Pull-Back Bundle i^*TZ_w

Let $i : G_w \hookrightarrow Z_w$ be the smooth embedding as above, and let i^*TZ_w be the pull-back vector bundle of the tangent bundle TZ_w to G_w. The tangent bundle TG_w of G_w is a subbundle of i^*TZ_w, and we have the following exact sequence of vector bundles:

$$0 \to TG_w \to i^*TZ_w$$

Note that the tangent bundles TG_w, TZ_w are algebraic vector bundles and the map i is an algebraic morphism, hence the bundle i^*TZ_w and the above bundle sequence are also algebraic.

The total space of the bundle i^*TZ_w is

$$\{(x, v) : x \in G_w, v \in T_xZ_w\},$$

and its fibre space $(i^*TZ_w)_x$ at a point $x \in G_w$ is exactly the tangent space T_xZ_w of Z_w at x. Given an algebraic vector field V on Z_w (a section of the bundle TZ_w), its restriction $V|_{G_w}$ to G_w is an algebraic section of the bundle i^*TZ_w.

Algebraic Sections of the Bundle i^*TZ_w

Let

$$\begin{aligned}\{X_1, \dots, X_k\} &= \text{a real basis of } \mathfrak{p}_0\\ \{Z_1, \dots, Z_l\} &= \text{a real basis of } \mathfrak{n}_{w0}^+\\ \{Y_1, \dots, Y_d\} &= \text{a real basis of } \mathfrak{n}_{w0}^-\end{aligned}$$

Let X_i^R be the *right* invariant vector fields on Z_w (or G) corresponding to X_i for all $i = 1, \dots, k$. Similarly, let $Z_i^L, i = 1, \dots, l$ and $Y_j^L, j = 1, \dots, d$ be the *left* invariant vector fields corresponding to Z_i, Y_j respectively. In Lemma 7.13, we have seen the

$$\{X_{i,x}^R, \dots, X_{k,x}^R\} \cup \{Z_{1,x}^L, \dots, Z_{l,x}^L\} \cup \{Y_{1,x}^L, \dots, Y_{d,x}^L\}$$

form a basis of the fibre space $(i^*TZ_w)_x = T_xZ_w$ of i^*TZ_w for every point $x \in G_w$, and the subset

$$\{X_{1,x}^R, \dots, X_{k,x}^R\} \cup \{Z_{1,x}^L, \dots, Z_{l,x}^L\}$$

is a basis of the subspace T_xG_w for every $x \in G_w$.

Lemma 7.16. *Under the above basis, every algebraic section s of the bundle i^*TZ_w on G_w is of the form*

$$s(x) = \sum_{i=1}^{k} A^i(x)X^R_{i,x} + \sum_{i=1}^{l} B^i(x)Z^L_{i,x} + \sum_{i=1}^{d} C^i(x)Y^L_{i,x}$$

for a unique set of regular functions $A^i, i = 1, \ldots, k$, $B^i, i = 1, \ldots, l$ and $C^i, i = 1, \ldots, d$. (If we write $x = pwn$ for unique $p \in P, n \in N_w$, the A^i, B^i, C^i are regular functions of p and n.)

The Coefficients A^i, B^i, C^i

Let $H \in \mathfrak{g}_0$ be an arbitrary element. For each $n \in N_w^+$, consider the conjugation $wnHn^{-1}w^{-1} \in \mathfrak{g}_0$. Since $\mathfrak{g}_0 = \mathfrak{p}_0 + \overline{\mathfrak{n}}_{P0}$, the $wnHn^{-1}w^{-1}$ is uniquely written as

$$wnHn^{-1}w^{-1} = H^{\mathfrak{p}_0} + H^{\overline{\mathfrak{n}}_{P0}}$$

where $H^{\mathfrak{p}_0} \in \mathfrak{p}_0, H^{\overline{\mathfrak{n}}_{P0}} \in \overline{\mathfrak{n}}_{P0}$. Therefore the H decomposes as

$$H = n^{-1}w^{-1}H^{\mathfrak{p}_0}wn + n^{-1}w^{-1}H^{\overline{\mathfrak{n}}_{P0}}wn.$$

Note that $n^{-1}w^{-1}H^{\overline{\mathfrak{n}}_{P0}}wn$ is in $n^{-1}w^{-1}\overline{\mathfrak{n}}_{P0}wn = n^{-1}\mathfrak{n}_{w0}n = \mathfrak{n}_{w0}$ (since $n \in N_w^+ \subset N_w$). Hence we can further decompose it as

$$n^{-1}w^{-1}H^{\overline{\mathfrak{n}}_{P0}}wn = H^+ + H^-$$

where $H^+ \in \mathfrak{n}_{w0}^+, H^- \in \mathfrak{n}_{w0}^-$. We denote by

$$H' = n^{-1}w^{-1}H^{\mathfrak{p}_0}wn.$$

Then H is a sum of the form

$$H = H' + H^+ + H^-$$

where $H' \in n^{-1}w^{-1}\mathfrak{p}_0wn$, $H^+ \in \mathfrak{n}_{w0}^+, H^- \in \mathfrak{n}_{w0}^-$. *Note that the three components H', H^+, H^- depend on n, hence one has different H', H^+, H^- at different point $n \in N_w^+$.*

Now let $pwn \in G_w$ be an arbitrary point (the $p \in P, n \in N_w^+$ are uniquely determined by the point), and $H \in \mathfrak{g}_0$ be an arbitrary element in the real Lie algebra. For the $n \in N_w^+$, let H', H^+, H^- be three components of H

as above, and let $(H')^L, (H^+)^L, (H^-)^L$ be the corresponding left invariant vector fields, then

$$H^L = (H')^L + (H^+)^L + (H^-)^L.$$

At the point $pwn \in G_w$, the tangent vector H^L_{pwn} decomposes into

$$\begin{aligned} H^L_{pwn} &= (H')^L_{pwn} + (H^+)^L_{pwn} + (H^-)^L_{pwn} \\ &= (pwnH'n^{-1}w^{-1}p^{-1})^R_{pwn} + (H^+)^L_{pwn} + (H^-)^L_{pwn} \end{aligned} \tag{7.11}$$

Since $H^+ \in \mathfrak{n}^+_{w0}, H^- \in \mathfrak{n}^-_{w0}$, one has $(H^+)^L_{pwn} \in L^{\mathfrak{n}^+_{w0}}_{pwn}$ and $(H^-)^L_{pwn} \in L^{\mathfrak{n}^-_{w0}}_{pwn}$. Also by $H' = n^{-1}w^{-1}H^{\mathfrak{p}_0}wn \in n^{-1}w^{-1}\mathfrak{p}_0 wn$, one has

$$pwnH'n^{-1}w^{-1}p^{-1} = pH^{\mathfrak{p}_0}p^{-1} \in \mathfrak{p}_0,$$

hence $(H')^L_{pwn} = (pwnH'n^{-1}w^{-1}p^{-1})^R_{pwn} \in R^{\mathfrak{p}_0}_{pwn}$.

Therefore the above decomposition (7.11) agrees with the tangent space decomposition in Lemma 7.13:

$$T_{pwn}Z_w = R^{\mathfrak{p}_0}_{pwn} \oplus L^{\mathfrak{n}^+_{w0}}_{pwn} \oplus L^{\mathfrak{n}^-_{w0}}_{pwn}.$$

In sum, given an element $H \in \mathfrak{g}_0$, and let H^L be the corresponding left invariant vector field on G. If we restrict H^L to the open submanifold Z_w, we get an algebraic section in $\Gamma(Z_w, TZ_w)$, and by abuse of notation, we still denote it by H^L. The restriction $H^L|_{G_w}$ of $H^L \in \Gamma(Z_w, TZ_w)$ to G_w is an algebraic section in $\Gamma(G_w, i^*TZ_w)$, and by Lemma 7.16, the $H^L|_{G_w}$ is uniquely written as

$$\begin{aligned} (H^L|_{G_w})(pwn) &= H^L_{pwn} && (7.12) \\ &= \sum_{i=1}^{k} A^i(pwn)X^R_{i,pwn} + \sum_{i=1}^{l} B^i(pwn)Z^L_{i,pwn} \\ &\quad + \sum_{i=1}^{d} C^i(pwn)Y^L_{i,pwn}, && (7.13) \end{aligned}$$

where X_i, Z_i, Y_i are the basis of subalgebras as above, and A^i, B^i, C^i are algebraic functions on G_w (of p and n).

Remark 7.17. Given an $H \in \mathfrak{g}_0$, we can find the coefficient functions $C^i(pwn)$ in the following way:

(1) conjugate H to nHn^{-1};

(2) decompose the nHn^{-1} into a sum

$$nHn^{-1} = (nHn^{-1})^{w^{-1}\mathfrak{p}_0 w} + (nHn^{-1})^{w^{-1}\bar{\mathfrak{n}}_{P0} w}$$

(3) conjugate the second component $(nHn^{-1})^{w^{-1}\bar{\mathfrak{n}}_{P0} w} \in w^{-1}\bar{\mathfrak{n}}_{P0} w$ back to

$$n^{-1}(nHn^{-1})^{w^{-1}\bar{\mathfrak{n}}_{P0} w} n$$

(4) then the $C^i(pwn)$ is the coefficient of the base vector Y^i_{pwn} in

$$(n^{-1}(nHn^{-1})^{w^{-1}\bar{\mathfrak{n}}_{P0} w} n)^L_{pwn}$$

7.2 The Inclusion $\mathcal{M} \subset \mathcal{K}_{[\mathfrak{n}_\emptyset \bullet]}$

In this section, we show the inclusion $\mathcal{M} \subset \mathcal{K}_{[\mathfrak{n}_\emptyset \bullet]}$ by showing

$$\mathcal{M}^k \subset \mathcal{K}_{[\mathfrak{n}_\emptyset \bullet]}$$

for all $k \geq 0$, since the $\{\mathcal{M}^k\}_{k\geq 0}$ form an exhaustive filtration of $\mathcal{M}$.

The distributions in $\mathcal{K} = \mathcal{U} \otimes U(\mathfrak{n}_w^-)$ are transverse derivatives of distributions on G_w, while the $\mathfrak{n}_\emptyset$ acts on such distributions as derivatives. Let $\Phi \in \mathcal{U} = \mathcal{S}(G_w, V)'$ and $u \in U_n(\mathfrak{n}_w^-)$, then the $\Phi \otimes u = \Phi \cdot u$ is a typical element in $\mathcal{K}$. Let $u' \in U(\mathfrak{n}_\emptyset)$, then the right multiplication is given by

$$(\Phi \cdot u) \cdot u' = \Phi \cdot (u \cdot u')$$

where $u \cdot u'$ is the multiplication in $U(\mathfrak{g})$ or in the larger algebra $\mathcal{D}(G_{\geq w})$ of algebraic differential operators on $G_{\geq w}$.

The elements in $U(\mathfrak{n}_w^-)$ "protects" the elements in $\mathcal{U}$ from elimination by $U(\mathfrak{n}_\emptyset)$. The essence of inclusion $\mathcal{M} \subset \mathcal{K}_{[\mathfrak{n}_\emptyset \bullet]}$ is: the $U(\mathfrak{n}_\emptyset)$ can always "peel off" the protection from $U(\mathfrak{n}_w^-)$, and annihilate the inner distributions in $\mathcal{U}$, if they are $\mathfrak{n}_\emptyset$-torsion.

The main idea is to use induction, and show one can peel off "one layer of the $U(\mathfrak{n}_w^-)$" at a time. Then after sufficiently many steps, the $U(\mathfrak{n}_\emptyset)$ can reach the $\mathcal{U}$ part of the distributions.

7.2.1 Some Algebraic Lemmas—I

We show some algebraic results before proving the inclusion $\mathcal{M} \subset \mathcal{K}_{[\mathfrak{n}_\emptyset \bullet]}$.

A Product Formula

Let $\mathfrak{h}$ be an arbitrary Lie algebra.

Lemma 7.18. *For arbitrary $n \in \mathbb{Z}_{>0}$, let $H, X_1, \ldots, X_n \in \mathfrak{h}$ be arbitrary elements. Then in $U(\mathfrak{h})$ we have the formula:*

$$X_1 \cdot X_2 \cdots X_n \cdot H = H \cdot X_1 \cdots X_n - \sum_{i=1}^{n} X_1 \cdots X_{i-1} \cdot [H, X_i] \cdot X_{i+1} \cdots X_n. \tag{7.14}$$

Proof. We prove the formula by induction on n. For $n = 1$, we have

$$X_1 \cdot H = H \cdot X_1 - [H, X_1],$$

which verifies (7.14).

Suppose (7.14) is true for $n = k$, then for $n = k + 1$, we have

$$\begin{aligned}
X_1 \cdots X_k \cdot X_{k+1} \cdot H &= X_1 \cdots X_k \cdot (X_{k+1} \cdot H) \\
&= X_1 \cdot X_k \cdot (H \cdot X_{k+1} - [H, X_{k+1}]) \\
&= (X_1 \cdots X_{k-1} \cdot X_k \cdot H) \cdot X_{k+1} - X_1 \cdots X_k \cdot [H, X_{k+1}] \\
&= (H \cdot X_1 \cdots X_k - \sum_{i=1}^{k} X_1 \cdots [H, X_i] \cdots X_k) \cdot X_{k+1} \\
&\quad - X_1 \cdots X_k \cdot [H, X_{k+1}] \\
&= H \cdot X_1 \cdots X_{k+1} - \sum_{i=1}^{k+1} X_1 \cdots [H, X_i] \cdots X_{k+1}
\end{aligned}$$

□

This lemma is easier to prove than Lemma 7.8, hence it is less powerful than Lemma 7.8. The advantage of Lemma 7.8 is: all Y_i's are on the right end of the product.

Torsion Submodule is Stable

Let $\mathfrak{h}$ be an arbitrary complex Lie algebra, and $\mathfrak{h}_1, \mathfrak{h}_2$ be two subalgebras of $\mathfrak{h}$ such that

$$\mathfrak{h} = \mathfrak{h}_1 + \mathfrak{h}_2$$

i.e. $\mathfrak{h}$ is the direct sum of $\mathfrak{h}_1, \mathfrak{h}_2$ as vector spaces. Suppose $\mathfrak{h}_1$ normalizes $\mathfrak{h}_2$:

$$[\mathfrak{h}_1, \mathfrak{h}_2] \subset \mathfrak{h}_2,$$

or equivalently $\mathfrak{h}_2$ is an ideal of $\mathfrak{h}$. We have

Lemma 7.19. *Let*

$$\begin{aligned} U &= \textit{a right } \mathfrak{h}\textit{-module, also a right } U(\mathfrak{h})\textit{-module,} \\ &\quad \textit{and we can regard it as a } \mathfrak{h}_2\textit{-module} \\ U_{[\mathfrak{h}_2^\bullet]} &= \textit{the } \mathfrak{h}_2\textit{-torsion subspace of } U \end{aligned}$$

Then each $\mathfrak{h}_2$-annihilator $U_{[\mathfrak{h}_2^k]}$ is a $\mathfrak{h}$-submodule of U. In particular, the $\mathfrak{h}_2$-torsion subspace $U_{[\mathfrak{h}_2^\bullet]}$ is a $\mathfrak{h}$-submodule of U.

Proof. We just need to show each $U_{[\mathfrak{h}_2^k]}$ is stable under the left $\mathfrak{h}$-action. Obviously it is stable under $\mathfrak{h}_2$-action, hence we just need to show it is stable under $\mathfrak{h}_1$-action.

Let $k \in \mathbb{Z}_{>0}$ be arbitrary (when $k = 0$, $U_{[\mathfrak{h}_2^0]} = U$, there is nothing to be proved). Let $u \in U_{[\mathfrak{h}_2^k]}$ be an arbitrary element, i.e. for any $X_1, \ldots, X_k \in \mathfrak{h}_2$, one has

$$u \cdot X_1 \cdots X_k = 0.$$

Let $H \in \mathfrak{h}_1$ be an arbitrary element, we show $u \cdot H \in U_{[\mathfrak{h}_2^k]}$, i.e. it is annihilated by products of k elements in $\mathfrak{h}_2$. For arbitrary $X_1, \ldots, X_k \in \mathfrak{h}_2$, by (7.14), we have

$$\begin{aligned} u \cdot H \cdot X_1 \cdots X_k &= u \cdot (H \cdot X_1 \cdots X_k) \\ &= u \cdot X_1 \cdots X_k \cdot H + \sum_{i=1}^{k} u \cdot X_1 \cdots [H, X_i] \cdots X_k \end{aligned}$$

The first term is zero since $u \cdot X_1 \cdots X_k = 0$. Every term in the second sum is zero, since $[H, X_i] \in \mathfrak{h}_2$. (Note that we assumed $\mathfrak{h}_2$ to be normalized by $\mathfrak{h}_1$.) Hence $u \cdot H \cdot X_1 \cdots X_k = 0$ for all $X_1, \ldots, X_k \in \mathfrak{h}_2$ and $H \in \mathfrak{h}_1$. $\square$

Torsion Submodule is Absorbing

Lemma 7.20. *Let*

$$\begin{aligned} \mathfrak{h} &= \textit{a complex Lie algebra} \\ U &= \textit{a right } \mathfrak{h}\textit{-module} \\ U_{[\mathfrak{h}^\bullet]} &= \textit{the } \mathfrak{h}\textit{-torsion submodule of } U \end{aligned}$$

Let $u \in U$ be an element, and assume there exists an integer $n > 0$ such that

$$u \cdot X_1 \cdots X_n \in U_{[\mathfrak{h}^\bullet]}$$

for all $X_1, \ldots, X_n \in \mathfrak{h}$. Then $u \in U_{[\mathfrak{h}^\bullet]}$.

Proof. Let u and n be as in the Lemma. Let $b_1, \dots, b_d$ be a basis of $\mathfrak{h}$.

Note that the n is "uniform" for arbitrary $X_1, \dots, X_n$. In particular we can pick X_i's from the basis $\{b_1, \dots, b_d\}$, and there are d^n choices. The d^n elements of the form $X_1 \cdot X_2 \cdots X_n$ (each X_i is from the above basis) generate the ideal $(\mathfrak{h}^n)$.

For each product $X_1 \cdots X_n$ (X_i from the above basis), the $u \cdot X_1 \cdots X_n$ is in $U_{[\mathfrak{h}\bullet]}$, hence

$$(u \cdot X_1 \cdots X_n) \cdot (\mathfrak{h}^m) = 0$$

for a $m > 0$ depending on the choices of X_i. Since there are only d^n choices of the product $X_1 \cdots X_n$, we pick the largest m and denote it by M, then

$$(u \cdot X_1 \cdots X_n) \cdot (\mathfrak{h}^M) = 0$$

for all d^n choices of $X_1, \dots, X_n$. Hence $u \cdot (\mathfrak{h}^{n+M}) = 0$ and $u \in U_{[\mathfrak{h}\bullet]}$. $\square$

The Quotient $\mathfrak{g}/\mathfrak{p}_\emptyset$ is a $\mathfrak{n}_\emptyset$-Torsion Module

The adjoint representation of $\mathfrak{n}_\emptyset$ on $\mathfrak{g}$ makes $\mathfrak{g}$ into a (finite dimensional) $\mathfrak{n}_\emptyset$-torsion module. The $\mathfrak{p}_\emptyset \subset \mathfrak{g}$ is a $\mathfrak{n}_\emptyset$-submodule (parabolic subalgebras are self-normalizing), hence the quotient module $\mathfrak{g}/\mathfrak{p}_\emptyset$ is a $\mathfrak{n}_\emptyset$-torsion module. Hence we have

Lemma 7.21. *Let $Y \in \mathfrak{g}$ be an arbitrary element. Then there exists a positive integer n (depending only on Y), such that*

$$[\cdots[[Y, X_1], X_2] \cdots X_n] \in \mathfrak{p}_\emptyset,$$

for all $X_1, \dots, X_n \in \mathfrak{n}_\emptyset$.

7.2.2 The Inclusion $\mathcal{M}^k \subset \mathcal{K}_{[\mathfrak{n}_\emptyset\bullet]}$

We show the following Lemma by induction on k:

Lemma 7.22. *For each $k \geq 0$, the subspace*

$$\mathcal{M}^k = \mathcal{U}_{[\mathfrak{n}_\emptyset\bullet]} \otimes U_k(\mathfrak{n}_w^-)$$

is contained in $\mathcal{K}_{[\mathfrak{n}_\emptyset\bullet]}$.

The case $k = 0$ is trivial. Namely the

$$\mathcal{M}^0 = \mathcal{U}_{[\mathfrak{n}_\emptyset\bullet]} \otimes \mathbb{C} = \mathcal{U}_{[\mathfrak{n}_\emptyset\bullet]} = \mathcal{S}(G_w, V)'_{[\mathfrak{n}_\emptyset\bullet]}$$

is contained in $\mathcal{K}_{[\mathfrak{n}_\emptyset \bullet]}$, since the inclusion $\mathcal{U} \hookrightarrow \mathcal{K}$ is a right $\mathfrak{n}_\emptyset$-homomorphism and the $\mathfrak{n}_\emptyset$-torsion functor is left exact (Lemma 3.36).

(Induction Hypothesis): We assume

$$\mathcal{M}^{k-1} \subset \mathcal{K}_{[\mathfrak{n}_\emptyset \bullet]}.$$

We show the $\mathcal{M}^k$ is contained in $\mathcal{K}_{[\mathfrak{n}_\emptyset \bullet]}$.

Proof. By the Poincare-Birkhoff-Witt theorem, every element in $\mathcal{M}^k$ is a sum of terms of the following form:

$$\Phi \otimes Y_1 \cdot Y_2 \cdots Y_i = \Phi \cdot Y_1 \cdots Y_i$$

where $\Phi \in \mathcal{U}_{[\mathfrak{n}_\emptyset \bullet]}$, $0 \leq i \leq k$ and $Y_1, \ldots, Y_i \in \mathfrak{n}_w^-$. If $i < k$, then by induction hypothesis, such elements are already contained in $\mathcal{K}_{[\mathfrak{n}_\emptyset \bullet]}$, since they are in $\mathcal{M}^i \subset \mathcal{M}^{k-1}$. We just need to show elements of the following form are contained in $\mathcal{K}_{[\mathfrak{n}_\emptyset \bullet]}$:

$$\Phi \cdot Y_1 \cdots Y_k,$$

where $\Phi \in \mathcal{U}_{[\mathfrak{n}_\emptyset \bullet]}$ and $Y_1, \ldots, Y_k \in \mathfrak{n}_w^-$.

First we write

$$\Phi \cdot Y_1 \cdots Y_k = (\Phi \cdot Y_1 \cdots Y_{k-1}) \cdot Y_k.$$

The $\Phi \cdot Y_1 \cdots Y_{k-1}$ is in $\mathcal{M}^{k-1}$ and by the induction hypothesis, it is $\mathfrak{n}_\emptyset$-torsion, hence there exists an integer $n_1 > 0$ such that

$$(\Phi \cdot Y_1 \cdots Y_{k-1}) \cdot ({\mathfrak{n}_\emptyset}^{n_1}) = \{0\}.$$

By Lemma 7.21, there exists an integer $n_2 > 0$ such that

$$[\cdots[[Y_k, X_1], X_2] \cdots], X_{n_2}] \in \mathfrak{p}_\emptyset,$$

for all $X_1, \ldots, X_{n_2} \in \mathfrak{n}_\emptyset$.

Now let $n = n_1 + n_2 + 1$, then for any n elements $X_1, \ldots, X_n \in \mathfrak{n}_\emptyset$, we have (by (7.10))

$$\begin{aligned}(\Phi \cdot Y_1 \cdots Y_k) \cdot X_1 \cdots X_n &= (\Phi \cdot Y_1 \cdots Y_{k-1}) \cdot (Y_k \cdot X_1 \cdots X_n) \\ &= \sum_{S \subset [1,n]} (\Phi \cdot Y_1 \cdots Y_{k-1}) \cdot X_S \cdot \langle Y_k, X_{S^c} \rangle\end{aligned}$$

If $|S| \geq n_1$, then $(\Phi \cdot Y_1 \cdots Y_{k-1}) \cdot X_S = 0$, since $X_S \in ({\mathfrak{n}_\emptyset}^{n_1})$ and we have $(\Phi \cdot Y_1 \cdots Y_{k-1}) \cdot ({\mathfrak{n}_\emptyset}^{n_1}) = \{0\}$ as above.

If $|S^c| \geq n_2$, then $\langle Y_k, X_{S^c}\rangle \in \mathfrak{p}_\emptyset$. By the induction hypothesis, the $\Phi \cdot Y_1 \cdots Y_{k-1} \in \mathcal{K}_{[\mathfrak{n}_\emptyset \bullet]}$, hence $(\Phi \cdot Y_1 \cdots Y_{k-1}) \cdot X_S \in \mathcal{K}_{[\mathfrak{n}_\emptyset \bullet]}$, since $\mathcal{K}_{[\mathfrak{n}_\emptyset \bullet]}$ is a $\mathfrak{n}_\emptyset$-submodule of $\mathcal{K}$ and is stable under multiplication of $U(\mathfrak{n}_\emptyset)$. Then by applying Lemma 7.19 to $\mathfrak{h} = \mathfrak{p}_\emptyset, \mathfrak{h}_1 = \mathfrak{m}_\emptyset, \mathfrak{h}_2 = \mathfrak{n}_\emptyset$, we know the $\mathcal{K}_{[\mathfrak{n}_\emptyset \bullet]}$ is $\mathfrak{p}_\emptyset$-stable, and

$$(\Phi \cdot Y_1 \cdots Y_{k-1}) \cdot X_S \cdot \langle Y_k, X_{S^c}\rangle \in \mathcal{K}_{[\mathfrak{n}_\emptyset \bullet]}.$$

Since we have chosen $n = n_1 + n_2 + 1$, then either $|S| \geq n_1$ or $|S^c| \geq n_2$. Hence the term

$$(\Phi \cdot Y_1 \cdots Y_{k-1}) \cdot X_S \cdot \langle Y_k, X_{S^c}\rangle$$

is either zero, or in $\mathcal{K}_{[\mathfrak{n}_\emptyset \bullet]}$. In sum, the summands $(\Phi \cdot Y_1 \cdots Y_{k-1}) \cdot X_S \cdot \langle Y_k, X_{S^c}\rangle$ are all in $\mathcal{K}_{[\mathfrak{n}_\emptyset \bullet]}$, hence

$$(\Phi \cdot Y_1 \cdots Y_k) \cdot X_1 \cdots X_n \in \mathcal{K}_{[\mathfrak{n}_\emptyset \bullet]}$$

for all $X_1, \ldots, X_n \in \mathfrak{n}_\emptyset$. Then by Lemma 7.20, we see

$$\Phi \cdot Y_1 \cdots Y_k \in \mathcal{K}_{[\mathfrak{n}_\emptyset \bullet]}.$$

$\square$

7.3 The Inclusion $\mathcal{K}_{[\mathfrak{n}_\emptyset \bullet]} \subset \mathcal{M}$

Similar to the proof of Lemma 3.52, the crucial idea to prove

$$(\mathcal{U} \otimes U(\mathfrak{n}_w^-))_{[\mathfrak{n}_\emptyset \bullet]} \subset \mathcal{U}_{[\mathfrak{n}_\emptyset \bullet]} \otimes U(\mathfrak{n}_w^-)$$

is to find a "good basis" of the enveloping algebra $U(\mathfrak{n}_w^-)$.

Let $Y_1, \ldots, Y_d$ be a set of basis of the Lie algebra $\mathfrak{n}_w^-$. Let $I = (i_1, \ldots, i_d)$ be a multi-index in $\mathbb{Z}_{\geq 0}^d$, and we denote by

$$Y^I = Y_1^{i_1} \cdot Y_2^{i_2} \cdots Y_d^{i_d}$$

the ordered product. Then the P-B-W Theorem tells us the $\{Y^I : I \in \mathbb{Z}_{\geq 0}^d\}$ form a basis of the enveloping algebra $U(\mathfrak{n}_w^-)$.

Therefore every element in $\mathcal{K}$ is uniquely written as a finite sum

$$\sum_{I \in \mathbb{Z}_{\geq 0}^d} \Phi_I \otimes Y^I = \sum_{I \in \mathbb{Z}_{\geq 0}^d} \Phi_I \cdot Y^I$$

where $\Phi_I \in \mathcal{U}$ and all but finitely many terms in the sum are zero. This element (sum) in $\mathcal{K}$ is in $\mathcal{M}$ if and only if each nonzero Φ_I is in $\mathcal{U}_{[\mathfrak{n}_\emptyset \bullet]}$.

As an analogue to the Lemma 3.52, *the main difficulty of the proof is: if the above sum is $\mathfrak{n}_\emptyset$-torsion, one cannot see whether each summand $\Phi_I \cdot Y^I$ is $\mathfrak{n}_\emptyset$-torsion.* The idea to prove the inclusion is very similar: we need to single out one summand at a time and show it is torsion.

Unlike the proof of Lemma 3.52, the tensor product $\mathcal{U} \otimes U(\mathfrak{n}_w^-)$ is not a tensor product of $\mathfrak{n}_\emptyset$-modules, and the multiplication of $\mathfrak{n}_\emptyset$ on it is extremely complicated, such that one cannot expect a simple multiplication formula as (3.2).

This requires us to arrange the PBW-basis Y^I by a linear order on the multi-index set, so that we can perform induction on this linear order to separate the summands. This order is constructed by the following two steps:

- First fix a basis of $\mathfrak{n}_w^-$ consists of root vectors, and arrange them as $Y_1, \ldots, Y_d$ (here d is the dimension of $\mathfrak{n}_w^-$) such that their "height" are non-decreasing (see 7.3.3).
- Second we define a "linear order" on the index set $\mathfrak{L}^d = \mathbb{Z}_{\geq 0}^d$ (see 7.3.1). The the above fixed basis $\{Y_1, \ldots, Y_d\}$ determines a linear order on the PBW-basis
$$\{Y^I = Y_1^{i_1} Y_2^{i_2} \cdots Y_d^{i_d} : I = (i_1, \ldots, i_d) \in \mathfrak{L}^d = \mathbb{Z}_{\geq 0}^d\}.$$

Combining the above two aspects, we have a good linear order "$\leq$" on the PBW-basis $\{Y^I : I \in \mathfrak{L}^d\}$. Let

$$U_I(\mathfrak{n}_w^-) = \text{the subspace spanned by } Y^J \text{ for all } J \leq I.$$

Then we define a filtration on the $\mathcal{K}$ and its subspace $\mathcal{K}_{[\mathfrak{n}_\emptyset \bullet]}$ by

$$\begin{aligned} \mathcal{K}^I &:= \mathcal{U} \otimes U_I(\mathfrak{n}_w^-) \\ \mathcal{K}^I_{[\mathfrak{n}_\emptyset \bullet]} &:= \mathcal{K}^I \cap \mathcal{K}_{[\mathfrak{n}_\emptyset \bullet]} \end{aligned}$$

The $\{\mathcal{K}^I_{[\mathfrak{n}_\emptyset \bullet]} : I \in \mathfrak{L}^d\}$ forms an exhaustive filtration of $\mathcal{K}_{[\mathfrak{n}_\emptyset \bullet]}$, and we just need to show each subspace $\mathcal{K}^I_{[\mathfrak{n}_\emptyset \bullet]}$ is included in $\mathcal{M}$.

The key points of the proof are

- Each subspace $\mathcal{K}^I_{[\mathfrak{n}_\emptyset \bullet]}$ in the filtration is $\mathfrak{n}_\emptyset$-stable (Lemma 7.44), hence is a $\mathfrak{n}_\emptyset$-submodule of $\mathcal{K}$.
- Each graded piece of the filtration $\{\mathcal{K}^I_{[\mathfrak{n}_\emptyset \bullet]} : I \in \mathfrak{L}^d\}$ is isomorphic to $\mathcal{U}$ as $\mathfrak{n}_\emptyset$-modules.

7.3.1 Orders on Multi-index Sets

For a positive integer d, let

$$\mathfrak{L}^d = \mathbb{Z}^d_{\geq 0} = \{(i_1, \ldots, i_d) : i_1, \ldots, i_d \in \mathbb{Z}_{\geq 0}\}$$

be the set of multi-indices with d components of non-negative integers. The $\mathfrak{L}^d$ is an additive semigroup. We denote a generic element in $\mathfrak{L}^d$ by $I = (i_1, \ldots, i_d)$, and we denote the sum of its components by $|I| = i_1 + \ldots + i_d$.

For each $m \in \mathbb{Z}_{\geq 0}$, we let

$$\mathfrak{L}^d_m = \{I \in \mathfrak{L}^d : |I| = m\} = \{(i_1, \ldots, i_d) : i_1 + \ldots + i_d = m\}$$

be the subset of multi-indices with sums of their components equal to m. Then $\mathfrak{L}^d = \bigcup_{m\geq 0} \mathfrak{L}^d_m$ is a disjoint union. It is easy to count the size of $\mathfrak{L}^d_m$:

$$|\mathfrak{L}^d_m| = C^m_{d+m-1}.$$

For each $m \in \mathbb{Z}_{\geq 0}$, we let

$$\mathfrak{L}^d_{\leq m} = \bigcup_{0\leq k\leq m} \mathfrak{L}^d_k.$$

Then $\mathfrak{L}^d = \bigcup_{m\geq 0} \mathfrak{L}^d_{\leq m}$.

The Reverse Lexicographic Order on $\mathfrak{L}^d_m$

Let $\mathfrak{L}^d_m = \{I \in \mathfrak{L}^d : |I| = m\}$ be as above. We define an order on $\mathfrak{L}^d_m$.

Definition 7.23 (The Reverse Lexicographic Order on $\mathfrak{L}^d_m$). Let $I = (i_1, \ldots, i_d), J = (j_1, \ldots, j_d)$ be two elements in $\mathfrak{L}^d_m$. We define

$$I < J$$

if there is an l, such that $0 \leq l \leq d$ and

$$i_d = j_d, i_{d-1} = j_{d-1}, \ldots, i_{l+1} = j_{l+1}, i_l < j_l.$$

This is an order called the **reverse lexicographic order on $\mathfrak{L}^d_m$**. We denote by $I \leq J$ if $I < J$ or $I = J$.

Remark 7.24. It is easy to see this order is a linear order (total order), and is a well-order since $\mathfrak{L}^d_m$ is finite and discrete. Actually the above reverse lexicographic order is defined on the entire $\mathfrak{L}^d_m$, and we just restrict it to each subset $\mathfrak{L}^d_m$.

Example 7.25. Consider the case $d = 3, m = 4$. It contains $C_6^4 = \frac{6!}{4!2!} = 15$ elements, and they are ordered as the following linear sequence:

$$\begin{aligned}(0,0,4) &> (0,1,3) > (1,0,3)\\ &> (0,2,2) > (1,1,2) > (2,0,2)\\ &> (0,3,1) > (1,2,1) > (2,1,1) > (3,0,1)\\ &> (0,4,0) > (1,3,0) > (2,2,0) > (3,1,0)\\ &> (4,0,0)\end{aligned}$$

The Linear Order on $\mathfrak{L}^d$

Since the $\mathfrak{L}^d$ is the disjoint union of $\mathfrak{L}^d_m, m \geq 0$, we can connect the linear orders on each $\mathfrak{L}^d_m$ to obtain a linear order on $\mathfrak{L}^d$.

Definition 7.26 (**linear order on $\mathfrak{L}^d$**)**.** Let $I = (i_1, \dots, i_d), J = (j_1, \dots, j_d)$ be two elements in $\mathfrak{L}^d$. We define

$$I < J$$

if $|I| < |J|$ or $|I| = |J| = m$ and $I < J$ under the reverse lexicographic order on $\mathfrak{L}^d_m$. We call this order the **linear order on $\mathfrak{L}^d$**.

Example 7.27. For $d = 2$, the linear order on $\mathfrak{L}^2$ is just the snake-like order parameterizing the rational numbers.

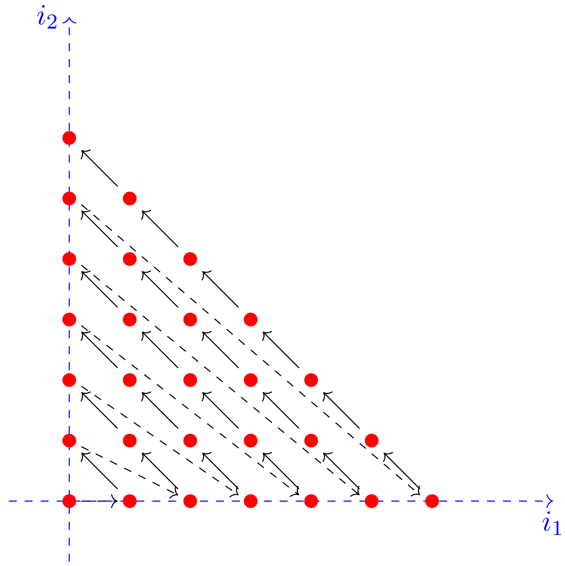
Remark 7.28. Note that the above linear order on $\mathfrak{L}^d$ is NOT the reverse lexicographic order. For example, let $d = 2$, then under the lexicographic order, one has $(3,0) < (1,1)$ by comparing the second component. However under our order, the $(3,0) > (1,1)$ since the sum $3 + 0 > 1 + 1$.

Why Linear Order?

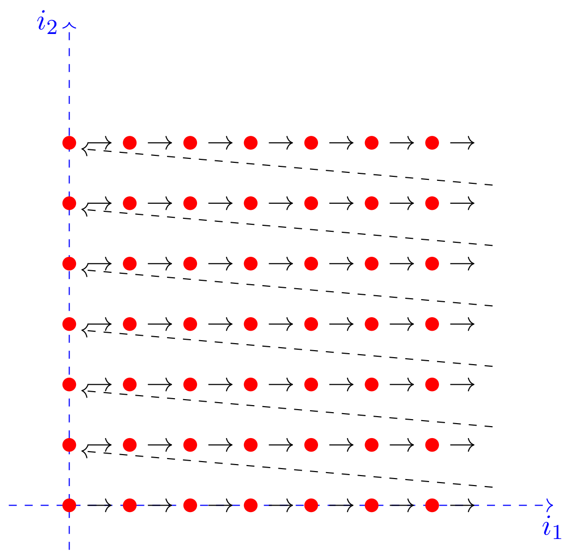
Why not use the reverse lexicographic order on $\mathfrak{L}^d$? Because we cannot perform induction on the reverse lexicographic order.

In Figure 7.2, we show the linear and reverse lexicographic order for $\mathfrak{L}^2$.. In both figures, each dot is an element in $\mathfrak{L}^2$, and each dot has an arrow pointing to the adjacent element which is larger than it in the corresponding order, i.e. one has an increasing sequence following the arrows.

As we can see from the Figure 7.2 (a), each dot has a unique adjacent dot "smaller" than it. Starting from an arbitrary dot, one can always descend to the initial dot $(0,0)$ after finitely many steps.



(a) Linear order $d = 2$



(b) Reverse lexicographic order $d = 2$

Figure 7.2: Linear order and Reverse lexicographic order (d=2)

However in Figure 7.2 (b), one can see the dot $(0, 1)$ has no adjacent element smaller than it. In other word, under the reverse lexicographic order, the subset $\{I \in \mathfrak{L}^2 : I < (0,1)\}$ contains all $(i,0), i \geq 0$, and this subset has no maximal element. Hence starting from $(0,1)$, one cannot descend to the initial point $(0,0)$ after finite steps, therefore one cannot perform induction on this order.

A Partial Order on $\mathfrak{L}^d$

We define a partial order on the $\mathfrak{L}^d$

Definition 7.29. Let $I = (i_1, \dots, i_d), J = (j_1, \dots, j_d)$ be two elements in $\mathfrak{L}^d$. We define

$$I \preceq J$$

if $i_1 \leq j_1, i_2 \leq j_2, \dots, i_d \leq j_d$. And we denote by $I \prec J$ if $I \preceq J$ but $I \neq J$. It is easy to see the $\preceq$ is a partial order on $\mathfrak{L}^d$, and we call it the **component order on $\mathfrak{L}^d$**.

Lemma 7.30. *For two $I, J \in \mathfrak{L}^d$, if $I \prec J$, then $|I| < |J|$ hence $I < J$ under the linear order on $\mathfrak{L}^d$.*

Definition 7.31. Let $I = (i_1, \dots, i_d), J = (j_1, \dots, j_d)$ and suppose $I \preceq J$. We define their **difference** by

$$J - I := (j_1 - i_1, \dots, j_d - i_d).$$

Obviously we have $J - I \in \mathfrak{L}^d$, and $J - I \preceq J$ and $|J - I| = |J| - |I|$.

The Fortified Formula of (7.9) and (7.10)

Let $\mathfrak{g}$ be a complex Lie algebra, and $Y_1, \dots, Y_d$ be d arbitrary elements in $\mathfrak{g}$. For a multi-index $I = (i_1, \dots, i_d) \in \mathfrak{L}^d$, we denote by

$$Y^I = Y_1^{i_1} Y_2^{i_2} \cdots Y_d^{i_d}.$$

And we denote by $\langle Y^I, - \rangle$ the following element in $\mathrm{Hom}_{\mathbb{C}}(U(\mathfrak{g}), U(\mathfrak{g}))$

$$\langle Y^I, u \rangle := (\mathrm{ad}Y_1)^{i_1} \circ (\mathrm{ad}Y_2)^{i_2} \circ \dots \circ (\mathrm{ad}Y_d)^{i_d}(u),$$

for all $u \in U(\mathfrak{g})$, and $\langle -, Y^I \rangle$ the following element in $\mathrm{Hom}_{\mathbb{C}}(U(\mathfrak{g}), U(\mathfrak{g}))$:

$$\langle u, Y^I \rangle := (-1)^{|I|} (\mathrm{ad}Y_d)^{i_d} \circ \dots \circ (\mathrm{ad}Y_1)^{i_1}(u).$$

By convention, if $I = (0, \dots, 0)$, then $\langle Y^I, - \rangle = \langle -, Y^I \rangle = \mathrm{id}$.

With the above partial order on $\mathfrak{L}^d$ and notations, we have the fortified version of the formulas (7.9) and (7.10).

Lemma 7.32. *Let $X \in \mathfrak{g}$ be an arbitrary element. Then we have the fortified formula of (7.9):*

$$
\begin{aligned}
Y^I \cdot X &= \sum_{J \preceq I} C_I^J \langle Y^J, X\rangle \cdot Y^{I-J} \qquad (7.15)\\
&= \sum_{j_1 \leq i_1, \ldots, j_d \leq i_d} C_I^J \langle Y^J, X\rangle \cdot Y_1^{i_1 - j_1} \cdots Y_d^{i_d - j_d}.
\end{aligned}
$$

and the fortified formula of (7.10):

$$
X \cdot Y^I = \sum_{J \preceq I} C_I^J Y^J \cdot \langle X, Y^{I-J}\rangle. \qquad (7.16)
$$

Here the coefficient C_I^J is the product of combinatoric coefficients:

$$
C_I^J = \binom{j_1}{i_1} \cdot \binom{j_2}{i_2} \cdots \binom{j_d}{i_d}.
$$

Proof. To show (7.15) we simply use the (7.9). By counting the multiplicities of repeating terms, we have the combinatoric coefficients in (7.15). The (7.16) is shown in the same way. $\square$

7.3.2 Some Algebraic Lemmas—II

The Poincare-Birkhoff-Witt Basis

Let $\mathfrak{h}$ be an arbitrary complex Lie algebra, with dimension d, and let $Y_1, \ldots, Y_d$ be an arbitrary basis of $\mathfrak{h}$. For a multi-index $I = (i_1, \ldots, i_d) \in \mathfrak{L}^d = \mathbb{Z}_{\geq 0}^d$, we let

$$
Y^I = Y_1^{i_1} \cdots Y_d^{i_d},
$$

then $\{Y^I : I \in \mathfrak{L}^d\}$ form a basis of the enveloping algebra $U(\mathfrak{h})$, called a **Poincare-Birkhoff-Witt basis of** $\mathfrak{h}$, or simply a **PBW basis**.

Lemma 7.33. *Let $I \in \mathfrak{L}^d$ be an arbitrary multi-index, and let Y^I be the PBW basis element as above. Let $Y \in \mathfrak{h}$ be an arbitrary element, then the $Y \cdot Y^I$ is contained in the linear span of*

$$
\{Y^J : |J| \leq |I| + 1\}.
$$

Equivalently, this means

$$
\mathfrak{h} \cdot U_n(\mathfrak{h}) \subset U_{n+1}(\mathfrak{h}).
$$

Proof. We just need to show the case when $Y = Y_k$ is a basis vector from $\{Y_1, \ldots, Y_d\}$. We prove by induction on $|I|$. If $|I| = 0$, then $Y^I = 1$ and the result is trivial. If $|I| = 1$, then Y^I is a single basis vector Y_l for some $1 \le l \le d$. Then

$$Y_k Y_l = \begin{cases} Y_k Y_l, & \text{if } k < l \\ Y_l^2, & \text{if } k = l \\ Y_l Y_k + [Y_k, Y_l], & \text{if } k > l \end{cases}$$

and the result is true.

For a general Y^I, we have

$$\begin{aligned} Y_k \cdot Y^I &= Y_k \cdot Y_1^{i_1} \cdots Y_d^{i_d} \\ &= (Y_k Y_1) Y_1^{i_1 - 1} Y_2^{i_2} \cdots Y_d^{i_d} \\ &= Y_1 (Y_k Y_1^{i_1 - 1} Y_2^{i_2} \cdots Y_d^{i_d}) + [Y_k, Y_1] Y_1^{i_1 - 1} Y_2^{i_2} \cdots Y_d^{i_d} \end{aligned}$$

By the induction hypothesis, the $Y_k Y_1^{i_1-1} Y_2^{i_2} \cdots Y_d^{i_d}$ is in the span of $\{Y^J : |J| \le |I| - 1 + 1\}$ and the $[Y_k, Y_1] Y_1^{i_1-1} Y_2^{i_2} \cdots Y_d^{i_d}$ is in the span of $\{Y^J : |J| \le |I| - 1 + 1\}$, hence the $Y_k \cdot Y^I$ is in the span of $\{Y^J : |J| \le |I| + 1\}$. $\square$

By iteration, i.e. by repeatedly using the above Lemma, we have

Lemma 7.34. *Let $X_1, \ldots, X_k$ be arbitrary element in $\mathfrak{h}$, and I, Y^I as above. Then the*

$$X_1 \cdots X_k \cdot Y^I$$

is a linear combination of Y^J such that $|J| \le |I| + k$.

A Formula on Lie Brackets in Enveloping Algebra

Let $\mathfrak{h}$ be an arbitrary complex Lie algebra, and $U(\mathfrak{h})$ be its enveloping algebra.

Lemma 7.35. *Let $X, Y_1, \ldots, Y_k$ be arbitrary elements in $\mathfrak{h}$. Then in $U(\mathfrak{h})$ we have*

$$[X, Y_1 \cdots Y_k] = \sum_{i=1}^{k} Y_1 \cdots Y_{i-1} [X, Y_i] Y_{i+1} \cdots Y_k \tag{7.17}$$

and similarly

$$[Y_1 \cdots Y_k, X] = \sum_{i=1}^{k} Y_1 \cdots Y_{i-1} [Y_i, X] Y_{i+1} \cdots Y_k \tag{7.18}$$

Proof. We prove the first one by induction on k, and the second equality is implied immediately by the first one.

For $k = 1$, the equality is trivial. We assume the equality holds for $k-1$, then

$$\begin{aligned}[X, Y_1 \cdots Y_k] &= XY_1 \cdots Y_k - Y_1 \cdots Y_k X \\ &= XY_1Y_2 \cdots Y_k - Y_1XY_2 \cdots Y_k + Y_1XY_2 \cdots Y_k - Y_1 \cdots Y_k X \\ &= [X, Y_1]Y_2 \cdots Y_k + Y_1(XY_2 \cdots Y_k - Y_2 \cdots Y_k X)\end{aligned}$$

By the induction hypothesis, we have

$$\begin{aligned}XY_2 \cdots Y_k - Y_2 \cdots Y_k X &= [X, Y_2 \cdots Y_k] \\ &= \sum_{i=2}^{k} Y_2 \cdots Y_{i-1}[X, Y_i]Y_{i+1} \cdots Y_k\end{aligned}$$

Hence

$$\begin{aligned}[X, Y_1 \cdots Y_k] &= [X, Y_1]Y_2 \cdots Y_k + Y_1 \sum_{i=2}^{k} Y_2 \cdots Y_{i-1}[X, Y_i]Y_{i+1} \cdots Y_k \\ &= \sum_{i=1}^{k} Y_1 \cdots Y_{i-1}[X, Y_i]Y_{i+1} \cdots Y_k\end{aligned}$$

□

By a straightforward application of the above Lemma, we have

Lemma 7.36. *Let $I = (i_1, \ldots, i_d) \in \mathfrak{L}^d$ be a multi-index, and $X, Y_1, \ldots, Y_d$ be arbitrary elements in $\mathfrak{h}$, and let $Y^I = Y_1^{i_1} \cdots Y_d^{i_d}$ be the ordered product. Then*

$$[X, Y^I] = \sum_{k=1}^{d} \sum_{l=0}^{i_k-1} Y_1^{i_1} \cdots Y_{k-1}^{i_{k-1}} Y_k^l [X, Y_k] Y_k^{i_k-l-1} Y_{k+1}^{i_{k+1}} \cdots Y_d^{i_d} \tag{7.19}$$

and similarly

$$[Y^I, X] = \sum_{k=1}^{d} \sum_{l=0}^{i_k-1} Y_1^{i_1} \cdots Y_{k-1}^{i_{k-1}} Y_k^l [Y_k, X] Y_k^{i_k-l-1} Y_{k+1}^{i_{k+1}} \cdots Y_d^{i_d} \tag{7.20}$$

7.3.3 Heights on Basis of $\mathfrak{n}_w^-$

The real Lie algebra $\mathfrak{n}_{w0}^- = w^{-1}\overline{\mathfrak{n}}_{P0}w \cap \overline{\mathfrak{n}}_{\emptyset 0}$ of the real group $N_w^- = w^{-1}\overline{N}_P w \cap \overline{N}_\emptyset$ is a direct sum of restricted (relative) root spaces. In particular, we can find a $\mathbb{R}$-basis of $\mathfrak{n}_{w0}^-$ (which is also a $\mathbb{C}$-basis of the complexified Lie algebra $\mathfrak{n}_w^-$) consisting of root vectors.

The roots occurring in $\mathfrak{n}_w^-$ (or $\mathfrak{n}_{w0}^-$) form the subset $\Sigma^- \cap w^{-1}(\Sigma^- - \Sigma_\Theta^-)$, and this subset is closed under addition. For a root α occurring in $\mathfrak{n}_w^-$, let $\mathfrak{g}_{0\alpha}$ be the root space, its complexification $\mathfrak{g}_\alpha$ is exactly the relative root space. The $\mathfrak{g}_{0\alpha}$ is an abelian subalgebra of $\mathfrak{g}_0$ if 2α is not a root. (Note that its dimension may not be one since the group $\mathbf{G}$ may not be split.)

Let $\Delta = \{\alpha_1, \ldots, \alpha_r\}$ be the fixed simple system of the relative (restricted) root system. Since $\Sigma^- \cap w^{-1}(\Sigma^- - \Sigma_\Theta^-)$ is contained in Σ^-, every root in it is uniquely written as an integral combination of roots in $-\Delta$. Let $\alpha \in \Sigma^- \cap w^{-1}(\Sigma^- - \Sigma_\Theta^-)$, and suppose it is of the form

$$\alpha = -\sum n_i \alpha_i$$

for $n_i \in \mathbb{Z}_{\geq 0}$. We define the **height of** α to be

$$\mathrm{Ht}(\alpha) = \sum_{i=1}^{r} n_i.$$

Remark 7.37. Note that this is not the ordinary height function in the theory of root system, since we are working with negative roots. The above "height" Ht is the height function under the base $-\Delta$ in the ordinary sense.

Ordered Basis of $\mathfrak{n}_w^-$

From this section through the entire chapter, we fix a $\mathbb{R}$-basis of $\mathfrak{n}_{w0}^-$ consisting of root vectors, and they are also $\mathbb{C}$-basis of the complexified Lie algebra $\mathfrak{n}_w^-$.

Let d be the dimension of the $\mathfrak{n}_w^-$, and we label the basis of root vectors as $Y_1, \ldots, Y_d$, and define the **height of** Y_i to be the height of the corresponding root. We label the Y_i's such that

$$\mathrm{Ht}(Y_1) \leq \mathrm{Ht}(Y_2) \leq \ldots \leq \mathrm{Ht}(Y_d),$$

i.e. the height of the corresponding roots of Y_i increase with i.

Lemma 7.38. *For two basis root vectors Y_i, Y_j $(i \leq j)$, we have*

$$[Y_i, Y_j] = \sum_{\substack{k \geq j \\ \mathrm{Ht}(Y_k) > \mathrm{Ht}(Y_j)}} c_k Y_k, \tag{7.21}$$

*i.e. the $[Y_i, Y_j]$ is a linear combination of basis vectors with **strictly larger heights**.*

In particular, the basis root vectors corresponding to the highest roots (e.g. the Y_d) are in the center of $\mathfrak{n}_w^-$.

Proof. For two roots α, β occurring in $\mathfrak{n}_w^-$, we have

$$[\mathfrak{g}_{0\alpha}, \mathfrak{g}_{0\beta}] \subset \begin{cases} \mathfrak{g}_{0(\alpha+\beta)}, & \text{if } \alpha + \beta \in \Sigma^- \cap w^{-1}(\Sigma^- - \Sigma_\Theta^-) \\ \{0\}, & \text{otherwise.} \end{cases}$$

The Lemma is implied immediately by this fact. □

7.3.4 The Coefficients C^i

Let Y_i be a base vector in the fixed basis $\{Y_1, \ldots, Y_d\}$ of $\mathfrak{n}_{w0}^-$ and suppose we label the basis such that their height are increasing. Without loss of generality, let $X \in \mathfrak{n}_{\emptyset 0}$ be a root vector in the real Lie algebra $\mathfrak{n}_{\emptyset 0}$. In this subsection, we study the element $[Y_i, X] \in \mathfrak{g}_0$, and its decomposition into linear combination of base vector fields (with coefficients in the ring of algebraic functions).

Remark 7.39. Note that $[Y_i, X]$ may not be in $\mathfrak{p}_{\emptyset 0}$ or $\mathfrak{n}_{w0}^-$ (neither tangent nor transverse), and it could be an element in $w^{-1}\mathfrak{p}_0 w \cap \overline{\mathfrak{n}}_{\emptyset 0}$.

Lemma 7.40. *Let C^k be the coefficient function of Y_k^L in the expression (7.11) of $[Y_i, X]$, then we have $C^k = 0$ for all $k \geq i$.*

Proof. By the assumption, both Y_i and X are root vector, we may assume that $[Y_i, X]$ is also a root vector. We go through the 4 steps in Remark 7.17 to compute C^k.

In the step (1) of Remark 7.17, we may assume $n = \exp(X)$ for some $X \in \mathfrak{n}_{w0}^+$, then by the Baker-Campbell-Hausdorff formula, we have

$$nHn^{-1} = H + [X, H] + \frac{1}{2!}[X, [X, H]] + \ldots$$

On the right-hand-side, every term has *strictly lower heights than Y_i*.

According to the step (2) of Remark 7.17, we need to decompose the nHn^{-1} into two components

$$nHn^{-1} = (nHn^{-1})^{w^{-1}\mathfrak{p}_0 w} + (nHn^{-1})^{w^{-1}\overline{\mathfrak{n}}_{P0} w}.$$

By the step (1) above, we see the second component $(nHn^{-1})^{w^{-1}\overline{\mathfrak{n}}_{P0} w}$ is a linear combination of root vectors with *strictly lower heights than* Y_i. (The coefficients are functions of n)

According to the step (3) of Remark 7.17, if we conjugate the second component back to

$$n^{-1}(nHn^{-1})^{w^{-1}\overline{\mathfrak{n}}_{P0} w} n$$

again by Baker-Campbell-Hausdorff, it is still a linear combination of root vectors with *strictly lower heights than* Y_i.

Therefore, all root vectors $Y_k, k \geq i$ cannot occur in the linear combination (i.e. coefficients are zero), hence $C^k(pwn) = 0$ for all $k \geq i$. $\square$

7.3.5 The Elements $Y_k \cdot Y^I$

Let $Y_1, \ldots, Y_d$ be a basis of $\mathfrak{n}_w^-$ consisting of root vectors and satisfying

$$\mathrm{Ht}(Y_1) \leq \mathrm{Ht}(Y_2) \leq \ldots \leq \mathrm{Ht}(Y_d).$$

For a multi-index $I = (i_1, \ldots, i_d) \in \mathfrak{L}^d$, we denote by Y^I the ordered product $Y_1^{i_1} \cdots Y_d^{i_d}$, and they form the PBW basis of $U(\mathfrak{n}_w^-)$.

Let Y_k be an arbitrary basis vector from $\{Y_1, \ldots, Y_d\}$, we consider the element $Y_k \cdot Y^I$. It is hard to find a expression of it as a combination of PBW-basis. However we can estimate the PBW-basis vectors Y^J occurring in the expression of $Y_k \cdot Y^I$, and have the following result which is a fortified version of Lemma 7.33:

Lemma 7.41. *For all $Y_k, 1 \leq k \leq d$, the $Y_k \cdot Y^I$ is a linear combination of Y^J such that*

$$J \leq (i_1, \ldots, i_{k-1}, i_k + 1, i_{k+1}, \ldots, i_d).$$

(under the linear order on $\mathfrak{L}^d$). In particular, all such J satisfy $|J| \leq |I|+1$, and this lemma implies the Lemma 7.33.

The product $Y_k Y^I = Y_k Y_1^{i_1} \cdots Y_d^{i_d}$ is in $U(\mathfrak{n}_w^-)$ and is a linear combination of PBW-basis $Y^J, J \in \mathfrak{L}^d$. This Lemma estimate the "leading term" in the expansion of $Y_k Y^I$ as sum of PBW-basis. Note that the following proof is a proof by the first principal, not by induction.

Proof. We first look at some simple cases. If $|I| = 0$, or equivalently $I = (0, \dots, 0)$, the statement in the Lemma is trivial.

Next we look at the simplest nontrivial cases when $|I| = 1$, and assume $I = (0, \dots, 0, 1, 0, \dots, 0)$, i.e. the ith component is 1 and all other components are zero. Then $Y^I = Y_i = Y_i^1$, and we have

$$Y_k Y^I = Y_k Y_i = \begin{cases} Y_k Y_i, & \text{if } k < i \\ Y_k^2 = Y_i^2, & \text{if } k = i \\ Y_i Y_k + [Y_k, Y_i], & \text{if } k > i \end{cases}$$

The cases $k < i, k = i$ obviously verify the Lemma. The interesting case is when $k > i$, where a new term $[Y_k, Y_i]$ is created. By Lemma 7.38, the $[Y_k, Y_i]$ is a linear combination of root vectors with strictly larger heights, i.e. it is a combination of Y_j such that $j > i$ and $j > k$. However, such Y_j's are still single elements, and they are of the form Y^J, where $J = (0, \dots, 1, \dots, 0)$ (i.e. the jth component is 1 and all the other components are zero). Such J still satisfies $J \le (i_1, \dots, i_k + 1, \dots, i_d)$, since $|J| = 1$. Hence the $Y_i Y_k + [Y_k, Y_i]$ is still spanned by Y^J with $J \le (i_1, \dots, i_k + 1, \dots, i_d)$.

Now we look at the general case $Y_k \cdot Y^I$, where $I = (i_1, \dots, i_d)$. We have

$$\begin{aligned} Y_k \cdot Y^I &= Y_k \cdot Y_1^{i_1} \cdots Y_{k-1}^{i_{k-1}} \cdot Y_k^{i_k} Y_{k+1}^{i_{k+1}} \cdots Y_d^{i_d} \\ &= Y_1^{i_1} \cdots Y_{k-1}^{i_{k-1}} \cdot Y_k^{i_k+1} \cdot Y_{k+1}^{i_{k+1}} \cdots Y_d^{i_d} \\ &+ [Y_k, Y_1^{i_1} \cdots Y_{k-1}^{i_{k-1}}] \cdot Y_k^{i_k} Y_{k+1}^{i_{k+1}} \cdots Y_d^{i_d} \end{aligned}$$

The first term is Y^J, where $J = (i_1, \dots, i_{k-1}, i_k + 1, i_{k+1}, \dots, i_d)$, i.e. it is obtained from the original I by adding 1 to its kth component. This term has J equals to the "upper bound" in the Lemma. We need to show the second term $[Y_k, Y_1^{i_1} \cdots Y_{k-1}^{i_{k-1}}] \cdot Y_k^{i_k} Y_{k+1}^{i_{k+1}} \cdots Y_d^{i_d}$ is a linear combination of Y^J such that J is *strictly smaller than* $(i_1, \dots, i_{k-1}, i_k + 1, i_{k+1}, \dots, i_d)$.

By (7.20) in Lemma 7.36, we have

$$[Y_k, Y_1^{i_1} \cdots Y_{k-1}^{i_{k-1}}] = \sum_{p=1}^{k-1} \sum_{l=0}^{i_p - 1} Y_1^{i_1} \cdots Y_p^l [Y_k, Y_p] Y_p^{i_p - l - 1} Y_{p+1}^{i_{p+1}} \cdots Y_{k-1}^{i_{k-1}}$$

We just need to show for each p such that $1 \le p \le k-1$ and l such that $0 \le l \le i_p - 1$, the product

$$Y_1^{i_1} \cdots Y_p^l [Y_k, Y_p] Y_p^{i_p - l - 1} Y_{p+1}^{i_{p+1}} \cdots Y_{k-1}^{i_{k-1}} Y_k^{i_k} Y_{k+1}^{i_{k+1}} \cdots Y_d^{i_d}$$

is a linear combination of Y^J such that $J < (i_1, \dots, i_k + 1, \dots, i_d)$.

We apply the Lemma 7.34 to the above elements. Note that the rear segment $Y_p^{i_p-l-1}Y_{p+1}^{i_{p+1}}\cdots Y_{k-1}^{i_{k-1}}Y_k^{i_k}Y_{k+1}^{i_{k+1}}\cdots Y_d^{i_d}$ is of the form Y^J where $J = (0,\ldots,0,i_p-l-1,i_{p+1},\ldots,i_k,\ldots,i_d)$, i.e. it is in the PBW-basis. And the front segment $Y_1^{i_1}\cdots Y_p^l[Y_k,Y_p]$ is a (sum of) products of $i_1+\ldots+i_{p-1}+l+1$ elements. Then in Lemma 7.34, we see the element

$$Y_1^{i_1}\cdots Y_p^l[Y_k,Y_p]Y_p^{i_p-l-1}Y_{p+1}^{i_{p+1}}\cdots Y_{k-1}^{i_{k-1}}Y_k^{i_k}Y_{k+1}^{i_{k+1}}\cdots Y_d^{i_d}$$

is a combination of Y^J where $|J| \le (i_p - l - 1 + i_{p+1} + \ldots i_d) + (i_1 + \ldots + i_{p-1} + l + 1) = i_1 + \ldots + i_d = |I| < |I| + 1$. Of course, these elements are linear combination of Y^J with $J < (i_1,\ldots,i_k+1,\ldots,i_d)$ because the lengths $|J| \le |I| < |I| + 1$. □

Remark 7.42. After proving this lemma, we found out this is a basic fact on enveloping algebra. One don't need the linear order on the PBW-basis, and one don't need to label the basis of $\mathfrak{n}_w^-$ according to the height either.

7.3.6 The Submodule $\mathcal{K}^I$

Let d be the dimension of the Lie algebra $\mathfrak{n}_w^-$, and let $Y_1,\ldots,Y_d$ be a real basis of $\mathfrak{n}_{w0}^-$ (which is also a $\mathbb{C}$-basis of $\mathfrak{n}_w^-$), and suppose we label them with increasing heights: $\mathrm{Ht}(Y_1) \le \ldots \le \mathrm{Ht}(Y_d)$. As in the last subsection, for a multi-index $I = (i_1,\ldots,i_d) \in \mathfrak{L}^d$, we denote by

$$Y^I = Y_1^{i_1}\cdot Y_2^{i_2}\cdots Y_d^{i_d}$$

the corresponding monomial PBW-basis.

Definition 7.43. For a multi-index $I = (i_1,\ldots,i_d) \in \mathfrak{L}^d$, the set $\{J \in \mathfrak{L}^d : J > I\}$ has a unique minimal element under the linear order, and the (finite) set $\{J \in \mathfrak{L}^d : J < I\}$ has a unique maximal element under the linear order. We denote them by

$$\begin{aligned} I^+ &= \text{the minimal element in } \{J \in \mathfrak{L}^d : J > I\} \\ I^- &= \text{the maximal element in } \{J \in \mathfrak{L}^d : J < I\} \end{aligned}$$

and we call I^+ the **the upper adjacent of** I and I^- the **lower adjacent of** I.

The Refined Filtrations on $U(\mathfrak{n}_w^-)$ and $\mathcal{K}$

For an arbitrary $I \in \mathfrak{L}^d$, we denote by

$$\mathfrak{L}^d{}_{\leq I} = \{J \in \mathfrak{L}^d : J \leq I\}$$
$$U_I(\mathfrak{n}_w^-) = \text{the subspace of } U(\mathfrak{n}_w^-) \text{ spanned by all } Y^J \text{ such that } J \leq I$$

Under the linear order on $\mathfrak{L}^d$, the $\{U_I(\mathfrak{n}_w^-) : I \in \mathfrak{L}^d\}$ form an exhaustive filtration of $U(\mathfrak{n}_w^-)$ by the PBW theorem. And for each $I \in \mathfrak{L}^d$, let I^- be its lower adjacent multi-index as above, then

$$\mathfrak{L}^d{}_{\leq I} = \{I\} \cup \mathfrak{L}^d{}_{\leq I^-},$$

and by the PBW theorem, the quotient space $U_I(\mathfrak{n}_w^-)/U_{I^-}(\mathfrak{n}_w^-)$ is an one dimensional space spanned by the Y^I.

We denote by

$$\mathcal{K}^I = \mathcal{U} \otimes U_I(\mathfrak{n}_w^-).$$

Obviously the $\{\mathcal{K}^I : I \in \mathfrak{L}^d\}$ form an exhaustive filtration of $\mathcal{K}$. Let $\mathcal{K}^I/\mathcal{K}^{I^-}$ be the quotient space and

$$\begin{aligned} \mathcal{K}^I &\to \mathcal{K}^I/\mathcal{K}^{I^-} \\ \sum_{J \leq I} \Phi_J \cdot Y^J &\mapsto \Phi_I \cdot Y^I \mathrm{mod} \mathcal{K}^{I^-} \end{aligned} \tag{7.22}$$

be the quotient map. Then the quotient space is linearly isomorphic to $\mathcal{U}$:

$$\begin{aligned} \mathcal{U} &\xrightarrow{\sim} \mathcal{K}^I/\mathcal{K}^{I^-} \\ \Phi &\mapsto \Phi \cdot Y^I \mathrm{mod} \mathcal{K}^{I^-} \end{aligned} \tag{7.23}$$

The $\mathcal{K}^I$ is a $\mathfrak{n}_\emptyset$-Submodule of $\mathcal{K}$

The following lemma is the most crucial result. It tells us: if a finite sum of the form $\sum_{J \leq I} \Phi_J \cdot Y^J$ is $\mathfrak{n}_\emptyset$-torsion, then its "leading term" $\Phi_I \cdot Y^I$ is torsion, and then every term is torsion by iteration.

Lemma 7.44. *For all $I \in \mathfrak{L}^d$, the subspace $\mathcal{K}^I$ of the right $U(\mathfrak{n}_\emptyset)$-module $\mathcal{K}$ is stable under the right multiplication of $\mathfrak{n}_\emptyset$. Hence they are right $U(\mathfrak{n}_\emptyset)$-submodule of $\mathcal{K}$.*

Therefore the quotient space $\mathcal{K}^I/\mathcal{K}^{I^-}$ is a right $U(\mathfrak{n}_\emptyset)$-module. The quotient map (7.22) is a right $U(\mathfrak{n}_\emptyset)$-homomorphism, and the isomorphism (7.23) is an isomorphism of right $U(\mathfrak{n}_\emptyset)$-modules.

Remark 7.45 (**(Sketch of the proof)**). We prove by induction on the linear order of $\mathfrak{L}^d$. For the initial case, when $I = (0, \dots, 0)$, the $\mathcal{K}^I = \mathcal{U}$ is obviously a $\mathfrak{n}_\emptyset$-submodule.

For an arbitrary I, suppose for all $J < I$ (equivalently $J \leq I^-$), the $\mathcal{K}^J$ is a right $U(\mathfrak{n}_\emptyset)$-submodule (induction hypothesis). To show the $\mathcal{K}^I$ is $\mathfrak{n}_\emptyset$-stable, we just need to show the $(\Phi \cdot Y^I) \cdot X$ (i.e. the leading term multiplied by X), is still in $\mathcal{K}^I$ for all $\Phi \in \mathcal{U}, X \in \mathfrak{n}_\emptyset$.

First, by the fortified formula (7.15) in Lemma 7.32, we have

$$\begin{aligned}(\Phi \cdot Y^I) \cdot X &= \Phi \cdot (Y^I \cdot X) \\ &= \sum_{J \preceq I} \Phi \cdot \langle Y^J, X \rangle \cdot Y^{I-J}\end{aligned}$$

We split the last sum into three parts:

$$\begin{aligned}\sum_{J \preceq I, |J|=0} \Phi \cdot \langle Y^J, X \rangle \cdot Y^{I-J} &+ \sum_{J \preceq I, |J|=1} \Phi \cdot \langle Y^J, X \rangle \cdot Y^{I-J} \\ &+ \sum_{J \preceq I, |J|>1} \Phi \cdot \langle Y^J, X \rangle \cdot Y^{I-J}.\end{aligned}$$

1. The first part is a single term

   $$\sum_{J \preceq I, |J|=0} \Phi \cdot \langle Y^J, X \rangle \cdot Y^{I-J} = \Phi \cdot X \cdot Y^I$$

   since $\langle Y^J, - \rangle = \mathrm{id}$. *This "leading term" is still in $\mathcal{K}^I$ but not in $\mathcal{K}^{I^-}$.*

2. For the third part $\sum_{J \preceq I, |J|>1} \Phi \cdot \langle Y^J, X \rangle \cdot Y^{I-J}$, we look at each summand. Since $|J| \geq 2$, we see $|I - J| = |I| - |J| \leq |I| - 2$. Therefore, *the third part is in $\mathcal{K}^{I^-}$.*

3. The second term is the hard part to deal with. We just need to show each

   $$\Phi \cdot \langle Y^J, X \rangle \cdot Y^{I-J}$$

   is in $\mathcal{K}^{I^-}$ for all $J \preceq I, |J| = 1$.

Proof. Let $I = (i_1, \dots, i_d)$. Assume $J = (0, \dots, 1, \dots, 0)$ be the multi-index with jth component equal to 1 and all the other component equal to 0. Then

$$Y^J = Y_j$$

is a single monomial element (the jth element from the basis $\{Y_1, \dots, Y_d\}$ of $\mathfrak{n}_w^-$).

We need to show

$$\Phi \cdot \langle Y^J, X\rangle \cdot Y^{I-J} = \Phi \cdot [Y_j, X] \cdot Y^{I-J}$$

is in the $\mathcal{K}^{I^-}$.

Without loss of generality, we assume X is a root vector in $\mathfrak{n}_\emptyset$. Note that the element $[Y_j, X]$ may not be an element in $\mathfrak{p}_\emptyset$ or $\mathfrak{n}_w^-$ (the worst case: it might be in $w^{-1}\mathfrak{p}w \cap \overline{\mathfrak{n}}_\emptyset$), hence it may be neither tangent to G_w nor transverse to G_w. The derivative $[Y_j, X]$ is regarded as a left invariant vector field $[Y_j, X]^L$ on G and also on the open subset $G_{\geq w}$. As in 7.1.5, we can write it as a pointwise "linear combination"

$$[Y_j, X]^L_x = D_x + \sum_{i=1}^{d} C^i(x) Y^L_{i,x}$$

where D_x is a tangent vector in $T_x G_w$ and $C^i(x)$ is an algebraic function of x (and we have seen it only depends on n if we write $x = pwn$ for $p \in P, n \in N_w^+$.)

By Lemma 7.40, we have seen the C^i are all zero if $i \geq j$, hence the above linear combination is written as

$$[Y_j, X] = D + \sum_{k<j} C^k Y_k.$$

Now the original $\Phi \cdot [Y_j, X] \cdot Y^{I-J}$ is written as

$$\begin{aligned}
\Phi \cdot [Y_j, X] \cdot Y^{I-J} &= \Phi \cdot D \cdot Y^{I-J} + \sum_{k<j} \Phi \cdot (C^k Y_k) \cdot Y^{I-J} \\
&= \Phi \cdot D \cdot Y^{I-J} + \sum_{k<j} (\Phi \cdot C^k) \cdot Y_k \cdot Y^{I-J}
\end{aligned}$$

The C^k are algebraic functions, hence $\Phi \cdot C^k$ are still distributions in $\mathcal{U} = \mathcal{S}(G_w, V)'$. The D is tangent to G_w therefore $\Phi \cdot D$ is also in $\mathcal{S}(G_w, V)'$.

By Lemma 7.41, the terms $Y_k \cdot Y^{I-J}$ are $\mathbb{C}$-linear combinations of elements of the form $Y_1^{l_1} Y_2^{l_2} \cdots Y_d^{l_d}$ where

$$(l_1, \dots, l_d) \leq (i_1, \dots, i_{k-1}, i_k + 1, i_{k+1}, \dots, i_{j-1}, i_j - 1, i_{j+1}, \dots, i_d)$$

and the latter index is *strictly less than* I.

In sum, the original $\Phi \cdot [Y_j, X] \cdot Y^{I-J}$ is in $\mathcal{K}^{I^-}$. $\square$

7.3.7 The Inclusion $\mathcal{K}_{[\mathfrak{n}_\emptyset \bullet]} \subset \mathcal{M}$

Let

$$\mathcal{K}^I_{[\mathfrak{n}_\emptyset \bullet]} = \mathcal{K}_{[\mathfrak{n}_\emptyset \bullet]} \cap \mathcal{K}^I.$$

Then the $\{\mathcal{K}^I_{[\mathfrak{n}_\emptyset \bullet]} : I \in \mathfrak{L}^d\}$ form an exhaustive filtration of $\mathcal{K}_{[\mathfrak{n}_\emptyset \bullet]}$. To show the inclusion $\mathcal{K}_{[\mathfrak{n}_\emptyset \bullet]} \subset \mathcal{M}$, we just need to show

$$\mathcal{K}^I_{[\mathfrak{n}_\emptyset \bullet]} \subset \mathcal{M}$$

for all I.

We show the inclusions $\mathcal{K}^I_{[\mathfrak{n}_\emptyset \bullet]} \subset \mathcal{M}$ by induction on the linear order of $I \in \mathfrak{L}^d$. First the initial case is obvious: let $I = (0, \ldots, 0)$, we see $\mathcal{K}^I_{[\mathfrak{n}_\emptyset \bullet]} = \mathcal{U} \cap \mathcal{K}_{[\mathfrak{n}_\emptyset \bullet]} = \mathcal{U}_{[\mathfrak{n}_\emptyset \bullet]} \subset \mathcal{M}$.

(Induction Hypothesis): For arbitrary I, assume $\mathcal{K}^J_{[\mathfrak{n}_\emptyset \bullet]} \subset \mathcal{M}$ for all $J < I$. We show $\mathcal{K}^I_{[\mathfrak{n}_\emptyset \bullet]} \subset \mathcal{M}$. Let

$$\sum_{J \le I} \Phi_J \cdot Y^J \in \mathcal{K}^I_{[\mathfrak{n}_\emptyset \bullet]}$$

be an arbitrary element, then we need to show all Φ_J are inside $\mathcal{U}_{[\mathfrak{n}_\emptyset \bullet]}$.

Consider the following commutative diagram

$$\begin{tikzcd}
\mathcal{K}^I \arrow[r, two heads] & \mathcal{K}^I/\mathcal{K}^{I^-} \arrow[r, "\simeq"] & \mathcal{U} \\
\mathcal{K}^I_{[\mathfrak{n}_\emptyset \bullet]} \arrow[u, hook] \arrow[r] & (\mathcal{K}^I/\mathcal{K}^{I^-})_{[\mathfrak{n}_\emptyset \bullet]} \arrow[u, hook] \arrow[r, "\simeq"] & \mathcal{U}_{[\mathfrak{n}_\emptyset \bullet]} \arrow[u, hook]
\end{tikzcd}$$

By Lemma 7.44, the linear isomorphism $\mathcal{U} \simeq \mathcal{K}^I/\mathcal{K}^{I^-}$ is actually an isomorphism of $U(\mathfrak{n}_\emptyset)$-modules, hence they have the same torsion submodules:

$$\mathcal{U}_{[\mathfrak{n}_\emptyset \bullet]} \simeq (\mathcal{K}^I/\mathcal{K}^{I^-})_{[\mathfrak{n}_\emptyset \bullet]}.$$

The image of $\sum_{J \le I} \Phi_J \cdot Y^J$ in the quotient $\mathcal{K}^I/\mathcal{K}^{I^-}$ is in the torsion submodule $(\mathcal{K}^I/\mathcal{K}^{I^-})_{[\mathfrak{n}_\emptyset \bullet]}$ since the quotient map is $U(\mathfrak{n}_\emptyset)$-linear by Lemma 7.44, hence $\Phi_I \cdot Y^I \mathrm{mod} \mathcal{K}^{I^-}$ is in the torsion submodule $(\mathcal{K}^I/\mathcal{K}^{I^-})_{[\mathfrak{n}_\emptyset \bullet]}$, then the $\Phi_I \in \mathcal{U}_{[\mathfrak{n}_\emptyset \bullet]}$.

Hence the leading term $\Phi_I \cdot Y^I$ of $\sum_{J \le I} \Phi_J \cdot Y^J$ is in $\mathcal{U}_{[\mathfrak{n}_\emptyset \bullet]} \subset \mathcal{M}$. By the "easy part": $\mathcal{M} \subset \mathcal{K}_{[\mathfrak{n}_\emptyset \bullet]}$, we see the leading term $\Phi_I \cdot Y^I$ is in $\mathcal{K}^I_{[\mathfrak{n}_\emptyset \bullet]}$, hence the following sum

$$\sum_{J < I} \Phi_J \cdot Y^J = \sum_{J \le I} \Phi_J \cdot Y^J - \Phi_I \cdot Y^I$$

is in the intersection $\mathcal{K}_{[\mathfrak{n}_\emptyset \bullet]} \cap \mathcal{K}^{I^-} = \mathcal{K}^{I^-}_{[\mathfrak{n}_\emptyset \bullet]}$. By the induction hypothesis, each Φ_J for $J < I$ are inside $\mathcal{U}_{[\mathfrak{n}_\emptyset \bullet]}$. In sum, all Φ_J in the sum $\sum_{J \leq I} \Phi_J \cdot Y^J$ are inside $\mathcal{U}_{[\mathfrak{n}_\emptyset \bullet]}$, hence

$$\sum_{J \leq I} \Phi_J \cdot Y^J \in \mathcal{M}.$$

Chapter 8

Shapiro's Lemma

Summary of This Chapter

In the last two chapters, we have shown the space

$$\mathrm{Ker}(\mathrm{Res}_w) = (I_{\geq w}/I_{>w})'$$

is isomorphic to $I'_w \otimes U(\mathfrak{n}_w^-)$, and its $\mathfrak{n}_\emptyset$-torsion subspace is exactly

$$[(I_{\geq w}/I_{>w})']^{[\mathfrak{n}_\emptyset{}^\bullet]} = (I'_w)^{[\mathfrak{n}_\emptyset{}^\bullet]} \otimes U(\mathfrak{n}_w^-).$$

This chapter is devoted to the study of the torsion subspace $(I'_w)^{[\mathfrak{n}_\emptyset{}^\bullet]}$, or more precisely the annihilators $(I'_w)^{[\mathfrak{n}_\emptyset{}^k]}$. Our aim is to find the explicit $M_\emptyset$-action on these annihilators.

- In 8.1, we use the "annihilator-invariant trick" to show the following isomorphism (see (8.1))

$$(I'_w)^{[\mathfrak{n}_\emptyset{}^k]} \xrightarrow{\sim} H^0(\mathfrak{n}_\emptyset, (I_w \otimes F_k)').$$

  Then we can study the kth annihilator by studying the zeroth $\mathfrak{n}_\emptyset$-cohomology on $(I_w \otimes F_k)'$.

- In 8.2, we show the following isomorphism

$$I_w \otimes F_k \xrightarrow{\sim} \mathcal{S}\mathrm{Ind}_{N_\emptyset \cap w^{-1}Pw}^{N_\emptyset}(\sigma^w \otimes \eta_k).$$

  Then we can replace the $(I_w \otimes F_k)'$ in the 0th cohomology by the dual of the Schwartz induction space $\mathcal{S}\mathrm{Ind}_{N_\emptyset \cap w^{-1}Pw}^{N_\emptyset}(\sigma^w \otimes \eta_k)$.

- In 8.3, we formulate Shapiro's Lemma and use it to compute the zeroth cohomology

$$H^0(N_\emptyset, [\mathcal{S}\mathrm{Ind}_{N_\emptyset \cap w^{-1}Pw}^{N_\emptyset}(\sigma^w \otimes \eta_k)]').$$

Combining all three sections, we have the following isomorphisms

$$\begin{aligned}(I'_w)^{[\mathfrak{n}_\emptyset{}^k]} &\simeq H^0(\mathfrak{n}_\emptyset, (I_w \otimes F_k)') \\ &\simeq H^0(N_\emptyset, [\mathcal{S}\mathrm{Ind}_{N_\emptyset \cap w^{-1}Pw}^{N_\emptyset}(\sigma^w \otimes \eta_k)]') \\ &\simeq H^0(N_\emptyset \cap w^{-1}Pw, (V_{\sigma^w} \otimes F_k)')\end{aligned}$$

Through these isomorphisms, we explicitly express the $M_\emptyset$-action on $(I'_w)^{[\mathfrak{n}_\emptyset{}^k]}$ by the $M_\emptyset$-action on $(V_{\sigma^w} \otimes F_k)'$. The *main result* of this chapter is Lemma 8.27: the $M_\emptyset$ acts on $(I'_w)^{[\mathfrak{n}_\emptyset{}^k]}$ by $\widehat{\sigma^w \otimes \eta_k} \otimes \gamma_w$, where σ^w is the twisted representation $\sigma \circ \mathrm{Ad}w$, η_k is the conjugation on the finite dimensional quotient $U(\mathfrak{n}_\emptyset)/(\mathfrak{n}_\emptyset{}^k)$, the "hat" means the dual representation, and the γ_w is the $M_\emptyset$-modular character on the quotient $N_\emptyset \cap w^{-1}Pw \backslash N_\emptyset$.

8.1 Preparation

8.1.1 Main Object of This Chapter

In Chapter 6, we have shown the following isomorphism

$$(I_{\geq w}/I_{>w})' = \mathrm{Ker}(\mathrm{Res}_w) \simeq I'_w \otimes U(\mathfrak{n}_w^-).$$

This isomorphism is only a linear isomorphism: the $U(\mathfrak{n}_w^-)$ is not stable under $P_\emptyset$-conjugation, hence the right-hand-side cannot be regarded as a tensor product of $P_\emptyset$-representation. However we see the $\mathfrak{n}_w^-$ is stable under $M_\emptyset$-conjugation, hence the $U(\mathfrak{n}_w^-)$ has the $M_\emptyset$-action on it. Then we have

Lemma 8.1. *The isomorphism* $\mathrm{Ker}(\mathrm{Res}_w) \simeq I'_w \otimes U(\mathfrak{n}_w^-)$ *is $M_\emptyset$-equivariant, when the right-hand-side is endowed with the tensor product $M_\emptyset$-action.*

Proof. Actually for arbitrary $\Phi \in I'_w, u \in U(\mathfrak{n}_w^-)$ and $m \in M_\emptyset$, we have

$$\begin{aligned}\langle m \cdot (\Phi \otimes u), \phi\rangle &= \langle \Phi \otimes u, R_{m^{-1}}\phi\rangle \\ &= \langle \Phi, R_u R_{m^{-1}}\phi\rangle \\ &= \langle \Phi, R_{m^{-1}} R_m R_u R_{m^{-1}}\phi\rangle \\ &= \langle \Phi, R_{m^{-1}} R_{\mathrm{Ad}m(u)}\phi\rangle \\ &= \langle m \cdot \Phi, R_{\mathrm{Ad}m(u)}\phi\rangle \\ &= \langle (m \cdot \Phi) \otimes (\mathrm{Ad}m(u)), \phi\rangle\end{aligned}$$

for all $\phi \in I_{\geq w}$. (Here $R_m R_u R_{m^{-1}} = R_{\mathrm{Ad}m(u)}$ because the Lie algebra action is obtained by differentiation of the Lie group action.) ☐

In Chapter 7, we have shown the following equality

$$[(I_{\geq w}/I_{>w})']^{[\mathfrak{n}_\emptyset{}^\bullet]} = (I'_w)^{[\mathfrak{n}_\emptyset{}^\bullet]} \otimes U(\mathfrak{n}_w^-).$$

Since the $\mathfrak{n}_\emptyset$-torsion subspaces are $M_\emptyset$-stable, this equality is also equivariant under $M_\emptyset$, when the right-hand-side is endowed with the tensor product $M_\emptyset$-action.

The main object of this chapter is to study the $M_\emptyset$-action on the torsion subspace $(I'_w)^{[\mathfrak{n}_\emptyset{}^\bullet]}$, or more precisely the $M_\emptyset$-action on the annihilators $(I'_w)^{[\mathfrak{n}_\emptyset{}^k]}$ for all $k \in \mathbb{Z}_{>0}$.

8.1.2 The Annihilator $(I'_w)^{[\mathfrak{n}_\emptyset{}^k]}$ and the Annihilator-Invariant Trick

In this subsection, we show the following isomorphism by an algebraic trick

$$(I'_w)^{[\mathfrak{n}_\emptyset{}^k]} \simeq H^0(\mathfrak{n}_\emptyset, (F_k \otimes I_w)'). \tag{8.1}$$

(See below for the meaning of F_k.) Moreover, this isomorphism is $P_\emptyset$-equivariant.

The $P_\emptyset$-Representation on $U(\mathfrak{n}_\emptyset)/(\mathfrak{n}_\emptyset{}^k)$

Let $(\mathfrak{n}_\emptyset{}^k)$ be the two-sided ideal of $U(\mathfrak{n}_\emptyset)$ generated by products of k elements in $\mathfrak{n}_\emptyset$. Let

$$F_k = U(\mathfrak{n}_\emptyset)/(\mathfrak{n}_\emptyset{}^k)$$

be the quotient space of $U(\mathfrak{n}_\emptyset)$ modulo the ideal $(\mathfrak{n}_\emptyset{}^k)$.

Since the $(\mathfrak{n}_\emptyset{}^k)$ is an ideal, it is a $U(\mathfrak{n}_\emptyset)$-submodule of $U(\mathfrak{n}_\emptyset)$, and the quotient space F_k has a natural structure of (left) quotient $U(\mathfrak{n}_\emptyset)$-modules ($\mathfrak{n}_\emptyset$-modules). More precisely, for a $u \in U(\mathfrak{n}_\emptyset)$, let $\overline{u} \in F_k$ be its image under the quotient map $U(\mathfrak{n}_\emptyset) \to F_k$. Then the left $U(\mathfrak{n}_\emptyset)$-action is given by

$$u' \cdot \overline{u} := \overline{u'u}, \quad \forall u, u' \in U(\mathfrak{n}_\emptyset).$$

The quotient module F_k has the following properties.

Lemma 8.2. *For the quotient spaces F_k, we have*

- *The F_k is a finite dimensional complex vector space.*

- *The natural conjugation of $P_\emptyset$ on $U(\mathfrak{n}_\emptyset)$ induces a $P_\emptyset$-action on F_k. More precisely, the $P_\emptyset$-action is given by*

$$p \cdot \overline{u} := \overline{p \cdot u},$$

where $p \cdot u = \mathrm{Ad}p(u)$ is the natural conjugation. And this action makes F_k into a finite dimensional continuous (hence smooth) $P_\emptyset$-representation. We denote this representation by (η_k, F_k).

- *The (η_k, F_k) is an algebraic representation of $P_\emptyset$.*

- *The differentiation of the $N_\emptyset$-action on F_k coincides with the $U(\mathfrak{n}_\emptyset)$-module structure.*

Remark 8.3. Since F_k is finite dimensional, the topology on it is the canonical topology. The strong dual F_k' is exactly the algebraic dual F_k^* (also with the canonical topology since F_k^* is also finite dimensional). All algebraic tensor product with F_k or F_k^* have a unique topology and they are automatically complete.

The Annihilator $(I_w')^{[{\mathfrak{n}_\emptyset}^k]}$

The annihilator $(I_w')^{[{\mathfrak{n}_\emptyset}^k]}$ is a $P_\emptyset$-stable subspace of the $P_\emptyset$-representation I_w' and a $\mathfrak{n}_\emptyset$-submodule of I_w'.

Given an element $\Phi \in (I_w')^{[{\mathfrak{n}_\emptyset}^k]}$, one has a map

$$\begin{aligned} L_\Phi : F_k &\to I_w' \\ \overline{u} &\mapsto u \cdot \Phi \end{aligned}$$

Here $\overline{u}$ is the image of an arbitrary $u \in U(\mathfrak{n}_\emptyset)$ in the quotient $F_k = U(\mathfrak{n}_\emptyset)/({\mathfrak{n}_\emptyset}^k)$. The L_Φ is a well-defined linear map and it is actually a left $U(\mathfrak{n}_\emptyset)$-homomorphism between the two left $U(\mathfrak{n}_\emptyset)$-modules F_k and I_w'. The following algebraic facts is easy to prove:

Lemma 8.4. *The map*

$$\begin{aligned} (I_w')^{[{\mathfrak{n}_\emptyset}^k]} &\to \mathrm{Hom}_{U(\mathfrak{n}_\emptyset)}(F_k, I_w') \\ \Phi &\mapsto L_\Phi \end{aligned} \tag{8.2}$$

is a linear isomorphism.

Proof. If $L_\Phi = 0$, then $u \cdot \Phi = 0$ for all $u \in U(\mathfrak{n}_\emptyset)$. In particular $1 \cdot \Phi = 0$ and $\Phi = 0$, hence (8.2) is injective.

Let $L \in \mathrm{Hom}_{U(\mathfrak{n}_\emptyset)}(F_k, I_w')$, we let $\Phi = L(\overline{1}) \in I_w'$. Obviously $({\mathfrak{n}_\emptyset}^k) \cdot \Phi = 0$, hence $\Phi \in (I_w')^{[{\mathfrak{n}_\emptyset}^k]}$ and it is easy to see $L = L_\Phi$, hence (8.2) is surjective. $\square$

The Diagonal $P_\emptyset$-Action on $\mathrm{Hom}_{\mathbb{C}}(F_k, I'_w)$

Let $\mathrm{Hom}_{\mathbb{C}}(F_k, I'_w)$ be the vector space of linear maps from F_k to I'_w. The F_k is a $P_\emptyset$-representation by Lemma 8.2, and the I'_w is the contragredient $P_\emptyset$-representation of the right regular $P_\emptyset$-representation on I_w. We thus have a abstract group action of $P_\emptyset$ on the vector space $\mathrm{Hom}_{\mathbb{C}}(F_k, I'_w)$.

More precisely, let $L \in \mathrm{Hom}_{\mathbb{C}}(F_k, I'_w)$, we define the $P_\emptyset$-action on it by

$$^{p}L(\overline{u}) = p \cdot L(p^{-1}\overline{u}), \quad \forall \overline{u} \in F_k.$$

We call this the **diagonal $P_\emptyset$-action on** $\mathrm{Hom}_{\mathbb{C}}(F_k, I'_w)$.

The subspace $\mathrm{Hom}_{U(\mathfrak{n}_\emptyset)}(F_k, I'_w)$ is stable under the diagonal $P_\emptyset$-action. Actually on both F_k and I'_w, the $\mathfrak{n}_\emptyset$-module structure is obtained by differentiating the smooth $N_\emptyset$-action. Hence the $P_\emptyset$-action and $\mathfrak{n}_\emptyset$-action on the F_k and I'_w are compatible in the sense (C-3) in Definition 2.26. More precisely, let $L \in \mathrm{Hom}_{U(\mathfrak{n}_\emptyset)}(F_k, I'_w), p \in P_\emptyset$, we have

$$\begin{aligned}
^{p}L(u'\overline{u}) &= p \cdot L[p^{-1} \cdot u' \cdot \overline{u}] \\
&= p \cdot L[(p^{-1} \cdot u) \cdot (p^{-1} \cdot \overline{u})] \\
&= p \cdot [(p^{-1} \cdot u') \cdot L(p^{-1} \cdot \overline{u})] \quad (L \text{ is } \mathfrak{n}_\emptyset\text{-linear}) \\
&= [p \cdot (p^{-1} \cdot u')] \cdot [p \cdot L(p^{-1} \cdot \overline{u})] \\
&= u' \cdot [p \cdot L(p^{-1} \cdot \overline{u})] \\
&= u' \cdot {}^{p}L(\overline{u})
\end{aligned}$$

for all $u' \in U(\mathfrak{n}_\emptyset), \overline{u} \in F_k$. Hence ^{p}L is still $\mathfrak{n}_\emptyset$-linear.

Lemma 8.5. *With the* $\mathrm{Hom}_{\mathbb{C}}(F_k, I'_w)$ *and* $\mathrm{Hom}_{U(\mathfrak{n}_\emptyset)}(F_k, I'_w)$ *endowed with the above diagonal $P_\emptyset$-actions, we have*

- *The isomorphism (8.2) in Lemma 8.4 is $P_\emptyset$-equivariant.*
- *The natural isomorphism*

  $$\mathrm{Hom}_{\mathbb{C}}(F_k, I'_w) \simeq F_k^* \otimes I'_w$$

  is $P_\emptyset$-equivariant, with the $\mathrm{Hom}_{\mathbb{C}}(F_k, I'_w)$ *endowed with the diagonal $P_\emptyset$-action, and $F_k^* \otimes I'_w$ endowed with the tensor product $P_\emptyset$-action.*
- *By differentiating the diagonal $P_\emptyset$-action, one has the $\mathfrak{n}_\emptyset$-action on* $\mathrm{Hom}_{\mathbb{C}}(F_k, I'_w)$*, and its $\mathfrak{n}_\emptyset$-invariant subspace is*

  $$H^0(\mathfrak{n}_\emptyset, \mathrm{Hom}_{\mathbb{C}}(F_k, I'_w)) = \mathrm{Hom}_{U(\mathfrak{n}_\emptyset)}(F_k, I'_w)$$

  and this equality is $P_\emptyset$-equivariant.

- *By differentiating the $P_\emptyset$-actions on* $\mathrm{Hom}_{\mathbb{C}}(F_k, I'_w)$ *and* $F_k^* \otimes I'_w$*, their* $\mathfrak{n}_\emptyset$*-invariant subspaces are isomorphic*

$$H^0(\mathfrak{n}_\emptyset, \mathrm{Hom}_{\mathbb{C}}(F_k, I'_w)) \simeq H^0(\mathfrak{n}_\emptyset, F_k^* \otimes I'_w),$$

and this isomorphism is $P_\emptyset$*-equivariant.*

Summary

Combining the above two Lemmas, we have the following sequence of isomorphisms

$$\begin{aligned}(I'_w)^{[\mathfrak{n}_\emptyset{}^k]} &\simeq \mathrm{Hom}_{U(\mathfrak{n}_\emptyset)}(F_k, I'_w) \quad \text{(Lemma 8.4)} \\ &= H^0(\mathfrak{n}_\emptyset, \mathrm{Hom}_{\mathbb{C}}(F_k, I'_w)) \quad \text{(Lemma 8.5)} \\ &\simeq H^0(\mathfrak{n}_\emptyset, F_k^* \otimes I'_w) \\ &\simeq H^0(\mathfrak{n}_\emptyset, (F_k \otimes I_w)')\end{aligned}$$

and all isomorphisms above are $P_\emptyset$-equivariant.

Remark 8.6. We can construct a linear map directly:

$$\begin{aligned}(I'_w)^{[\mathfrak{n}_\emptyset{}^k]} &\to (F_k \otimes I_w)' \\ \Phi &\mapsto \widetilde{\Phi}\end{aligned} \tag{8.3}$$

where $\widetilde{\Phi}$ is given by

$$\widetilde{\Phi}(\overline{u} \otimes \phi) := \Phi(R_u \phi).$$

And it is easy to see this map is well-defined, $P_\emptyset$-equivariant, with image inside $H^0(\mathfrak{n}_\emptyset, (F_k \otimes I_w)')$, and is exactly the composition of the above sequence of isomorphisms, hence is an isomorphism.

In sum, to study the annihilator $(I'_w)^{[\mathfrak{n}_\emptyset{}^k]}$ and the $M_\emptyset$-action on it, we just need to compute the 0th cohomology $H^0(\mathfrak{n}_\emptyset, (F_k \otimes I_w)')$. We will first show the $F_k \otimes I_w$ is isomorphic to a Schwartz induction space, then use Shapiro Lemma to compute this 0th cohomology.

8.2 The I_w and The Tensor Product $I_w \otimes F_k$

The main object of this section is to study the tensor product $I_w \otimes F_k$, and the $M_\emptyset$-action on it.

- In 8.2.1, we quickly recall the structure of the local Schwartz induction $I_w = \mathcal{S}\mathrm{Ind}_P^{PwP_\emptyset}\sigma$.
- In 8.2.2, we define the Schwartz induction $\mathcal{S}\mathrm{Ind}_{N_\emptyset\cap w^{-1}Pw}^{N_\emptyset}\sigma^w$ where σ^w is the twisted representation $\sigma \circ \mathrm{Ad}w$ of $N_\emptyset \cap w^{-1}Pw$ on V. We show there is a natural isomorphism ((8.4) or (8.5)):

$$I_w \xrightarrow{\sim} \mathcal{S}\mathrm{Ind}_{N_\emptyset\cap w^{-1}Pw}^{N_\emptyset}\sigma^w.$$

- In 8.2.3, we describe the $M_\emptyset$-action on the right-hand-side

$$\mathcal{S}\mathrm{Ind}_{N_\emptyset\cap w^{-1}Pw}^{N_\emptyset}\sigma^w$$

  induced from the $M_\emptyset$-representation I_w by the above isomorphism.
- In 8.2.4, we recall the notion of external tensor products of representations, and the following basic property of Schwartz inductions:

$$\mathcal{S}\mathrm{Ind}_{P_1}^{G_1}\sigma_1 \widehat{\otimes} \mathcal{S}\mathrm{Ind}_{P_2}^{G_2}\sigma_2 \xrightarrow{\sim} \mathcal{S}\mathrm{Ind}_{P_1\times P_2}^{G_1\times G_2}\sigma_1 \boxtimes \sigma_2.$$

- In 8.2.5, we apply the above basic property to show the following isomorphism (8.17):

$$(\mathcal{S}\mathrm{Ind}_{N_\emptyset\cap w^{-1}Pw}^{N_\emptyset}\sigma^w) \otimes F_k \xrightarrow{\sim} \mathcal{S}\mathrm{Ind}_{N_\emptyset\cap w^{-1}Pw}^{N_\emptyset}(\sigma^w \otimes \eta_k|_{N_\emptyset\cap w^{-1}Pw}).$$

In sum, the $I_w \otimes F_k$ is isomorphic to the Schwartz induction

$$\mathcal{S}\mathrm{Ind}_{N_\emptyset\cap w^{-1}Pw}^{N_\emptyset}(\sigma^w \otimes \eta_k|_{N_\emptyset\cap w^{-1}Pw}),$$

and we can write down the explicit $M_\emptyset$-action on it.

8.2.1 Revisiting the I_w

Recall that the notation I_w means the local Schwartz induction:

$$I_w = \mathcal{S}\mathrm{Ind}_P^{G_w}\sigma,$$

where $G_w = PwP_\emptyset$ is the double coset corresponding to the representative $w \in [W_\Theta\backslash W]$.

Structure of the Double Coset $G_w = PwP_\emptyset$

By abuse of notation, we denote the fixed representative of $w \in [W_\Theta \backslash W]$ in G (actually in $N_G(S)$) by the same w.

We regard G_w as an abstract real variety. For every $p \in P, q \in P_\emptyset$, we denote the point $pwq \in PwP_\emptyset$ by

$$x_{pwq} \in G_w.$$

Remark 8.7. The reader can treat it as the point $pwq \in G$, but we want to emphasize that the expression pwq is not unique.

We have the following easy facts:

$$\begin{aligned} PwP_\emptyset &= PwN_\emptyset \\ &= Pw(N_\emptyset \cap w^{-1}Pw) \cdot (N_\emptyset \cap w^{-1}\overline{N}_P w) \\ &= Pw(N_\emptyset \cap w^{-1}\overline{N}_P w) \\ &= PwN_w^+ \end{aligned}$$

(Recall that N_w^+ is exactly $N_\emptyset \cap w^{-1}\overline{N}_P w$ as defined in Chapter 6.)

Lemma 8.8. *The last expression in the above align is unique. More precisely, the map*

$$\begin{aligned} P \times N_w^+ &\to G_w = PwP_\emptyset \\ (p, n) &\mapsto pwn \end{aligned}$$

is an isomorphism of real affine varieties and smooth manifolds. In particular, every element in G_w *is uniquely written as* x_{pwn} *for some* $p \in P$ *and* $n \in N_w^+$.

Functions in I_w

The $I_w = \mathcal{S}\mathrm{Ind}_P^{G_w}\sigma$ is defined to be the image space of the following integration map

$$\mathcal{S}(G_w, V) \to C^\infty(G_w, V, \sigma), f \mapsto f^\sigma,$$

where $C^\infty(G_w, V, \sigma) = \{f \in C^\infty(G_w, V) : f(px) = \sigma(p)f(x), \forall p \in P, x \in G_w\}$ and $f^\sigma(x) := \int_P \sigma(p^{-1})f(px)dp$ is the σ-mean value function.

The topology on $\mathcal{S}\mathrm{Ind}_P^{G_w}\sigma$ is the quotient topology, and in particular it is a nuclear Fréchet space, and one has the following surjective homomorphism of TVS and $P_\emptyset$-representations:

$$\mathcal{S}(G_w, V) \twoheadrightarrow \mathcal{S}\mathrm{Ind}_P^{G_w}\sigma.$$

By the above definition, the $I_w = \mathcal{S}\mathrm{Ind}_P^{G_w}\sigma$ is contained in $C^\infty(G_w, V, \sigma)$. In particular, a function ϕ in I_w is a smooth function and satisfies the "σ-rule":

$$\phi(px) = \sigma(p)\phi(x), \quad \forall p \in P, x \in G_w.$$

8.2.2 The I_w is Isomorphic to $\mathcal{S}\mathrm{Ind}_{N_\emptyset \cap w^{-1}Pw}^{N_\emptyset}\sigma^w$

We denote by (σ^w, V) the following representation of $N_\emptyset \cap w^{-1}Pw$ on V:

$$\sigma^w(u)v := \sigma(wuw^{-1})v, \quad \forall v \in V, u \in N_\emptyset \cap w^{-1}Pw.$$

(Note that $\sigma \circ \mathrm{Ad}w$ is a twisted representation of $w^{-1}Pw$ on V, the above σ^w is its restriction to the subgroup $N_\emptyset \cap w^{-1}Pw$.)

The representation σ^w is actually a (real) algebraic representation of $N_\emptyset \cap w^{-1}Pw$ on V. In particular, it is of moderate growth, and one can define as in Chapter 4 the Schwartz induction

$$\mathcal{S}\mathrm{Ind}_{N_\emptyset \cap w^{-1}Pw}^{N_\emptyset}\sigma^w,$$

which is the image of $\mathcal{S}(N_\emptyset, V)$ under the σ^w-mean value map:

$$\begin{aligned} \mathcal{S}(N_\emptyset, V) &\to \mathcal{S}\mathrm{Ind}_{N_\emptyset \cap w^{-1}Pw}^{N_\emptyset}\sigma^w \\ f &\mapsto f^{\sigma^w} \end{aligned}$$

where f^{σ^w} is given by

$$f^{\sigma^w}(n) = \int_{N_\emptyset \cap w^{-1}Pw} \sigma^w(u^{-1})f(un)du, \forall n \in N_\emptyset.$$

Similar to the I_w, a function ψ in $\mathcal{S}\mathrm{Ind}_{N_\emptyset \cap w^{-1}Pw}^{N_\emptyset}\sigma^w$ is a smooth function on $N_\emptyset$, satisfying the σ^w-rule:

$$\psi(un) = \sigma^w(u)\psi(n), \quad \forall u \in N_\emptyset \cap w^{-1}Pw, n \in N_\emptyset.$$

The Map from I_w to $\mathcal{S}\mathrm{Ind}_{N_\emptyset \cap w^{-1}Pw}^{N_\emptyset}\sigma^w$

We can define the following map

$$\begin{aligned} I_w = \mathcal{S}\mathrm{Ind}_P^{G_w}\sigma &\to C^\infty(N_\emptyset, V, \sigma^w) \\ \phi &\mapsto \widehat{\phi} \end{aligned} \tag{8.4}$$

where the $\widehat{\phi}$ is given by

$$\widehat{\phi}(n) = \phi(x_{wn}), \forall n \in N_\emptyset.$$

The $\widehat{\phi}$ is obviously a smooth V-valued function on $N_\emptyset$, and satisfying the σ^w-rule: for all $u \in N_\emptyset \cap w^{-1}Pw, n \in N_\emptyset$, one has

$$\begin{aligned} \widehat{\phi}(un) &= \phi(x_{wun}) \\ &= \phi(x_{wuw^{-1}wn}) \\ &= \sigma(wuw^{-1})\phi(x_{wn}) \qquad (\phi \text{ satisfies the } \sigma\text{-rule}) \\ &= \sigma^w(u)\widehat{\phi}(n) \end{aligned}$$

The Image of (8.4)

We show

Lemma 8.9. *The image of the map (8.4) is in the Schwartz induction space $\mathcal{S}\mathrm{Ind}_{N_\emptyset \cap w^{-1}Pw}^{N_\emptyset}\sigma^w$. Therefore, we have an isomorphism (of TVS):*

$$\begin{aligned} I_w &\xrightarrow{\simeq} \mathcal{S}\mathrm{Ind}_{N_\emptyset \cap w^{-1}Pw}^{N_\emptyset}\sigma^w \\ \phi &\mapsto \widehat{\phi} \end{aligned} \tag{8.5}$$

First we have the following diagram

$$\begin{tikzcd}
 & C^\infty(N_\emptyset, V, \sigma^w) & \\
I_w \arrow[ur, "(8.4)"] \arrow[rr, dashed, "?"] \arrow[dr, "\simeq", "(4.37)"'] & & \mathcal{S}\mathrm{Ind}_{N_\emptyset \cap w^{-1}Pw}^{N_\emptyset}\sigma^w \arrow[ul] \arrow[dl, "\simeq"', "(4.37)"] \\
 & \mathcal{S}(N_w^+, V) &
\end{tikzcd}$$

In this diagram, the upper-right arrow is the inclusion, and the two down-slash arrows are isomorphisms given by the Lemma 4.92, since both G_w and $N_\emptyset$ are isomorphic (as real algebraic varieties) to direct products:

$$\begin{aligned} G_w &\simeq P \times N_w^+ \\ N_\emptyset &\simeq (N_\emptyset \cap w^{-1}Pw) \times N_w^+ \end{aligned}$$

To show the image of (8.4) is in $\mathcal{S}\mathrm{Ind}_{N_\emptyset \cap w^{-1}Pw}^{N_\emptyset} \sigma^w$, we just need to show the composition of the isomorphism $I_w \to \mathcal{S}(N_w^+, V)$ and the inverse isomorphism $\mathcal{S}(N_w^+, V) \to \mathcal{S}\mathrm{Ind}_{N_\emptyset \cap w^{-1}Pw}^{N_\emptyset} \sigma^w$ is exactly the map (8.4).

Proof. The map $I_w \to \mathcal{S}(N_w^+, V)$ is given by (4.37) in Lemma 4.92. Pick a $\phi \in I_w$, its image in $\mathcal{S}(N_w^+, V)$ is exactly

$$\underline{\phi}(n) := \phi(x_{wn}) = \phi(wn).$$

The proof of Lemma 4.92 gives us an explicit inverse of the isomorphism $\mathcal{S}\mathrm{Ind}_{N_\emptyset \cap w^{-1}Pw}^{N_\emptyset} \sigma^w \to \mathcal{S}(N_w^+, V)$. Namely, let $\underline{\phi} \in \mathcal{S}(N_w^+, V)$ be an arbitrary Schwartz function on N_w^+, let $\gamma \in C_c^\infty(N_\emptyset \cap w^{-1}Pw)$ be an arbitrary $\mathbb{C}$-valued smooth function with compact support such that $\gamma(e) = 1$ and $\int_{N_\emptyset \cap w^{-1}Pw} \gamma(u)du = 1$. Then one can first define a function $\Phi \in \mathcal{S}(N_\emptyset, V)$ by

$$\Phi(un) = \gamma(u)\sigma^w(u)\underline{\phi}(n), \quad \forall u \in N_\emptyset \cap w^{-1}Pw, n \in N_w^+.$$

(Here note $N_\emptyset = (N_\emptyset \cap w^{-1}Pw) \cdot N_w^+$ is a direct product of manifolds.) It is easy to see the Φ is Schwartz on $N_\emptyset$. Now we consider the σ^w-mean value function of Φ:

$$\Phi^{\sigma^w}(z) = \int_{N_\emptyset \cap w^{-1}Pw} \sigma^w(a^{-1})\Phi(az)da, \quad \forall z \in N_\emptyset$$

It is easy to check the $\Phi^{\sigma^w} \in \mathcal{S}\mathrm{Ind}_{N_\emptyset \cap w^{-1}Pw}^{N_\emptyset} \sigma^w$ is independent of the choice of γ as in the proof of Lemma 4.92. Actually for all $u \in N_\emptyset \cap$

$w^{-1}Pw, n \in N_w^+$, one has

$$\begin{aligned}
\Phi^{\sigma^w}(un) &= \int_{N_\emptyset \cap w^{-1}Pw} \sigma^w(a^{-1})\Phi(aun)da \\
&= \int_{N_\emptyset \cap w^{-1}Pw} \sigma^w(a^{-1})[\gamma(au)\sigma^w(au)\underline{\phi}(n)]da \\
&= \int_{N_\emptyset \cap w^{-1}Pw} \gamma(au)\sigma^w(u)\underline{\phi}(n)da \\
&= [\int_{N_\emptyset \cap w^{-1}Pw} \gamma(au)da]\sigma^w(u)\underline{\phi}(n) \\
&= \sigma^w(u)\underline{\phi}(n)
\end{aligned}$$

Obviously $\Phi^{\sigma^w}|_{N_w^+} = \underline{\phi}$, hence Φ^{σ^w} is indeed the image under the inverse isomorphism $\mathcal{S}(N_w^+, V) \to \mathcal{S}\mathrm{Ind}_{N_\emptyset \cap w^{-1}Pw}^{N_\emptyset}\sigma^w$.

Now the composition of the two maps $I_w \to \mathcal{S}(N_w^+, V)$ and $\mathcal{S}(N_w^+, V) \to \mathcal{S}\mathrm{Ind}_{N_\emptyset \cap w^{-1}Pw}^{N_\emptyset}\sigma^w$ sends the ϕ to Φ^{σ^w}. It is easy to see

$$\Phi^{\sigma^w} = \widehat{\phi} \text{ on } N_\emptyset$$

Therefore the composition of the two isomorphisms $I_w \to \mathcal{S}(N_w^+, V)$ and $\mathcal{S}(N_w^+, V) \to \mathcal{S}\mathrm{Ind}_{N_\emptyset \cap w^{-1}Pw}^{N_\emptyset}\sigma^w$ is exactly the map (8.4). $\square$

8.2.3 The Isomorphism (8.5) is $M_\emptyset$-Equivariant

Note that the local Schwartz induction $I_w = \mathcal{S}\mathrm{Ind}_P^{G_w}\sigma$ is a representation of $P_\emptyset$ under the right regular $P_\emptyset$-action, denoted by

$$R_p\phi, \quad \forall \phi \in I_w, p \in P_\emptyset.$$

The isomorphism (8.5) transport the right regular $P_\emptyset$-action on I_w to a $P_\emptyset$-action on $\mathcal{S}\mathrm{Ind}_{N_\emptyset \cap w^{-1}Pw}^{N_\emptyset}\sigma^w$. We can explicitly write down the $P_\emptyset$-action on $\mathcal{S}\mathrm{Ind}_{N_\emptyset \cap w^{-1}Pw}^{N_\emptyset}\sigma^w$.

The $N_\emptyset$-Action on $\mathcal{S}\mathrm{Ind}_{N_\emptyset \cap w^{-1}Pw}^{N_\emptyset}\sigma^w$

The $\mathcal{S}\mathrm{Ind}_{N_\emptyset \cap w^{-1}Pw}^{N_\emptyset}\sigma^w$ is obviously a $N_\emptyset$-representation with right regular $N_\emptyset$-action. The following lemma is trivial.

Lemma 8.10. *With both sides endowed with the right regular $N_\emptyset$-actions, the isomorphism (8.5) is $N_\emptyset$-equivariant.*

The $M_\emptyset$-Action on $\mathcal{S}\mathrm{Ind}_{N_\emptyset \cap w^{-1}Pw}^{N_\emptyset}\sigma^w$

We compute the $M_\emptyset$-action on $\mathcal{S}\mathrm{Ind}_{N_\emptyset \cap w^{-1}Pw}^{N_\emptyset}\sigma^w$. Let $\phi \in I_w$ be arbitrary, and let $\widehat{\phi} \in \mathcal{S}\mathrm{Ind}_{N_\emptyset \cap w^{-1}Pw}^{N_\emptyset}\sigma^w$ be its image under the isomorphism (8.5). For arbitrary $m \in M_\emptyset, n \in N_\emptyset$, we have

$$\begin{aligned}
\widehat{R_m\phi}(n) &= (R_m\phi)(x_{wn}) \\
&= \phi(x_{wnm}) \\
&= \phi(x_{wmw^{-1}wm^{-1}nm}) \\
&= \sigma(wmw^{-1})\phi(x_{wm^{-1}nm}) \\
&= \sigma^w(m)\widehat{\phi}(m^{-1}nm)
\end{aligned}$$

In sum, the $M_\emptyset$-action on $\mathcal{S}\mathrm{Ind}_{N_\emptyset \cap w^{-1}Pw}^{N_\emptyset}\sigma^w$ is given by

$$m \cdot \psi = \sigma^w(m) \circ \psi \circ \mathrm{Ad}m^{-1} \tag{8.6}$$

for all $\psi \in \mathcal{S}\mathrm{Ind}_{N_\emptyset \cap w^{-1}Pw}^{N_\emptyset}\sigma^w, m \in M_\emptyset$. Actually this $M_\emptyset$-action is well-defined on the entire $C^\infty(N_\emptyset, V, \sigma^w)$, and we have just shown the subspace $\mathcal{S}\mathrm{Ind}_{N_\emptyset \cap w^{-1}Pw}^{N_\emptyset}\sigma^w$ is stable under this $M_\emptyset$-action.

Lemma 8.11. *With the right regular $M_\emptyset$-action on I_w and the $M_\emptyset$-action on $\mathcal{S}\mathrm{Ind}_{N_\emptyset \cap w^{-1}Pw}^{N_\emptyset}\sigma^w$ defined in (8.6), the isomorphism (8.5) is $M_\emptyset$-equivariant.*

8.2.4 External Tensor Products

For $i = 1, 2$, let

$$\begin{aligned}
G_i &= \text{a real algebraic groups} \\
P_i &= \text{an algebraic subgroup of } G_i \\
(\sigma_i, V_i) &= \text{a smooth nuclear Fréceht representations of } P_i
\end{aligned}$$

External Tensor Products of Representations

Definition 8.12. The following representation of $P_1 \times P_2$ on $V_1 \widehat{\otimes} V_2$ is called **the external tensor product of** σ_1 **and** σ_2, and is denoted by $(\sigma_1 \boxtimes \sigma_2, V_1 \widehat{\otimes} V_2)$:

$$\sigma_1 \boxtimes \sigma_2(p_1, p_2)(v_1 \otimes v_2) := [\sigma_1(p_1)v_1] \otimes [\sigma_2(p_2)v_2],$$

for all $p_i \in P_i, v_i \in V_i, i = 1, 2$.

Remark 8.13. When $P_1 = P_2$, the restriction of $\sigma_1 \boxtimes \sigma_2$ to the diagonal subgroup of $P_1 \times P_2$ is exactly the tensor product $\sigma_1 \otimes \sigma_2$.

It is easy to see

Lemma 8.14. *The representation $\sigma_1 \boxtimes \sigma_2$ is a smooth NF-representation. Moreover, if σ_1, σ_2 are of moderate growth, then so is $\sigma_1 \boxtimes \sigma_2$.*

Tensor Product of Schwartz Inductions

Suppose σ_1, σ_2 are of moderate growth, then one has three Schwartz induction spaces:

$$\mathcal{S}\mathrm{Ind}_{P_1}^{G_1}\sigma_1, \quad \mathcal{S}\mathrm{Ind}_{P_2}^{G_2}\sigma_2, \quad \mathcal{S}\mathrm{Ind}_{P_1\times P_2}^{G_1\times G_2}\sigma_1 \boxtimes \sigma_2.$$

For $i = 1, 2$, let $\phi_i \in \mathcal{S}\mathrm{Ind}_{P_i}^{G_i}\sigma_i$ be two arbitrary functions, then one can define a smooth function on $G_1 \times G_2$ with values in $V_1 \widehat{\otimes} V_2$:

$$\phi_1 \boxtimes \phi_2(g_1, g_2) := \phi_1(g_1) \otimes \phi_2(g_2), \quad \forall (g_1, g_2) \in G_1 \times G_2. \tag{8.7}$$

Then we have

Lemma 8.15. *The function $\phi_1 \boxtimes \phi_2$ is in $\mathcal{S}\mathrm{Ind}_{P_1\times P_2}^{G_1\times G_2}\sigma_1 \boxtimes \sigma_2$, and the map $\phi_1 \otimes \phi_2 \mapsto \phi_1 \boxtimes \phi_2$ extends to an isomorphism of TVS:*

$$\begin{aligned} \mathcal{S}\mathrm{Ind}_{P_1}^{G_1}\sigma_1 \widehat{\otimes} \mathcal{S}\mathrm{Ind}_{P_2}^{G_2}\sigma_2 &\to \mathcal{S}\mathrm{Ind}_{P_1\times P_2}^{G_1\times G_2}\sigma_1 \boxtimes \sigma_2 \\ \phi_1 \otimes \phi_2 &\mapsto \phi_1 \boxtimes \phi_2 \end{aligned} \tag{8.8}$$

8.2.5 Tensor Product of Schwartz Inductions

In this subsection, we show a Lemma about tensor product of Schwartz induction. We will apply this Lemma to the tensor product $I_w \otimes F_k$. The analogue of this Lemma in the ordinary induction picture is well-known.

Let G, P be the same as in 4.6, and let

$$\begin{aligned} (\sigma, V) &= \text{a nuclear Fréchet representation of } P \text{ of moderate growth} \\ (\eta, F) &= \text{a finite dimensional algebraic representation of } G \\ (\eta|_P, F) &= \text{the restriction of } \eta \text{ to the subgroup } P \\ \mathcal{S}\mathrm{Ind}_P^G\sigma &= \text{the Schwartz induction space of } \sigma \end{aligned}$$

In particular, the η and $\eta|_P$ are of moderate growth.

Let $\phi \in \mathcal{S}\mathrm{Ind}_P^G\sigma, v \in F$, we define the following function in $C^\infty(G, V \otimes F)$:

$$\phi_v(g) := \phi(g) \otimes \eta(g)v, \quad \forall g \in G. \tag{8.9}$$

The main result of this subsection is:

Lemma 8.16. *The function ϕ_v is in $\mathcal{S}\mathrm{Ind}_P^G(\sigma \otimes \eta|_P)$. And we have the following isomorphism of TVS:*

$$\begin{aligned} \mathcal{S}\mathrm{Ind}_P^G\sigma \otimes F &\to \mathcal{S}\mathrm{Ind}_P^G(\sigma \otimes \eta|_P) \\ \phi \otimes v &\mapsto \phi_v \end{aligned} \tag{8.10}$$

This map is an isomorphism of G-representations, where the left hand side is the tensor product representation and right hand side is the right regular representation.

We first show the function ϕ_v is indeed in $\mathcal{S}\mathrm{Ind}_P^G(\sigma \otimes \eta|_P)$, by finding its preimage in the Schwartz function space $\mathcal{S}(G, V \otimes F)$. Then we show the above map is an isomorphism.

Tensor Product on Schwartz Function Spaces

For a $f \in \mathcal{S}(G, V)$ and a $v \in F$, we can define the following smooth function on G with values in $V \otimes F$

$$f_v(g) := f(g) \otimes \eta(g)v, \quad \forall g \in G. \tag{8.11}$$

We claim:

Lemma 8.17. *The function f_v is in $\mathcal{S}(G, V \otimes F)$.*

To prove this Lemma, we just need to apply the following Lemma to $(\rho, U) = (1_G^V \otimes \eta, V \otimes F)$ where 1_G^V is the trivial representation of G on V.

Lemma 8.18. *Let G be a real point group of a linear algebraic group, and (ρ, U) be a representation of G of moderate growth. For an arbitrary $f \in \mathcal{S}(G, U)$, we define the function*

$$^{\rho}f(g) := \rho(g)f(g), \quad \forall g \in G. \tag{8.12}$$

Then $^{\rho}f$ is in $\mathcal{S}(G, U)$.

Proof. One can easily check, for all $u \in U(\mathfrak{g})$, the

$$[R_u(^{\rho}f)](g) = \rho(g)[R_u f(g)].$$

Since ρ is of moderate growth, the $R_u{}^{\rho}f$ is bounded on entire G. Since $U(\mathfrak{g})$ generates the entire ring of Nash differential operators (and algebraic differential operators), we see $^{\rho}f$ is rapidly decreasing on G. □

The ϕ_v is in $\mathcal{S}\mathrm{Ind}_P^G(\sigma \otimes \eta|_P)$

We show the first part of Lemma 8.16: for all $\phi \in \mathcal{S}\mathrm{Ind}_P^G\sigma$, and $v \in F$, the ϕ_v defined by (8.9) is in the Schwartz induction space $\mathcal{S}\mathrm{Ind}_P^G(\sigma \otimes \eta|_P)$.

By definition of the Schwartz induction, one can find a Schwartz function $f \in \mathcal{S}(G, V)$ such that $f^\sigma = \phi$. Then we consider the function $f_v \in \mathcal{S}(G, V \otimes F)$ defined in (8.11), and its $\sigma \otimes \eta$-mean value function $(f_v)^{\sigma\otimes\eta}$, and we see

$$\begin{aligned}
(f_v)^{\sigma\otimes\eta}(g) &= \int_P (\sigma \otimes \eta)(p^{-1}) f_v(pg) dp \\
&= \int_P (\sigma \otimes \eta)(p^{-1})[f(pg) \otimes \eta(pg)v] dp \\
&= \int_P [\sigma(p^{-1}) f(pg)] \otimes [\eta(p^{-1})\eta(pg)v] dp \\
&= \int_P [\sigma(p^{-1}) f(pg)] \otimes [\eta(g)v] dp \\
&= [\int_P \sigma(p^{-1}) f(pg) dp] \otimes [\eta(g)] \\
&= f^\sigma(g) \otimes \eta(g)v \\
&= \phi(g) \otimes \eta(g)v \\
&= \phi_v(g)
\end{aligned}$$

Therefore, the ϕ_v is the image of f_v under the mean value map $\mathcal{S}(G, V \otimes F) \to \mathcal{S}\mathrm{Ind}_P^G(\sigma \otimes \eta|_P)$.

Relation With External Tensor Product

For a $\psi \in \mathcal{S}\mathrm{Ind}_P^G(\sigma \otimes \eta|_P)$, one can define the following smooth function on $G \times G$ with values in $V \otimes F$:

$$\phi^\wedge(g_1, g_2) := [\mathrm{id} \otimes \eta(g_2 g_1^{-1})]\psi(g_1), \quad \forall (g_1, g_2) \in G \times G. \tag{8.13}$$

Also for a $\Psi \in \mathcal{S}\mathrm{Ind}_{P\times G}^{G\times G}\sigma \boxtimes \eta$, one can define the following smooth function on G with values in $V \otimes F$:

$$\Psi^\vee(g) := \Psi(g, g), \quad \forall g \in G. \tag{8.14}$$

Then we have

Lemma 8.19. *With the above notations,*

(1) The $\psi^\wedge$ is in $\mathcal{S}\mathrm{Ind}_{P\times G}^{G\times G}\sigma \boxtimes \eta$.

(2) *The* $\Psi^\vee$ *is in* $\mathcal{S}\mathrm{Ind}_P^G(\sigma\otimes\eta|_P)$.

(3) *The two maps* $\psi\mapsto\psi^\wedge$ *and* $\Psi\mapsto\Psi^\vee$ *are mutually inverse, therefore we have the following isomorphism of TVS:*

$$\mathcal{S}\mathrm{Ind}_P^G(\sigma\otimes\eta|_P)\xrightarrow{\sim}\mathcal{S}\mathrm{Ind}_{P\times G}^{G\times G}\sigma\boxtimes\eta \tag{8.15}$$

(4) *The isomorphism (8.15) is G-equivariant, when the left-hand-side has the right regular G-action, and right-hand-side has the diagonal right regular G-action.*

The Map (8.10) is an Isomorphism

We show the (8.10) is an isomorphism.

First we note $F\simeq\mathcal{S}\mathrm{Ind}_G^G\eta$ by Lemma 4.93, and this isomorphism combined with the Lemma 8.15, gives the following isomorphism

$$\begin{aligned}\mathcal{S}\mathrm{Ind}_P^G\sigma\otimes F&\xrightarrow{\simeq}\mathcal{S}\mathrm{Ind}_{P\times G}^{G\times G}\sigma\boxtimes\eta\\ \phi\otimes v&\mapsto\phi\boxtimes v\end{aligned}\tag{8.16}$$

where the function $\phi\boxtimes v$ is given by

$$\phi\boxtimes v(g_1,g_2):=\phi(g_1)\otimes\eta(g_2)v,\quad\forall(g_1,g_2)\in G\times G.$$

We then have the following diagram

$$\begin{tikzcd}
\mathcal{S}\mathrm{Ind}_P^G\sigma\otimes F \arrow[rr, "(8.16)"] \arrow[dr, "(8.10)"'] & & \mathcal{S}\mathrm{Ind}_{P\times G}^{G\times G}\sigma\boxtimes\eta \\
& \mathcal{S}\mathrm{Ind}_P^G(\sigma\otimes\eta|_P) \arrow[ur, "(8.15)"'] &
\end{tikzcd}$$

And we have seen the (8.15) and (8.16) are isomorphisms, we just need to verify the above diagram commutates, i.e. $(\phi_v)^\wedge=\phi\boxtimes v$. Actually,

$$\begin{aligned}(\phi_v)^\wedge(g_1,g_2)&=[\mathrm{id}\otimes\eta(g_2g_1^{-1})]\phi_v(g_1)\\&=[\mathrm{id}\otimes\eta(g_2g_1^{-1})]\phi(g_1)\otimes\eta(g_1)v\\&=\phi(g_1)\otimes\eta(g_2)v\\&=\phi\boxtimes v(g_1,g_2)\end{aligned}$$

Application to $I_w \otimes F_k$

Applying the Lemma 8.16, to

$$\begin{aligned} G &= N_\emptyset \\ P &= N_\emptyset \cap w^{-1}Pw \\ (\sigma, V) &= (\sigma^w, V) \\ (\eta, F) &= (\eta_k, F_k) \end{aligned}$$

where $F_k = U(\mathfrak{n}_\emptyset)/(\mathfrak{n}_\emptyset{}^k)$, we have the following Lemma

Lemma 8.20. *For any $\phi \in \mathcal{S}\mathrm{Ind}_{N_\emptyset \cap w^{-1}Pw}^{N_\emptyset}\sigma^w$ and $v \in F_k$, the function*

$$\phi_v(n) := \phi(n) \otimes \eta_k(n)v, \quad \forall n \in N_\emptyset,$$

is in $\mathcal{S}\mathrm{Ind}_{N_\emptyset \cap w^{-1}Pw}^{N_\emptyset}(\sigma^w \otimes \eta_k)$. The map

$$\begin{aligned} (\mathcal{S}\mathrm{Ind}_{N_\emptyset \cap w^{-1}Pw}^{N_\emptyset}\sigma^w) \otimes F_k &\to \mathcal{S}\mathrm{Ind}_{N_\emptyset \cap w^{-1}Pw}^{N_\emptyset}(\sigma^w \otimes \eta_k) \\ \phi \otimes v &\mapsto \phi_v \end{aligned} \tag{8.17}$$

is an isomorphism of TVS.

By abuse of notation, we write η_k instead of $\eta_k|_{N_\emptyset \cap w^{-1}Pw}$, to keep all equations short. This will not cause any ambiguity.

The $M_\emptyset$-Action on $\mathcal{S}\mathrm{Ind}_{N_\emptyset \cap w^{-1}Pw}^{N_\emptyset}(\sigma^w \otimes \eta_k)$

Combining the isomorphisms (8.5) in Lemma 8.9 and (8.17) in Lemma 8.20, we have the following isomorphism

$$\begin{aligned} I_w \otimes F_k &\xrightarrow{\sim} \mathcal{S}\mathrm{Ind}_{N_\emptyset \cap w^{-1}Pw}^{N_\emptyset}(\sigma^w \otimes \eta_k) \\ \phi \otimes v &\mapsto \widehat{\phi}_v \end{aligned} \tag{8.18}$$

where the $\widehat{\phi}$ is defined by $\widehat{\phi}(n) = \phi(x_{wn})$, and the $\widehat{\phi}_v(n)$ is defined by $\widehat{\phi}_v(n) = \widehat{\phi}(n) \otimes \eta_k(n)v$.

The left-hand-side $I_w \otimes F_k$ has the tensor product $M_\emptyset$-action, where the $M_\emptyset$ acts on I_w by right regular action, and on F_k by the conjugation which is also denoted by η_k.

Now we want to describe the corresponding $M_\emptyset$-action on the right-hand-side $\mathcal{S}\mathrm{Ind}_{N_\emptyset \cap w^{-1}Pw}^{N_\emptyset}(\sigma^w \otimes \eta_k)$. Obviously, it is given by

$$\begin{aligned} m \cdot \widehat{\phi}_v &= (\widehat{R_m \phi})_{\eta_k(m)v} \\ &= \widehat{R_m \phi}(n) \otimes \eta_k(n)\eta_k(m)v \\ &= R_m\phi(x_{wn}) \otimes \eta_k(n)\eta_k(m)v \end{aligned}$$

We should write the $(\widehat{R_m\phi})_{\eta_k(m)v}$ in a more explicit form. The first component is given by (8.6):

$$\widehat{R_m\phi}(n) = \sigma^w(m)\widehat{\phi}(\mathrm{Ad}m^{-1}n).$$

The second component is given by

$$\eta_k(n)\eta_k(m)v = \eta_k(m)\eta_k(\mathrm{Ad}m^{-1}n)v.$$

Therefore, we have

$$m \cdot \widehat{\phi}_v = (\sigma^w \otimes \eta_k)(m) \cdot \widehat{\phi}_v(\mathrm{Ad}m^{-1}n).$$

And in general, we have

Lemma 8.21. *The $M_\emptyset$-action on the $\mathcal{S}\mathrm{Ind}_{N_\emptyset \cap w^{-1}Pw}^{N_\emptyset}(\sigma^w \otimes \eta_k)$ induced by the isomorphism (8.18) is given by*

$$m \cdot \Phi = (\sigma^w \otimes \eta_k)(m) \circ \Phi \circ \mathrm{Ad}m^{-1}, \tag{8.19}$$

for all $\Phi \in \mathcal{S}\mathrm{Ind}_{N_\emptyset \cap w^{-1}Pw}^{N_\emptyset}(\sigma^w \otimes \eta_k)$ and $m \in M_\emptyset$.

8.3 Shapiro's Lemma

In the last two sections, we have shown the isomorphism

$$(I_w')^{[\mathfrak{n}_\emptyset{}^k]} \simeq H^0(\mathfrak{n}_\emptyset, (I_w \otimes F_k)')$$

and the following isomorphism between $I_w \otimes F_k$ and a Schwartz induction space

$$I_w \otimes F_k \simeq \mathcal{S}\mathrm{Ind}_{N_\emptyset \cap w^{-1}Pw}^{N_\emptyset}(\sigma^w \otimes \eta_k).$$

Therefore, the computation of the kth annihilator $(I_w')^{[\mathfrak{n}_\emptyset{}^k]}$, is reduced to the computation of the space of $\mathfrak{n}_\emptyset$-invariant distributions on the Schwartz induction $\mathcal{S}\mathrm{Ind}_{N_\emptyset \cap w^{-1}Pw}^{N_\emptyset}(\sigma^w \otimes \eta_k)$, namely the following space

$$H^0(\mathfrak{n}_\emptyset, [\mathcal{S}\mathrm{Ind}_{N_\emptyset \cap w^{-1}Pw}^{N_\emptyset}(\sigma^w \otimes \eta_k)]').$$

- In 8.3.1, we formulate a version of Shapiro's lemma (Theorem 8.23). Let G, H be unimodular real algebraic groups, and (ρ, U) be a smooth nuclear representation of H of moderate growth. Let $(\widehat{\rho}, U')$ be the dual representation of (ρ, U), and $\mathcal{S}\mathrm{Ind}_H^G \rho$ be the Schwartz induction. Then the natural map (see (8.21))

$$H^0(H, \widehat{\rho}) \to H^0(G, (\mathcal{S}\mathrm{Ind}_H^G \rho)')$$

is an isomorphism.

- In 8.3.2, we apply Shapiro's lemma to the Schwartz induction

$$\mathcal{S}\mathrm{Ind}_{N_\emptyset \cap w^{-1}Pw}^{N_\emptyset}(\sigma^w \otimes \eta_k).$$

Since the group $N_\emptyset$ and its subgroup $N_\emptyset \cap w^{-1}Pw$ are unimodular and cohomological trivial, we can apply the Shapiro's Lemma which tells us the $H^0(\mathfrak{n}_\emptyset, (I_w \otimes F_k)')$ is exactly the following space

$$H^0(\mathfrak{n}_\emptyset \cap w^{-1}\mathfrak{p}w, (V_{\sigma^w} \otimes F_k)').$$

8.3.1 Shapiro's Lemma

In this subsection, we state the version of Shapiro's Lemma which we will apply. To simplify the understanding of Shapiro Lemma, we state a weaker version of it, and the general version could be reduced to this weaker version. This means we put some strong conditions on the groups, and in this subsection, we temporarily stick to the following notations:

$$\begin{aligned}
G &= \text{a real algebraic group which is unimodular} \\
H &= \text{a (unimodular) closed algebraic subgroup of } G \\
(\rho, U) &= \text{smooth nuclear Fréchet representation of } H \\
&\quad\ \text{which is of moderate growth} \\
\mathcal{S}(G, U) &= \text{the space of Schwartz } V\text{-valued functions on } G \\
\mathcal{S}\mathrm{Ind}_H^G \rho &= \text{the Schwartz induction space}
\end{aligned}$$

Since G, H are both unimodular, one has a right G-invariant measure on the quotient $H\backslash G$, denoted by dx. Further more, we assume the H has a complement in G, i.e. G has a real subvariety Q such that the multiplication map $H \times Q \to G$ is an isomorphism of real varieties and manifolds.

Let $(\widehat{\rho}, U')$ be the contragredient H-representation on the strong dual U', and let $H^0(H, U')$ be the space of H-invariant vectors in U'. Given a vector $\lambda \in H^0(H, U')$ and a $\phi \in \mathcal{S}\mathrm{Ind}_H^G \rho$, we consider the following function on G:

$$\langle \lambda, \phi(-) \rangle : g \mapsto \langle \lambda, \phi(g) \rangle, \quad \forall g \in G,$$

where $\langle , \rangle : U' \times U \to \mathbb{C}$ is the pairing between U' and U. This function has the following properties:

Lemma 8.22. *The $\langle \lambda, \phi(-) \rangle$ is constant on each H-coset, hence it factors through a smooth function on the quotient manifold $H \backslash G$, which is denoted by*

$$\langle \lambda, \phi \rangle : x_g \mapsto \langle \lambda, \phi(g) \rangle, \forall x_g \in H \backslash G.$$

Here x_g means the point on $H \backslash G$ represented by the group element $g \in G$.

Moreover, the function $\langle \lambda, \phi \rangle$ is integrable on $H \backslash G$, and the map:

$$\begin{aligned} \Upsilon_\lambda : \mathcal{S}\mathrm{Ind}_H^G \rho &\to \mathbb{C} \\ \phi &\mapsto \int_{H \backslash G} \langle \lambda, \phi \rangle (x_g) dx_g \end{aligned} \tag{8.20}$$

is a continuous linear functional on $\mathcal{S}\mathrm{Ind}_H^G \rho$, which is G-invariant.

Proof. The $\langle \lambda, \phi(g) \rangle$ is constant on each double coset because λ is invariant under H. The $\langle \lambda, \phi \rangle$ is integrable because it is a Schwartz function on $H \backslash G$. The functional Υ_λ is G-invariant because the measure dx is right G-invariant. □

Therefore, we have a linear map

$$\begin{aligned} H^0(H, U') &\to H^0(G, (\mathcal{S}\mathrm{Ind}_H^G \rho)') \\ \lambda &\mapsto \Upsilon_\lambda \end{aligned} \tag{8.21}$$

and it is an isomorphism by the following version of Shapiro's Lemma

Theorem 8.23 (Casselman)**.** *The map (8.21) is an isomorphism.*

The proof is unpublished now, and we omit it.

Remark 8.24. The traditional Shapiro's Lemma says the above isomorphism also holds for higher cohomology. We will only apply the special case of zeroth cohomology, which is actually a variation of Frobenius reciprocity.

Remark 8.25. Our original work applied the version of Shapiro's Lemma in [2] (p182 Theorem 4.0.13). But the version there requires the representation σ to be a Nash representation, in particular the σ need to be finite dimensional, which cannot be generalized to infinite dimensional σ. This is a strong confinement of our work, therefore we abandon the old proof and switch to the current version of Shapiro's Lemma which also applies to infinite dimensional σ.

8.3.2 Applying Shapiro's Lemma

We need to study the $M_\emptyset$-structure on the annihilators $(I'_w)^{[\mathfrak{n}_\emptyset{}^k]}$, and we have shown it is linearly isomorphic to

$$H^0(\mathfrak{n}_\emptyset, (I_w \otimes F_k)') \simeq H^0(\mathfrak{n}_\emptyset, [\mathcal{S}\mathrm{Ind}_{N_\emptyset \cap w^{-1}Pw}^{N_\emptyset}(\sigma^w \otimes \eta_k)]').$$

Since the $N_\emptyset$ and its subgroup $N_\emptyset \cap w^{-1}Pw$ are cohomologically trivial, (i.e. they are isomorphic to Euclidean spaces by the exponential maps), we have the isomorphisms of functors

$$H^0(N_\emptyset, -) = H^0(\mathfrak{n}_\emptyset, -)$$
$$H^0(N_\emptyset \cap w^{-1}Pw, -) = H^0(\mathfrak{n}_\emptyset \cap w^{-1}\mathfrak{p}w, -)$$

In particular, we have

$$H^0(\mathfrak{n}_\emptyset, [\mathcal{S}\mathrm{Ind}_{N_\emptyset \cap w^{-1}Pw}^{N_\emptyset}(\sigma^w \otimes \eta_k)]') = H^0(N_\emptyset, [\mathcal{S}\mathrm{Ind}_{N_\emptyset \cap w^{-1}Pw}^{N_\emptyset}(\sigma^w \otimes \eta_k)]')$$

Combining this with Shapiro's Lemma (Theorem 8.23):

$$H^0(N_\emptyset, [\mathcal{S}\mathrm{Ind}_{N_\emptyset \cap w^{-1}Pw}^{N_\emptyset}(\sigma^w \otimes \eta_k)]') \simeq H^0(N_\emptyset \cap w^{-1}Pw, (V_{\sigma^w} \otimes F_k)'),$$

(Here the V_{σ^w} means the V with the group action through the σ^w) we have

$$\begin{aligned}
(I'_w)^{[\mathfrak{n}_\emptyset{}^k]} &\simeq H^0(\mathfrak{n}_\emptyset, (I_w \otimes F_k)') \\
&\simeq H^0(\mathfrak{n}_\emptyset, [\mathcal{S}\mathrm{Ind}_{N_\emptyset \cap w^{-1}Pw}^{N_\emptyset}(\sigma^w \otimes \eta_k)]') \\
&= H^0(N_\emptyset, [\mathcal{S}\mathrm{Ind}_{N_\emptyset \cap w^{-1}Pw}^{N_\emptyset}(\sigma^w \otimes \eta_k)]') \\
&\simeq H^0(N_\emptyset \cap w^{-1}Pw, (V_{\sigma^w} \otimes F_k)')
\end{aligned}$$

In sum, we can describe the elements in the annihilators $(I'_w)^{[\mathfrak{n}_\emptyset{}^k]}$ by the cohomologies

$$H^0(N_\emptyset \cap w^{-1}Pw, (V_{\sigma^w} \otimes F_k)').$$

And the above sequence of isomorphisms also transplant the natural $M_\emptyset$-action on $(I'_w)^{[\mathfrak{n}_\emptyset{}^k]}$ onto $H^0(N_\emptyset \cap w^{-1}Pw, (V_{\sigma^w} \otimes F_k)')$. The last step is to describe the $M_\emptyset$-action on $H^0(N_\emptyset \cap w^{-1}Pw, (V_{\sigma^w} \otimes F_k)')$ in terms of $(V_{\sigma^w} \otimes F_k)'$.

Remark 8.26. The $(V_{\sigma^w} \otimes F_k)'$ is the dual of $V_{\sigma^w} \otimes F_k$, which is a tensor product of two $M_\emptyset$-representations: the $M_\emptyset$ acts on V_{σ^w} through σ^w (since $M_\emptyset \subset w^{-1}Pw$), and on the F_k through η_k. *However we will see the $M_\emptyset$-action obtained from $(I'_w)^{[\mathfrak{n}_\emptyset{}^k]}$ through the above isomorphisms, is NOT the dual of $\sigma^w \otimes \eta_k$.*

The $M_\emptyset$-Action From $(I'_w)^{[\mathfrak{n}_\emptyset{}^k]}$

We have the shown two isomorphisms to $H^0(N_\emptyset, [\mathcal{S}\mathrm{Ind}_{N_\emptyset \cap w^{-1}Pw}^{N_\emptyset}(\sigma^w \otimes \eta_k)]')$:

$$(I'_w)^{[\mathfrak{n}_\emptyset{}^k]} \xrightarrow{\sim} H^0(N_\emptyset, [\mathcal{S}\mathrm{Ind}_{N_\emptyset \cap w^{-1}Pw}^{N_\emptyset}(\sigma^w \otimes \eta_k)]')$$
$$H^0(N_\emptyset \cap w^{-1}Pw, (V_{\sigma^w} \otimes F_k)') \xrightarrow{\sim} H^0(N_\emptyset, [\mathcal{S}\mathrm{Ind}_{N_\emptyset \cap w^{-1}Pw}^{N_\emptyset}(\sigma^w \otimes \eta_k)]')$$

Therefore the $M_\emptyset$-structure on $(I'_w)^{[\mathfrak{n}_\emptyset{}^k]}$ could be described by the $M_\emptyset$-structure on $(V_{\sigma^w} \otimes F_k)'$. Unfortunately, the inverse of the isomorphism (8.21) is not explicit. Thus to compare the $M_\emptyset$-actions on $(I'_w)^{[\mathfrak{n}_\emptyset{}^k]}$ and $H^0(N_\emptyset \cap w^{-1}Pw, (V_{\sigma^w} \otimes F_k)')$, we have to send these two $M_\emptyset$-actions to $H^0(N_\emptyset, [\mathcal{S}\mathrm{Ind}_{N_\emptyset \cap w^{-1}Pw}^{N_\emptyset}(\sigma^w \otimes \eta_k)]')$ and compare on this space.

First we look at the $M_\emptyset$-action from $(V_{\sigma^w} \otimes F_k)'$. Let $\lambda \in H^0(N_\emptyset \cap w^{-1}Pw, (V_{\sigma^w} \otimes F_k)') \subset (V_{\sigma^w} \otimes F_k)'$, and $m \in M_\emptyset$ be arbitrary element. Let $\Upsilon_{m\cdot\lambda} \in H^0(N_\emptyset, [\mathcal{S}\mathrm{Ind}_{N_\emptyset \cap w^{-1}Pw}^{N_\emptyset}(\sigma^w \otimes \eta_k)]')$ be the integration distribution introduced in Lemma 8.22, then for all $\Phi \in \mathcal{S}\mathrm{Ind}_{N_\emptyset \cap w^{-1}Pw}^{N_\emptyset}(\sigma^w \otimes \eta_k)$, we have

$$\begin{aligned}\langle \Upsilon_{m\cdot\lambda}, \Phi\rangle &= \int_{N_\emptyset \cap w^{-1}Pw \backslash N_\emptyset} \langle m\cdot\lambda, \Phi(n)\rangle dx_n \\ &= \int_{N_\emptyset \cap w^{-1}Pw \backslash N_\emptyset} \langle \lambda, [(\sigma^w \otimes \eta_k)(m^{-1})]\cdot\Phi(n)\rangle dx_n\end{aligned}$$

Second we look at the $M_\emptyset$-action from $(I'_w)^{[\mathfrak{n}_\emptyset{}^k]}$, we have $(\lambda, m, \Upsilon_\lambda, \Phi$ as above)

$$\begin{aligned}\langle m\cdot\Upsilon_\lambda, \Phi\rangle &= \langle \Upsilon, m^{-1}\cdot\Phi\rangle \\ &= \int_{N_\emptyset \cap w^{-1}Pw \backslash N_\emptyset} \langle \lambda, [(\sigma^w \otimes \eta_k)(m^{-1})]\cdot\Phi(mnm^{-1})\rangle dx_n\end{aligned}$$

Let $n' = mnm^{-1}$, then we have

$$\begin{aligned}\langle m\cdot\Upsilon_\lambda, \Phi\rangle &= \int_{N_\emptyset \cap w^{-1}Pw \backslash N_\emptyset} \langle \lambda, [(\sigma^w \otimes \eta_k)(m^{-1})]\cdot\Phi(n')\rangle dx_{m^{-1}n'm} \\ &= \gamma_w(m)\int_{N_\emptyset \cap w^{-1}Pw \backslash N_\emptyset} \langle \lambda, [(\sigma^w \otimes \eta_k)(m^{-1})]\cdot\Phi(n')\rangle dx_{n'} \\ &= \gamma_w(m)\langle \Upsilon_{m\cdot\lambda}, \Phi\rangle\end{aligned}$$

where the γ_w is a character of $M_\emptyset$ given by

$$\gamma_w(m) = \det(\mathrm{Ad}_{\mathfrak{n}_\emptyset \cap w^{-1}\mathfrak{p}w \backslash \mathfrak{n}_\emptyset}(m^{-1})), \quad \forall m \in M_\emptyset. \tag{8.22}$$

In particular, the γ_w restricted to the $A_\emptyset$ is given by the following character of $A_\emptyset$:

$$\delta_w(a) = \prod_{\substack{\alpha \in \Sigma^+ \\ \alpha \in w^{-1}(\Sigma^- - \Sigma^-_\Theta)}} \alpha(a)^{m_\alpha}, \quad \forall a \in A_\emptyset, \tag{8.23}$$

where m_α is the multiplicity of α (Note that the roots are restricted roots, with multiplicities).

Now we can see the difference between two $M_\emptyset$-actions on $H^0(N_\emptyset \cap w^{-1}Pw, (V_{\sigma^w} \otimes F_k)')$: the $M_\emptyset$-action on $H^0(N_\emptyset, [\mathcal{S}\mathrm{Ind}^{N_\emptyset}_{N_\emptyset \cap w^{-1}Pw}(\sigma^w \otimes \eta_k)]')$ is expressed by the $M_\emptyset$-action on $(V_{\sigma^w} \otimes F_k)'$ twisted by γ_w.

Lemma 8.27. *The $M_\emptyset$ acts on $(I'_w)^{[\mathfrak{n}_\emptyset{}^k]}$, through $\widehat{\sigma^w \otimes \eta_k} \otimes \gamma_w$, where γ_w is the character in (8.22).*

Chapter 9

Application: Finite Dimensional (σ, V)

Summary of This Chapter

In this chapter, we combine tools developed in previous chapters, to reproduce irreducibility theorems (Theorem 7.2a on page 193 and Theorem 7.4 on page 203) in [15].

Setting and notations

For groups, we keep the notations as previous chapters. In particular, for the parabolic subgroup $P = M_P N_P$, we also write it as $P_\Theta = M_\Theta N_\Theta$ when we need to emphasize the subset Θ. In this chapter, let

$$(\tau, V) = \text{a irreducible unitary representation of } M_P$$

and we extend it trivially to the entire P. The τ is obviously a Harish-Chandra representation, in particular the V is nuclear.

Let

$$\sigma = \tau \otimes \delta_P^{1/2}$$

be the representation of P on V obtained by twisting the τ by $\delta_P^{1/2}$. Then the normalized unitary (Hilbert) induction of $\mathrm{Ind}_P^G \tau$ is infinitesimally equivalent to:

$$I = C^\infty \mathrm{Ind}_P^G \sigma = \mathcal{S}\mathrm{Ind}_P^G \sigma.$$

We will apply the results in Chapter 5, 6, 7 and 8 to the $\sigma = \tau \otimes \delta_P^{1/2}$.

9.1 Degenerate Principal Series

In this section, we prove an analogue results (Theorem 9.6) of Bruhat's Theorem 7.4 in [15], on the irreducibilities of degenerate principal series.

- In 9.1.1, we show the restriction of τ to the $A_\emptyset$ is a direct sum of $A_\emptyset$-characters that are W_Θ-conjugate to each other.
- In 9.1.2, we formulate our analogous version of Bruhat's Theorem 7.4, and give an outline of the proof.
- In 9.1.3, we complete the first step of the proof, by showing

$$\mathrm{Hom}_P(V', (I/I_{>e})') = \mathbb{C}.$$

  The Theorem 9.8 also tells us the interesting phenomenon occurs on non-identity double cosets when the I is reducible.

The second step to prove the Theorem 9.6 is completed in the next section.

9.1.1 $A_\emptyset$-Spectrum on (τ, V)

In this subsection, let (τ, V) be a *finite dimensional* irreducible unitary representation of M_P. We follow the [15] to describe the $A_\emptyset$-spectrum on the representation (τ, V).

The main result in this subsection is Lemma 9.2, which says the restriction of τ to the $A_\emptyset$ splits into a direct sum of unitary characters of $A_\emptyset$, moreover these $A_\emptyset$-characters are all W_Θ-conjugate to each other.

For the parabolic subgroup $P = P_\Theta = M_\Theta N_\Theta$, let $P_\Theta = {}^\circ M_\Theta A_\Theta N_\Theta$ be its Langlands decomposition, i.e. N_Θ is the unipotent radical, A_Θ is the split component, and ${}^\circ M_\Theta$ is the intersection of kernels of real characters on M_Θ. The Levi component is the direct product of ${}^\circ M_\Theta$ and A_Θ:

$$M_\Theta = {}^\circ M_\Theta \times A_\Theta,$$

and A_Θ is exactly the center of M_Θ.

For the irreducible unitary representation (τ, V), we know it is trivial on N_Θ. Let

$$\begin{aligned} {}^\circ\tau &= \text{the restriction of } \tau \text{ to } {}^\circ M_\Theta \\ \chi_c &= \text{the restriction of } \tau \text{ to } A_\Theta \end{aligned}$$

Then the ${}^\circ\tau$ is an *irreducible* unitary representation of ${}^\circ M_\Theta$ on V, the χ_c is a unitary character of A_Θ (called the **restricted character of** τ), and the τ is written as

$$\tau = {}^\circ\tau \otimes \chi_c \otimes 1. \tag{9.1}$$

The Restriction of τ to ${}^\circ M_\Theta{}^0$

Let ${}^\circ M_\Theta{}^0$ be the identity component of ${}^\circ M_\Theta$, and let

$$ {}^\circ\tau^0 = \text{the restriction of } \tau \text{ to } {}^\circ M_\Theta{}^0. $$

In general, the ${}^\circ\tau^0$ is a unitary representation of ${}^\circ M_\Theta{}^0$, but it may not be irreducible.

Since the ${}^\circ\tau^0$ is a finite dimensional unitary representation of ${}^\circ M_\Theta{}^0$, it decomposes into a direct sum of finite dimensional irreducible unitary representations of ${}^\circ M_\Theta{}^0$. The ${}^\circ M_\Theta{}^0$-irreducible constituents on V are ${}^\circ M_\Theta$-conjugate to each other.

Actually, we just need to apply the following Lemma to $G = {}^\circ M_\Theta, H = {}^\circ M_\Theta{}^0$:

Lemma 9.1. *Let G be a Lie group, H be a normal subgroup of G with finite index. Let (π, V) be a finite dimensional irreducible unitary representation of G. Then the restriction $(\pi|_H, V)$ is a finite direct sum of irreducible unitary representations of H, and the H-irreducible constituents are G-conjugate to each other.*

Proof. Since V is finite dimensional, one can find a minimal nonzero H-invariant subspace of V, denoted by V_0. Then V_0 is obviously a irreducible representation of H. Let

$$ G(V_0) = \{g \in G : \pi(g)V_0 \subset V_0\}. $$

Then the $G(V_0)$ is a subgroup of G containing H, and H is also normal in $G(V_0)$.

Let $\{g_0 = e, g_1, \ldots, g_l\}$ be a set of representatives of the right cosets in $G/G(V_0)$. Then obviously each $\pi(g_i)V_0$ is a H-invariant subspace of V (of the same dimension as V_0) since H is normal, and they are irreducible representations of H.

It is easy to see: for $i \neq j$, either $\pi(g_i)V_0 \cap \pi(g_j)V_0 = \{0\}$ or $\pi(g_i)V_0 = \pi(g_j)V_0$. The latter cannot happen, otherwise $g_i^{-1}g_j \in G^\pi$ contradicting to $i \neq j$. Hence one has a direct sum

$$ \bigoplus_{i=0}^{l} \pi(g_i)V_0, $$

which is G-invariant subspace of V, hence equals to the entire V since V is G-irreducible. $\square$

By applying the above Lemma, we may denote the irreducible constituents of $({}^\circ\tau|_{{}^\circ M_\Theta{}^0}, V)$ by $(\rho_0, V_0), \ldots, (\rho_l, V_l)$:

$$ {}^\circ\tau^0 = {}^\circ\tau|_{{}^\circ M_\Theta{}^0} = \bigoplus_{i=0}^{l} (\rho_i, V_i). $$

As above, let ${}^\circ M_\Theta(V_0)$ be the subgroup of ${}^\circ M_\Theta$ fixing the subspace V_0. And for each $i \neq 0$, there is an element $g_i \in {}^\circ M_\Theta$ representing its ${}^\circ M_\Theta(V_0)$-coset in ${}^\circ M_\Theta$, such that $\rho_i = \rho_0 \circ \mathrm{Ad} g_i$.

The $A_\emptyset$-Spectrum on (τ, V)

As in [15], the restriction of the ${}^\circ M_\Theta{}^0$-irreducible constituent (ρ_0, V_0) to the subgroup ${}^\circ M_\Theta \cap A_\emptyset$ is a unitary character of ${}^\circ M_\Theta \cap A_\emptyset$, denoted by χ_0. Similarly, for each $i = 1, \ldots, l$, the restriction of ρ_i to ${}^\circ M_\Theta \cap A_\emptyset$ is a unitary character χ_i, we thus have $l + 1$-unitary characters of ${}^\circ M_\Theta \cap A_\emptyset$: $\chi_0, \chi_1, \ldots, \chi_l$.

We have seen the ρ_i are all ${}^\circ M_\Theta$-conjugate to each other, i.e. for each $i = 1, \ldots, l$ there is a $g_i \in {}^\circ M_\Theta$ such that $\rho_i = \rho_0 \circ \mathrm{Ad} g_i$. Therefore, for the characters χ_i, one also has

$$ \chi_i = \chi_0 \circ \mathrm{Ad} g_i, i = 0, 1, \ldots, l. $$

Moreover, one can choose the g_i from the normalizer $N_G(A_\emptyset)$, and let w_i be its image in the Weyl group W. One can see the w_i is actually in W_Θ since it centralizes the A_Θ. Therefore, we see the characters χ_i are all W_Θ-conjugate to each other:

$$ \chi_i = \chi_0 \circ \mathrm{Ad} w_i, \quad i = 0, \ldots, l. $$

Since the $A_\emptyset$ is the direct product of A_Θ and ${}^\circ M_\Theta \cap A_\emptyset$:

$$ A_\emptyset = ({}^\circ M_\Theta \cap A_\emptyset) \times A_\Theta, $$

each χ_i combined with χ_c gives a character of $A_\emptyset$. We denote it by

$$ \chi^i = \chi_i \cdot \chi_c. $$

It is easy to see χ^i is exactly the w_i-conjugation of χ^0:

$$ \chi^i = \chi^0 \circ \mathrm{Ad} w_i, \quad i = 0, 1, \ldots, l. $$

(Note that all $w_i \in W_\Theta$ fix the subgroups A_Θ and χ_c.) In sum, we have the following description of the $A_\emptyset$-spectrum on (τ, V):

Lemma 9.2. *The restriction of τ to the $A_\emptyset$ is a direct sum of unitary characters (with the same multiplicities). Let*

$$\mathrm{Spec}(A_\emptyset, \tau) = \{\chi^0, \chi^1, \dots, \chi^l\}$$

be the set of unitary characters of $A_\emptyset$ occurring on (τ, V), i.e.

$$\tau|_{A_\emptyset} = \bigoplus_{\chi \in \mathrm{Spec}(A_\emptyset,\tau)} d\chi = \bigoplus_{i=0}^{l} d\chi^i,$$

where d is the dimension of the irreducible ${}^\circ M_\Theta{}^0$-representation ρ_0. Then the χ^i are conjugate to each other by elements in W_Θ. More precisely, there is a element $w_i \in W_\Theta$ (which may not be unique), such that $\chi^i = \chi^0 \circ \mathrm{Ad}w_i$ (and one can choose w_0 to be identity).

Remark 9.3. The explicit description of the w_i depends on the concrete representation τ, and there is no uniform way to describe them.

9.1.2 Formulating an Analogue of Bruhat's Theorem 7.4

We formulate the Theorem 7.4 in Bruhat's [15]. We keep the setting as in the last subsection, and still assume (τ, V) to be finite dimensional, irreducible and unitary. Let $\mathrm{Spec}(A_\emptyset, \tau)$ be the finite set of $A_\emptyset$-characters occurring in τ as in Lemma 9.2.

Definition 9.4. A character $\chi \in \mathrm{Spec}(A_\emptyset, \tau)$ is called **regular** if

$$\chi \neq \chi \circ \mathrm{Ad}w$$

for all $w \in W - W_\Theta$, i.e. the w-conjugation of χ is not identically equal to χ for all $w \notin W_\Theta$.

If one (hence all) $A_\emptyset$-characters in $\mathrm{Spec}(A_\emptyset, \tau)$ is regular, we say the representation τ is **regular**.

Remark 9.5. If one character in $\mathrm{Spec}(A_\emptyset, \tau)$ is regular in the above sense, then every character in $\mathrm{Spec}(A_\emptyset, \tau)$ are regular. Actually, let $\chi \in \mathrm{Spec}(A_\emptyset, \tau)$ be a regular character, then the other characters in $\mathrm{Spec}(A_\emptyset, \tau)$ are of the form $\chi \circ \mathrm{Ad}s^{-1}$ for some $s \in W_\Theta$. If the $\chi \circ \mathrm{Ad}s^{-1}$ is not regular, then there is a $w \in W - W_\Theta$ such that $\chi \circ \mathrm{Ad}s^{-1} \circ \mathrm{Ad}w = \chi \circ \mathrm{Ad}s^{-1}$. Hence $\chi \circ \mathrm{Ad}(s^{-1}ws) = \chi$. But $s^{-1}ws$ is not in W_Θ (otherwise $w \in W_\Theta$ a contradiction), this contradicts the assumption that χ is regular.

Now we state the theorem in Bruhat's thesis, which gives a sufficient condition of irreducibility of unitary parabolic induction of finite dimensional representations:

Theorem 9.6 (Bruhat [15] Theorem 7.4)**.** *Let (τ, V) be a irreducible finite dimensional unitary representation of P_Θ (therefore of M_Θ), and let* $\mathrm{Spec}(A_\emptyset, \tau)$ *be the finite set of $A_\emptyset$-character occurring on τ. If one (hence all) character in* $\mathrm{Spec}(A_\emptyset, \tau)$ *is regular in the sense of Definition 9.4, then the normalized parabolic induction* $\mathrm{Ind}_P^G \tau$ *is irreducible.*

Outline of the Proof of Theorem 9.6

By the Remark 5.22, we just need to show the Schwartz induction (smooth induction) I satisfies $\mathrm{Hom}_G(I, I) = \mathbb{C}$. By the Remark 5.26, we will show the irreducibility in by the following two steps:

1. show the $\mathrm{Hom}_P(V', (I/I_{>e})') = \mathbb{C}$.
2. *find a subset $\Omega \subset \Theta$ such that for all $w \in [W_\Theta \backslash W / W_\Omega], w \neq e$,* the $\mathrm{Hom}_{P_\Omega}(V', (I^\Omega_{\geq w}/I^\Omega_{>w})') = \{0\}$.

As in Remark 5.26, these two steps correspond to the two cases that $\mathrm{supp} D \subset P$ or $\mathrm{supp} D \nsubseteq P$. The first step is done in the next subsection 9.1.3. For the second step, we will choose $\Omega = \emptyset$ and show the Hom spaces are zero in section 9.2.

9.1.3 The Space $\mathrm{Hom}_P(V', (I/I_{>e})')$

In this subsection, *we only assume (τ, V) to be irreducible unitary, without assuming it to be finite dimensional.*

By Lemma 5.24, the space of intertwining distributions with supports contained in P is linearly isomorphic to the following space

$$\mathrm{Hom}_P(V', (I/I_{>e})').$$

By the Theorem 6.1 in Chapter 6, we have the following isomorphism

$$(I/I_{>e})' \simeq I'_e \otimes U(\mathfrak{n}_e^-)$$

(Note that for $w = e$, the transverse subalgebra $\mathfrak{n}_w^-$ is exactly $\overline{\mathfrak{n}}_P$.) Also by Lemma 4.93, we see the $I_e = \mathcal{S}\mathrm{Ind}_P^P \sigma$ is exactly the P-representation (σ, V). Therefore, we have

$$(I/I_{>e})' \simeq V' \otimes U(\overline{\mathfrak{n}}_P) \tag{9.2}$$

(The $(I/I_{>e})'$ is isomorphic to a generalized Verma module.)

Therefore we have the following inclusion

$$\begin{aligned}\mathrm{Hom}_P(V', (I/I_{>e})') &\simeq \mathrm{Hom}_P(V', V' \otimes U(\overline{\mathfrak{n}}_P)) \\ &\subset \mathrm{Hom}_M(V', V' \otimes U(\overline{\mathfrak{n}}_P))\end{aligned}$$

where the $V' \otimes U(\overline{\mathfrak{n}}_P)$ has the tensor product M-action by the same argument as in Lemma 8.1. More precisely, we have

$$\begin{aligned}\mathrm{Hom}_P(V', (I/I_{>e})') &\subset \mathrm{Hom}_M(\widehat{\sigma}, \widehat{\sigma} \otimes U(\overline{\mathfrak{n}}_P)) \\ &= \mathrm{Hom}_M(\widehat{\tau} \otimes \delta_P^{-1/2}, \widehat{\tau} \otimes \delta_P^{-1/2} \otimes U(\overline{\mathfrak{n}}_P))\end{aligned}$$

(Here the "hat" means dual representation.)

Note that a $M = M_\Theta$-equivariant map from $\widehat{\tau} \otimes \delta_P^{-1/2}$ to $\widehat{\tau} \otimes \delta_P^{-1/2} \otimes U(\overline{\mathfrak{n}}_P)$ has to be A_Θ-equivariant. The τ restricted to A_Θ is a single unitary character χ_c of A_Θ (the restricted character discussion in the last subsection), while the other two components $\delta_P^{-1/2}$ and $U(\overline{\mathfrak{n}}_P)$ are real rational representations. Hence the image of a A_Θ-equivariant map from $\widehat{\tau} \otimes \delta_P^{-1/2}$ to $\widehat{\tau} \otimes \delta_P^{-1/2} \otimes U(\overline{\mathfrak{n}}_P)$ must send the $\widehat{\tau}$ to $\widehat{\tau}$. Therefore we have

$$\begin{aligned}&\mathrm{Hom}_M(\widehat{\tau} \otimes \delta_P^{-1/2}, \widehat{\tau} \otimes \delta_P^{-1/2} \otimes U(\overline{\mathfrak{n}}_P)) \\ &\subset \mathrm{Hom}_M(\widehat{\tau}, \widehat{\tau}) \\ &= \mathbb{C}\end{aligned}$$

Remark 9.7. In the above argument, we don't need the $\sigma = \tau \otimes \delta_P^{1/2}$ to be finite dimensional. We only need the τ to be irreducible unitary, since all such representations have restricted A_Θ-characters.

To summarize the above discussion, we see the $\mathrm{Hom}_P(V', (I/I_{>e})') = \mathbb{C}$ when V is an irreducible unitary representation. We have the following theorem:

Theorem 9.8. *Let (τ, V) be an irreducible unitary representation of M (and P by trivial extension), then the intertwining distributions with support contained in P are exactly the scalar intertwining distributions.*

Remark 9.9. This theorem tells us: even when the parabolic induction is reducible, *its non-scalar intertwining distributions are not supported in P.*

This theorem has the following easy corollary

Corollary 9.10. *Let (τ, V) be an irreducible unitary representation of M (and P by trivial extension), then the normalized Schwartz induction $I = \mathcal{S}\mathrm{Ind}_P^G(\tau \otimes \delta_P^{1/2})$ is irreducible, if and only if all intertwining distributions have their supports contained in P. Equivalently, the I is irreducible if and only if there is no intertwining distribution with its support containing a double coset other than P.*

9.2 The Space $\mathrm{Hom}_{P_\emptyset}(V', (I_{\geq w}/I_{>w})')$

In this section, we complete the proof of the Theorem 9.6. *We still assume (τ, V) to be a finite dimensional and irreducible unitary* representation.

We choose the $\Omega = \emptyset$, and show the following Hom spaces are zero:

$$\mathrm{Hom}_{P_\emptyset}(V', (I_{\geq w}/I_{>w})') = \{0\}, \quad \text{for all } w \neq e, w \in [W_\Theta \backslash W].$$

In a word, we prove these spaces are zero, by comparing the $A_\emptyset$-Spectrum on V' and $[(I_{\geq w}/I_{>w})']^{[\mathfrak{n}_\emptyset{}^\bullet]}$.

9.2.1 The Reason to Study $\mathfrak{n}_\emptyset$-Torsion Subspaces

The condition "(τ, V) is finite dimensional" is a very strong condition and it largely simplify the proof of irreducibility of $\mathrm{Ind}_P^G\tau$. More precisely, the smooth representation V is $\mathfrak{n}_P$-trivial, and is $\mathfrak{m}_P \cap \mathfrak{n}_\emptyset$-torsion since it is finite dimensional. Therefore the V is torsion as a $\mathfrak{n}_\emptyset$-module:

$$V^{[\mathfrak{n}_\emptyset{}^\bullet]} = V,$$

and so is its dual V': $(V')^{[\mathfrak{n}_\emptyset{}^\bullet]} = V'$.

For an arbitrary $\Phi \in \mathrm{Hom}_{P_\emptyset}(V', (I_{\geq w}/I_{>w})')$, since it is $P_\emptyset$-equivariant, it is also $\mathfrak{n}_\emptyset$-equivariant. In particular, it maps the $V' = (V')^{[\mathfrak{n}_\emptyset{}^\bullet]}$ to the $\mathfrak{n}_\emptyset$-torsion subspace of $(I_{\geq w}/I_{>w})'$:

$$\Phi \in \mathrm{Hom}_{\mathbb{C}}(V', [(I_{\geq w}/I_{>w})']^{[\mathfrak{n}_\emptyset{}^\bullet]}).$$

Moreover, since the $\mathfrak{n}_\emptyset$-torsion subspace is $M_\emptyset$-stable, and the Φ is a $M_\emptyset$-equivariant map from V' to $[(I_{\geq w}/I_{>w})']^{[\mathfrak{n}_\emptyset{}^\bullet]}$. In sum, we have the following Lemma

Lemma 9.11. *If (τ, V) is a finite dimensional representation, then*

$$\mathrm{Hom}_{P_\emptyset}(V', (I_{\geq w}/I_{>w})') = \mathrm{Hom}_{M_\emptyset}(V', [(I_{\geq w}/I_{>w})']^{[\mathfrak{n}_\emptyset{}^\bullet]}). \tag{9.3}$$

By this lemma, we only need to show the right-hand-side is zero, by showing the $A_\emptyset$-spectrum on $[(I_{\geq w}/I_{>w})']^{[\mathfrak{n}_\emptyset{}^\bullet]}$ is disjoint from the $A_\emptyset$-spectrum on $(\widehat{\tau} \otimes \delta_P^{-1/2}, V')$. The Chapter 6, 7 and 8 are devoted to the study of $A_\emptyset$-spectrum on $[(I_{\geq w}/I_{>w})']^{[\mathfrak{n}_\emptyset{}^\bullet]}$:

- In Chapter 6, we have shown

$$(I_{\geq w}/I_{>w})' \simeq I'_w \otimes U(\mathfrak{n}_w^-), \text{ as } M_\emptyset\text{-spaces.}$$

- In Chapter 7, we have shown (under the above isomorphism)

$$[I'_w \otimes U(\mathfrak{n}_w^-)]^{[\mathfrak{n}_\emptyset{}^\bullet]} = [I'_w]^{[\mathfrak{n}_\emptyset{}^\bullet]} \otimes U(\mathfrak{n}_w^-).$$

- In Chapter 8, we use Shapiro's Lemma to compute the $M_\emptyset$-structure (hence also the $A_\emptyset$-spectrum) on $[I'_w]^{[\mathfrak{n}_\emptyset{}^\bullet]}$.

9.2.2 Comparison of the $A_\emptyset$-Spectrums

We compare the $A_\emptyset$-Spectrums on V' and $[(I_{\geq w}/I_{>w})']^{[\mathfrak{n}_\emptyset{}^\bullet]}$. Given an arbitrary $\Phi \in \mathrm{Hom}_{M_\emptyset}(V', [(I_{\geq w}/I_{>w})']^{[\mathfrak{n}_\emptyset{}^\bullet]})$, by the Lemma 9.11 and the main theorems in Chapter 6, 7, the Φ is reduced to a $M_\emptyset$-equivariant map

$$\Phi : V' \to (I'_w)^{[\mathfrak{n}_\emptyset{}^\bullet]} \otimes U(\mathfrak{n}_w^-).$$

Since the V' is finite dimensional, the image of Φ is thus contained in

$$(I'_w)^{[\mathfrak{n}_\emptyset{}^k]} \otimes U_n(\mathfrak{n}_w^-)$$

for some $k > 0$ and $n \geq 0$ (large enough). To show the

$$\mathrm{Hom}_{M_\emptyset}(V', [(I_{\geq w}/I_{>w})']^{[\mathfrak{n}_\emptyset{}^\bullet]})$$

is $\{0\}$ or equivalently the above arbitrary Φ is zero, we just need to show

$$\mathrm{Hom}_{M_\emptyset}(V', (I'_w)^{[\mathfrak{n}_\emptyset{}^k]} \otimes U_n(\mathfrak{n}_w^-)) = \{0\}$$

for all $k > 0, n \geq 0$. This requires us to compare the $A_\emptyset$-spectrum on V' and $(I'_w)^{[\mathfrak{n}_\emptyset{}^k]} \otimes U_n(\mathfrak{n}_w^-)$.

The $A_\emptyset$-spectrum on V' is the following finite set of $A_\emptyset$-characters:

$$\mathrm{Spec}(A_\emptyset, \widehat{\tau \otimes \delta_P^{1/2}}) = \{\chi^{-1} \otimes \delta_P^{-1/2} : \chi \in \mathrm{Spec}(A_\emptyset, \tau)\} \tag{9.4}$$

We are left to study the $A_\emptyset$-spectrum on $(I'_w)^{[\mathfrak{n}_\emptyset{}^k]} \otimes U_n(\mathfrak{n}_w^-)$.

The $A_\emptyset$-Spectrum on $(I'_w)^{[\mathfrak{n}_\emptyset{}^k]} \otimes U_n(\mathfrak{n}_w^-)$

By Lemma 8.1, the tensor product $(I'_w)^{[\mathfrak{n}_\emptyset{}^k]} \otimes U_n(\mathfrak{n}_w^-)$ is actually a tensor product of $M_\emptyset$-representations hence also $A_\emptyset$-representations. We just need to find the $A_\emptyset$-spectrum on the annihilator $(I'_w)^{[\mathfrak{n}_\emptyset{}^k]}$.

The $M_\emptyset$-action on $(I'_w)^{[\mathfrak{n}_\emptyset{}^k]}$ is studied in Chapter 8 and summarized in Lemma 8.27, i.e. the $M_\emptyset$ acts on $(I'_w)^{[\mathfrak{n}_\emptyset{}^k]}$ by the representation

$$[(\tau \otimes \delta_P^{1/2})^w \otimes \eta_k]^\wedge \otimes \gamma = \widehat{\tau^w} \otimes \widehat{(\delta_P^{1/2})^w} \otimes \widehat{\eta_k} \otimes \gamma.$$

(Note that the $\sigma = \tau \otimes \delta_P^{1/2}$ in the concrete case.) In particular, by restricting the $M_\emptyset$-action to the $A_\emptyset$, we have the following description of the $A_\emptyset$-spectrum:

Lemma 9.12. *The $A_\emptyset$-spectrum on $(I'_w)^{[\mathfrak{n}_\emptyset{}^k]}$ consists of $A_\emptyset$-characters of the following form:*

$$(\chi^w)^{-1} \otimes (\delta_P^{-1/2})^w \otimes \delta_w \otimes \mu$$

where $\chi \in \mathrm{Spec}(A_\emptyset, \tau)$, δ_w is the $A_\emptyset$-character in (8.23), and μ is a $A_\emptyset$-character on the $\widehat{\eta_k}$, of the following form

$$\prod_{\alpha \in \Sigma^+} \alpha^{-k_\alpha} \tag{9.5}$$

where k_α are non-negative integers such that $\sum k_\alpha = j$ for some $j \leq k-1$ (The μ comes from the $A_\emptyset$-spectrum on η_k).

The $A_\emptyset$-spectrum on the transverse derivative space $U_n(\mathfrak{n}_w^-)$ consists of $A_\emptyset$-characters of the form

$$\prod_{\substack{\alpha \in \Sigma^- \\ \alpha \in w^{-1}(\Sigma^- - \Sigma_\Theta^-)}} \alpha^{l_\alpha} \tag{9.6}$$

where l_α are non-negative integers such that $\sum l_\alpha \leq n$.

In sum, we have:

Lemma 9.13. *The $A_\emptyset$-spectrum on the $(I'_w)^{[\mathfrak{n}_\emptyset{}^k]} \otimes U_n(\mathfrak{n}_w^-)$ consists of $A_\emptyset$-characters as follows:*

$$(\chi^w)^{-1} \otimes (\delta_P^{-1/2})^w \otimes \delta_w \otimes \mu \otimes \nu \tag{9.7}$$

where $\chi \in \mathrm{Spec}(A_\emptyset, \tau)$, δ_w is the $A_\emptyset$-character in (8.23), μ is the $A_\emptyset$-character as in (9.5) and ν is the $A_\emptyset$-character as in (9.6).

The Spectrum $\mathrm{Spec}(A_\emptyset, \tau^w)$ of Regular τ

Recall that in Definition 9.4, the τ is called regular, if any $A_\emptyset$-character occurring in τ is regular. Given the (τ, V) and an element $w \in W$, we denote by

$$\tau^w = \tau \circ \mathrm{Ad}w$$

the twisted representation of $w^{-1} M_P w$ on V. And similarly, for a character χ of $A_\emptyset$, we denote by χ^w the $A_\emptyset$-character $\chi \circ \mathrm{Ad}w$.

For a regular τ, we have the following lemma about the $A_\emptyset$-spectrum of τ^w for $w \notin W_\Theta$:

Lemma 9.14. *Suppose the finite dimensional irreducible unitary representation τ is regular, and $w \in W - W_\Theta$. Then the finite set $\mathrm{Spec}(A_\emptyset, \tau^w)$ is disjoint from $\mathrm{Spec}(A_\emptyset, \tau)$.*

Proof. Obviously, one has

$$\mathrm{Spec}(A_\emptyset, \tau^w) = \{\chi^w : \chi \in \mathrm{Spec}(A_\emptyset, \tau)\}.$$

Suppose a $\chi^w \in \mathrm{Spec}(A_\emptyset, \tau^w)$ is also in $\mathrm{Spec}(A_\emptyset, \tau)$, then there is a $s \in W_\Theta$ such that $\chi^w = \chi^s$. Then one has $\chi^{ws^{-1}} = \chi$. Since τ is regular, so is χ, this implies $ws^{-1} \in W_\Theta$, which further implies $w \in W_\Theta$, a contradiction. $\square$

Comparison of The $A_\emptyset$-Spectrum and Proof of Bruhat's Theorem 7.4

Since

$$\mathrm{Hom}_{M_\emptyset}(V', (I'_w)^{[\mathfrak{n}_\emptyset{}^k]} \otimes U_n(\mathfrak{n}_w^-)) \subset \mathrm{Hom}_{A_\emptyset}(V', (I'_w)^{[\mathfrak{n}_\emptyset{}^k]} \otimes U_n(\mathfrak{n}_w^-)),$$

to show the first $M_\emptyset$-Hom space is zero, we just need to show the second $A_\emptyset$-Hom space is zero. In (9.4) and (9.7), we have written down the $A_\emptyset$-spectrum on V' and $(I'_w)^{[\mathfrak{n}_\emptyset{}^k]} \otimes U_n(\mathfrak{n}_w^-)$. We just need to show every $A_\emptyset$-character occurring on V' cannot occur on $(I'_w)^{[\mathfrak{n}_\emptyset{}^k]} \otimes U_n(\mathfrak{n}_w^-)$. For simplicity, *we write the characters additively.*

We just need to show the $A_\emptyset$-character of the form $-\chi - \frac{1}{2}\delta_P$ and the $A_\emptyset$-character of the form $-w^{-1}\chi' - \frac{1}{2}w^{-1}\delta_P + \delta_w + \mu + \nu$ are never equal. (Here χ, χ' are $A_\emptyset$-characters in $\mathrm{Spec}(A_\emptyset, \tau)$ which may not be equal.)

Assume these two $A_\emptyset$-characters are equal:

$$-\chi - \frac{1}{2}\delta_P = -w^{-1}\chi' - \frac{1}{2}w^{-1}\delta_P + \delta_w + \mu + \nu.$$

Then we must have

$$\begin{aligned} \chi &= w^{-1}\chi' \\ -\frac{1}{2}\delta_P &= -\frac{1}{2}w^{-1}\delta_P + \delta_w + \mu + \nu \end{aligned} \tag{9.8}$$

This is because the χ and $w^{-1}\chi'$ are unitary characters with values on the unit circle, and all the other characters are real characters taking values in $\mathbb{R}_{>0}$. Therefore for the two $A_\emptyset$-characters to be equal, their pure imaginary and real parts have to be equal respectively.

Now since the w is a non-identity element in the representative set $[W_\Theta \backslash W]$, it is not in the subgroup W_Θ. If τ is regular as in Bruhat's theorem, by Lemma 9.14, we see the χ and $w^{-1}\chi'$ cannot be equal. Therefore every $A_\emptyset$-character occurring on $(\widehat{\tau \otimes \delta_P^{1/2}}, V')$ cannot occur in $(I'_w)^{[\mathfrak{n}_\emptyset{}^k]} \otimes U_n(\mathfrak{n}_w^-)$, and the proof is completed.

9.3 Minimal Principal Series

In this section, we prove our analogous result (Theorem 9.15) of Bruhat's Theorem 7.2a in [15]. In this section, we let

$$P = P_\emptyset$$

be the minimal parabolic subgroup (i.e. $\Theta = \emptyset$, then the only choice of Ω is the empty set). The representative set $[W_\Theta \backslash W]$ is exactly W, which is in one-to-one correspondence with the $(P_\emptyset, P_\emptyset)$-double cosets.

- In 9.3.1, we formulate and prove our analogue (Theorem 9.15) of the Theorem 7.2a in [15].
- In 9.3.2, we study the real parts of the $A_\emptyset$-spectrum on $[I'_w]^{[\mathfrak{n}_\emptyset{}^\bullet]} \otimes U(\mathfrak{n}_w^-)$. We will show the real parts of $A_\emptyset$-spectrum on V' and $[I'_w]^{[\mathfrak{n}_\emptyset{}^k]} \otimes U_n(\mathfrak{n}_w^-)$ agree only when $k = 1, n = 0$. Then for certain special cases, (e.g. split groups), we can actually write down the local intertwining distributions as integrations by Shapiro's Lemma.

9.3.1 An Analogue of Bruhat's Theorem 7.2a

As above, let $P = P_\emptyset$ be the minimal parabolic subgroup. In the Langlands decomposition ${}^\circ M_\emptyset A_\emptyset N_\emptyset$, the component ${}^\circ M_\emptyset$ is a compact group, hence the irreducible unitary representation (τ, V) of $M_\emptyset = {}^\circ M_\emptyset \times A_\emptyset$ has to be finite dimensional.

We now formulate the analogue of Theorem 7.2a in [15]:

Theorem 9.15. *Let $P = P_\emptyset$ be the minimal parabolic as above. If the representation $\tau^w = \tau \circ \mathrm{Ad}w$ is not equivalent to τ for all $w \in W, w \neq e$, then the induced representation $\mathrm{Ind}_{P_\emptyset}^G \tau$ is irreducible.*

Proof. Similar to the proof of Theorem 9.6, we just need to show

$$\mathrm{Hom}_{P_\emptyset}(V', (I_{\geq w}/I_{>w})') = \{0\}$$

for all $w \neq e$.

Since V' is finite dimensional, we have

$$\mathrm{Hom}_{P_\emptyset}(V', (I_{\geq w}/I_{>w})') = \mathrm{Hom}_{M_\emptyset}(V', [(I_{\geq w}/I_{>w})']^{[\mathfrak{n}_\emptyset{}^\bullet]})$$

i.e. the image of each $\Phi \in \mathrm{Hom}_{P_\emptyset}(V', (I_{\geq w}/I_{>w})')$ has its image in the torsion subspace. Combining the results in Chapter 6 and 7, we have

$$\mathrm{Hom}_{M_\emptyset}(V', [(I_{\geq w}/I_{>w})']^{[\mathfrak{n}_\emptyset{}^\bullet]}) = \mathrm{Hom}_{M_\emptyset}(V', [I'_w]^{[\mathfrak{n}_\emptyset{}^\bullet]} \otimes U(\mathfrak{n}_w^-)).$$

Again, since V' is finite dimensional, there exists $k > 0, n \geq 0$ (large enough), such that

$$\mathrm{Hom}_{M_\emptyset}(V', [I'_w]^{[\mathfrak{n}_\emptyset{}^\bullet]} \otimes U(\mathfrak{n}_w^-)) = \mathrm{Hom}_{M_\emptyset}(V', [I'_w]^{[\mathfrak{n}_\emptyset{}^k]} \otimes U_n(\mathfrak{n}_w^-)).$$

To summarize the above steps, we have

$$\begin{aligned}\mathrm{Hom}_{P_\emptyset}(V', (I_{\geq w}/I_{>w})') &= \mathrm{Hom}_{M_\emptyset}(V', [(I_{\geq w}/I_{>w})']^{[\mathfrak{n}_\emptyset{}^\bullet]}) \\ &= \mathrm{Hom}_{M_\emptyset}(V', [I'_w]^{[\mathfrak{n}_\emptyset{}^\bullet]} \otimes U(\mathfrak{n}_w^-)) \\ &= \mathrm{Hom}_{M_\emptyset}(V', [I'_w]^{[\mathfrak{n}_\emptyset{}^k]} \otimes U_n(\mathfrak{n}_w^-))\end{aligned}$$

We just need to show $\mathrm{Hom}_{M_\emptyset}(V', [I'_w]^{[\mathfrak{n}_\emptyset{}^k]} \otimes U_n(\mathfrak{n}_w^-)) = \{0\}$ by comparing the $M_\emptyset$-actions on V' and $[I'_w]^{[\mathfrak{n}_\emptyset{}^k]} \otimes U_n(\mathfrak{n}_w^-)$.

On the V', the $M_\emptyset$ acts by $\widehat{\tau} \otimes \delta_P^{-1/2}$, while on the $[I'_w]^{[\mathfrak{n}_\emptyset{}^k]} \otimes U_n(\mathfrak{n}_w^-)$, the $M_\emptyset$ acts by $\widehat{\tau^w} \otimes w^{-1}\delta_P^{-1/2} \otimes \widehat{\eta_k} \otimes \gamma_w \otimes U(\mathfrak{n}_w^-)$. Let $\Phi \in \mathrm{Hom}_{M_\emptyset}(V', [I'_w]^{[\mathfrak{n}_\emptyset{}^k]} \otimes U_n(\mathfrak{n}_w^-))$ be an arbitrary element. Since the $\widehat{\tau}$ is unitary, the image of Φ has to lay inside the $\widehat{\tau^w}$. This is because Φ is also $A_\emptyset$-equivariant, the only unitary $A_\emptyset$-eigensubspace in $[I'_w]^{[\mathfrak{n}_\emptyset{}^k]} \otimes U_n(\mathfrak{n}_w^-)$ is $\widehat{\tau^w}$. Therefore, the Φ is inside $\mathrm{Hom}_{M_\emptyset}(\widehat{\tau}, \widehat{\tau^w})$. By Schur's Lemma, this space is zero, if τ satisfies the condition $\tau^w \neq \tau$ for all $w \neq e$. Therefore the Φ has to be zero, and $\mathrm{Hom}_{M_\emptyset}(V', [I'_w]^{[\mathfrak{n}_\emptyset{}^k]} \otimes U_n(\mathfrak{n}_w^-)) = \{0\}$.

□

9.3.2 Real Parts of the $A_\emptyset$-Spectrum

In the proof of Bruhat's theorem 7.4, we show the $A_\emptyset$-characters that occurring on V' cannot occur on $(I'_w)^{[\mathfrak{n}_\emptyset{}^k]} \otimes U_n(\mathfrak{n}_w^-)$, by showing their *pure imaginary parts* are never equal on all the other double cosets other than P. Actually, the *real parts* of the $A_\emptyset$-characters occurring on $(I'_w)^{[\mathfrak{n}_\emptyset{}^k]} \otimes U_n(\mathfrak{n}_w^-)$ also give us interesting results. We will study the real parts of the $A_\emptyset$-spectrum on $(I'_w)^{[\mathfrak{n}_\emptyset{}^k]} \otimes U_n(\mathfrak{n}_w^-)$, for $P = P_\emptyset$ the minimal parabolic subgroup.

The real parts of the $A_\emptyset$-characters occurring on $(I'_w)^{[\mathfrak{n}_\emptyset{}^k]} \otimes U_n(\mathfrak{n}_w^-)$ are of the form:

$$-\frac{1}{2}w^{-1}\delta_P + \delta_w + \mu + \nu,$$

where δ_w, μ, ν are $A_\emptyset$-character as follows:

$$\begin{aligned}
\delta_w &= -\sum_{\substack{\alpha\in\Sigma^+ \\ \alpha\in w^{-1}(\Sigma^- - \Sigma_\Theta^-)}} m_\alpha\alpha \\
\mu &= -\sum_{\alpha\in\Sigma^+} k_\alpha\alpha \\
\nu &= \sum_{\substack{\alpha\in\Sigma^- \\ \alpha\in w^{-1}(\Sigma^- - \Sigma_\Theta^-)}} l_\alpha\alpha
\end{aligned}$$

Here all characters are written additively, m_α are the multiplicity of the (restricted) root α, k_α are non-negative integers such that $\sum k_\alpha \leq k-1$, l_α are non-negative integers such that $\sum l_\alpha \leq n$. One can see the δ_w, μ, ν are all integral combinations of negative roots.

The real parts of $A_\emptyset$-spectrum on V' are all equal to $-\frac{1}{2}\delta_P$, we may ask when does this equal to the real parts $-\frac{1}{2}w^{-1}\delta_P + \delta_w + \mu + \nu$.

Lemma 9.16. *For all $w \in W$, one has*

$$-\frac{1}{2}\delta_P = -\frac{1}{2}w^{-1}\delta_P + \delta_w. \tag{9.9}$$

Proof. This is elementary. Note that we have assumed $P = P_\emptyset$, therefore

$$\begin{aligned}
\delta_P &= \sum_{\alpha\in\Sigma^+} m_\alpha\alpha \\
\delta_w &= -\sum_{\substack{\alpha\in\Sigma^+ \\ \alpha\in w^{-1}\Sigma^-}} m_\alpha\alpha
\end{aligned}$$

(Note that the δ_w depends on w.)

First we decompose

$$\Sigma^+ = \{\alpha \in \Sigma^+ : w^{-1}\alpha \in \Sigma^+\} \coprod \{\alpha \in \Sigma^+ : w^{-1}\alpha \in \Sigma^-\}.$$

Then we have

$$\begin{aligned}
\frac{1}{2}w^{-1}\delta_P &= \frac{1}{2}w^{-1} \sum_{\substack{\alpha\in\Sigma^+ \\ w^{-1}\alpha\in\Sigma^+}} m_\alpha \alpha + \frac{1}{2}w^{-1} \sum_{\substack{\alpha\in\Sigma^+ \\ w^{-1}\alpha\in\Sigma^-}} m_\alpha \alpha \\
&= \frac{1}{2} \sum_{\substack{\alpha\in\Sigma^+ \\ w\alpha\in\Sigma^+}} m_\alpha \alpha + \frac{1}{2} \sum_{\substack{\alpha\in\Sigma^- \\ w\alpha\in\Sigma^+}} m_\alpha \alpha
\end{aligned}$$

Then

$$\begin{aligned}
\frac{1}{2}w^{-1}\delta_P - \frac{1}{2}\delta_P &= \frac{1}{2} \sum_{\substack{\alpha\in\Sigma^+ \\ w\alpha\in\Sigma^+}} m_\alpha \alpha + \frac{1}{2} \sum_{\substack{\alpha\in\Sigma^- \\ w\alpha\in\Sigma^+}} m_\alpha \alpha - \frac{1}{2} \sum_{\alpha\in\Sigma^+} m_\alpha \alpha \\
&= \frac{1}{2} \sum_{\substack{\alpha\in\Sigma^- \\ w\alpha\in\Sigma^+}} m_\alpha \alpha - \frac{1}{2} \sum_{\substack{\alpha\in\Sigma^+ \\ w\alpha\in\Sigma^-}} m_\alpha \alpha \\
&= \frac{1}{2} \sum_{\substack{\alpha\in\Sigma^- \\ \alpha\in w^{-1}\Sigma^+}} m_\alpha \alpha - \frac{1}{2} \sum_{\substack{\alpha\in\Sigma^+ \\ \alpha\in w^{-1}\Sigma^-}} m_\alpha \alpha \\
&= -\frac{1}{2} \sum_{\substack{\alpha\in\Sigma^+ \\ \alpha\in w^{-1}\Sigma^-}} m_\alpha \alpha - \frac{1}{2} \sum_{\substack{\alpha\in\Sigma^+ \\ \alpha\in w^{-1}\Sigma^-}} m_\alpha \alpha \\
&= - \sum_{\substack{\alpha\in\Sigma^+ \\ \alpha\in w^{-1}\Sigma^-}} m_\alpha \alpha \\
&= \delta_w
\end{aligned}$$

□

By this Lemma, we see if the real parts of $A_\emptyset$-characters occurring in V' and $(I'_w)^{[\mathfrak{n}_\emptyset{}^k]} \otimes U_n(\mathfrak{n}_w^-)$ are equal, i.e $-\frac{1}{2}\delta_P = -\frac{1}{2}w^{-1}\delta_P + \delta_w + \mu + \nu$, then we must have

$$\mu = 0, \nu = 0.$$

Equivalently, this could only happen when $k = 1, n = 0$. Recall that the μ comes from the $\widehat{\eta_k} = F_k^*$ which is only non-trivial on higher annihilators, and the ν comes from the transverse derivatives in $U(\mathfrak{n}_w^-)$. Then $\mu = 0, \nu = 0$ implies the distribution has neither higher annihilators nor transverse derivatives. This means the image of $\Phi \in \mathrm{Hom}_{M_\emptyset}(V', (I'_w)^{[\mathfrak{n}_\emptyset{}^k]} \otimes U_n(\mathfrak{n}_w^-))$ is inside $(I'_w)^{[\mathfrak{n}_\emptyset{}^1]}$.

Therefore, we have

$$\begin{aligned}\mathrm{Hom}_{P_\emptyset}(V', (I_{\geq w}/I_{>w})') &= \mathrm{Hom}_{M_\emptyset}(V', (I'_w)^{[\mathfrak{n}_\emptyset{}^1]}) \\ &= \mathrm{Hom}_{M_\emptyset}(V', H^0(\mathfrak{n}_\emptyset, I'_w)) \\ &= \mathrm{Hom}_{M_\emptyset}(\widehat{\tau} \otimes \delta_P^{-1/2}, \widehat{\tau^w} \otimes w^{-1}\delta_P^{-1/2} \otimes \gamma_w)\end{aligned} \tag{9.10}$$

(The last step is by Shapiro's Lemma.)

Although we don't have any immediate consequence, at least we know the local intertwining distributions do not have any transverse derivatives in their expression, and these local intertwining distributions are given by integrations on the quotient spaces $N_\emptyset \cap w^{-1}P_\emptyset w \backslash N_\emptyset$.

Chapter 10

General Results and Future Work

In this chapter, we sketch the proof of the general results (for arbitrary $\Omega \subset \Theta$), and discuss the topics we will study in the future.

10.1 Generalization of Chapter 6

We keep the setting as in the introductory subsection 1.1.3, e.g. the notations $\mathbf{G}, \mathbf{P} = \mathbf{P}_\Theta, G, P = P_\Theta, \tau, \sigma = \tau \otimes \delta_P^{1/2}, I = \mathcal{S}\mathrm{Ind}_P^G \sigma$ have the same meaning as there. For each subset Ω of Θ, and w in the set $[W_\Theta \backslash W / W_\Omega]$ of minimal representatives, we let

$$
\begin{aligned}
G_w^\Omega &= P w P_\Omega \\
G_{\geq w}^\Omega &= \coprod_{P w P_\Omega \subset \overline{P x P_\Omega}} P x P_\Omega \\
G_{>w}^\Omega &= G_{\geq w}^\Omega - G_w^\Omega
\end{aligned}
$$

and let $I_w^\Omega, I_{\geq w}^\Omega, I_{>w}^\Omega$ be their corresponding local Schwartz inductions.

10.1.1 Formulating the Theorem

As in Chapter 6, the dual quotient $(I_{\geq w}^\Omega / I_{>w}^\Omega)'$ is exactly the kernel of the following restriction map

$$\mathrm{Res}_w^\Omega : (I_{\geq w}^\Omega)' \to (I_{>w}^\Omega)'.$$

Let

$$\mathfrak{t}_w^\Omega := \bar{\mathfrak{n}}_\Omega \cap w^{-1} \bar{\mathfrak{n}}_P w = \bar{\mathfrak{n}}_\Omega \cap w^{-1} \bar{\mathfrak{n}}_\Theta w$$

be the transverse subalgebra.

As in the 6.1.1, since $G_{\geq w}^\Omega$ is open in G and G_w^Ω is closed in $G_{\geq w}^\Omega$, the $U(\mathfrak{g})$-derivatives of distributions on G_w^Ω are supported in G_w^Ω (vanishing on $G_{>w}^\Omega$). As a generalization of Theorem 6.1, we have the following theorem:

Theorem 10.1. *We have the following linear maps as in (6.1) and (6.2):*

$$\mathcal{S}(G_w^\Omega, V)' \otimes U(\mathfrak{t}_w^\Omega) \to \mathrm{Ker}\{\mathcal{S}(G_{\geq w}^\Omega, V)' \to \mathcal{S}(G_{>w}^\Omega, V)'\}$$
$$(I_w^\Omega)' \otimes U(\mathfrak{t}_w^\Omega) \to \mathrm{Ker}\{(I_{\geq w}^\Omega)' \to (I_{>w}^\Omega)'\}$$

i.e. they are given by $U(\mathfrak{t}_w^\Omega)$-derivatives of distributions on G_w^Ω. The above two maps are isomorphisms.

10.1.2 Sketch of the Proof

The key points of the proof are

- Change of neighoubrhood: Lemma 4.38 and Lemma 4.80.
- The Z_w is a tubular neighbourhood of G_w^Ω:
$$Z_w \simeq G_w^\Omega \times (\overline{N}_\Omega \cap w^{-1}\overline{N}_P w).$$
- The tensor product property, i.e. (E-6) in Proposition 4.30.
- The distribution with point support, i.e. the kernel of the restriction map
$$\mathcal{S}(\overline{N}_\Omega \cap w^{-1}\overline{N}_P w, V)' \to \mathcal{S}(\overline{N}_\Omega \cap w^{-1}\overline{N}_P w - \{e\}, V)'$$
is exactly the enveloping algebra $U(\overline{\mathfrak{n}}_\Omega \cap w^{-1}\overline{\mathfrak{n}}_P w) = U(\mathfrak{t}_w^\Omega)$.

The G_w^Ω and the Tubular Neighbourhood Z_w

As in 6.1.2, we let

$$Z_w = P\overline{N}_P w = P_\Theta \overline{N}_\Theta w$$

for all $w \in W$. This is an Zariski open subset of G, and the Lemma 6.3 still holds.

For each $\Omega \subset \Theta$ and $w \in [W_\Theta \backslash W / W_\Omega]$, the $G_w^\Omega = PwP_\Omega$ is isomorphic to $P \times \{w\} \times \frac{P_\Omega}{P_\Omega \cap w^{-1}Pw}$. The canonical map

$$(M_\Omega \cap w^{-1}\overline{N}_P w) \cdot (N_\Omega \cap w^{-1}\overline{N}_P w) \to P_\Omega \cap w^{-1}Pw \backslash P_\Omega \tag{10.1}$$

is an isomorphism of manifolds and real algebraic varieties. Similar to the Lemma 8.8, we have the following Lemma on the structure of G_w^Ω:

Lemma 10.2. *The map*

$$P \times (M_\Omega \cap w^{-1}\overline{N}_P w) \cdot (N_\Omega \cap w^{-1}\overline{N}_P w) \to G_w^\Omega = PwP_\Omega \tag{10.2}$$
$$(p, mn) \mapsto pwmn$$

is an isomorphism of manifolds and real algebraic varieties. The G_w^Ω is a nonsingular closed subvariety of Z_w and a closed regular submanifold of Z_w. The Z_w is a tubular neighbourhood of G_w^Ω:

$$Z_w \simeq P \times w^{-1}\overline{N}_P w \simeq G_w^\Omega \times (\overline{N}_\Omega \cap w^{-1}\overline{N}_P w). \tag{10.3}$$

As in 6.1.3, by the Lemma 4.38 and 4.80, we have the following isomorphisms between kernels of restriction maps of distributions:

$$\mathrm{Ker}\{\mathcal{S}(G_{\geq w}^\Omega, V)' \to \mathcal{S}(G_{>w}^\Omega, V)'\} \simeq \mathrm{Ker}\{\mathcal{S}(Z_w, V)' \to \mathcal{S}(Z_w - G_w^\Omega, V)'\}$$
$$\mathrm{Ker}\{(I_{\geq w}^\Omega)' \to (I_{>w}^\Omega)'\} \simeq \mathrm{Ker}\{(\mathcal{S}\mathrm{Ind}_P^{Z_w}\sigma)' \to (\mathcal{S}\mathrm{Ind}_P^{Z_w - G_w^\Omega}\sigma)'\}$$

Distributions on Schwartz Function Spaces

We first show the following isomorphism on distributions on Schwartz functions:

$$\mathcal{S}(G_w^\Omega, V)' \otimes U(\mathfrak{t}_w^\Omega) \xrightarrow{\sim} \mathrm{Ker}\{\mathcal{S}(Z_w, V)' \to \mathcal{S}(Z_w - G_w^\Omega, V)'\}.$$

First by the above Lemma 10.2 and the (E-6) of Proposition 4.30, we have

$$\mathcal{S}(Z_w, V) \simeq \mathcal{S}(G_w^\Omega, V) \mathbin{\widehat{\otimes}} \mathcal{S}(\overline{N}_\Omega \cap w^{-1}\overline{N}_P w, \mathbb{C})$$
$$\mathcal{S}(Z_w - G_w^\Omega, V) \simeq \mathcal{S}(G_w^\Omega, V) \mathbin{\widehat{\otimes}} \mathcal{S}(\overline{N}_\Omega \cap w^{-1}\overline{N}_P w - \{e\}, \mathbb{C})$$

Since all spaces are nuclear, we take the strong dual of the above isomorphisms, and we see the $\mathrm{Ker}\{\mathcal{S}(Z_w, V)' \to \mathcal{S}(Z_w - G_w^\Omega, V)'\}$ is isomorphic to

$$\mathcal{S}(G_w^\Omega, V)' \mathbin{\widehat{\otimes}} \mathrm{Ker}\{\mathcal{S}(\overline{N}_\Omega \cap w^{-1}\overline{N}_P w, \mathbb{C})' \to \mathcal{S}(\overline{N}_\Omega \cap w^{-1}\overline{N}_P w - \{e\}, \mathbb{C})\}.$$

By the Lemma 6.7, we see the above space is exactly

$$\mathcal{S}(G_w^\Omega, V)' \otimes U(\overline{\mathfrak{n}}_\Omega \cap w^{-1}\overline{\mathfrak{n}}_P w) = \mathcal{S}(G_w^\Omega, V)' \otimes U(\mathfrak{t}_w^\Omega).$$

Distributions on Schwartz Induction Spaces

We show the isomorphism

$$(I_w^\Omega)' \otimes U(\mathfrak{t}_w^\Omega) \xrightarrow{\sim} \operatorname{Ker}\{(\mathcal{S}\mathrm{Ind}_P^{Z_w}\sigma)' \to (\mathcal{S}\mathrm{Ind}_P^{Z_w - G_w^\Omega}\sigma)'\}.$$

By the Lemma 4.91, we have

$$\begin{aligned}
&\mathcal{S}\mathrm{Ind}_P^{Z_w}\sigma \\
&\simeq \mathcal{S}\mathrm{Ind}_P^P\sigma \mathbin{\widehat{\otimes}} \mathcal{S}((M_\Omega \cap w^{-1}\overline{N}_P w)\cdot(N_\Omega \cap w^{-1}\overline{N}_P w), \mathbb{C}) \\
&\quad \mathbin{\widehat{\otimes}} \mathcal{S}(\overline{N}_\Omega \cap w^{-1}\overline{N}_P w, \mathbb{C}) \\
&\simeq \mathcal{S}\mathrm{Ind}_P^{G_w^\Omega}\sigma \mathbin{\widehat{\otimes}} \mathcal{S}(\overline{N}_\Omega \cap w^{-1}\overline{N}_P w, \mathbb{C}) \\
&\mathcal{S}\mathrm{Ind}_P^{Z_w - G_w^\Omega}\sigma \\
&\simeq \mathcal{S}\mathrm{Ind}_P^P\sigma \mathbin{\widehat{\otimes}} \mathcal{S}((M_\Omega \cap w^{-1}\overline{N}_P w)\cdot(N_\Omega \cap w^{-1}\overline{N}_P w), \mathbb{C}) \\
&\quad \mathbin{\widehat{\otimes}} \mathcal{S}(\overline{N}_\Omega \cap w^{-1}\overline{N}_P w - \{e\}, \mathbb{C}) \\
&\simeq \mathcal{S}\mathrm{Ind}_P^{G_w^\Omega}\sigma \mathbin{\widehat{\otimes}} \mathcal{S}(\overline{N}_\Omega \cap w^{-1}\overline{N}_P w - \{e\}, \mathbb{C})
\end{aligned}$$

Since all spaces are nuclear, by taking their dual, we see the kernel $\operatorname{Ker}\{(\mathcal{S}\mathrm{Ind}_P^{Z_w}\sigma)' \to (\mathcal{S}\mathrm{Ind}_P^{Z_w - G_w^\Omega}\sigma)'\}$ is isomorphic to

$$(\mathcal{S}\mathrm{Ind}_P^{G_w^\Omega}\sigma)' \mathbin{\widehat{\otimes}} \operatorname{Ker}\{\mathcal{S}(\overline{N}_\Omega \cap w^{-1}\overline{N}_P w, \mathbb{C})' \to \mathcal{S}(\overline{N}_\Omega \cap w^{-1}\overline{N}_P w - \{e\}, \mathbb{C})\},$$

which is exactly $(I_w^\Omega)' \otimes U(\mathfrak{t}_w^\Omega)$.

10.2 Generalization of Chapter 7

10.2.1 Formulating the Theorem

The following theorem is a generalization of the main result (Theorem 7.1) in Chapter 7:

Theorem 10.3. *The $\mathfrak{n}_\Omega$-torsion subspace on the kernels of the restriction map $\mathcal{S}(G_{\geq w}^\Omega, V)' \to \mathcal{S}(G_{>w}^\Omega, V)'$ and $(I_{\geq w}^\Omega)' \to (I_{>w}^\Omega)'$ are given by*

$$\begin{aligned}
[\mathcal{S}(G_w^\Omega, V)' \otimes U(\mathfrak{t}_w^\Omega)]^{[\mathfrak{n}_\Omega^\bullet]} &= [\mathcal{S}(G_w^\Omega, V)']^{[\mathfrak{n}_\Omega^\bullet]} \otimes U(\mathfrak{t}_w^\Omega) \\
[(I_w^\Omega)' \otimes U(\mathfrak{t}_w^\Omega)]^{[\mathfrak{n}_\Omega^\bullet]} &= [(I_w^\Omega)']^{[\mathfrak{n}_\Omega^\bullet]} \otimes U(\mathfrak{t}_w^\Omega)
\end{aligned}$$

By the same argument as in 7.1.1, we just need to show the first equality. by the same argument as in 7.1.2, we regard all distribution spaces as right modules over enveloping algebras.

We use the same notation as in Chapter 7, let

$$\begin{aligned}\mathcal{U} &= \mathcal{S}(G_w^\Omega, V)' \\ \mathcal{K} &= \mathcal{S}(G_w^\Omega, V)' \otimes U(\mathfrak{t}_w^\Omega) \\ \mathcal{M} &= [\mathcal{S}(G_w^\Omega, V)']^{[\mathfrak{n}_\Omega^\bullet]} \otimes U(\mathfrak{t}_w^\Omega)\end{aligned}$$

For each $n \geq 0$, we let

$$\begin{aligned}\mathcal{K}^n &= \mathcal{S}(G_w^\Omega, V)' \otimes U_n(\mathfrak{t}_w^\Omega) \\ \mathcal{M}^n &= [\mathcal{S}(G_w^\Omega, V)']^{[\mathfrak{n}_\Omega^\bullet]} \otimes U_n(\mathfrak{t}_w^\Omega)\end{aligned}$$

We have $\mathcal{K}^0 = \mathcal{U}, \mathcal{M}^0 = \mathcal{U}_{[\mathfrak{n}_\Omega^\bullet]}$. We need to show

$$\mathcal{M} = \mathcal{K}_{[\mathfrak{n}_\Omega^\bullet]}.$$

10.2.2 Sketch of the Proof

The Easy Part $\mathcal{M} \subset \mathcal{K}_{[\mathfrak{n}_\Omega^\bullet]}$

We prove

$$\mathcal{M}^k \subset \mathcal{K}_{[\mathfrak{n}_\Omega^\bullet]}$$

by induction on k. The case $k = 0$ is obvious. Assuming $\mathcal{M}^{k-1} \subset \mathcal{K}_{[\mathfrak{n}_\Omega^\bullet]}$, we show $\mathcal{M}^k \subset \mathcal{K}_{[\mathfrak{n}_\Omega^\bullet]}$. By the same argument as in 7.2.2, we just need to show the elements of the following form are in the torsion subspace $\mathcal{K}_{[\mathfrak{n}_\Omega^\bullet]}$:

$$\Phi \cdot Y_1 \cdots Y_k$$

where $\Phi \in \mathcal{U}_{[\mathfrak{n}_\Omega^\bullet]}, Y_1, \ldots, Y_k \in \mathfrak{t}_w^\Omega$.

By Lemma 7.20, we just need to show there exists a large n such that

$$\Phi \cdot Y_1 \cdots Y_k \cdot (\mathfrak{n}_\Omega^n) \subset \mathcal{K}_{[\mathfrak{n}_\Omega^\bullet]}.$$

Actually, by writing

$$\Phi \cdot Y_1 \cdots Y_k = (\Phi \cdot Y_1 \cdots Y_{k-1}) \cdot Y_k,$$

we see there exists a $n_1 \geq 0$, such that $(\Phi \cdot Y_1 \cdots Y_{k-1}) \cdot (\mathfrak{n}_\Omega^{n_1}) = \{0\}$ since the $\Phi \cdot Y_1 \cdots Y_{k-1}$ is in the $\mathfrak{n}_\Omega$-torsion subspace by the induction hypothesis. Also there exists a $n_2 \geq 0$ large enough such that

$$[\ldots[[Y_k, X_1], X_2] \ldots X_{n_2}] \in \mathfrak{n}_\Omega,$$

for all $X_i \in \mathfrak{n}_\Omega$.

Therefore let $n = n_1 + n_2 + 1$, and use the formula in Lemma 7.8, we see

$$\Phi \cdot Y_1 \cdots Y_k \cdot (\mathfrak{n}_\Omega^n) \subset \mathcal{K}_{[\mathfrak{n}_\Omega^\bullet]}.$$

Linear Order on PBW-Basis and the Linear Filtration of $\mathcal{K}$

As in 7.3, we fix a basis $\{Y_1, \ldots, Y_d\}$ of $\mathfrak{t}_w^\Omega$ consisting of root vectors and satisfying

$$\mathrm{Ht}(Y_1) \leq \mathrm{Ht}(Y_2) \leq \ldots \leq \mathrm{Ht}(Y_d).$$

Recall that if Y_i is a root vector of the restricted root α, and α is uniquely written as non-positive integral combinations of simple roots $\alpha_1, \ldots, \alpha_r$:

$$\alpha = -\sum_{i=1}^{r} n_i \alpha_i, \quad n_i \geq 0,$$

then the height of Y_i is defined as $\mathrm{Ht}(Y_i) = \sum_{i=1}^r n_i$.

With the above labeling on the basis of $\mathfrak{t}_w^\Omega$, we thus have a PBW-basis of the enveloping algebra

$$\{Y^I : I \in \mathfrak{L}^d\}$$

where $\mathfrak{L}^d = \mathbb{Z}_{\geq 0}^d$ is the set of multi-index, and for a $I = (i_1, i_2, \ldots, i_d)$, the Y^I means the product $Y_1^{i_1} Y_2^{i_2} \cdots Y_d^{i_d}$ in $U(\mathfrak{t}_w^\Omega)$ and the Y^I form a PBW-basis. As in 7.3.1, we choose the linear order on the index set $\mathfrak{L}^d$ thus the above labeling of $Y_1, \ldots, Y_d$ gives a linear order on the PBW-basis Y^I of $U(\mathfrak{t}_w^\Omega)$.

As in 7.3.6, for each multi-index $I \in \mathfrak{L}^d$, let I^- be its lower adjacent (see Definition 7.43), and let

$$U_I(\mathfrak{t}_w^\Omega) = \text{the subspace of } U(\mathfrak{t}_w^\Omega) \text{ spanned by } Y^J, J \leq I,$$

and let

$$\mathcal{K}^I = \mathcal{U} \otimes U_I(\mathfrak{t}_w^\Omega).$$

Then the $\{\mathcal{K}^I : I \in \mathfrak{L}^d\}$ form an exhaustive filtration of $\mathcal{K}$.

The Decompositions of Vector Fields

As in 7.1.5, given an element $X \in \mathfrak{g}$ (complexified Lie algebra), its corresponding left invariant vector field is denoted by X^L, and right invariant vector field is denoted by X^R, and for a point $x \in G$, the tangent vector of the vector field X^L (resp. X^R) at x is denoted by X_x^L (resp. X_x^R).

Let

$$L_x^{\mathfrak{t}_w^\Omega} := \mathrm{span}_{\mathbb{R}}\{X_x^L : X \in \mathfrak{t}_w^\Omega\}.$$

This is a subspace of the tangent space $T_xG = T_xZ_w$, and one has

$$T_xG = T_xZ_w = T_xG_w^\Omega \oplus L_x^{\mathfrak{t}_w^\Omega}$$

for all $x \in G_w^\Omega$, i.e. the subalgebra $\mathfrak{t}_w^\Omega$ is transverse to the submanifold G_w^Ω at every point on it.

Given an element $H \in \mathfrak{g}$, let H^L be the corresponding left invariant vector field on G, let $H^L|_{G_w^\Omega}$ be the restriction of the vector field to the submanifold G_w^Ω, then this restriction is uniquely written as

$$(H^L|_{G_w^\Omega})_x = \sum_{i=1}^{k} A^i(x)X_{i,x}^R + \sum_{i=1}^{l} B^i(x)Z_{i,x}^L + \sum_{i=1}^{d} C^i(x)Y_{i,x}^L$$

where $\{X_1, \ldots, X_k\}$ is an arbitrary basis of $\mathfrak{p}$, $\{Z_1, \ldots, Z_l\}$ is an arbitrary basis of $\mathfrak{m}_\Omega \cap w^{-1}\mathfrak{p}w + \mathfrak{n}_\Omega \cap w^{-1}\mathfrak{p}w$, and $\{Y_1, \ldots, Y_d\}$ is an arbitrary basis of $\mathfrak{t}_w^\Omega$. The A^i, B^i, C^i are algebraic functions on the variety G_w^Ω.

By the same argument as in Lemma 7.40, we can show

Lemma 10.4. *Let $Y_1, \ldots, Y_d$ be a basis of $\mathfrak{t}_w^\Omega$ with non-decreasing heights. Let $X \in \mathfrak{n}_\Omega$ be an arbitrary element, and let $[Y_j, X]$ be the Lie algebra bracket. Then in the above decomposition of $[Y_j, X]^L|_{G_w^\Omega}$, we have all $C^k \equiv 0$ on G_w^Ω, for all $k \geq j$.*

The $\mathcal{K}^I$ are $\mathfrak{n}_\Omega$-Submodules

The most crucial results to prove the inclusion $\mathcal{M} \supset \mathcal{K}_{[\mathfrak{n}_\Omega^\bullet]}$ is: each $\mathcal{K}^I$ is $\mathfrak{n}_\Omega$-stable, hence is a $\mathfrak{n}_\Omega$-submodule of $\mathcal{K}$. Namely we have

Lemma 10.5. *For each multi-index $I \in \mathfrak{L}^d$, the subspace $\mathcal{K}^I$ of the right $\mathfrak{n}_\Omega$-module $\mathcal{K}$ is stable under the right multiplication of $\mathfrak{n}_\Omega$.*

The quotient space $\mathcal{K}^I/\mathcal{K}^{I^-}$ is isomorphic to $\mathcal{U}$ as $\mathfrak{n}_\Omega$-modules.

As in the Remark 7.45, we just need to show the following element is in $\mathcal{K}^{I^-}$:

$$\Phi \cdot [Y_j, X] \cdot Y^{I-J}$$

where $\Phi \in \mathcal{U}, X \in \mathfrak{n}_\Omega$ and the multi-index $J = (0, \ldots, 0, 1, 0, \ldots, 0)$ (with jth entry equal to 1 and all other entries zero). This is easy to see by the above Lemma on the coefficients C^i and the Lemma 7.41.

The Hard Part $\mathcal{M} \supset \mathcal{K}_{[\mathfrak{n}_\Omega^\bullet]}$

To show the inclusion $\mathcal{K}_{[\mathfrak{n}_\Omega^\bullet]} \subset \mathcal{M}$, we just need to show

$$\mathcal{K}^I_{[\mathfrak{n}_\Omega^\bullet]} \subset \mathcal{M}$$

by induction on the linear order of I. The case $I = (0, \ldots, 0)$ is clear. For an arbitrary I, let I^- be its lower adjacent (Definition 7.43), and we assume $\mathcal{K}^{I^-}_{[\mathfrak{n}_\Omega^\bullet]} \subset \mathcal{M}$.

For an arbitrary element $\sum_{J \le I} \Phi_J \cdot Y^J$ of $\mathcal{K}^I_{[\mathfrak{n}_\Omega^\bullet]}$, first its image in the quotient $\mathcal{K}^I/\mathcal{K}^{I^-}$ is exactly $\Phi_I \cdot Y^I \mathrm{mod} \mathcal{K}^{I^-}$. Since the original element is $\mathfrak{n}_\Omega$-torsion, we see the image $\Phi_I \cdot Y^I \mathrm{mod} \mathcal{K}^{I^-}$ is also $\mathfrak{n}_\Omega$-torsion. The quotient $\mathcal{K}^I/\mathcal{K}^{I^-}$ is isomorphic to $\mathcal{U}$ as $\mathfrak{n}_\Omega$-modules, therefore the $\Phi_I \in \mathcal{U}$ has to be $\mathfrak{n}_\Omega$-torsion.

By the (easy part) inclusion $\mathcal{M} \subset \mathcal{K}_{[\mathfrak{n}_\Omega^\bullet]}$, we see the leading term $\Phi_I \cdot Y^I$ is in $\mathcal{K}_{[\mathfrak{n}_\Omega^\bullet]}$. Therefore the tail sum is also torsion:

$$\sum_{J<I} \Phi_J \cdot Y^J \in \mathcal{K}_{[\mathfrak{n}_\Omega^\bullet]}.$$

By the induction hypothesis the tail term $\sum_{J<I} \Phi_J \cdot Y^J$ is in $\mathcal{M}$, and the leading term $\Phi_I \cdot Y^I$ is in $\mathcal{M}$ (since Φ_I is torsion), therefore the entire sum is in $\mathcal{M}$:

$$\sum_{J \le I} \Phi_J \cdot Y^J \in \mathcal{M}.$$

10.3 Generalization of Chapter 8

We keep the notations and setting as the last two sections. We can generalize some of the results in Chapter 8 to arbitrary subset $\Omega \subset \Theta$. But a crucial problem is: the generalization of Lemma 8.1 is not true.

The Lemma 8.1 says, the isomorphism $(I_{\ge w}/I_{>w})' \simeq I'_w \otimes U(\mathfrak{n}_w^-)$ is $M_\emptyset$-equivariant when the right-hand-side is endowed with the tensor product $M_\emptyset$-action. However for the general Ω, the tensor product $(I_w^\Omega)' \otimes U(\mathfrak{t}_w^\Omega)$ is not a tensor product of M_Ω-representations, as one can see the subalgebra $\mathfrak{t}_w^\Omega = \bar{\mathfrak{n}}_\Omega \cap w^{-1}\bar{\mathfrak{n}}_\Theta w$ is not stable under M_Ω-conjugation.

However, to make the results as general as possible, we state the generalizations some results in Chapter 8. We omit the proof since they are proved by exactly the same way as the corresponding Lemma in Chapter 8

Lemma 10.6 (Annihilator-Invariant Trick)**.** *The kth annihilator space on $(I_w^\Omega)'$ is isomorphic to*

$$[(I_w^\Omega)']^{[\mathfrak{n}_\Omega^k]} \simeq H^0(\mathfrak{n}_\Omega, (I_w^\Omega \otimes F_k^\Omega)'),$$

where $F_k^\Omega = U(\mathfrak{n}_\Omega)/(\mathfrak{n}_\Omega^k)$ is a finite dimensional representation of N_Ω (or P_Ω).

This is prove exactly the same way as in 8.1.2, by algebraic tricks.

Lemma 10.7. *The local Schwartz induction $I_w^\Omega = \mathcal{S}\mathrm{Ind}_P^{G_w^\Omega}\sigma$ is isomorphic to the Schwartz induction*

$$\mathcal{S}\mathrm{Ind}_{P_\Omega \cap w^{-1}Pw}^{P_\Omega}\sigma^w$$

where $\sigma^w = \sigma \circ \mathrm{Ad}w$ is the representation of $w^{-1}Pw$ and regarded as a representation of $P_\Omega \cap w^{-1}Pw$ by restriction.

This is proved by the same way as Lemma 8.9 since both spaces are (linearly) isomorphic to the Schwartz function space $\mathcal{S}((M_\Omega \cap w^{-1}\overline{N}_P w) \cdot (N_\Omega \cap w^{-1}\overline{N}_P w), V)$.

Lemma 10.8 (Tensor Product Trick)**.** *The tensor product $I_w^\Omega \otimes F_k^\Omega$ is isomorphic to*

$$\mathcal{S}\mathrm{Ind}_{P_\Omega \cap w^{-1}Pw}^{P_\Omega}(\sigma^w \otimes F_k^\Omega).$$

This is proved by the same way as Lemma 8.20.

10.4 Future Works

As mentioned in the section 1.2, some topic of our future work are:

- **Irreducibility**: Reproduce the result in [40] about complex groups.
- **Globalization of local intertwining distributions**, namely, find the original $D \in \mathrm{Hom}_P(I, V)$ from its restrictions to open subsets. (e.g. for $\mathbf{SL}(3, \mathbb{C})$, reproduce all intertwining operators studied in [37]).

However in this final section, we want to show our plan on how to generalize the methods to study the parabolic inductions of infinite dimensional representations.

As we have seen in Chapter 9, the condition "V is finite dimensional" largely simplify the proof: the entire V' is $\mathfrak{n}_\emptyset$-torsion and the image of an arbitrary $\Phi \in \mathrm{Hom}_{P_\emptyset}(V', (I_{\geq w}/I_{>w})')$ is contained in the $\mathfrak{n}_\emptyset$-torsion subspace of $(I_{\geq w}/I_{>w})'$. Then we have

$$\mathrm{Hom}_{P_\emptyset}(V', (I_{\geq w}/I_{>w})') = \mathrm{Hom}_{M_\emptyset}(V', [(I_{\geq w}/I_{>w})']^{[\mathfrak{n}_\emptyset{}^\bullet]})$$

and we just need to study the $M_\emptyset$-structure on $[(I_{\geq w}/I_{>w})']^{[\mathfrak{n}_\emptyset{}^\bullet]}$.

In general, when the V is infinite dimensional, the V and V' are only $\mathfrak{n}_P$-invariant (or $\mathfrak{n}_P$-torsion) if we assume (τ, V) to be irreducible. Therefore, instead of considering $\mathfrak{n}_\emptyset$-torsion subspaces, we need to study $\mathfrak{n}_P$-torsion subspaces. Unlike in Chapter 6, 7 and 8, we will choose the subset $\Omega = \Theta$.

For simplicity, we still let

$$(\tau, V) = \text{an irreducible unitary representation of } P.$$

As shown in Theorem 9.8, the interesting phenomenon only occur on non-identity double cosets. To show the irreducibilities of $I = \mathcal{S}\mathrm{Ind}_P^G \sigma$ where $\sigma = \tau \otimes \delta_P^{1/2}$, we need to show

$$\mathrm{Hom}_P(V', (I^\Theta_{\geq w}/I^\Theta_{>w})') = \{0\}.$$

As in the main body of the thesis, we are required to study the quotient dual $(I^\Theta_{\geq w}/I^\Theta_{>w})'$ and its $\mathfrak{n}_P = \mathfrak{n}_\Theta$-torsion subspace.

10.4.1 A Conjecture

For each $w \in [W_\Theta \backslash W / W_\Theta]$, by applying the Theorem 10.1, we have the following isomorphism

$$(I^\Theta_{\geq w}/I^\Theta_{>w})' \simeq (I^\Theta_w)' \otimes U(\mathfrak{t}^\Theta_w).$$

By the Theorem 10.3, we can identify its $\mathfrak{n}_P = \mathfrak{n}_\Theta$-torsion subspace:

$$[(I^\Theta_w)' \otimes U(\mathfrak{t}^\Theta_w)]^{[\mathfrak{n}^\bullet_P]} = [(I^\Theta_w)']^{[\mathfrak{n}^\bullet_P]} \otimes U(\mathfrak{t}^\Theta_w).$$

However, as we have seen above or in the introductory subsection 1.1.3, the first obstacle we meet is: the above tensor product is not a tensor product of M_Θ-representations as the $\mathfrak{t}^\Theta_w = \overline{\mathfrak{n}}_P \cap w^{-1}\overline{\mathfrak{n}}_P w$ is not stable under $M_\Theta = M_P$ conjugation.

Fortunately, if the w normalizes M_P (or equivalently normalizing the subsystem $\langle\Theta\rangle$ spanned by Θ), the $\mathfrak{t}^\Theta_w$ is stable under the M_P-conjugation, and the above tensor product is indeed a tensor product of representations of M_P.

As suggested by all examples we have known, we propose the following conjecture:

Conjecture 10.9. If w does not normalize the $M_P = M_\Theta$, then we have

$$\mathrm{Hom}_{M_P}(V', (I^\Theta_{\geq w}/I^\Theta_{>w})') = \{0\}.$$

10.4.2 Another Conjecture of Casselman

As we have seen in the introductory subsection 1.1.3, the second obstacle we meet is: we cannot apply Shapiro's lemma to the Schwartz induction

$$\mathcal{S}\mathrm{Ind}^P_{P\cap w^{-1}Pw}(\sigma^w \otimes F^\Theta_k).$$

The main difficulty is: the P and $P \cap w^{-1}Pw$ are neither unimodular nor cohomologically trivial. More importantly, the quotient

$$P \cap w^{-1}Pw\backslash P$$

is not N_P-transitive. Therefore we cannot apply Shapiro's Lemma directly. To overcome this obstacle, Casselman has proposed a conjecture, which will be formulated below.

Let $P_{\Theta\cap w\Theta}$ be the standard real parabolic subgroup corresponding to the subset $\Theta \cap w\Theta$. It is contained in P_Θ, and we denote its image under the quotient map

$$P_\Theta \to M_\Theta \simeq P_\Theta/N_\Theta$$

by Q_w. This Q_w is a parabolic subgroup in $M_\Theta = M_P$, and let $M_{Q_w}N_{Q_w}$ be its Levi decomposition.

For the representation (ξ, H) of $M_\Theta = M_P$, let (ξ^∞, H^∞) be its Harish-Chandra module. By a theorem of Hecht-Schmid, the quotient space

$$H^\infty/\mathfrak{n}_{Q_w}H^\infty$$

is a finitely generated Harish-Chandra module of M_{Q_w}, we denote its Harish-Chandra globalization (in the sense of Casselman's [17]) by

$$(\rho, U).$$

The canonical map (of Harish-Chandra modules) $V^\infty \to V^\infty/\mathfrak{n}_{Q_w}V^\infty$ extends to a map

$$(\xi, H) \to (\rho, U)$$

and this map is a M_{Q_w}-map.

We can then define the following map by integration

$$\mathcal{S}\mathrm{Ind}^{P_\Theta}_{P_\Theta\cap w^{-1}P_\Theta w}(\xi^w) \to \mathcal{S}\mathrm{Ind}^{M_\Theta}_{w^{-1}M_{Q_w}w}(\rho^w).$$

This map induces a map

$$[\mathcal{S}\mathrm{Ind}_{w^{-1}M_{Q_w}w}^{M_\Theta}(\rho^w)]' \to H^0(\mathfrak{n}_P, [\mathcal{S}\mathrm{Ind}_{P_\Theta\cap w^{-1}P_\Theta w}^{P_\Theta}(\xi^w)]').$$

Casselman proposed the following conjecture:

Conjecture 10.10 (Casselman)**.** The above map is an isomorphism.

By applying this conjecture to $(\xi, H) = (\sigma^w \otimes F_k^\Theta, V \otimes F_k^\Theta)$, we might be able to compute the $M_P = M_\Theta$-action on the $H^0(\mathfrak{n}_P, (I_w^\Theta \otimes F_w^\Theta)')$. But we are still working on this conjecture.

Bibliography

[1] Avraham Aizenbud and Dmitry Gourevitch. Schwartz functions on Nash manifolds. *International Mathematics Research Notices*, 2008-1:rnm155, January 2008.

[2] Avraham Aizenbud and Dmitry Gourevitch. The de-Rham theorem and Shapiro lemma for Schwartz functions on Nash manifolds. *Israel Journal of Mathematics*, 177(1):155–188, 2010.

[3] Dan Barbasch. The unitary dual for complex classical Lie groups. *Inventiones mathematicae*, 96(1):103–176, 1989.

[4] Dan Barbasch and David A Vogan Jr. Unipotent representations of complex semisimple groups. *Annals of Mathematics*, 121(1):41–110, 1985.

[5] Valentine Bargmann. Irreducible unitary representations of the Lorentz group. *Annals of Mathematics*, 48(3):568–640, 1947.

[6] Jacek Bochnak, Michel Coste, and Marie-Françoise Roy. *Real Algebraic Geometry*, volume 36 of *A Series of Modern Survey in Mathematics*. Springer, 2013.

[7] Armand Borel. Groupes linéaires algébriques. *Annals of mathematics*, 64(1):20–82, 1956.

[8] Armand Borel. Linear algebraic groups. In *Algebraic Groups and Discontinuous Subgroups*, volume 9 of *Proceedings of Symposia in Pure Mathematics*, pages 3–19. American Mathematical Society, 1966.

[9] Armand Borel. *Représentations de Groupes Localement Compacts*, volume 276 of *Lecture Notes in Mathematics*. Springer, 1972.

[10] Armand Borel. *Linear Algebraic Groups, second edition*, volume 126 of *Graduate Texts in Mathematics*. Springer, 2012.

[11] Armand Borel and Jacques Tits. Groupes réductifs. *Publications Mathématiques de l'IHÉS*, 27(1):55–150, 1965.

[12] Armand Borel and Nolan R Wallach. *Continuous Cohomology, Discrete Subgroups, and Representations of Reductive Groups*, volume 67 of *Mathematical Surveys and Monographs*. American Mathematical Soc., 2013.

[13] Nicolas Bourbaki. Lie groups and Lie algebras, Chapters 4–6. Elements of mathematics, 2002.

[14] Nicolas Bourbaki. *Topological Vector Spaces*. Springer, 2003.

[15] François Bruhat. Sur les représentations induites des groupes de Lie. *Bulletin de la Société Mathématique de France*, 84:97–205, 1956.

[16] Bill Casselman. Introduction to admissible representations of p-adic groups. *unpublished notes*, 1974.

[17] William Casselman. Canonical extensions of Harish-Chandra modules to representations of g. *Canadian Journal of Mathematics*, 41(3):385–438, 1989.

[18] William Casselman, Henryk Hecht, and Dragan Milicic. Bruhat filtrations and whittaker vectors for real groups. In *Proceedings of Symposia in Pure Mathematics*, volume 68, pages 151–190. Providence, RI; American Mathematical Society; 1998, 2000.

[19] Charles W Curtis. On Lusztig's isomorphism theorem for Hecke algebras. *Journal of Algebra*, 92(2):348–365, 1985.

[20] Fokko Du Cloux. Sur les représentations différentiables des groupes de Lie algébriques. *Annales scientifiques de l'Ecole normale supérieure*, 24(3):257–318, 1991.

[21] Thomas Farmer. On the reduction of certain degenerate principal series representations of $\mathrm{Sp}(n, \mathbb{C})$. *Pacific Journal of Mathematics*, 84(2):291–303, 1979.

[22] Thomas A Farmer. Irreducibility of certain degenerate principal series representations of $\mathrm{Sp}(n, \mathbb{R})$. *Proceedings of the American Mathematical Society*, 83(2):411–420, 1981.

[23] Lars Gårding. Note on continuous representations of Lie groups. *Proceedings of the National Academy of Sciences*, 33(11):331–332, 1947.

[24] I Gelfand. On one-parametrical groups of operators in a normed space. In *CR (Doklady) Acad. Sci. URSS (NS)*, volume 25, pages 713–718, 1939.

[25] Kenneth I Gross. The dual of a parabolic subgroup and a degenerate principal series of $\mathrm{Sp}(n, \mathbb{C})$. *American Journal of Mathematics*, 93(2):398–428, 1971.

[26] Alexandre Grothendieck. Produits tensoriels topologiques et espaces nucléaires. *Memoir of American Mathematics Society*, 16, 1955.

[27] Harish-Chandra. Discrete series for semisimple Lie groups II. *Acta Math*, 116(1):1–111, 1966.

[28] Harish-Chandra. Harmonic analysis on real reductive groups. 1. theory of constant term. *Journal of Functional Analysis*, 19(2):104–204, 1975.

[29] Christophe Hohlweg and Mark Skandera. A note on Bruhat order and double coset representatives. *arXiv preprint math/0511611*, 2005.

[30] Anthony W Knapp. *Representation Theory of Semisimple Groups: An Overview Based on Examples*. Princeton University Press, 1986.

[31] J.A.C Kolk and V.S. Varadarajan. On the transverse symbol of vectorial distributions and some applications to harmonic analysis. *Indagationes Mathematicae*, 7(1):67–96, 1996.

[32] George Lusztig. *Characters of Reductive Groups over a Finite Field*, volume 107 of *Annals of Mathematics Studies*. Princeton University Press, 1984.

[33] George D Mostow. Self-adjoint groups. *Annals of Mathematics*, 62(1):44–55, 1955.

[34] Helmut H Schaefer and Manfred P Wolff. *Topological Vector Spaces*. Springer, 1999.

[35] Laurent Schwartz. *Théorie des Distributions*, volume 1,2. Hermann Paris, 1959.

[36] François Trèves. *Topological Vector Spaces, Distributions and Kernels*, volume 25. Dover Publications, 2006.

[37] Masao Tsuchikawa. On the representations of $\mathrm{SL}(3, \mathbb{C})$ I. *Proceedings of the Japan Academy*, 43(9):852–855, 1967.

[38] David A Vogan. *Representations of Real Reductive Lie Groups*, volume 15 of *Progress in Mathematics*. Birkhauser, 1981.

[39] Nolan Wallach. Asymptotic expansions of generalized matrix entries of representations of real reductive groups. In *Lie group representations I*, volume 1024 of *Lecture Notes in Mathematics*, pages 287–369. Springer, 1983.

[40] Nolan R Wallach. Cyclic vectors and irreducibility for principal series representations. *Transactions of the American Mathematical Society*, 158(1):107–113, 1971.

[41] Nolan R Wallach. *Harmonic Analysis on Homogeneous Spaces*. Marcel Dekker, 1973.

[42] Garth Warner. *Harmonic analysis on semi-simple Lie groups I, II*, volume 188 of *Grundlehren der Mathematischen Wissenshaften*. Springer, 1972.