Essays on analysis

The Gamma function

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My principal references here are [Schwartz:1965]. and [Tate:1950/1967].

Contents

1. Characters as distributions

The Schwartz space $\mathcal{S}(\mathbb{R})$ is the space of all smooth functions f on $\mathbb{R}$ such that

$$f^{(n)}(x) \ll \big(1+|x|\big)^{-N}$$

for all n, $N \geq 0$ or, equivalently, for which

$$\|f\|_{N,n} = \sup_{\mathbb{R}} \big(1+|x|\big)^N \big|f^{(n)}(x)\big| < \infty$$

for all non-negative integers N, n. It is a Fréchet space with these semi-norms. It contains as closed subspace the space $\mathcal{S}(0,\infty)$ of functions that vanish identically for $x \leq 0$.

[s0infty] **Proposition 1.1.** *The space $\mathcal{S}(0,\infty)$ is the same as that of all f in $C^\infty(0,\infty)$ such that*

$$x^N f^{(n)}(x)$$

is bounded on $(0,\infty)$ for all $n \geq 0$, $N \in \mathbb{Z}$.

Proof. Suppose f in $\mathcal{S}(0,\infty)$. Since f is by definition in $\mathcal{S}(\mathbb{R})$, $x^N f^{(n)}(x)$ is bounded for $N \geq 0$. But also by definition, the Taylor series at 0 of a function f in $\mathcal{S}(0,\infty)$ vanishes identically. This implies that for every n, $N \geq 0$.

$$f^{(n)}(x) = x^N \, \frac{f^{(n)}(\theta x)}{n!}$$

for some $0 \leq \theta < 1$. So the condition on f is necessary.

As for sufficiency, it must be shown that if this equation holds for all n, $N \geq 0$ then f extends to a function smooth on all of $\mathbb{R}$ vanishing on $(-\infty, 0]$. This is immediate from the definition of smoothness. $\square$

Let D be the multiplicative derivative xd/dx.

[s0infty-cor] **Corollary 1.2.** *The space $\mathcal{S}(0,\infty)$ is the same as that of all f in $C^\infty(0,\infty)$ such that*

$$x^N(D^n f)(x)$$

is bounded on $(0,\infty)$ for all $n \ge 0$, $N \in \mathbb{Z}$.

[mult-xs] **Corollary 1.3.** *For any s in $\mathbb{C}$ multiplication by x^s is an isomorphism of $\mathcal{S}(0,\infty)$ with itself.*

Proof. This follows from Leibniz's formula for $(x^s f)^{(n)}$. ∎

For every s in $\mathbb{C}$ the integral

$$\langle \Phi_s, f\rangle = \int_0^\infty x^s f(x)\,\frac{dx}{x}$$

defines therefore a continuous linear functional on $\mathcal{S}(0,\infty)$—in effect a distribution.

The multiplicative group $\mathbb{R}^{\mathrm{pos}}$ of positive real numbers acts on both of the spaces $\mathcal{S}(\mathbb{R})$ and $\mathcal{S}(0,\infty)$, as well as on their continuous linear duals, by the formula:

$$\mu_a f(x) = f(a^{-1}x), \quad \langle \mu_a\Phi, f\rangle = \langle \Phi, \mu_{a^{-1}} f\rangle\ .$$

The scale factor a^{-1} rather than a is chosen so that the general linear group acts on the left on functions by means of the analogous formula for $\mathbb{R}^n$. The Lie algebra of $\mathbb{R}^{\mathrm{pos}}$ is spanned by the differential operator $D = xd/dx$, and the representations are smooth in the sense that

$$\lim_{h\to 0}\frac{\mu_{1+h}f - f}{h} = -Df\ .$$

in $\mathcal{S}(\mathbb{R})$. The $-$ sign here comes about because of the choice of a^{-1} rather than a. It will continue to annoy.

[xs-distribution] **Proposition 1.4.** *The distribution Φ_s on $\mathcal{S}(0,\infty)$ is an eigendistribution for μ_a with eigencharacter $|a|^{-s}$.*

Proof. We have

$$\begin{aligned}\langle \mu_a\Phi_s, f\rangle &= \langle \Phi_s, \mu_{a^{-1}} f\rangle \\ &= \int_0^\infty x^s f(ax)\,\frac{dx}{x} \\ &= \int_0^\infty (y/a)^s f(y)\,\frac{dy}{y} \\ &= a^{-s}\int_0^\infty y^s f(y)\,\frac{dy}{y} \\ &= a^{-s}\langle \Phi_s, f\rangle\end{aligned}$$

so that $\mu_a\Phi_s = a^{-s}\Phi_s$ as a distribution. ∎

Differential operators act on distributions. We have

$$\langle \Phi', f\rangle = -\langle \Phi, f'\rangle$$

and hence

$$\langle L\Phi, f\rangle = \langle \Phi, L^*\rangle$$

for any differential operator L, where L^* is its formal adjoint. The distribution Φ_s is an eigendistribution of the operator D as well. One can see this directly:

$$\begin{aligned}\langle D\Phi_s, f\rangle &= -\langle \Phi_s, Df\rangle \\ &= -\int_0^\infty x^s f'(x)\,dx \\ &= s\int_0^\infty x^{s-1} f(x)\,dx \\ &= s\langle \Phi_s, f\rangle\end{aligned}$$

leading to $(D-s)\Phi_s = 0$.

There is a converse to this claim, and there is also a uniqueness theorem for eigendistributions, but I want to put these claims in a slightly wider context. Any smooth representation of $\mathbb{R}^{\text{pos}}$ also determines a representation of its Lie algebra, or in effect a module over the polynomial ring $\mathbb{C}[D]$.

[gp-lalg] **Lemma 1.5.** *The category of smooth finite dimensional representations of $\mathbb{R}^{\text{pos}}$ is equivalent to the category of finite-dimensional modules over the ring $\mathbb{C}[D]$.*

Proof. There are a number of assertions implicit in this. The most important is that if (π, U) is a finite-dimensional module over $\mathbb{C}[D]$, there exists a smooth representation of $\mathbb{R}^{\text{pos}}$ on U with this as its derivative representation. This amounts to a differential equation, and has as solution the matrix exponential:

$$\pi\big(\exp(t)\big) = \exp\big(t\pi(D)\big)\,.$$

Others include these claims, which I leave as exercises:

(a) any D-stable subspace of U is also stable under $\mathbb{R}^{\text{pos}}$;
(b) if $\pi(D)u = su$ then u is an eigenvector of $\mathbb{R}^{\text{pos}}$ with respect to the character $|a|^s$. $\square$

In our situation, there are many likely finite-dimensional D-stable subspaces lurking around, namely the spaces of distributions annihilated by various polynomials $P(D)$, and in particular the polynomial $(D-s)$. We can construct some of these easily. For each s and $m \geq 0$ define a distribution by the formula

$$\langle \Phi_{s,m}, f\rangle = \int_0^\infty x^s (\log x)^m\, f(x)\,\frac{dx}{x}\,.$$

Then

$$\begin{aligned}\langle D\Phi_{s,k}, f\rangle &= -\langle \Phi_{s,m}, Df\rangle \\ &= -\int_0^\infty x^s \log^m x\; Df(x)\,\frac{dx}{x} \\ &= -\int_0^\infty x^s \log^m x\; f'(x)\,dx \\ &= \int_0^\infty (x^s \log^m x)' f(x)\,dx \\ &= \int_0^\infty (sx^{s-1}\log^m x + mx^{s-1}\log^{m-1} x) f(x)\,dx\end{aligned}$$

leading to

$$(D-s)\Phi_{s,m} = m\Phi_{s,m-1}$$

and then

$$(D-s)^{m+1}\Phi_{s,m} = 0$$

which means that $\Phi_{s,m}$ vanishes on the image of $(D+s)^{m+1}$ in $\mathcal{S}(0,\infty)$, since $-D-s$ is the adjoint of $D-s$.

[S-fd] **Lemma 1.6.** *Suppose s to be in $\mathbb{C}$. If V is the space $\mathcal{S}(0,\infty)$, then $(D+s)^m V$ is the subspace of all f such that*

$$\langle \Phi_{s,k}, f\rangle = 0$$

for all $0 \le k < m$.

Since it is the kernel of the map into the array $(\langle \Phi_{s,k}, f\rangle)$ $(k < m)$ the image of $(D+s)^m$ in V is finite-dimensional.

Proof. We have already seen that these $\Phi_{s,k}$ vanish on $(D+s)^m$. Suppose conversely that $\langle \Phi_{s,k}, f\rangle = 0$ for all $k < m$. Let's suppose first that $s = 0$. If

$$\int_0^\infty f(x)\,\frac{dx}{x} = 0$$

then

$$f_1(x) = \int_0^x f(t)\,\frac{dt}{t}$$

will lie in $\mathcal{S}(0,\infty)$ and $Df_1 = f$. We can continue:

$$\int_0^\infty \log x\, f(x)\,\frac{dx}{x} = \int_0^\infty f_1(x)\,\frac{dx}{x}$$

so that if the left hand integral vanishes we can set $f_1 = Df_2$, and so on. This proves the claim for $s = 0$. But for an arbitrary s we deduce that $x^s f = D^m(x^s f_m)$ for some f_m, which is equivalent to $(D+s)^m f_m = f$. ▯

[D-mu] **Lemma 1.7.** *If (π, V) is any Fréchet smooth representation of $\mathbb{R}^{\mathrm{pos}}$ with $V/(D-s)V$ finite-dimensional, then for Φ in $\widehat{V}$ the two conditions (a) $\pi(a)\Phi = |a|^s\Phi$ and (b) $D\Phi = s\Phi$ are equivalent.*

Proof. The group $\mathbb{R}^{\mathrm{pos}}$ acts smoothly on $\widehat{V}[D-s]$ (space of $(D-s)$-torsion in $\widehat{V}$), which is a finite-dimensional space on which $D-s$ acts trivially. From this observation the proof is an easy exercise. ▯

[s-unique] **Proposition 1.8.** *The distribution Φ_s on $\mathcal{S}(0,\infty)$ is up to scalar multiples the only distribution satisfying either (a) $\mu_a\Phi = |a|^{-s}\Phi$ or (b) $D\Phi = s\Phi$.*

Now define $\mathcal{S}[0,\infty)$ to be the space of restrictions to the closed half-line $[0,\infty)$ of functions in $\mathcal{S}(\mathbb{R})$. It may be identified with the quotient $\mathcal{S}(\mathbb{R})/\mathcal{S}(-\infty,0)$. The space $\mathcal{S}(0,\infty)$ is embedded in it, and again the multiplicative group acts smoothly on it.

Does there exist an eigendistribution on $\mathcal{S}[0,\infty)$ extending Φ_s?

For $\Re(s) > 0$ the integral

$$\int_0^\infty x^s f(x)\,\frac{dx}{x}$$

converges and defines a continuous extension of Φ_s to this space, also an eigendistribution. Integration by parts give us

$$\begin{aligned}
\langle \Phi_s, f\rangle &= \int_0^\infty x^s f(x)\,\frac{dx}{x} \\
&= \int_0^\infty x^{s-1} f(x)\,dx \\
&= \left[\frac{f(x)x^s}{s}\right]_0^\infty - \frac{1}{s}\int_0^\infty x^s f'(x)\,dx \\
&= -\frac{1}{s}\int_0^\infty x^{s+1} f'(x)\,\frac{dx}{x} \\
&= -\frac{1}{s}\langle \Phi_{s+1}, f'\rangle
\end{aligned}$$

and then continuing

$$\begin{aligned}\langle \Phi_s, f\rangle &= -\frac{1}{s}\langle \Phi_{s+1}, f'\rangle \\ &= \frac{1}{s(s+1)}\langle \Phi_{s+2}, f''\rangle \\ &= \frac{(-1)^{n+1}}{s(s+1)\dots(s+n)}\langle \Phi_{s+(n+1)}, f^{(n+1)}\rangle\end{aligned}$$

so that Φ_s may be defined on $\mathcal{S}[0,\infty)$ for all s not in $-\mathbb{N}$.

So for every s not in $-\mathbb{N}$ we have an eigendistribution with eigencharacter $|x|^{-s}$. *Is it unique? What happens for $s = -n$?*

Let's see what happens at $s = -n$. It follows from the calculation above that

$$\langle \Phi_s, f\rangle = \frac{(-1)^{n+1}}{s(s+1)\dots(s+n)}\int_0^\infty x^{s+n} f^{(n+1)}(x)\,dx\,.$$

Thus $(s+n)\langle \Phi_s, f\rangle$ as $s \to -n$ has limit

$$-\frac{1}{n!}\int_0^\infty f^{(n+1)}(x)\,dx = \frac{f^{(n)}(0)}{n!}\,.$$

The distribution δ_0 is defined to take f to $f(0)$. Its derivative $\delta_0^{(n)}$ takes f to $(-1)^n f^{(n)}(0)$. The residue of Φ_s at $s = -n$ is therefore $(-1)^n\delta_0^{(n)}/n!$.

[delta-eigen] **Lemmma 1.9.** *The distribution $\delta_0^{(n)}$ is an eigendistribution for the character a^n.*

Proof. Since $f^{(n)}(ax) = a^n f^{(n)}(ax)$. 0

In other words, the character Φ_s fails to be defined precisely when another eigencharacter arises. Is this an accident? One way to understand the situation is by considering the short exact sequence

$$0 \to \mathcal{S}(0,\infty) \to \mathcal{S}[0,\infty) \to \mathbb{C}[[x]] \to 0$$

where the last map is that taking f to its Taylor series at 0, surjective by a classic theorem of Émile Borel. The dual sequence of spaces of distributions is this:

$$0 \to \mathcal{D}_0 \to \widehat{\mathcal{S}}[0,\infty) \to \widehat{\mathcal{S}}(0,\infty) \to 0$$

where $\mathcal{D}_0$ is the space spanned by the Dirac δ_0 and its derivatives. We would like to introduce a long exact sequence of cohomology, but there are some technical difficulties. It is best to replace the group $\mathbb{R}^{\mathrm{pos}}$ by its Lie algebra, which in this case is a one-dimensional vector space spanned by the differential operator $D = x d/dx$. It is easy to see that for each of the outer spaces in the first sequence the image of $D + s$ has finite codimension, which implies that the same is true of the inner one. Thus in looking for an eigendistribution of $\mathbb{R}^{\mathrm{pos}}$ we can look for one of D. The original dual sequence leads to a long exact sequence

$$0 \to \mathcal{D}_0[D-s] \to \widehat{\mathcal{S}}[0,\infty)[D-s] \to \widehat{\mathcal{S}}(0,\infty)[D-s] \to \mathcal{D}_0/(D-s)\mathcal{D}_0 \to \cdots$$

The distribution Φ_s lies in the third term in this sequence, and we want to know (a) is it the image of something in the second? (b) Is it unique? The first is answered affirmatively when something in the fourth term is not 0, the second when the first term is 0. The first space is non-trivial precisely when $s = -n$ for some $n \geq 0$, and these are precisely the cases when both the first and fourth terms are non-zero. So we recover quite formally that Φ_s may be defined for s not in $\mathbb{N}$.

Summarizing:

[delta-res] **Proposition 1.10.** *The distribution Φ_s on $\mathcal{S}[0,\infty)$ is meromorphic on all of $\mathbb{C}$ with residue $(-1)^n\delta_0^{(n)}/n!$ at n. For each s not in $-\mathbb{N}$ it is the unique eigendistribution on $\mathcal{S}[0,\infty)$ for the character a^s. For s in $-\mathbb{N}$ the distribution $\delta_0^{(n)}$ spans the space of eigendistributions for a^n.*

Any function in $\mathcal{S}[0,\infty)$ corresponds to the function (sometimes called its **Mellin transform**)

$$\widehat{f}(s) = \langle \Phi_s, f\rangle \, .$$

[ft-closed] **Proposition 1.11.** *The function $\widehat{f}$ is meromorphic on $\mathbb{C}$ with simple poles on $-\mathbb{N}$. In any bounded vertical strip $|\Re(s)| \le C$ away from the real axis it is rapidly decreasing as a function of $\Im(s)$.*

This is because $D\Phi_s = s\Phi_s$. It is not hard to show that, conversely, any function satisfying these conditions is $\widehat{f}$ for some f in $\mathcal{S}[0,\infty)$.

The full multiplicative group $\mathbb{R}^\times$ acts on its own Schwartz space $\mathcal{S}(\mathbb{R}^\times)$, the subspace of functions in $\mathcal{S}(\mathbb{R})$ whose Taylor series at 0 vanish. We now have distributions

$$\langle \Phi_s^{[m]}, f\rangle = \int_{-\infty}^{\infty} f(x)|x|^s \operatorname{sgn}^m(x)\,\frac{dx}{|x|}$$

for $\Re(s) > 0$ and $m = 0, 1$, which are again eigen-distributions:

$$\mu_a \Phi_s^{[m]} = \operatorname{sgn}^m(a)|a|^{-s}\Phi_s^{[m]} \, .$$

We can express

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)|x|^s \operatorname{sgn}^m(x)\,\frac{dx}{|x|} &= (-1)^m \int_{-\infty}^{0} x^s f(x)\,\frac{dx}{|x|} + \int_0^\infty x^s f(x)\,\frac{dx}{x} \\
&= (-1)^m \int_0^\infty x^s f(-x)\,\frac{dx}{x} + \int_0^\infty x^s f(x)\,\frac{dx}{x} \\
&= \langle \Phi_s, f\rangle + (-1)^m \langle \Phi_s, f^-\rangle \\
&= \frac{(-1)^n}{s(s+1)\dots(s+n-1)} \left\langle \Phi_{s+n}, f^{(n)} + (-1)^m (f^-)^{(n)} \right\rangle
\end{aligned}$$

which means that $\Phi_s^{[m]}$ extends covariantly and meromorphically to $\mathcal{S}(\mathbb{R})$ over all of $\mathbb{C}$ with residue

$$\big((-1)^m + (-1)^n\big)\,\frac{\delta_0^{(n)}}{n!}$$

at $-n$. In particular, there is no pole if the parity of m is different from the parity of n. In this case, we get as value at $-n$

$$\frac{1}{n!}\int_0^\infty \left[\frac{f^{(n)}(x) - f^{(n)}(-x)}{x}\right] dx$$

which always makes sense because the integrand is still a smooth function. For reasons we'll see in a moment this is called the **finite part** of $1/x^{n+1}$.

The integral defining $\Phi_s^{[m]}$ in this case is, when restricted to $\mathcal{S}(\mathbb{R}^\times)$,

$$\int_{\mathbb{R}} |x|^{-n-1}\operatorname{sgn}^{n-1}(x) f(x)\,dx = \int_{\mathbb{R}} x^{-(n+1)} f(x)\,dx \, .$$

2. Finite parts

I want to explore in more detail in this section what happens for values of s in $-\mathbb{N}$.

The first important observation is that the Dirac distributions are eigendistributions.

[dirac-eigen] **Proposition 2.1.** *For $n \ge 0$*

$$\mu_a \delta_0^{(n)} = a^n \delta_0^{(n)}$$

and

$$D\delta_0^{(n)} = -n\,\delta_0^{(n)}\,.$$

Proof. The first is immediate. For the second, if

$$f = f_0 + f_1 x + f_2 x^2 + \cdots + f_m x^m + \cdots$$

then

$$Df = f_1 x + 2f_2 x^2 + \cdots + m f_m x^m + \cdots$$

so

$$\begin{aligned}\langle D\delta_0^{(n)}, f\rangle &= -\langle \delta_0^{(n)}, Df\rangle \\ &= -(-1)^n (Df)^{(n)}(0) \\ &= -(-1)^n n f^{(n)}(0) \\ &= \langle -n\,\delta_0^{(n)}, f\rangle\,. \ \square\end{aligned}$$

There is, however, another distribution Φ such that $D\Phi = -n\Phi$. For $\varepsilon > 0$ and f in $\mathcal{S}$ we have

$$\begin{aligned}\int_\varepsilon^\infty \frac{f(x)}{x^{n+1}}\,dx &= \int_\varepsilon^1 \frac{f(x)}{x^{n+1}}\,dx + \int_1^\infty \frac{f(x)}{x^{n+1}}\,dx \\ &= \int_\varepsilon^1 \frac{f_0 + f_1x + f_2x^2 + \cdots + f_nx^n + \text{ something divisible by } x^{n+1}}{x^{n+1}}\,dx \\ &\qquad + \text{ something finite} \\ &= \big(nf_0\varepsilon^{-n} + (n-1)f_1\varepsilon^{-(n-1)} + \cdots + f_n \log\varepsilon\big) + \text{ something finite}\end{aligned}$$

where $f^{(k)}(0)/k! = f_k$. The limit

$$\lim_{\varepsilon\to 0} \int_\varepsilon^\infty \frac{f(x)}{x^{n+1}}\,dx - \left(nf(0)\varepsilon^{-n} + (n-1)f'(0)\varepsilon^{-(n-1)} + \cdots + \frac{f^{(n)}(0)\log\varepsilon}{n!}\right)$$

therefore exists, and defines a distribution called the **partie finie** of $1/x^{n+1}$ on $(0,\infty)$, or $\mathrm{Pf}(1/x^{n+1})_{x>0}$. Similarly, the limit

$$\lim_{\varepsilon\to 0} \int_{-\infty}^{-\varepsilon} \frac{f(x)}{x^{n+1}}\,dx + \left(nf(0)(-\varepsilon)^{-n} + (n-1)f'(0)(-\varepsilon)^{-(n-1)} + \cdots + \frac{f^{(n)}(0)\log\varepsilon}{n!}\right)$$

exists, and defines a distribution called the *partie finie* of $1/x^{n+1}$ on $(-\infty,0)$, or $\mathrm{Pf}(1/x^{n+1})_{x<0}$. Define Pf itself to be $\mathrm{Pf}_{x>0} + \mathrm{Pf}_{x<0}$. Thus

$$\begin{aligned}\langle \mathrm{Pf}(1/x^{n+1}), f\rangle &= \lim_{\varepsilon\to 0} \int_{|x|\ge\varepsilon} \frac{f(x)}{x^{n+1}}\,dx \\ &\quad - \Big(nf(0)\varepsilon^{-n}\big(1-(-1)^{-n}\big) + (n-1)f'(0)\varepsilon^{-(n-1)}\big(1-(-1)^{-(n-1)}\big) + \cdots + 2f^{(n-1)}(0)\varepsilon^{-1}\Big)\end{aligned}$$

[D-pf] **Proposition 2.2.** *We have*

$$\begin{aligned}\mu_a \mathrm{Pf}(1/x^{n+1}) &= a^n \mathrm{sgn}(a) \mathrm{Pf}(1/x^{n+1}) \\ D\,\mathrm{Pf}(1/x^{n+1}) &= -n\,\mathrm{Pf}(1/x^{n+1}) \\ (d/dx)\mathrm{Pf}(1/x^n) &= -n\,\mathrm{Pf}(1/x^{n+1})\,.\end{aligned}$$

Proof. Left as exercise. ∎

In summary, the two distributions $\delta_0^{(n)}$ and $\mathrm{Pf}(1/x^{n+1})$ span the space of eigendistributions Φ on $\mathbb{R}$ such that $D\Phi = -m\Phi$, or (equivalently) $\mu_a\Phi = a^m\Phi$, but they are distinguished by what μ_{-1} does to them:

$$\mu_{-1}\delta_0^{(n)} = (-1)^n\delta_0^{(n)}, \quad \mu_{-1}\mathrm{Pf}(1/x^{n+1}) = -(-1)^n\mathrm{Pf}(1/x^{n+1})\,.$$

It is relatively easy to check that

$$\langle \mathrm{Pf}(1/x^{n+1}), f\rangle = \frac{1}{n!}\int_0^\infty \left[\frac{f^{(n)}(x) - f^{(n)}(-x)}{x}\right]\frac{dx}{x}$$

which is an expression we saw in the previous section. This has to be, of course, since a cohomological argument like the one we saw earlier shows that there is at most one $\mathbb{R}^\times$-covariant extension to all of $\mathcal{S}(\mathbb{R})$ of the distribution which on $\mathcal{S}(\mathbb{R}^\times)$ is given by the formula

$$\int_{-\infty}^\infty \frac{f(x)}{x^{n+1}}\,dx\,.$$

3. The Gamma function

One function in $\mathcal{S}[0,\infty)$ is the restriction of $f(x) = e^{-x}$ to $[0,\infty)$. The Gamma function is defined to be the integral

$$\Gamma(s) = \int_0^\infty x^s e^{-x}\,\frac{dx}{x} = \langle \Phi_s, e^{-x}\rangle\,.$$

for $\Re(s) > 0$. The argument extending Φ_s in the last section is classical in this case. Since here $f'(x) = -f(x)$, we have the functional equation

$$\Gamma(s+1) = s\,\Gamma(s)$$

and since $\Gamma(1) = 1$, we see by induction that if s is a positive integer n

$$\Gamma(n) = (n-1)!$$

The extension formula can be rewritten as

$$\Gamma(s) = \frac{\Gamma(s+1)}{s}$$

so that we can extend the definition of $\Gamma(s)$ to the region $\Re(s) > -1$, except for $s = 0$. And so on. More explicitly we have

$$\Gamma(s) = \frac{\Gamma(s+n+1)}{(s+n)(s+n-1)\ldots(s+1)s}$$

which allows $\Gamma(s)$ to be defined for $\Re(s) > -n-1$, except at the negative integers, where it will have simple poles (of order one).

Proposition. *For $n \geq 0$ the residue of $\Gamma(s)$ at $-n$ is $(-1)^n/n!$*

Another formula for $\Gamma(s)$ can be obtained by a change of variables $t = x^2$:

$$\Gamma(s) = 2\int_0^\infty e^{-x^2} x^{2s-1}\,dx$$

which can also be written as

$$\Gamma\left(\frac{s}{2}\right) = \int_{-\infty}^\infty |x|^s e^{-x^2}\,\frac{dx}{|x|}\,.$$

Making a change of variables this becomes

$$\pi^{-s/2}\,\Gamma\left(\frac{s}{2}\right) = \int_{-\infty}^\infty |x|^s e^{-\pi x^2}\,\frac{dx}{|x|}\,.$$

This function of s is often expressed as $\zeta_{\mathbb{R}}(s)$ because of its role in functional equations of ζ functions.

4. The volumes and areas of spheres

If we set $s = 1$ in the formula for $\zeta_{\mathbb{R}}$ at the end of the last section, we get

$$\pi^{1/2}\Gamma\left(\frac{1}{2}\right) = \int_{-\infty}^\infty e^{-\pi x^2}\,dx\,.$$

The integral on the right cannot be evaluated as an improper integral, but there is a well known trick one can use to evaluate the infinite integral. We move into two dimensions. We can shift to polar coordinates and get

$$\begin{aligned}
\left(\int_{\mathbb{R}} e^{-\pi x^2}\,dx\right)^2 &= \int_{\mathbb{R}} e^{-\pi x^2}\,dx\cdot\int_{\mathbb{R}} e^{-\pi y^2}\,dy \\
&= \int_{\mathbb{R}^2} e^{-\pi(x^2+y^2)}\,dx\,dy \\
&= \int_0^\infty dr\int_0^{2\pi} e^{-\pi r^2} r\,d\theta \\
&= \int_0^\infty 2\pi r e^{-\pi r^2}\,dr \\
&= \int_0^\infty 2\sqrt{\pi} r e^{-\pi r^2}\,dr \\
&= \int_0^\infty e^{-\pi r^2}(2\pi r)\,dr \\
&= \int_0^\infty e^{-s}\,ds \\
&= 1\,,
\end{aligned}$$

so $\pi^{-1/2}\Gamma(1/2) = 1$, and $\Gamma(1/2) = \sqrt{\pi}$.

We can use this formula and the same trick to find a formula for the volumes of spheres in n dimensions. Let S_{n-1} be the volume of the unit sphere in $\mathbb{R}^n$. Then

$$\begin{aligned}\left(\int_{\mathbb{R}} e^{-\pi x^2}\,dx\right)^n &= 1 \\ &= \int_{\mathbb{R}^n} e^{-\pi r^2}\,dx_1\,\ldots\,dx_n \\ &= \int_0^\infty S_{n-1}\,r^{n-1}e^{-\pi r^2}\,dr \\ &= \int_0^\infty S_{n-1}\,r^n e^{-\pi r^2}\,\frac{dr}{r} \\ &= S_{n-1}\frac{1}{2}\pi^{-n/2}\Gamma(n/2)\,. \\ S_{n-1} &= \frac{2\pi^{n/2}}{\Gamma(n/2)}\,.\end{aligned}$$

For example, the area of the two-sphere in $\mathbb{R}^3$ is

$$S_2 = \frac{2\pi^{3/2}}{\Gamma(3/2)} = \frac{2\pi^{3/2}}{\pi/2} = 4\pi\,.$$

The volume of the n-ball of radius R in $\mathbb{R}^n$ is

$$V_n(R) = \int_0^R S_{n-1}r^{n-1}\,dr = \frac{S_{n-1}R^n}{n}\,.$$

5. Tate's functional equation

Now I introduce the Fourier transform and its interaction with the multiplicative group. For f in $\mathcal{S}(\mathbb{R})$ its Fourier transform is

$$\widehat{f}(\lambda) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i\lambda x}\,dx$$

and this defines an isomorphism of $\mathcal{S}(\mathbb{R})$ with itself. The inverse is

$$f(x) = \int_{-\infty}^{\infty} \widehat{f}(x)e^{2\pi i\lambda x}\,dx$$

Another way to express this is that $\widehat{\widehat{f}} = \mu_{-1}f$.

How do the Fourier transform and the multiplication operators interact?

[tate-ft] **Proposition 5.1.** *For $a \neq 0$*

$$\widehat{\mu_a f} = |a|\,\mu_{a^{-1}}\widehat{f}\,.$$

Proof. Because

$$\begin{aligned}\widehat{\mu_a f}(\lambda) &= \int_{-\infty}^{\infty} [\mu_a f](x) e^{-2\pi i\lambda x}\,dx \\ &= \int_{-\infty}^{\infty} f(a^{-1}x) e^{-2\pi i\lambda x}\,dx \\ &= |a| \int_{-\infty}^{\infty} f(y) e^{-2\pi i\lambda a y}\,dy \\ &= |a|\mu_{a^{-1}}\widehat{f}(\lambda)\,. \;\blacksquare\end{aligned}$$

The Fourier transform $\widehat{\Phi}$ of a distribution Φ is defined by

$$\langle \widehat{\Phi}, f\rangle = \langle \Phi, \widehat{f}\rangle\,.$$

This, as an easy calculation will show, agrees with the definition the Fourier transform on $\mathcal{S}(\mathbb{R})$.

Suppose χ to be a multiplicative character. The distribution $\Phi = \Phi_\chi$ is defiend by

$$\langle \Phi_\chi, \rangle = \int_{\mathbb{R}} \chi(x) f(x)\,\frac{dx}{x}$$

defined by convergence for certain χ and extended meromorphically. What is the Fourier transform of Φ? Since $\mu_a\Phi_\chi = \chi^{-1}(a)\Phi_\chi$ we have

$$\begin{aligned}\langle \mu_a\widehat{\Phi}, f\rangle &= \langle \widehat{\Phi}, \mu_{a^{-1}} f\rangle \\ &= \langle \Phi, \widehat{\mu_{a^{-1}} f}\rangle \\ &= \langle \Phi, |a|^{-1}\mu_a \widehat{f}\rangle \\ &= |a|^{-1}\langle \mu_{a^{-1}}\Phi, \widehat{f}\rangle \\ &= |a|^{-1}\chi(a)\langle \Phi, \widehat{f}\rangle \\ &= |a|^{-1}\chi(a)\langle \widehat{\Phi}, f\rangle\end{aligned}$$

so because of uniqueness $\widehat{\Phi}$ must be a scalar multiple $\gamma_\chi\Phi_{\widetilde{\chi}}$ where $\widetilde{\chi}(a) = |a|\chi^{-1}(a)$. To calculate the scalar γ_χ explicitly, we calculate first the Fourier transform of some particular functions.

[ft-hermite] **Lemma 5.2.** *The Fourier transform of $e^{-\pi x^2}$ is itself.*

Proof. Let $f(x) = e^{-\pi x^2}$. Then

$$\begin{aligned}\widehat{f}(\lambda) &= \int_{-\infty}^{\infty} e^{-2\pi i\lambda x - \pi x^2}\,dx \\ &= e^{-\pi\lambda^2}\int_{-\infty}^{\infty} e^{\pi\lambda^2 - 2\pi i\lambda x - \pi x^2}\,dx \\ &= e^{-\pi\lambda^2}\int_{-\infty}^{\infty} e^{-\pi(x-i\lambda)^2}\,dx \\ &= e^{-\pi\lambda^2}\int_{-\infty}^{\infty} e^{-\pi x^2}\,dx \\ &= e^{-\pi\lambda^2}\,. \;\blacksquare\end{aligned}$$

Let now $\chi(x) = |x|^s$, $\Phi = \Phi_\chi$. Then

$$\begin{aligned}
\langle \widehat{\Phi}, e^{-\pi x^2} \rangle &= \langle \Phi, e^{-\pi x^2} \rangle \\
&= \int_{-\infty}^{\infty} |x|^s e^{-\pi x^2} \, \frac{dx}{|x|} \\
&= \pi^{-s/2} \int_{-\infty}^{\infty} |x|^s e^{-x^2} \, \frac{dx}{|x|} \\
&= \pi^{-s/2} \Gamma(s/2) \\
&= \zeta_{\mathbb{R}}(s) \, .
\end{aligned}$$

Since $\widetilde{\chi} = |x|^{1-s}$:

[tate-1] **Proposition 5.3.** *We have*

$$\widehat{\Phi}_{s,0} = \gamma_s \Phi_{1-s,0}$$

where

$$\gamma_s = \frac{\zeta_{\mathbb{R}}(s)}{\zeta_{\mathbb{R}}(1-s)} \, .$$

This formula isn't quite right for values of s where $\Phi_{s,0}$ or $\Phi_{1-s,0}$ have poles. The simplest way to formulate things is to observe that $\Phi_{s,0}/\zeta(s)$ is entire, and that this formula says that the Fourier transform of $\Phi_{s,0}/\zeta_{\mathbb{R}}(s)$ is $\Phi_{1-s,0}/\zeta_{\mathbb{R}}(1-s)$.

We can reason similarly for $|x|^s \mathrm{sgn}(x)$ with $xe^{\pi x^2}$.

[ft-hermite2] **Proposition 5.4.** *The Fourier transform of $xe^{-\pi x^2}$ is $-i\lambda e^{-\pi\lambda^2}$.*

Proof. Differentiate the equation

$$\int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i \lambda x} \, dx = e^{-\pi\lambda^2}$$

with respect to λ. ∎

Therefore

$$\begin{aligned}
\langle \widehat{\Phi_{s,1}}, xe^{-\pi x^2} \rangle &= \langle \Phi_{s,1}, -ixe^{-\pi x^2} \rangle \\
&= -i \int_{\mathbb{R}} |x|^{s-1} \mathrm{sgn}(x) x e^{-\pi x^2} \, dx \\
&= -i \int_{\mathbb{R}} |x|^s e^{-\pi x^2} \, dx \\
&= -i \zeta_{\mathbb{R}}(1+s) \\
\langle \Phi_{1-s,1}, xe^{-\pi x^2} \rangle &= \zeta_{\mathbb{R}}\big(1 + (1-s)\big)
\end{aligned}$$

and hence:

[tate-1] **Proposition 5.5.** *We have*

$$\widehat{\Phi}_{s,1} = \lambda_s \Phi_{1-s,1}$$

where

$$\lambda_s = -i \, \frac{L_{\mathbb{R}}(s)}{L_{\mathbb{R}}(1-s)}, \quad L_{\mathbb{R}}(s) = \pi^{-(s+1)/2} \, \Gamma\left(\frac{s+1}{2}\right) \, .$$

I conclude with a useful calculation, then I examine some special cases.

[useful-ft] **Proposition 5.6.** *Suppose Φ to be a tempered distribution on $\mathbb{R}$. Then*

(a) the Fourier transform of Φ' *is* $2\pi i\lambda\widehat{\Phi}$*;*
(b) the Fourier transform of $x\Phi$ *is* $\widehat{\Phi}'/(-2\pi i)$.

Proof. First assume Φ to be in $\mathbb{S}(\mathbb{R})$. The first assertion follows from integration by parts, the second by differentiating

$$\int_{-\infty}^{\infty} \Phi(x)e^{-2\pi i\lambda x}\,dx = \widehat{\Phi}(\lambda)$$

with respect to λ. Proving the assertion for distributions follows from this simpler case. ▯

The distributions defined by integrals

$$\int_{\mathbb{R}} x^n f(x)\,dx, \quad \int_{\mathbb{R}} x^n \mathrm{sgn}(x) f(x)\,dx$$

are of particular importance.

[ft-special] **Proposition 5.7.** *For* $n \geq 0$

(a) the Fourier transform of x^n *is* $\delta_0^{(n)}/(-2\pi i)^n$*;*
(b) the transform of $x^n\mathrm{sgn}(x)$ *is*

$$\frac{2\,n!}{(2\pi i)^{n+1}}\,\mathrm{Pf}(1/x^{n+1})\,.$$

As for the first, calculation shows that the transform of 1 is δ_0. But then by the previous lemma the transform of x^n is $\delta_0^{(n)}/(-2\pi i)^n$.

For the second, we can write $x^n\mathrm{sgn}(x)$ as $|x|^{n+1}\mathrm{sgn}^{n+1}(x)/|x|$, so its transform will be an eigendistribution for $|x|^{-n}\mathrm{sgn}^{n+1} = x^n\mathrm{sgn}(x)$, which means that it is a multiple of $\mathrm{Pf}(1/x^{n+1})$. To compute the constant, let's look at $n=0$, where we want the Fourier transform of $\mathrm{sgn}(x)$ itself. Here

$$\begin{aligned}\langle \widehat{\mathrm{sgn}}, xe^{-\pi x^2}\rangle &= \frac{-i}{\pi} = \frac{2}{2\pi i}\\ \langle \mathrm{Pf}(1/x), xe^{-\pi x^2}\rangle &= 1\end{aligned}$$

so that the transform of sgn is $(2/2\pi i)\mathrm{Pf}(1/x)$. Then

$$\widehat{x^n\mathrm{sgn}(x)} = \frac{1}{(-2\pi i)^n}\,\frac{2}{2\pi i}\big(\mathrm{Pf}(1/x)\big)^{(n)} = \frac{2n!}{(2\pi i)^{n+1}}\,\mathrm{Pf}(1/x^{n+1})$$

since $\mathrm{Pf}(1/x^n)' = -n\,\mathrm{Pf}(1/x^{n+1})$.

6. The Beta function

The Gamma function appears in a wide variety of integration formulas. One of the most useful is:

[beta] **Proposition 6.1.** *We have*

$$\int_0^\infty \frac{t^\alpha}{(1+t^2)^\beta}\,dt = \frac{1}{2}\,\frac{\Gamma\left(\dfrac{\alpha+1}{2}\right)\Gamma\left(\beta-\dfrac{\alpha+1}{2}\right)}{\Gamma(\beta)}$$

Proof. Start with

$$\Gamma(s) = 2\int_0^\infty e^{-x^2}x^{2s-1}\,dx\;.$$

Moving to two dimensions and switching to polar coordinates:

$$\begin{aligned}
\Gamma(u)\Gamma(v) &= 4\int\!\!\int_{s\ge 0,t\ge 0} e^{-s^2-t^2}s^{2u-1}t^{2v-1}\,ds\,dt \\
&= 4\int\!\!\int_{r\ge 0,0\le\theta\le\pi/2} e^{-r^2}r^{2(u+v)-1}\cos^{2u-1}\theta\sin^{2v-1}\theta\;dr\,d\theta \\
&= 4\int_{r\ge 0} e^{-r^2}r^{2(u+v)-1}\,dr\;\int_0^{\pi/2}\cos^{2u-1}(\theta)\sin^{2v-1}(\theta)\,d\theta \\
&= \Gamma(u+v)B(u,v) \\
B(u,v) &= \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}\;,
\end{aligned}$$

where

$$B(u,v) = 2\int_0^{\pi/2}\cos^{2u-1}(\theta)\;\sin^{2v-1}(\theta)\,d\theta\;.$$

If we change variables to $t=\tan(\theta)$ we get

$$\begin{aligned}
\theta &= \arctan(t) \\
d\theta &= dt/(1+t^2) \\
\cos(\theta) &= 1/\sqrt{1+t^2} \\
\sin(\theta) &= t/\sqrt{1+t^2}
\end{aligned}$$

leading to

$$\int_0^\infty \frac{t^\alpha}{(1+t^2)^\beta}\,dr = \frac{1}{2}\,\frac{\Gamma\left(\dfrac{\alpha+1}{2}\right)\Gamma\left(\beta-\dfrac{\alpha+1}{2}\right)}{\Gamma(\beta)}\;,$$

and in particular

$$\Gamma^2(1/2) = \int_{-\infty}^\infty \frac{dr}{1+r^2} = \pi,\quad \Gamma(1/2) = \sqrt{\pi}\;.$$

7. The limit product formula

The exponential function e^{-t} can be approximated by finite products.

Lemma. *For any real t we have*

$$e^{-t} = \lim_{n\to\infty} \left(1 - \frac{t}{n}\right)^n .$$

This can be seen most easily by taking logarithms since for $0 \le t < n$

$$\begin{aligned}
\log\left(1 - \frac{t}{n}\right)^n &= n \log\left(1 - \frac{t}{n}\right) \\
&= n\left(-\left(\frac{t}{n}\right) - \frac{1}{2}\left(\frac{t}{n}\right)^2 - \frac{1}{3}\left(\frac{t}{n}\right)^3 - \dots\right) \\
&= -t - \frac{t^2}{2n} - \frac{t^3}{3n^2} - \dots \\
&= -t - T
\end{aligned}$$

where

$$T = \frac{t^2}{2n} + \frac{t^3}{3n^2} + \dots .$$

which converges to 0 as $n \to \infty$.

Another way of putting this is to define

$$\varphi_n(t) = \begin{cases} (1 - t/n)^n & 0 \le t \le n \\ 0 & t > n \end{cases}$$

and then define for each n an approximation $\Gamma_n(s)$ to $\Gamma(s)$:

$$\begin{aligned}
\Gamma_n(s) &= \int_0^\infty t^{s-1}\varphi_n(t)\,dt \\
&= \int_0^n t^{s-1}\left(1 - \frac{t}{n}\right)^n\,dt
\end{aligned}$$

On the one hand, this can be explicitly calculated through repeated integration by parts:

$$\int_0^n t^{s-1}\left(1 - \frac{t}{n}\right)^n dt = \frac{1}{s}\,\frac{n-1}{n(s+1)}\,\frac{n-2}{n(s+2)}\cdots\frac{1}{n(s+n-1)}\int_0^n t^{s+n-1}\,dt = \frac{n!\,n^s}{s(s+1)\dots(s+n)}$$

On the other, since for all fixed t the limit of $\varphi_n(t)$ as $n \to \infty$ is equal to e^{-t}, and both $\varphi_n(t)$ and e^{-t} are small at ∞, this is at least plausible:

Proposition. *For any s with $\Re(s) > 1$ the limit of $\Gamma_n(s)$ as $n \to \infty$ is equal to $\Gamma(s)$. In other words, for any s in $\mathbb{C}$*

$$\Gamma(s) = \lim_{n\to\infty} \frac{n!\,n^s}{s(s+1)\dots(s+n)} .$$

The **Euler constant** γ is defined to be the limit

$$\gamma = \lim_{n\to\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots \frac{1}{n-1}\right) - \log n.$$

The limit product formula implies immediately a limit formula for $1/\Gamma(s)$:

$$\frac{1}{\Gamma(s)} = \lim_{n\to\infty} \left[s\left(1+\frac{s}{1}\right)\left(1+\frac{s}{2}\right)\ldots\left(1+\frac{s}{n-1}\right) n^{-s}\right]$$

but

$$n^{-s} = e^{-s\log n} = e^{-s\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n-1}\right)+s\gamma_n}$$

where $\gamma_n \to \gamma$. Therefore:

Proposition. *The inverse Gamma function has the product expansion*

$$\frac{1}{\Gamma(s)} = se^{\gamma s}\prod_1^\infty \left(1+\frac{s}{n}\right)e^{-s/n}$$

where γ is Euler's constant.

The limit product formula also implies Legendre's duplication formula:

$$\Gamma\left(\frac{1}{2}\right)\Gamma(s) = 2^{s-1}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)$$

Explicitly

$$\begin{aligned}\Gamma\left(\frac{s}{2}\right) &= \lim_{n\to\infty}\frac{2^{n+1}n!\,n^{s/2}}{s(s+2)\ldots(s+2n)}\\ \Gamma\left(\frac{s+1}{2}\right) &= \lim_{n\to\infty}\frac{2^{n+1}n!\,n^{s+1/2}}{(s+1)\ldots(s+2n+1)}\end{aligned}$$

so

$$\begin{aligned}&2^s\,\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)\Big/\Gamma(s)\\ &\qquad= \lim_{n\to\infty} 2^s\frac{2^{n+1}n!\,n^{s/2}}{s(s+2)\ldots(s+2n)}\frac{2^{n+1}n!\,n^{(s+1)/2}}{(s+1)(s+3)\ldots(s+2n+1)}\frac{s(s+1)(s+2)\ldots(s+2n)}{(2n)!\,(2n)^s}\\ &\qquad= \lim_{n\to\infty}\frac{(n!)^2}{(2n)!}\frac{2^{2n+2}\,n^{1/2}}{(s+2n+1)}\\ &\qquad= \lim_{n\to\infty}\frac{(n!)^2 2^{2n+1}}{(2n)!\,\sqrt{n}}\end{aligned}$$

but this last does not depend on s, and is finite since the limit on the left hand side exists, so we may set $s = 1/2$ to see that it is equal to $2\sqrt{\pi}$.

8. The reflection formula

The formula for the Beta function gives us

$$\begin{aligned}\Gamma(s)\Gamma(1-s) &= \int_0^1 u^{s-1}(1-u)^{-s}\,du \\ &= \int_0^\infty \frac{v^{s-1}}{1+v}\,dv \quad (u = v/1+v, v = (u/1-u), du/(1-u) = (1+v)dv)\end{aligned}$$

We can calculate this last integral by means of a contour integral in $\mathbb{C}$. Let C be the path determined by these four segments: (1) along the positive real axis, or just above it, from ϵ to R; (2) around the circle of radius R, counter-clockwise, to the point just below R; (3) along and just below the real axis to ϵ; (4) around the circle of radius ϵ, clockwise, to just above ϵ. We want to calculate the limit of the integral

$$\int_C \frac{z^{s-1}}{1+z}\,dz$$

as $\epsilon \to 0$ and $R \to \infty$.

On the one hand the integrals over the different components converge to

$$\int_0^\infty \frac{z^{s-1}}{1+z}\,dz + 0 - e^{2\pi is}\int_0^\infty \frac{z^{s-1}}{1+z}\,dz + 0 = (1-e^{2\pi is})\int_0^\infty \frac{z^{s-1}}{1+z}\,dz$$

But on the other there is exactly one pole inside the curves C, so the integral is also equal to $-2\pi i e^{\pi is}$. Therefore

$$\Gamma(s)\Gamma(1-s) = \int_0^\infty \frac{z^{s-1}}{1+z}\,dz = \frac{-2\pi i e^{\pi is}}{1-e^{2\pi is}} = \frac{\pi}{\sin \pi s}$$

Incidentally, combined with the product formula for $\Gamma(s)$ this gives the product formula for $\sin \pi s$

$$\sin \pi s = \pi s \prod_1^\infty \left(1 - \frac{s^2}{n^2}\right)$$

9. The Euler-Maclaurin formula

Define a sequence of polynomials

$$\begin{aligned}B_0(x) &= 1 \\ B_1(x) &= x - 1/2 \\ B_2(x) &= x^2 - x + 1/6 \\ &\ldots\end{aligned}$$

recursively determined by

$$B'_{n+1}(x) = nB_n(x), \qquad \int_0^1 B_n(x)\,dx = 0\,.$$

These are the **Bernoulli polynomials**. They determine in turn functions ψ_n by extension to all of $\mathbb{R}$ of period 1.

The following is a simple version of the much more interesting Euler-Maclaurin sum formula:

Proposition. *Suppose f to be a function on the interval $[k, \ell]$ which has continuous second derivatives. Then*

$$f(k) + f(k+1) + \ldots f(\ell-1) = \int_k^\ell f(x)\,dx - \frac{1}{2}\big(f(\ell) - f(k)\big) + \frac{1}{12}\big(f'(\ell) - f'(k)\big) + R_2$$

where

$$R_2 = -\frac{1}{2}\int_k^\ell f''(x)\psi_2(x)\,dx\,.$$

The proof is very simple, a repetition of integration by parts. Suppose m to be an integer with f defined and continuously differentiable on $[m, m+1]$. Then since $\psi_0 = 1$ and $\psi_1' = \psi_0$

$$\begin{aligned}\int_m^{m+1} f(x)\,dx &= \int_m^{m+1} f(x)\psi_0(x)\,dx \\ &= [f(x)\psi_1(x)]_m^{m+1} - \int_m^{m+1} f'(x)\psi_1(x)\,dx \\ &= \frac{1}{2}\big(f(m) + f(m+1)\big) - \int_m^{m+1} f'(x)\psi_1(x)\,dx\end{aligned}$$

since $\psi_1'(x) = 1$, and of course we look at the limit of ψ_1 from above at m, the limit from below at $m+1$.

Then we sum this equation over all the unit sub-intervals of $[k, \ell]$, using the periodicity of ψ_1.

$$\int_k^\ell f(x)\,dx = (1/2)f(k) + f(k+1) + \ldots + f(\ell-1) + (1/2)f(\ell) - \int_k^\ell f'(x)\psi_1(x)\,dx$$

We can rewrite this and apply integration by parts successively:

$$\begin{aligned}&f(k) + f(k+1) + \ldots + f(\ell-1) \\ &\quad= \int_k^\ell f(x)\,dx - \frac{1}{2}\big(f(\ell) - f(k)\big) + \int_k^\ell f'(x)\psi_1(x)\,dx \\ &\quad= \int_k^\ell f(x)\,dx - \frac{1}{2}\big(f(\ell) - f(k)\big) + \frac{1}{2}\big(\psi_2(\ell)f'(\ell) - \psi_2(k)f'(k)\big) - \frac{1}{2}\int_k^\ell f''(x)\psi_2(x)\,dx \\ &\quad= \int_k^\ell f(x)\,dx - \frac{1}{2}\big(f(\ell) - f(k)\big) + \frac{1}{12}\big(f'(\ell) - f'(k)\big) - \frac{1}{2}\int_k^\ell f''(x)\psi_2(x)\,dx\end{aligned}$$

The calculations can be continued to obtain an infinite asymptotic expansion involving the polynomials and their constant terms, the **Bernoulli numbers**.

10. Stirling's formula

We know that the Gamma function can be evaluated as a limit product

$$\begin{aligned}\Gamma(s) &= \lim_{n\to\infty} \frac{(n-1)!\,(n-1)^s}{s(s+1)\dots(s+n-1)} \\ &= \lim_{n\to\infty} \frac{(n-1)!\,n^s}{s(s+1)\dots(s+n-1)} \left(\frac{n-1}{n}\right)^s \\ &= \lim_{n\to\infty} \frac{(n-1)!\,n^s}{s(s+1)\dots(s+n-1)}\end{aligned}$$

We have proven this for s in the domain of convergence of the integral defining $\Gamma(s)$, but in fact the limit exists and defines an analytic function for all s except $s = -n$ with n a non-negative integer, so that by the principle of analytic continuation it must be valid wherever $\Gamma(s)$ is defined. As a consequence

$$\log \Gamma(s) = \lim_{n\to\infty} S_{n-1}(1) - S_n(s) + s\log n$$

where

$$S_n(s) = \log s + \log(s+1) + \dots + \log(s+n-1)$$

We can evaluate $S_n(s)$ by the Euler-Maclaurin formula

$$\begin{aligned}&f(0) + f(1) + \dots + f(n-1) \\ &\qquad = \int_0^n f(x)\,dx - \frac{1}{2}\big(f(n) - f(0)\big) + \frac{\beta_2}{2}\big(f'(n) - f'(0)\big) - \frac{1}{2}\int_0^n f^{(2)}(x)\psi_2(x)\,dx\end{aligned}$$

with

$$f(x) = \log(s+x), \quad f'(x) = \frac{1}{s+x}, \quad f^{(2)}(x) = -\frac{1}{(s+x)^2}$$

so

$$\begin{aligned}&\log s + \log(s+1) + \dots + \log(s+n-1) \\ &\quad = \int_0^n \log(s+x)\,dx - \frac{1}{2}\left[\log(s+n) - \log s\right] + \frac{1}{12}\left[\frac{1}{s+n} - \frac{1}{s}\right] + \frac{1}{2}\int_0^n \frac{\psi_2(x)}{(s+x)^2}\,dx \\ &\quad = \left[x\log x - x\right]_s^{s+n} - \frac{1}{2}\left[\log(s+n) - \log s\right] + \frac{1}{12}\left[\frac{1}{s+n} - \frac{1}{s}\right] + \frac{1}{2}\int_0^n \frac{\psi_2(x)}{(s+x)^2}\,dx \\ &\quad = (s+n-1/2)\log(s+n) - (s-1/2)\log s - n + \frac{1}{12}\left[\frac{1}{s+n} - \frac{1}{s}\right] + \frac{1}{2}\int_0^n \frac{\psi_2(x)}{(s+x)^2}\,dx\end{aligned}$$

and setting $s = 1$, $n-1$ for n:

$$\begin{aligned}&\log 1 + \log 2 + \dots + \log n \\ &\qquad = (n-1/2)\log n - (n-1) + \frac{1}{12}\left[\frac{1}{n} - 1\right] + \frac{1}{2}\int_1^n \frac{\psi_2(x)}{x^2}\,dx \\ &\qquad = (n-1/2)\log n - n + \frac{11}{12} + \frac{1}{12n} + \frac{1}{2}\int_1^n \frac{\psi_2(x)}{x^2}\,dx \\ &\qquad = (n-1/2)\log n - n + C + \frac{1}{12n} - \frac{1}{2}\int_n^\infty \frac{\psi_2(x)}{x^2}\,dx\end{aligned}$$

where we define the constant

$$C = \frac{11}{12} + \frac{1}{2}\int_1^\infty \frac{\psi_2(x)}{x^2}\,dx.$$

Taking limits, therefore

$$\log\Gamma(s) = \left(s - \frac{1}{2}\right)\log s - s + C + \frac{1}{12s} - \frac{1}{2}\int_0^\infty \frac{\psi_2(x)}{(s+x)^2}\,dx.$$

This is valid for all s not on the negative real axis, and gives immediately the generalization of Stirling's formula

$$\Gamma(s) \sim \frac{e^C}{\sqrt{s}}\left(\frac{s}{e}\right)^s$$

as s goes to infinity in any region

$$-\pi + \delta \le \arg(s) \le \pi - \delta$$

since the remainder will have a uniform estimate in this region. The constant C can be evaluated by letting $t \to \pm\infty$ in the reflection formula. On the one hand

$$\begin{aligned}\Gamma(it)\Gamma(-it) &= -\frac{\pi}{it\sin\pi it} \\ &= -\frac{2\pi i}{it[e^{-\pi t} - e^{\pi t}]} \\ &\sim 2\pi t^{-1}e^{-\pi t}\end{aligned}$$

while on the other

$$\begin{aligned}\Gamma(it)\Gamma(-it) &\sim \frac{e^C}{\sqrt{it}}\left(\frac{it}{e}\right)^{it}\frac{e^C}{\sqrt{-it}}\left(\frac{-it}{e}\right)^{-it} \\ &= \frac{e^{2C}}{t}(i)^{it}(-i)^{-it} \\ &= \frac{e^{2C}}{t}e^{(it)(\pi i)/2}e^{(-it)(-\pi i)/2} \\ &= \frac{e^{2C}}{t}e^{-\pi t}\end{aligned}$$

I recall that

$$x^y = e^{y\log x}$$

where $\log$ is given its principal value. This gives

$$C = \log\sqrt{2\pi}$$

and finally the explicit version

Proposition. *(Stirling's asymptotic formula) As s goes to ∞ in the region*

$$-\pi + \delta \le \arg(s) \le \pi - \delta$$

we have the asymptotic estimate

$$\Gamma(s) \sim \sqrt{\frac{2\pi}{s}}\left(\frac{s}{e}\right)^s$$

11. References

Laurent Schwartz, **Méthodes mathématiques pour les sciences physiques**, Hermann, 1965. Also available in English. This is very readable, although with few rigourous proofs.

John Tate, 'Fourier analysis in number fields and Hecke's zeta functions', pp. 305–347 in in **Algebraic number theory**, edited by J. W. S. Cassels and A. Fröhlich, Thompson, 1967. This is his Princeton Ph. D. thesis.