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Essays on harmonic analysis in number theory

Fourier series

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Contents

1. The group Z

In this section, let G be the additive group of integers $\mathbb{Z}$, in many ways the simplest group beyond the finite ones. Assign to each point of G measure 1, so the integral of $f(n)$ over G becomes just the sum $\sum_{\mathbb{Z}} f(n)$. There are several candidates for the analogue of what was $\mathbb{C}[G]$ in the case when G was finite. We shall work mainly with three of them:

(a) the **Schwartz space** $\mathcal{S}(\mathbb{Z})$ consisting of all complex-valued functions $f(n)$ that are rapidly decreasing at infinity:

$$|f|_N = \sup_n |f(n)|(1+|n|)^N < \infty$$

for all $N > 0$;
(b) the space of functions F of moderate growth:

$$|||F|||_N = \sup_n \frac{|f(n)|}{(1+|n|)^N} < \infty$$

for some $N > 0$;
(c) the Hilbert space $\mathrm{L}^2(\mathbb{Z})$ consisting of all functions with

$$\|f\|^2 = \sum |f(n)|^2 < \infty\,.$$

On this space we have also the Hermitian inner product

$$f \bullet h = \sum_n f(n)\overline{h(n)}\,.$$

These are all infinite-dimensional vector spaces. It is important that they are all **topological** vector spaces—that is to say, in each case we can say when one vector is close to another, or when a sequence of vectors convergences to a vector. For the Schwartz space, $f_n \to f$ if $|f - f_n| \to 0$ as $n \to \infty$, for all positive integers n. In the third, $f_n \to f$ if $\|f - f_n\| \to 0$ as $n \to \infty$. In the second case, defining the topology is a bit tricky, and we won't need it, anyway.

For F in the second space and f in $\mathcal{S}(\mathbb{Z})$ the infinite sum

$$\langle F, f \rangle = \sum_{\mathbb{Z}} F(n) f(n)$$

is defined. This pairing between the two spaces is continuous in the sense that if $|||F|||_N < \infty$ then

$$\langle F, f \rangle \le |||F|||_N |f|_{N+2} \sum \frac{1}{(1+|n|)^2}\,.$$

Conversely:

[z-dual] **Proposition 1.1.** *If Φ: $f \mapsto \Phi(f)$ is any linear map from $\mathcal{S}(\mathbb{Z})$ to $\mathbb{C}$ such that*

$$|\Phi(f)| \le C\,|f|_N$$

for some N, $C > 0$ then there exists F of moderate growth such that $\Phi(f) = \langle F, f \rangle$.

Thus the functions of moderate growth make up the full continuous dual of the Schwartz space. For this reason, I'll write the second space as $\widehat{\mathcal{S}}(\mathbb{Z})$.

Proof. Define $F(n) = \langle \Phi, \varepsilon_n \rangle$ where

$$\varepsilon_n(k) = \begin{cases} 1 & k = n \\ 0 & \text{otherwise.} \end{cases}$$

The growth condition on F is immediate from the continuity of Φ. We leave it as an exercise to show that $\Phi(f) = \langle F, f \rangle$. ∎

Incidentally, a function in $L^2(\mathbb{Z})$ is bounded, so we may embed $L^2(\mathbb{Z})$ into $\widehat{\mathcal{S}}(\mathbb{Z})$.

A **character** of $\mathbb{Z}$ is a homomorphism ψ into $\mathbb{C}^\times$. It is unitary if the image is in $\mathbb{S}^1$. Either set forms a group under multiplication. The map $\psi \mapsto \psi(1)$ is a bijection of characters with $\mathbb{C}^\times$, or of unitary characters with $\mathbb{S}^1$. Define the group $\mathbb{Z}^*$ dual to $\mathbb{Z}$ to be that of ita unitary characters.

2. The group S

In this section we look at the group $G = \mathbb{S}^1$, the unit circle in $\mathbb{C}$. It may be identified with the quotient of the imaginary axis by the discrete subgroup $2\pi i\mathbb{Z}$ via $x \mapsto e^x$, or with $\mathbb{R}/\mathbb{Z}$ vis $x \mapsto e^{2\pi i x}$. It is sometimes important to keep in mind that this last identification depends on a choice of $\sqrt{-1}$.

A character of G is a continuous homomorphism into $\mathbb{C}^\times$.

[unitary] **Proposition 2.1.** *Every character of G has image in $\mathbb{S}^1$—it is unitary.*

Proof. If ψ is a character of G then so is $|\psi|$. The image of G must be a compact subgroup of the multiplicative group of positive real numbers. The only possibility is 1. ∎

The group G^* dual to G is therefore the group of continuous homomorphisms into $\mathbb{S}^1$. There is an obvious family of characters of G—to each n corresponds the character $z \mapsto z^n$. We shall see later that every continuous character is one of these, but for the monet let's see a weak version of this.

[weak-chars] **Proposition 2.2.** *Any differentiable character of G is of the form $z \mapsto z^n$.*

Proof. Let ψ be a differentiable character of G. For x in $\mathbb{R}$ let $f(x) = \psi(e^{2\pi i x})$. Then $f(x+y) = f(x) \cdot f(y)$ and

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= f(x) \lim_{h \to 0} \frac{f(h) - f(0)}{h} \\ &= f(x) \cdot f'(0)\,. \end{aligned}$$

All solutions to this differential equation are of the form $Ce^{f'(0)x}$ for some constant C. Since $f(0) = 1$, $C = 1$; and since f has period 1 we must have $f'(0)$ in $2\pi i \mathbb{Z}$. ∎

3. Fourier series

Assign $G = \mathbb{S}^1$ the rotation-invariant measure of total measure 1. Thus

$$\int_G f(z)\,dz = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta})\,d\theta\,.$$

[int-null] **Proposition 3.1.** *If χ is a character of G then*

$$\int_G \chi(x)\,dx = \begin{cases} 1 & \text{if } \chi = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. For g in G, since the integral is rotation invariant, we have

$$\int_G \chi(x)\,dx = \int_G \lambda_g \chi(x)\,dx = \int_G \chi(xg^{-1})\,dx = \chi^{-1}(g) \int_G \chi(x)\,dx$$

so that if we can choose g with $\chi(g) \neq 1$ the integral must be 0. ∎

If $\chi(x) = z^n$ with $n \neq 0$, this comes from the calculation of the integral

$$\int_0^1 e^{2\pi i n x}\,dx = \left[\frac{e^{2\pi i n x}}{2\pi i n}\right]_0^1,$$

and in fact this is the only case we'll need, but we are not assuming yet that all characters are of this form.

[orth-chars] **Corollary 3.2.** *If χ, ρ are characters of G then*

$$\chi \bullet \rho = \begin{cases} 1 & \text{if } \chi = \rho \\ 0 & \text{otherwise.} \end{cases}$$

The characters z^n thus form an orthonormal set within the space of all continuous functions on G. We shall shwo that it is a basis of functions on G in some way, which means that for functions f of various kinds there exists a series $\sum f_m z^m$ converging to f in some sense. If this convergence is reasonable, we then expect

$$\begin{aligned} f \bullet z^{-n} &= \sum f_m z^{-n} \bullet z^m \\ &= f_n\,. \end{aligned}$$

Based on this, we define the **Fourier transform** of an integrable function f on G to be the sequence $\mathcal{F}f = (f_m)$ where

$$f_m = f \bullet z^{-m} .$$

This may be identified with the function $m \mapsto f_m$ on $\mathbb{Z}$. The **Fourier coefficients** are the individual terms in the sequence. and inquire in what circumstances the series

$$\sum f_m z^m$$

converges to f. This is a subtle question, but we shall require an answer only in a simple case. A **piece-wise continuous** function on G is one that is continuous on a finite number of arcs partitioning G. It is trivial to see that the Fourier coefficients of such a function are bounded, but in fact something stronger is true.

[riemann-lebesgue] **Lemma 3.3.** (Riemann-Lebesgue Lemma) *If $f(x)$ is in $\mathrm{L}^2(G)$ then $|f_n|$ has limit 0 as $|n|$ goes to infinity.*

This is the most elementary version of the Riemann-Lebesgue Lemma.

Proof. We have

$$\begin{aligned}\Big\| f(x) - \sum_{|n|\le R} (f \bullet z^n)\, z^n \Big\|^2 &= \Big(f(x) - \sum_{|n|\le R} (f \bullet z^n)\, z^n\Big) \bullet \Big(f(x) - \sum_{|n|\le R} (f \bullet z^n)\, z^n\Big) \\ &= \|f\|^2 - 2 \sum_{|n|\le R} \big|f \bullet z^n\big|^{2`} + \sum_{|m|,|n|\le R} \big(f \bullet z^n\big)\big(f \bullet z^m\big)(z^n \bullet z^m) \\ &= \|f\|^2 - \sum_{|n|\le R} \big|f \bullet z^n\big|^2\end{aligned}$$

Therefore

$$\sum_{n\in\mathbb{Z}} |f \bullet z^n|^2 \le \|f\|^2 \quad \text{(Bessel's inequality)},$$

which implies the Lemma. ▯

We shalls ee later that Bessel's inequality becomes an equality.

[trans-z] **Lemma 3.4.** *For any integrable function on G and g in G*

$$\mathcal{F}\lambda_g f = z^{-n} \cdot \mathcal{F}f .$$

This is a straightforward calculation.

For x in $\mathbb{R}$, set

$$z(x) = e^{2\pi i x} .$$

[fs-convergence] **Theorem 3.5.** *Let f be a piece-wise continuous function on G and $f_m = \mathcal{F}f(m)$. Then*

$$\lim_{N\to\infty} \Big|f(z) - \sum_{|n|\le N} f_n z^n\Big| = 0$$

whenever z is a point where f is differentiable.

Proof. Applying the previous Lemma, we may assume $z = 1$ Then

$$
\begin{aligned}
f(1) - \sum_{|n| \le N} f_m z^m &= \sum_{|n| \le N} \int_0^1 \big(f(0) - f(z)\big) z^{-m}\, dx \\
&= \int_0^1 \big(f(0) - f(z)\big) \Big(\sum_{|n| \le N} z^{-m} \Big)\, dx \\
&= \int_0^1 \big(f(0) - f(z)\big) \frac{z^{-N} - z^{N+1}}{1 - z}\, dx \\
&= \int_0^1 \frac{f(z) - f(0)}{z - 1} \big(z^{-N} - z^{N+1}\big)\, dx\,.
\end{aligned}
$$

But the function

$$\frac{f(z) - f(0)}{z - 1}$$

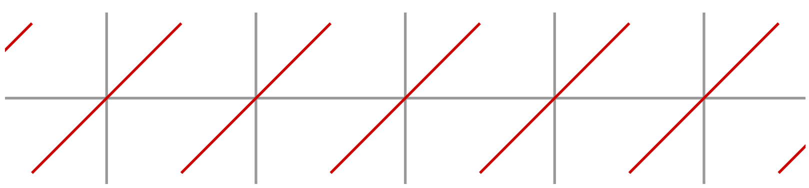
which is *a priori* defined for $z \ne 1$ is in fact continuous with at 1 since f is differentiable there. According
♣ [riemann-lebesgue] to Lemma 3.3, the right hand side converges to 0.

Exercise. *Show that the Fourier series of f converges to f uniformly on any interval in the neighbourhood of which f is C^1.*

Example. For x in $\mathbb{R}$, let $\{x\}$ be x modulo 1. Suppose

$$f = x - 1/2\,.$$

Its graph on $\mathbb{R}$ is therefore:



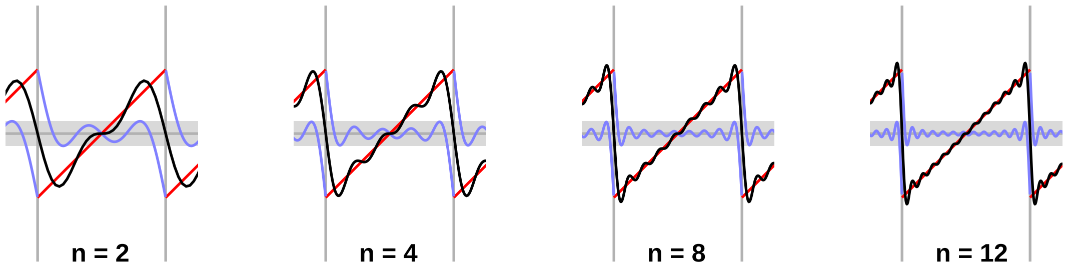
Its Fourier coefficients are

$$\int_0^1 (x - 1/2) e^{-2\pi i n x}\, dx = \begin{cases} 0 & \text{if } n = 0 \\ \dfrac{-1}{2\pi i n} & \text{otherwise.} \end{cases}$$

The terms for $e^{\pm 2\pi i n x}$ coalesce to become

$$-\frac{\sin 2\pi n x}{\pi n}$$

The convergence of Fourier series in the neighbourhood of points of discontuity is peculiar, because of something called **Gibb's phenomenon**. The Fourier approximations oscillate around such a point. The
♣ [fs-convergence] oscillations diusappear, as they have to according to Theorem 3.5, but thyat is because the oscaillations hoft towards the singularity, while oscillating at a fixed magnitude. The following pictures demonstrate this. In them, the red graph is the original function, the black one is the Fourier series sum of n terms, and the blue line shows the difference. That the magnitude of oscillations remains constant is shown by the grey band.



We shall see later a way in which Fourier series converge other than by point-wise convergence of functions.

4. Fourier series of smooth functions

Let D be the derivative $f \mapsto f'(x)$.

[fs-diff] **Proposition 4.1.** *If f on $\mathbb{R}/\mathbb{Z}$ has a piece-wise continuous derivative then*

$$\mathcal{F}Df = 2\pi in\mathcal{F}f\,.$$

This is a straightforward integration by parts.

[fs-schwartz] **Corollary 4.2.** *The Fourier transform of a smooth function on $G = \mathbb{S}^1$ lies in the Schwartz space of $\mathbb{Z}$.*

The converse is also true:

[fs-converse] **Proposition 4.3.** *If (f_m) is in the Schwartz space of $\mathbb{Z}$ then the series*

$$\sum f_m z(x)^m$$

converges uniformly to a smooth function on G.

[fs-iso] **Proposition 4.4.** *The map taking f in $C^\infty(\mathbb{R}/\mathbb{Z})$ to its Fourier transform is a continuous bijection of $C^\infty(\mathbb{R}/\mathbb{Z})$ with $\mathcal{S}(\mathbb{Z})$.*

5. The Plancherel theorem

[parseval-fs] **Proposition 5.1.** *For f, h in $C^\infty(\mathbb{R}/\mathbb{Z})$ we have*

$$f \bullet h = \mathcal{F}f \bullet \mathcal{F}h\,.$$

Term by term integration of Fourier series always works.

6. Distributions

The space of distributions on $G = \mathbb{S}^1$ is the **topological dual** of the space $V^\infty(G)$. Every integrable function F, for example, defines by integration a distribution

$$\langle F, f\rangle = \int_0^1 F(x)f(x)\,dx\,.$$

But there are many more—most notably the Dirac delta

$$\langle \delta_x, f\rangle = f(x)$$

for x in G.

If F, f both lie in $C^\infty(G)$ then

$$\int_0^1 F'(x)f(x)\,dx = [F(x)f(x)]_0^1 - \int_0^1 F(x)f'(x)\,dx\,.$$

In other words, for F in $C^\infty(G)$ and in fact for any F with a piece-wise continuous derivative

$$\langle DF, f\rangle = - < F, Df > \,.$$

which suggests defining the derivative of any distribution:

$$\langle D\Phi, f\rangle = -\langle \Phi, Df\rangle\,.$$

The Fourier transform identifies the space of distributions on $\mathbb{R}/\mathbb{Z}$ with the dual $\widehat{\mathcal{S}}(\mathbb{Z})$—the space of functions of moderate growth on $\mathbb{Z}$.

For any function (f_m) in $\widehat{\mathcal{S}}(\mathbb{Z})$ the series

$$\sum f_m z(x)^m$$

converges to a distribution on $\mathbb{R}/\mathbb{Z}$.

7. Bernoulli polynomials

The most interesting Fourier series are those of functions that are not smooth.

for each x in $\mathbb{R}$ let $\{x\}$ be x modulo 1, or equivalently $x - \lfloor x\rfloor$ (where $\lfloor x\rfloor$ is the integer n such that $n \le x < n+1$). If f is any function on $[0,1]$, it determines the periodic function $f(\{x\})$. We have alreday seen in this regard the function $\{x\} - 1/2$.

Certain polynomials, the **Bernoulli polynomials** $B_n(x)$, play a special role in this treatment. A first version are defined by simple recursion:

$$\begin{aligned} \beta_0(x) &= 1 \\ \beta_n'(x) &= \beta_{n-1}(x) \\ \int_0^1 \beta_n(x)\,dx &= 0 (n > 0)\,. \end{aligned}$$

The condition on the derivative of β_n specifies it up to a constant of integration, and the condition of vanishing integral then fixes it unambiguously.

The first few are

$$\begin{aligned}\beta_0(x) &= 1\\ \beta_1(x) &= x - \frac{1}{2}\\ \beta_2(x) &= \frac{x^2}{2} - \frac{x}{2} + \frac{1}{24}\end{aligned}$$

and it is easy to see that we have

$$\beta_n(x) = \frac{x^n}{n!} + \text{ terms of lower degree.}$$

This suggests that it is a good idea to work instead with the polynomials

$$B_n = n!\beta_n(x)$$

which are the Bernoulli polynomials.

Thus

$$\begin{aligned}\boldsymbol{B}_n(x) &= n!\,\beta_n(x)\\ \boldsymbol{B}_n(x) &= n\int \boldsymbol{B}_{n-1}(x)\,dx, \quad \int_0^1 \boldsymbol{B}_n(x)\,dx = 0 \quad (n \ge 1)\end{aligned}$$

Here are the first several:

$$\begin{aligned}\boldsymbol{B}_0 &= 1\\ \boldsymbol{B}_1 &= x - {}^1/{}_2\\ \boldsymbol{B}_2 &= x^2 - x + {}^1/{}_6\\ \boldsymbol{B}_3 &= x^3 - {}^3/{}_2\,x^2 + {}^1/{}_2\,x\\ \boldsymbol{B}_4 &= x^4 - 2x^3 + x^2 - {}^1/{}_{30}\\ \boldsymbol{B}_5 &= x^5 - {}^5/{}_4\,x^4 + {}^5/{}_3\,x^3 - {}^1/{}_6\,x\\ \boldsymbol{B}_6 &= x^6 - 3x^5 + {}^5/{}_2\,x^4 - {}^1/{}_2\,x^2 + {}^1/{}_{42}\\ &\ldots\end{aligned}$$

The **Bernoulli numbers** are the constants of the Bernoulli polynomials: $B_n = \boldsymbol{B}_n(0)$. Thje first several are:

$$\begin{aligned}B_0 &= 1\\ B_1 &= -1/2\\ B_2 &= 1/6\\ B_3 &= 0\\ B_4 &= -1/30\\ B_5 &= 0\\ B_6 &= 1/42\\ B_7 &= 0\\ B_8 &= -1/30\end{aligned}$$

as one can more or less easily calculate.

[brer-props] **Proposition 7.1.** *We have*

$$\boldsymbol{B}_n(1-x) = (-1)^n \boldsymbol{B}_n(x)$$
$$\boldsymbol{B}_n(x) = \sum_0^n \binom{n}{k} B_k x^p$$
$$B_n = -\frac{1}{n+1}\sum_{k=1}^n \binom{n+1}{k} B_{n-k}$$

[ber-vanishing] **Proposition 7.2.** *For odd* $n > 1$, $B_n = 0$.

The early Bernoulli numbers give little idea of what the rest are like:

$$\begin{aligned}
B_{10} &= 5/66 \\
B_{12} &= -691/2730 \\
B_{14} &= 7/6 \\
B_{16} &= -3617/510 \\
B_{18} &= 43867/798 \\
B_{20} &= -174611/330 \\
B_{22} &= 854513/138 \\
B_{24} &= -236364091/2730 \\
B_{26} &= 8553103/6 \\
B_{28} &= -23749461029/870 \\
B_{30} &= 8615841276005/14322 \\
B_{32} &= -7709321041217/510 \\
B_{34} &= 2577687858367/6 \\
B_{36} &= -26315271553053477373/1919190
\end{aligned}$$

We shall see in a moment an asymptotic estimate of these curious numbers.

[ber-fs] **Proposition 7.3.** *For* $m \geq 1$ *the Fourier series for* $\boldsymbol{B}_n$ *converges conditionally on* $(0,1)$*:*

$$\boldsymbol{B}_n(x) = \frac{-m!}{(2\pi i)^m} \sum_{n \neq 0} \frac{1}{n^m}\, e^{2\pi i n x}$$

For $n > 1$ *it converges uniformly and absolutely on the whole interval* $[0,1]$.

Proof. We already know that

$$\boldsymbol{B}_1(x) = -\left(\frac{\sin 2\pi x}{\pi} + \frac{\sin 4\pi x}{2\pi} + \frac{\sin 6\pi x}{3\pi} + \cdots\right)$$

which implies the Proposition by the definition of $\boldsymbol{B}_n$ and term-by-term integration. ///

8. L-functions

The zeta function $\zeta(s)$ is defined by the series

$$\zeta(s) = \sum_{1}^{\infty} \frac{1}{n^s}$$

which converges, by a comparison to the integral

$$\int_1^\infty \frac{dx}{x^s}$$

for $\mathrm{RE}(s) > 1$. If $N > 0$ is a positive integer and χ is a primitive character of $(\mathbb{Z}/N)^\times$, it may be extended to all of $\mathbb{N}$ by setting it equal to 0 off the units modulo N. The associated series

$$L_\chi(s) = \sum \frac{\chi(n)}{n^s} = \sum_{(n,q)=1} \frac{\chi(n)}{n^s}$$

also converges for $\mathrm{RE}(s) > 1$, and is called the **Dirichlet series** of χ. The ζ function is the special case with $\chi = 1$ and $N = 1$.

A character χ is said to be **even** if $\chi(-1) = 1$ and **odd** if $\chi(-1) = 1$. The function $L_\chi(s)$ may evaluated at even positive integers if χ is even and odd ones if χ is odd. The simplest case is:

[zeta-even] **Proposition 8.1.** *For any integer $m > 1$*

$$\zeta(2m) = \frac{(-1)^m (2\pi)^{2m}}{2(2m)!} B_m \, .$$

This follows immediately from the calculation of the Fourier series of $\boldsymbol{B}_m(x)$.

As a consequence:

[asymptotic-ber] **Corollary 8.2.** *We have the asymptotic estimate*

$$|B_m| \sim \frac{2(2m)!}{(2\pi)^{2m}}$$

for $m \to \infty$.

Suppose now φ to be any function on $\mathbb{Z}/N$ with $\varphi(-x) = (-1)^m \varphi(x)$. Let $\mathcal{F}\varphi$ be its finite Fourier transform:

$$\mathcal{F}\varphi(y) = \frac{1}{\sqrt{N}} \sum_{x \in \mathbb{Z}/N} \varphi(x) e^{-2\pi i x y/N}$$

and

$$\varphi(x) = \frac{1}{\sqrt{N}} \sum_{y \in \mathbb{Z}/N} \mathcal{F}\varphi(y) e^{2\pi i x y/N}$$

Assume further, in case 0, that $\mathcal{F}\varphi(0) = 0$ or, equivalently, that

$$\sum_{x \in \mathbb{Z}/N} \varphi(x) = 0 \, .$$

I then define

$$L(\varphi, s) = \sum_{n > 0} \frac{\varphi(n)}{n^s} \, .$$

Since we can write this as

$$L(\varphi, s) = \frac{1}{\sqrt{N}} \sum_{n>0} \frac{\sum_{y \in \mathbb{Z}/N} \mathcal{F}\varphi(y) e^{2\pi n y}}{n^s}$$

Because of the assum,ption on the sum of values, it will be analytic in a region $\mathrm{RE}(s) > 0$, and according ♣ [ber-fs] to Proposition 7.3 it can be evaluated at even positive integers if m is even, and at odd ones if m is odd.