SOME RECENT RESULTS ABOUT (TERNARY) QUADRATIC FORMS

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Introduction

This paper is meant to be a survey of some recent results in the theory of quadratic forms, specifically the question of representing integers by a positive definite form in 3 variables. The question of representability by a ternary form is particularly interesting because it exhibits both analytic and algebraic features, in contrast to the case of forms in ≥ 4 variables.

While we will briefly discuss the theory for forms in $n \geq 3$ variables, we will not do justice to its long and rich history. For a more complete picture of the history of the problem we recommend the books [Co, Gr, Ca], as well as the survey papers [Du2, Iw4, Sa2] which focus on the analytic approach used to resolve it.

We would especially like to thank William Duke for his excellent survey paper [Du2] which introduced the author to the subject, and hope this article will be a natural continuation of that paper.

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§1 Main questions

We will be interested in studying an integral quadratic form Q in n variables, by which we mean a homogeneous polynomial

$$Q(\vec{x}) = \sum_{1 \leq i \leq j \leq n} a_{ij} x_i x_j$$

of degree 2 with coefficients a_{ij} in $\mathbb{Z}$. Specifically, we will be interested in which integers $m \in \mathbb{Z}$ are **represented** by Q, meaning that we can find some vector $\vec{x} \in \mathbb{Z}^n$ so that $Q(\vec{x}) = m$. Some familiar answers to this question can be found in the following theorems:

Theorem (Lagrange, 1772). *An integer m can be written as a sum of four integer squares $\iff m \geq 0$.*

Theorem (Legendre, 1798). *An integer m can be written as a sum of three integer squares $\iff m \geq 0$ and $m \neq 4^a(8k+7)$ for any $a, k \in \mathbb{Z}$.*

Theorem (Fermat, 1640). *A prime number $p > 2$ can be written as a sum of two integer squares $\iff p \equiv 1 \pmod 4$.*

While we can ask about which integers are represented by a given quadratic form Q, it is also interesting to ask in *how many ways* is m represented by Q? If we let $r_Q(m)$ count the number of ways of representing m by Q (i.e., the number of vectors $\vec{x}$), then we are asking for a description of the function $r_Q(m)$. In these terms, the question of which m are represented asks when $r_Q(m)$ is > 0. In some instances one can actually find exact formulas for $r_Q(m)$, though as we will see later such formulas are quite rare. However, one familiar example of an exact formula is given by:

Theorem (Jacobi, 1828). *If $Q = x^2 + y^2 + z^2 + w^2$ and $m \in \mathbb{Z} > 0$, then*

$$r_Q(m) = 8 \sum_{\substack{d \mid m \\ 4 \nmid d > 0}} d.$$

Given the scarcity of exact formulas for $r_Q(m)$, one might instead hope to understand the *approximate size* of $r_Q(m)$. This somewhat more modest goal is still quite useful since a reasonable answer provides enough information to determine which numbers are represented by Q. In the next few sections, we will focus on answering these two main questions:

Question 1: *Can we describe which integers m are represented by Q?*
(When is $r_Q(m) > 0$?)

Question 2: *In about how many ways is m represented by Q?*
(How big is $r_Q(m)$?)

§2 Some local considerations

One early idea to understand which numbers m could be represented by Q was to look at which numbers are represented modulo M for every $M > 0$ and also to notice restrictions on the sign of m. By the Chinese Remainder Theorem, it is enough to consider congruence conditions modulo p^α for all powers α of every prime number p, and also conditions on the sign of m.

These conditions are called "local" conditions and can be thought of as conditions associated either to a prime number p (for p-power congruences) or to ∞ (for the sign). This language comes from the different ways of completing the integers to fill in the gaps, and can be thought of as conditions at each of these completed places. The choice of a prime p defines the p-adic integers $\mathbb{Z}_p$, and the other choice ∞ gives the real numbers $\mathbb{Z}_\infty = \mathbb{R}$. In general, we refer to statements over $\mathbb{Z}_v$ as **local** and the corresponding statements over $\mathbb{Z}$ as **global**. Much of our effort will be in understanding the passage between these two realms, referred to as a **local-global principle**.

When Q is the sum of 2, 3, and 4 squares, this idea recovers the conditions of the first three theorems. It turns out that in those cases the local conditions are actually enough to guarantee that m is represented by Q, but this is a very special occurence. In fact, it is a similar kind of local characterization that allows one to write down exact formulas for $r_Q(m)$. We state this as our first strategy towards solving Question 1.

Strategy 1: *Check if m is represented by Q locally.*
(Can we solve $Q(\vec{x}_v) = m$ with $\vec{x}_v$ in $\mathbb{Z}_v$ at all places v?)

When this approach succeeds, it often solves both Questions 1 and 2. However, the problem is that by looking at Q locally we may not in general be able to distinguish it from other quadratic forms. Therefore any local statement would at best be making a statement about all of the forms which locally look the same. We call the set $\mathrm{Gen}(Q)$ of such forms the **genus** of Q.

To understand this better, let's look at how many forms are in the genus. One important thing to notice is that two quadratic forms Q_1 and Q_2 look the same (so we will call them **equivalent**) if they differ from each other by some invertible (linear) change of variables with integer coefficients. For convenience, we write this equivalence as $Q_1 \sim_{\mathbb{Z}} Q_2$. Notice that if Q_1 and Q_2 are equivalent then $r_{Q_1}(m) = r_{Q_2}(m)$ for all integers m. (Earlier when we said that Q_1 and Q_2 are locally equivalent, we meant that $Q_1 \sim_{\mathbb{Z}_v} Q_2$ at all places v.) It will also be important to keep track of how many ways a form Q_1 is equivalent to itself. We call these self-equivalences the **automorphisms** of Q_1, and collectively refer to them as $\mathrm{Aut}(Q_1)$.

Since for answering our two main questions we cannot distinguish between equivalent quadratic forms, it makes sense to divide the genus into **classes** of equivalent forms. The following fundamental result in the theory of quadratic forms tells us that our local strategy reduces us to asking about only finitely many possibilities.

Theorem (Hermite). *There are only finitely many classes of forms Q_i in the genus of Q.*

With this, we are tempted to think of our local conditions as making a statement jointly about all forms in the genus of Q. However, we have yet to exhibit a concrete connection between local representability by Q and (global) representability by some form Q_i in the genus. By trying to establish just such a quantitative link, Siegel proved the following remarkable result which allows us to think of our local information as a *weighted average* of (global) information over the classes in the genus. For simplicity, we only state his theorem assuming Q is positive definite.[1]

Theorem (Siegel). *If the genus of Q is a (disjoint) union of classes represented by forms Q_i, then*

$$r_{\mathrm{Gen}(Q)}(m) := \frac{\sum_{Q_i} \frac{r_{Q_i}(m)}{\#\mathrm{Aut}(Q_i)}}{\sum_{Q_i} \frac{1}{\#\mathrm{Aut}(Q_i)}} = \prod_v \beta_v(m) \tag{2.1}$$

where $\beta_v(m)$ are local representation densities at the place v which describe how many local representations m has in $\mathbb{Z}_v$.

Siegel's Theorem allows us to conclude that Strategy 1 succeeds whenver there is only one class in the genus of Q, since if m is locally represented then the local representation densities $\beta_v(m)$ will all be > 0.[2]

Corollary 1. *If the genus of Q has only one class, then*

$$\begin{array}{c}\textit{m is locally}\\ \textit{represented by } Q\end{array} \iff \begin{array}{c}\textit{m is (globally)}\\ \textit{represented by } Q.\end{array}$$

Since the genus of the sum of 2, 3, and 4 squares forms all have only one class, Corollary 1 instantly proves the first 3 theorems of this section. In fact, by computing the local densities at all places, we can also recover Jacobi's exact formula for $r_Q(m)$ when Q is the sum of four squares. However, this happy coincidence between the genus and the class of Q is quite infrequent. In fact, there are only finitely many positive definite forms Q where this holds.[3] Therefore while Strategy 1 is a good first step, some further tools are needed to answer our two main questions in general.

§3 Theta functions and modular forms

One useful approach used to understand a sequence of numbers is to think of them as coefficients of some appropriate series, and to study the transformation properties of the series *as a whole.* For our purposes we are interested in the numbers $r_Q(m)$ for some positive definite quadratic form Q, so it is natural for us to study its **theta function**, defined by the Fourier series

$$\Theta_Q(z) := \sum_{m \geq 0} r_Q(m) e^{2\pi i m z},$$

which is a holomorphic function in the complex upper half-plane $\mathfrak{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$. Since $\mathfrak{H}$ has a natural action of the group $\mathrm{SL}_2(\mathbb{Z})$, given by $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \mapsto \frac{az+b}{cz+d} =: \gamma z$, we can ask about how $\Theta_Q(\gamma z)$ is related to $\Theta_Q(z)$. It turns

[1] Since Q is positive definite, then so are all forms Q_i in its genus, hence $\#\mathrm{Aut}(Q_i)$ is finite.

[2] Since $\prod_v \beta_v(m)$ is an infinite product, we cannot immediately conclude that the product is non-zero when each $\beta_v(m) > 0$. However by computing the local factors $\beta_v(m)$ almost everywhere one can check that it does converge when $n \geq 2$ (though conditionally when $n = 2$ or 3), so our conclusion is justified.

[3] The program of determining all genera with one class has been extensively pursued by Watson in a series of papers starting with [Wa2].

out that there is a "congruence" subgroup $\Gamma_0(N)$ of finite index in $\mathrm{SL}_2(\mathbb{Z})$ where the theta function has the particularly simple multiplicative transformation property

$$\Theta_Q(\gamma z) = \chi_Q(d)(cz+d)^{n/2}\Theta_Q(z), \tag{3.2}$$

for some character χ_Q on $(\mathbb{Z}/N\mathbb{Z})^\times$.[4]

Rather than study $\Theta_Q(z)$ by itself, it is more convenient to study $\Theta_Q(z)$ as one of many functions with similar transformation properties. In particular, $\Theta_Q(z)$ is an example of a "modular form" of weight $n/2$ for $\Gamma_0(N)$, and we now turn our attention to describing these in some detail.[5] This digression may appear unwarranted, but the general features and subtleties of modular forms, especially those of half-integral weight, will play a central role in answering our main questions about quadratic forms in small numbers of variables.

Modular forms for $\Gamma_0(N)$.

Let $\Gamma_0(N)$ denote the subgroup of $\mathrm{SL}_2(\mathbb{Z})$ whose reduction mod N looks like $\left[\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right]$. We say that a holomorphic function $f(z)$ on $\mathfrak{H}$ is a **modular form** of **weight** k and **level** N for some character χ on $(\mathbb{Z}/N\mathbb{Z})^\times$ if:

1) $f(\gamma z) = \chi(d)(cz+d)^k f(z)$ for all $\gamma \in \Gamma_0(N)$,
2) $f(z)$ is holomorphic at all "cusps" and has "mild growth" near them.

The first condition is just the transformation formula (3.2), and the second condition has to do with the growth properties of $f(z)$ as $\mathrm{Im}(z) \to \infty$ and as $z \to \alpha \in \mathbb{Q}$. In particular, condition 2 implies that we can assign a value to $f(z)$ at the boundary points $\mathbb{Q} \cup \infty$ of $\mathfrak{H}$. By condition 1, we see that these values are determined by specifying them at one point in each $\Gamma_0(N)$-orbit of the boundary. We refer to this finite collection of representative boundary points as the **cusps** of $\Gamma_0(N)$.

We now quickly sketch some important features of the space of modular forms of fixed weight k and level N:

Finite-dimensionality: The space of modular forms of fixed weight and level form a finite dimensional vector space over $\mathbb{C}$.

Fourier expansions: Since $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right] \in \Gamma_0(N)$, we see that $f(z) = f(z+1)$ from condition 1, so $f(z)$ has a Fourier expansion $f(z) = \sum_{m\geq 0} a(m)e^{2\pi i m z}$.

Cusp forms and Eisenstein series: There is a subspace of functions vanishing at all cusps called **cusp forms** and a naturally perpendicular space of functions called **Eisenstein series**. The Fourier coefficients of an Eisenstein series tend to be larger than those of a cusp form. Specifically, the Fourier coefficients $a_E(m)$ of an Eisenstein series are either zero or about as big as m^{k-1}, while the Fourier coefficients $a(m)$ of a cusp form are bounded by some constant multiple of $m^{k/2}$. (The bound $|a(m)| << m^{k/2}$ is often referred to as the **trivial bound** for cusp forms.) Also, if $f(z)$ is a cusp form then $a(0) = 0$.

[4]Since $n/2$ is not an integer when n is odd, this relationship only makes sense with some appropriate choice of square root of $cz+d$ for each $\gamma \in \Gamma_0(N)$, so that $\Theta(\gamma_1(\gamma_2 z)) = \Theta((\gamma_1\gamma_2)z)$. If such a consistent choice of square roots exists, we call it a **multiplier system** for $\Gamma_0(N)$. It is a non-obvious fact that multiplier systems exist for all $\Gamma_0(N)$ mentioned above.

[5]For a more detailed introduction to modular forms of integral weight on $\Gamma_0(N)$ one can refer to [Mi], [Sh2], [Iw2], and [Ko]. Half-integral weight forms are treated from an analytic point of view in [Iw2, Ch 1-5] and from an arithmetic point of view in [Ko, Ch IV].

Hecke operators: For every $m \in \mathbb{N}$ with $\gcd(m, N) = 1$, we have a linear operator T_m acting on the space of cusp forms. These **Hecke operators** have the following two very useful properties:

(1) T_m is a normal operator.
(2) If $\gcd(m_1, m_2) = 1$, then $T_{m_1 m_2} = T_{m_1} T_{m_2}$.

As a result, we can find a basis of (Hecke) eigenforms $\{f_i(z)\}$ for all T_m and can write any cusp form $f(z)$ as a linear combination $f(z) = \sum_i \gamma_i f_i(z)$.

If $f_i(z)$ has integral weight (i.e. $k \in \mathbb{Z}$), then its Hecke eigenvalues are connected to the Fourier coefficients by $a_i(1)\, T_m f_i(z) = a_i(m) f_i(z)$. If $f_i(z)$ has half-integral weight, then the Hecke operators T_m are identically zero unless m is a square. In this case, for the non-zero Hecke operators T_{m^2} we still have $a_i(t)\, T_{m^2} f_i(z) = a_i(tm^2) f_i(z)$ for any square-free number t.

Since the eigenforms can be freely multiplied by any constant, we will always assume that they are **normalized** in some appropriate sense. For integral weight forms we require $a_i(1) = 1$, while for half-integral weight we require the forms to have norm 1 (under the natural inner product).[6]

The analytic approach when $n \geq 5$.

Now that we have some basic facts about modular forms in general, we can apply this knowledge to better understand our specific modular form $\Theta_Q(z)$. Firstly, from the general decomposition into cusp forms and Eisenstein series, we can write

$$\Theta_Q(z) = E(z) + f(z)$$

as the sum of an Eisenstein series $E(z)$ and a cusp form $f(z)$.[7] By looking at the m^{th} Fourier coefficients of each, we have

$$r_Q(m) = a_E(m) + a(m).$$

Since we expect $a_E(m) = 0$ or $\sim m^{n/2-1}$ and we know $|a(m)| << m^{n/4}$, when $n > 4$ these combine to show that $r_Q(m) > 0$ when $a_E(m) \neq 0$ and m is sufficiently large. This is the heart of the analytic apprach to this problem, which has its origins in the circle method of Ramanujan, Hardy, and Littlewood. Therefore, if we could prove that the Eisenstein series had large enough Fourier coefficients, we could answer our main questions when $n \geq 5$. Thankfully, Siegel has given us a beautiful interpretation of the coefficients $a_E(m)$ in terms of the local behavior of Q when $n \geq 5$. He showed that $a_E(m)$ is exactly given by the weighted average/infinite product expression in (2.1). By using this interpretation of $a_E(m)$, we can prove the following theorem:

Theorem (Tartakowsky). *Suppose Q is a positive definite integral quadratic form in $n \geq 5$ variables. Then an integer m is represented by Q if m is locally represented by Q and sufficiently large.*

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[6]Strictly speaking, for integral weight the "theory of newforms" only guarantees that $a_i(1) \neq 0$ if $f_i(z)$ does not arise as $f'(rz)$ for some form $f'(z)$ of lower level. However, for such (old)forms we can just normalize the associated (newform) $f'(z)$ of smallest level using $a'(1) = 1$.

[7]The Eisenstein series $E(z)$ is actually a linear combination of the "standard" Eisenstein series naturally associated to the cusps of $\Gamma_0(N)$.

§3.1 The analytic approach when $n = 4$

One can try to modify the analytic approach above to prove a similar type of theorem when $n = 4$, however there are several new complications and features which must be dealt with. We again only discuss positive definite forms, though we will revisit indefinite forms in Section 4.

The existence of anisotropic primes.

The first notable difference between forms in 4 variables and their higher dimensional counterparts is that under certain circumstances the Eisenstein coefficients $a_E(m)$ don't necessarily grow as m grows. Using Siegel's interpretation of $a_E(m)$ in (2.1) (which still holds when $n = 4$ as we will see below), we can analyze the growth of $a_E(m)$ locally at each of the local factors $\beta_v(m)$. What we find is that there are at most finitely many "anisotropic" primes $p \mid N$ where $\beta_p(m) = \beta_p(p^2 m)$ when m is sufficiently p-divisible. The presence of these primes presents a problem for our analytic approach, since allowing m to grow by introducing powers of anisotropic primes p will not cause our main term to grow.

The term **anisotropic prime** comes from the fact that Q is anisotropic over the p-adic numbers $\mathbb{Q}_p$, meaning that the only representation $Q(\vec{x}) = 0$ with $\vec{x} \in \mathbb{Q}_p^n$ is when $\vec{x} = \vec{0}$. This geometric property means that when m is sufficiently divisible by p (i.e. p-adically close enough to zero) that there are no new representations of m, causing the local factor not to grow as it become more p-divisible. Notice that the condition for a prime to be anisotropic is harder to satisfy as n grows; in fact when $n \geq 5$ there are no anisotropic primes.

The presence of anisotropic primes when $n = 4$ requires us to consider m which are large but not too highly divisible by these primes. This will add the additional condition "a priori bounded divisibility at the anisotropic primes" to any analytic results we may obtain.

Convergence of the Eisenstein series.

Siegel's identification of the coefficients $a_E(m)$ with the expression (2.1) when $n \geq 5$ depends on having a convergent series expression for the weight k Eisenstein series $E_k(z) := E(z)$. For small weights the defining series does not converge, but one can recover $E_k(z)$ as the value at $s = k = \frac{n}{2}$ of the generalized Eisenstein series $E(z, s)$ whose extra parameter $s \in \mathbb{C}$ is related to the weight. The series defining $E(z, s)$ is only absolutely convergent when $\mathrm{Re}(s) > 2$, however it is known to have meromorphic continuation to all $s \in \mathbb{C}$.

When $n = 4$ the Eisenstein series $E_2(z)$ no longer lives in the region of absolute convergence, so more care is needed. However, since it is on the edge of the region of absolute convergence, one can use limiting arguments to show that the result holds in this case as well.[8]

Need for an improved error estimate.

Even after understanding the growth of $a_E(m)$ via Siegel's formula, there is a further challenge when $n = 4$. Our estimates show that, avoiding growth at the anisotropic primes, we have $a_E(m) = 0$ or $\sim m^{n/2-1}$ and $|a(m)| << m^{n/4}$. However when $n = 4$ this shows that $a_E(m) \sim m$ and $|a(m)| << m$, which is not very useful since the bound is as large as the main term as m grows.

[8] In fact (2.1) is still meaningful when $n = 3$. This was justified explicitly in [SP2, Korollar 1, p289], and follows more generally from the work of Kudla and Rallis [K-R1, K-R2] on the Siegel-Weil formula.

To salvage the situation, we are led to ask if one can improve on the trivial bound for $a(m)$. The best possible bound for the Fourier coefficients of a normalized cusp form of integral weight $k = \frac{n}{2}$ was conjectured by Ramanujan and Petersson to be $|a(m)| \leq \tau(m)m^{(k-1)/2}$, and this was established by Deligne [De] using deep methods in algebraic geometry.[9] However historically, an improvement over the trivial bound was first achieved by Kloosterman using his stronger version of the circle method. This leads to the following theorem:

Theorem (Kloosterman). *Suppose Q is a positive definite integral quadratic form in 4 variables. Then an integer m is represented by Q if it is locally represented by Q and sufficiently large with a priori bounded divisibility at the anisotropic primes.*

§4 Refined local considerations

At the outset, we focused our attention on two main questions about the integers m represented by a quadratic form Q. Our first strategy for answering them was to check the representability of m locally at all places, then by using modular forms we were able to bound the error in our local strategy. However in this process we made the simplifying assumption that Q was positive definite, which ignores the question of representability by an indefinite form. In this section we will refine our initial strategy, and in the process we recover information about indefinite forms as well.

From the point of view of quadratic forms as homogeneous polynomials of degree 2, it may seem that Strategy 1 is the best one can do when speaking about local equivalence. We defined a quadratic form as a particular polynomial, and then spoke of *equivalent* forms. In this language, the class and genus of a quadratic form are its equivalence classes under the actions of the groups $\mathrm{GL}_n(\mathbb{Z})$ and $\prod_v \mathrm{GL}_n(\mathbb{Z}_v)$ respectively.

Another equally good (and perhaps better) point of view is to think of a quadratic form as a lattice $L \cong \mathbb{Z}^n$ embedded in a fixed vector space V equipped with a fixed quadratic form φ (in the first sense). This point of view has two advantages. The first is that equivalences of lattices inside (V, φ) are defined by an action of the orthogonal group O^φ carrying one to another. The second advantage is that over a number field K we can speak of a lattice even when it does not have a basis over the ring of integers. Such a basis is required to write down a homogeneous polynomial, but may not exist when the class number of K is > 1.

Since we lose nothing by doing so, we will instead take the determinant 1 groups above, so we are looking at equivalences under the groups SL_n and SO^φ respectively. From either point of view we can define the notion of a class and genus of forms by using the relevant group over $\mathbb{Z}$ or over $\mathbb{Z}_v$ for all places v. An interesting feature of the second point of view is that SO^φ is not simply connected when $n \geq 3$, and admits a "double cover" called the **spin group** and denoted by Spin^φ. This is particularly interesting because this gives us a natural geometric refinement of our local equivalence by using Spin^φ instead of SO^φ. This refined equivalence is called **spinor equivalence** and its equivalence classes are called **spinor genera** which are intermediate to our notions of class and genus.

[9] This was first established in weight $k = 2$ by Eichler [E2], Shimura [Sh1], and Igusa [Ig].

Since the spinor genus is also geometrically defined and local, it is still relatively simple to answer representability questions for the spinor genus (though it is more involved than using the genus). This gives the following refined local strategy:

Strategy 2: *Check if m is represented by the spinor genus of Q.*

For this to be as useful as Strategy 1, we need a version of Siegel's Theorem for the (weighted) average number of representations $r_{\mathrm{Spn}(Q)}(m)$ of m by the spinor genus, defined analogously to $r_{\mathrm{Gen}(Q)}(m)$ of (2.1). One approach to understanding the $r_{\mathrm{Spn}(Q)}(m)$ is to see how they behave as we vary over all forms Q_i in a genus. To do this, one considers not only spinor equivalence of lattices, but also of the representations themselves. These can be geometrically realized as vectors in the associated lattice, which are also naturally acted upon by the spin group.

When $n \geq 4$ there are enough orthogonal transformations fixing a given representation vector to show that any representation by the genus can be transformed into a representation by any of the spinor genera it contains. Concretely, this means that $r_{\mathrm{Spn}(Q)}(m)$ is the same for all spinor genera in a genus, which together with (2.1) implies $r_{\mathrm{Spn}(Q)}(m) = r_{\mathrm{Gen}(Q)}(m)$. When $n = 3$ there are fewer spin transformations and the genus breaks into two half-genera (each containing half of the spinor genera) where $r_{\mathrm{Spn}(Q)}(m)$ is constant. In fact there is a quadratic character $\psi = \psi_m$ which distinguishes between these two half-genera (for any given m).

By incorporating this character ψ into Siegel's weighted average, Schulze-Pillot [SP1] has shown:

Theorem (SP). *Suppose Q is a positive definite integral quadratic form in 3 variables. Then*

$$r_{\mathrm{Spn}(Q)}(m) - r_{\mathrm{Gen}(Q)}(m) = \prod_v \widetilde{\beta}_v(m) \tag{2.2}$$

where $\widetilde{\beta}_v(m)$ are local spinor representation densities at the place v which describe how many local representations m has by the spinor genus in $\mathbb{Z}_v$.

This, together with Siegel's Theorem, allows us to compute $r_{\mathrm{Spn}(Q)}(m)$ based purely on local information, and to ask about spinor genera, which may be smaller than the genus given by Strategy 1. If the spinor genus happens to have only one class then we can completely answer our main questions. While this is not true in general, it is true for indefinite forms in $n \geq 3$ variables:[10,11]

Theorem (Eichler). *If Q is an indefinite quadratic form in 3 or more variables, then its spinor genus consists of only one class.*

§5 The connection with the Shimura lift

In dealing with our main questions when n is odd, we are led naturally to study the Fourier coefficients of modular forms of half-integral weight. By using Siegel's Theorem and the general decomposition of modular forms into Eisenstein series and cusp forms, we can further restrict our attention to understanding the coefficients of cusp forms.

[10] This is a consequence of the strong approximation property for the associated spin groups.

[11] A nice overview of the arithmetic of indefinite forms and spinor genera is given in [Hs].

Because of the close connection between the Fourier coefficients of eigenforms and their Hecke eigenvalues, it seems reasonable to study the (non-zero) Hecke operators T_{m^2} when $\gcd(m, N) = 1$. While doing this, Shimura noticed that the relationships between the Hecke operators T_{m^2} acting on half-integral weight forms closely resembled the relationships between the Hecke operators T_m for integral weight forms. He further showed that (for every square-free number t) there was in fact a way of *lifting* an eigenform of half-integral weight k to a form of integral weight $2k-1$, and that this commutes with the action of the Hecke operators (in the sense just described). Also, when $n > 3$ this lifting is actually a cusp form. This **Shimura lift** allows one to apply Deligne's theorem to obtain a sharp upper bound for the growth of the Fourier coefficients of an eigenform within a given square class $t\mathbb{Z}^2$.

For our purposes, the issue of deciding whether the Shimura lift is a cusp form when $n = 3$ is of great importance, since Eisenstein series have much larger Fourier coefficients than cusp forms. Using a variety of techniques, Flicker, Niwa, Cipra, and others have characterized the subspace whose Shimura lift is an Eisenstein series as linear combinations of particularly simple cusp forms

$$h_\psi(tz) = \sum_{m\in\mathbb{Z}} \psi(m) m\, e^{2\pi i t m^2 z}$$

for some odd Dirichlet character ψ, whose Fourier coefficents vanish outside of the square class $t\mathbb{Z}^2$. However, this does not answer the question of whether these simple cusp forms appear in the theta function $\Theta_Q(z)$. In fact they do appear, and this feature is closely related to the appearance of half-genera in the genus of Q.

More precisely, when $n = 3$ we can write $\Theta_Q(z)$ as

$$\Theta_Q(z) = E(z) + H(z) + f(z)$$

where $E(z)$ is an Eisenstein series, both $H(z)$ and $f(z)$ are cusp forms, and the Shimura lifts $\mathrm{Shi}(H;t)$ and $\mathrm{Shi}(f;t)$ (for any square-free t) are respectively an Eisenstein series and a cusp form. At the level of Fourier coeficients, this says

$$r_Q(m) = a_E(m) + a_H(m) + a(m).$$

The connection between these Fourier coefficients and the original quadratic form Q is given by the following (local) theorem, due to Schulze-Pillot [SP2].

Theorem (SP). *Suppose Q is a positive definite quadratic form in 3 variables. Then*

$$a_E(m) = r_{\mathrm{Gen}(Q)}(m) \qquad \text{and} \qquad a_H(m) = r_{\mathrm{Spn}(Q)}(m) - r_{\mathrm{Gen}(Q)}(m).$$

The local nature of this result can be seen from (2.1) and (2.2), and its proof involves explicitly computing the action of the Hecke operators.

§6 Analytic estimates for Fourier coefficients

While the Hecke eigenvalues will play an important role in answering our main questions, they are substantially weaker tools for understanding coefficients of half-integral weight forms than for their integral weight counterparts. In particular, we only have the (non-zero) Hecke operators T_{m^2} which leaves completely open the question of understanding the square-free Fourier coefficients. To establish upper bounds for these, the only available techniques are analytic in nature, and involve estimating the size of exponential sums or special values of L-functions. In this section we will discuss some features of this approach, and some results related to our main questions.

As we mentioned when discussing the case $n = 4$, Kloosterman needed to improve the trivial bound for weight 2 forms to say something about the numbers represented by Q. Since in these estimates we are more interested in the exponents than the constants, it suffices to only make a statement about a *basis* for the space of cusp forms. For arithmetic purposes one is most interested in the basis of Hecke eigenforms, but for analytic pursuits it is often more convenient to work with a basis of **Poincaré series**. The advantage of doing this is that their Fourier coefficients can be expressed as sums involving the **Kloosterman sums**[10]

$$K(u, v, c) = \sum_{d \in (\mathbb{Z}/c\mathbb{Z})^\times} \vartheta(\gamma)\, e^{2\pi i \frac{(ud + v\bar{d})}{c}} \qquad \text{where } \gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma_0(N).$$

The trivial estimate $|K(u, v, c)| \leq c$ for individual Kloosterman sums recovers the trivial bound $|a(m)| << m^{\frac{k}{2}+\varepsilon}$ for Fourier coefficients of a weight k cusp form. Kloosterman's improvement was equivalent to showing that $|K(u, v, c)| << c^{\frac{3}{4}+\varepsilon}$, which was further refined by Weil and Salié to $|K(u, v, c)| << c^{\frac{1}{2}+\varepsilon}$. This refinement is best possible, and it produces the bound $|a(m)| << m^{\frac{k}{2}-\frac{1}{4}+\varepsilon}$. When $k = \frac{3}{2}$, we see that this bound is achieved by the simple cusp forms $h_\psi(tz)$, so no further progress can be made without somehow excluding these forms. We therefore restrict our attention to the coefficients $a(t)$ where t is square-free. To obtain an improved estimate for $|a(t)|$, we must exhibit some further cancellation between the individual Kloosterman sums. This is particularly important for our main questions when $n = 3$ since the Eisenstein series $E(z)$ has coefficients $a_E(m) \sim m^{\frac{1}{2}}$, so only some slight improvement is needed to ensure this is the main term.

This breakthrough was accomplished by Iwaniec for weights $k \geq \frac{5}{2}$ in [Iw1], proving

$$|a(t)| << t^{\frac{k}{2}-\frac{2}{7}+\varepsilon},$$

and was extended to weight $\frac{3}{2}$ by Duke [Du1]. Together with the Hecke eigenvalue bound from the Shimura lift, this provides an estimate for the coefficients $a(m)$ of the cusp form $f(z)$ for all m. This bound is sufficient to settle the case $n = 3$ provided m is not in one of the finitely many **exceptional-type**[11] square classes where the coefficients of $H(z)$ are non-zero and as large as $E(z)$. This gives the following result:

Theorem (Duke-SP). *Suppose Q is a positive definite integral ternary quadratic form. Then every sufficiently large integer m locally represented by Q with a priori bounded divisibility at the anisotropic primes not lying within one of finitely many exceptional-type square classes is represented by Q.*

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[10] Here the factor $\vartheta(\gamma)$ is related to the multiplier system at γ, and $\bar{d} \in \mathbb{Z}$ is a representative for the class $d^{-1} \pmod{c}$. (Multiplier systems are defined in footnote 4.)

[11] The term "exceptional-type" is synonymous with the phrase "spinor exceptional" often found in the literature.

§7 The Exceptional-type square classes

While the analytic methods of the previous section provide an adequate answer to our main questions for most numbers when $n = 3$, they fail to make any statement about the representability of numbers within an exceptional-type square class $t\mathbb{Z}^2$. There are two methods of investigating this question, each with slightly different merits, which give the following theorem:

Theorem (SP, Hanke). *Suppose Q is a positive definite integral quadratic form in 3 variables. Then any sufficiently large integer m locally represented by the spinor genus of Q with bounded divisibility at the anisotropic primes is represented by Q except for at most finitely many infinite sequences $m = Mp^2$, where M is anti-exceptional*[12] *with $r_Q(M) = 0$ and p runs over all primes $p \nmid NM$ with $\psi_t(p) = -1$.*

The main feature of all of our theorems so far has been that all but finitely many numbers which are locally represented by Q are actually represented by Q. In slightly different language, one would say that a local-global principle[13] holds for all but finitely many numbers. Interestingly, our latest theorem casts some doubt as to whether this is still true when $n = 3$, and under certain circumstances (if they ever happen) guarantees that even a refined local-global principle based on the spinor genus fails infinitely often.

§7.1 Primitive spinor representations.

In this approach, one keeps track not only of the number $r_{\mathrm{Spn}(Q)}(m)$ of representations by the spinor genus, but also the number $r^*_{\mathrm{Spn}(Q)}(m)$ of **primitive representations** by $\mathrm{Spn}(Q)$, meaning that they are representations of m which do not come from stretching a representation of some smaller integer m/r^2. This is also a local quantity, and one can establish local product formulas analagous to (2.1) and (2.2) whose factors count local primitive representations of m by $\mathrm{Spn}(Q)$.

Since the growth of $r_{\mathrm{Spn}(Q)}(m)$ comes primarily from these primitive representations, it is useful to know which m have no primitive representations by some spinor genus (hence half-genus) in $\mathrm{Gen}(Q)$. These are called **primitive spinor exceptions**, and they can be characterized by local conditions on the image of the spinor norm map. Using these, one can show that if m is a primitive spinor exception, then so is mp^2 for all primes $p \nmid Nm$ with $\psi_t(p) = -1$. By keeping track of which spinor genera (half-genus) has no primitive representations, one can show that if m is not represented by Q then neither is mp^2 (since such a representation would be primitive). This approach was carried out in [SP3].[14]

The advantage of this approach is that it is purely local, so it provides a conceptually simple way of answering our first question. However, in principle it could be extended to provide asymptotics for $r_Q(m)$.

§7.2 The structure of the Shimura lift.

In the second approach, one tries to show that the term $a_E(m) + a_H(m)$ grows more quickly than $a(m)$ as m gets large. This is done by computing local densities

[12] The term "anti-exceptional" is defined in §7.2.

[13] In this context we mean that a local-global principle for representation over the integers as described in §2. If we instead speak about rational representations, then Hasse (in 1923–24) has showed that a *rational* local-global principle holds for *all* (rational) numbers.

[14] Partial representation results based on primitive representability by the spinor genus within a square class have been established by many authors. Most recently, analytically in [Du-SP, Thrm 3] and arithmetically in [Hs-J, §4, Thrm].

$\beta_v(m)$ at *all* places to produce a lower bound for $a_E(m)$, then using the explicit Fourier expansion for $H(z)$ to establish a lower bound for the main term. Notice that since $a_E(m)$ and $|a_H(m)|$ are both $\sim m^{\frac{1}{2}}$, significant cancellation may occur.

To describe the situation more fully, we say that m is **exceptional** if $a_E(m) + a_H(m) = 0$, and **anti-exceptional** if $a_E(m) = a_H(m)$.[15] From our local computations, we see that the main term either grows more quickly than $|a(m)|$ or is constant, and it is constant when either m stays exceptional (where it is zero, hence not represented by the spinor genus) or for numbers mp^2 where m is anti-exceptional and p runs over primes $p \nmid Nm$ with $\psi_t(p) = -1$.

When the main term fails to grow, the behavior of $r_Q(m)$ depends only on the Fourier coefficients of the cusp form $f(z)$, which a priori seems difficult to understand. (In general $f(z)$ is not even an eigenform!) However, by taking advantage of the fact that $r_Q(m) \geq 0$, one can show that the coefficients of $f(z)$ behave reasonably.

Since half-integral weight forms are difficult to work with, it is much more convenient to consider the weight 2 Shimura lift $g(z) := \mathrm{Shi}(f;t)$. This can be written as a linear combination of eigenforms $g_i(z)$, and by an argument with the associated Galois representations ρ_i one shows that the eigenforms $g_i(z)$ appear together as a difference of a form and its twist by ψ_t. This shows that many of the Fourier coefficients of $g(z)$ vanish, and when translated back to $f(z)$, shows that $a(m) = a(mp^2)$ when m is anti-exceptional and $\psi_t(p) = -1$. This shows that when m is anti-exceptional and not represented by Q, then neither is mp^2 for all primes $p \nmid Nm$ with $\psi_t(p) = -1$. This approach was carried out in [Ha2].

Because of its analytic nature, this approach answers both the question of representability and provides an estimate of the size of $r_Q(m)$. It also describes some of the structure of the cusp form $f(z)$ and its Shimura lift $g(z)$. However it is technically more complicated and involves global tools such as the Shimura lift and Galois representations (or alternatively some weak form of the Sato-Tate Conjecture).

§7.3 An example.

We now consider an example first described by Jones and Pall in 1939. Consider the genus of forms

$$S_1 \begin{cases} Q_1 = x^2 + 48y^2 + 144z^2 \\ Q_2 = 4x^2 + 48y^2 + 49z^2 + 4xz + 48yz \end{cases}$$
$$S_2 \begin{cases} Q_3 = 9x^2 + 16y^2 + 48z^2 \\ Q_4 = 16x^2 + 25y^2 + 25z^2 + 16xz + 16xy + 14yz \end{cases}$$

of level $N = 576 = 2^6 3^2$ and determinant $D = 6912 = 2^8 3^3$. This genus breaks into two spinor genera S_i which are their own half-genera. One can show that the only exceptional-type square class is $\mathbb{Z}^2$, whose associated quadratic character is $\psi_1(\cdot) = \left(\frac{-tD}{\cdot}\right) = \left(\frac{-3}{\cdot}\right)$. Also, this genus is anisotropic only at 3 and ∞. We now apply the above theorem to understand the representation behavior of the forms in this genus.

In the spinor genus S_1, one can check that only the numbers 1 and 4 are anti-exceptional. Since Q_1 represents both of them, a local-global principle holds for all

[15] In general we have the inequality $|a_H(m)| \leq a_E(m)$, so the terms "exceptional" and "anti-exceptional" characterize the extremal cases.

but finitely many integers m. However Q_2 misses the number 1, so $r_{Q_2}(p^2) = 0$ for all primes $p \neq 2$ with $p \equiv 2 \pmod 3$. *This gives an example of a failure of the local-global principle for infinitely many numbers m!*

In the spinor genus S_2 there are no anti-exceptional numbers, and 1 and 4 are exceptional. Therefore, for each of these forms a local-global principle holds for all but finitely many numbers.

It is interesting to notice that by global arithmetic methods Jones and Pall showed that for the forms Q_1 and Q_3, the local global principle

$$r_{\mathrm{Spn}(Q)}(m) > 0 \implies r_Q(m) > 0$$

holds *for all integers*, though they gave no results for Q_2 and Q_4. For more precise results about them one would need an effective version of the theorem above. Unfortunately, making it effective depends on some rather deep and unproven conjectures about L-functions, which follow from the Riemann hypothesis for these L-functions.

§8 Applications and generalizations

In closing, we describe some further questions and results related to our main themes.

Effective lower bounds

In our goal to characterize which numbers are represented by a quadratic form Q, we have been satisfied with answers applying to "sufficiently large" integers. However, this vague kind of answer becomes less satisfying when one seeks to explicitly characterize or list those numbers represented by a given Q. For this purpose, the statement "sufficiently large" begs the question "how large?". To establish this more precise (effective) answer, one must care not just about the exponents in our bounds, but also about the constants.

When $n \geq 4$ one can establish effective bounds by making precise local computations and noting the size of the coefficients of the eigenforms in the cusp form, however there is an additional subtlety when $n = 3$. The problem here becomes that the main term $a_E(t)$ (for square-free numbers t) is essentially as large as $L(1, \chi_t)$, for which we know only Siegel's ineffective lower bound $L(1, \chi_t) >> t^{\varepsilon}$. This issue is equivalent to the possibility of a "Siegel zero" for the associated Dirichlet L-functions. This is one of the major outstanding problems in analytic number theory, and would follow from the Riemann hypothesis for such L-functions. Since this is only an issue for square-free numbers, by allowing m to grow within only finitely many square classes $t\mathbb{Z}^2$ we may again establish an effective bound.

When $n \geq 5$ effective bounds have been established by Watson [Wa1] and Hsia-Icaza [Hs-Ic] by algebraic methods. When $n \geq 4$ one can apply the analytic ideas above to make Kloosterman's original argument effective [Kl], or use the modular forms directly as in [SP4] and [Ha1]. This was carried out in [Ha1] to show that $x^2 + 3x^2 + 5z^2 + 7w^2$, a form which has resisted algebraic methods, represents all positive integers except 2 and 22. When $n = 3$, by assuming various Riemann hypotheses, Ono and Soundararajan [O-S] have precisely characterized all numbers represented by the form $Q = x^2 + y^2 + 10z^2$.

Generalizations of the 15-theorem

One interesting application of sharp effective results for quadratic forms is that they are useful in proving finiteness theorems for quadratic forms. The original theorem along these lines is the following theorem of Conway and Schneeberger:[16]

15-Theorem. *Suppose Q is a positive definite classically integral quadratic form.*[17] *Then*

$$\begin{matrix} \textit{Q represents all} \\ \textit{positive integers} \end{matrix} \iff \begin{matrix} \textit{Q represents the numbers} \\ 1, 2, 3, 5, 6, 7, 10, 14, 15. \end{matrix}$$

Recently, Bhargava [Bh2] has shown that these finiteness theorems are quite common, and that in fact for any set $\mathbb{S} \subset \mathbb{N}$ there is a unique minimal finite subset $\mathbb{S}_0$ such that

$$Q \text{ represents } \mathbb{S} \iff Q \text{ represents } \mathbb{S}_0.$$

Finding the set $\mathbb{S}_0$ requires one to understand which numbers are represented by a certain finite list of quadratic forms (depending on $\mathbb{S}$). When $\mathbb{S} = \mathbb{N}$ only 1,850 forms arise and one can use various arithmetic methods to understand them, however in general these are not sufficient to settle the question. In some cases, it is reasonable to use analytic methods to supplement this approach.

As an example, by reformulating the result when $\mathbb{S} = 2\mathbb{N}$, one can hope to find the finite set $\mathbb{T}_0 \subset \mathbb{N}$ for the following theorem:

290-Theorem. *Suppose Q is a positive definite integral quadratic form. Then*

$$\begin{matrix} \textit{Q represents all} \\ \textit{positive integers} \end{matrix} \iff \textit{Q represents } \mathbb{T}_0.$$

It is likely that the largest number in $\mathbb{T}_0$ is 290, and the calculations required to prove this are currently underway [Bh-Ha]. In general, the complexity of the problem grows quickly with the number of small integers not in $\mathbb{S}$, thus for very thin sets the computations are not very feasible. The special case of these theorems for forms in 4 variables is known as "Ramanujan's universal quaternary forms problem", and its resolution is a crucial step in proving precise versions of the previous two theorems.[18]

Binary forms

On the surface, it may seem as though we have been unfair in postponing discussion of our main questions for quadratic foms in 2 variables until this point. However the nature of the question in this case is purely algebraic, and requires completely different techniques. There is currently no completely satisfactory answer to this question, though for **principal forms** (i.e., those corresponding to the identity of the ideal class group of a quadratic extension) one can answer this in terms of the Hilbert class field of the associated quadratic number field.[19]

[16] A sketch of the original proof appears in [Sch], however a simpler proof is described in [Bh1].

[17] Q is **classically integral** if Q is integral and the numbers a_{ij} are required to be even integers when $i \neq j$.

[18] A positive definite integral quadratic form is said to be **universal** if it represents all positive integers. Briefly, this problem starts with Ramanujan's 1916 paper [Ra] enumerating all (positive definite) universal diagonal quaternary forms (with one mistake). Later Willerding [Wi], in her 1948 thesis, listed 168 classically integral universal quaternary forms. This list was corrected and completed in 2000 by Conway, Schneeberger, and Bhargava [Bh1] to a final list of 204 such forms.

[19] For a complete and very readable description of the situation for principal forms, see [Co].

From an analytic point of view, the theta function $\Theta_Q(z)$ is a modular form of weight 1. We can still decompose this into an Eisenstein $E(z)$ and cuspidal part $f(z)$, however we expect the Fourier coefficients of these to be bounded by some constant (since $a_E(m) \sim m^{k-1}$ and $|a(m)| << m^{\frac{k-1}{2}}$). Since neither grows as m grows, we cannot make any statements about "sufficiently large" numbers.

Quadratic forms over number fields

Given our success with integral quadratic forms over $\mathbb{Q}$, one might be tempted to ask about quadratic forms over a number field F. In this case, the definition of integral requires the coefficients a_{ij} lie in its ring of integers $\mathfrak{o}_F$. As with $\mathbb{Q}$, the main distinction to be made is between definite and indefinite forms, however here there are more places v "at infinity". If Q is definite at all of these, we say Q is **totally definite**, otherwise Q is said to be **indefinite**.

The theories for totally definite and indefinite forms are the same as when $F = \mathbb{Q}$, however we note that Q can only by totally definite if F is **totally real** (i.e. at all infinite places $F_v = \mathbb{R}$). Local computations in this setting are very similar to those over $\mathbb{Q}$, with the additional technical complication that the primes $\mathfrak{p}$ in $\mathfrak{o}_F$ over 2 may be ramified. When Q is totally *positive* definite[20] the associated $\Theta_Q(z)$ is a **Hilbert modular form** which is now a function on a finite product of upper half-planes. One can pursue all of our previous analytic strategies for Hilbert modular forms, but it becomes technically more complicated. As a consequence, results in that setting are less complete. For instance, the best possible bound for Hecke eigenvalues has not been established in full generality (see [Br-La]), and only very recently was a subconexity estimate established for square-free Fourier coefficients of half-integral weight forms [C, C-PS-S]. However, Shimura has generalized his lifting to this setting [Sh4, Sh5].

Representing forms by forms

One very interesting feature of our main questions is that they can be reformulated in the much broader context of embedings of quadratic forms into each other. From this perspective one fixes a quadratic form Q in n variables and some positive integer $r < n$, and is interested in the number of ways it represents all smaller forms q in r variables. This question was first investigated by Siegel, and led to the theory of **Siegel modular forms** for congruence subgroups of the symplectic group $\mathrm{Sp}_{2r}(\mathbb{Z})$.

In this context there are (local) analogues of (2.1) and (2.2), and the features found when $n = 3$ now appear in the "codimension 2" case $n = r + 2$. As expected, the local calculations are still feasible though there are some difficulties when $p = 2$.

The asymptotic approach is also tractable in this setting, though current results there are analogous to the case $n \geq 5$ and here the statement "m is sufficiently large" is replaced by requiring "the minimal length vector in q is sufficiently large". (See [Hs-Ki-Kn, Ki1, Ki3] for details.)

The main new analytic feature in this case is that Fourier coefficients here are symmetric matrices and the Eisenstein series are stratified into $E_i(z)$ defined by the vanishing of Fourier coefficients of rank $\leq i$. The local information is contained completely in $E_0(z)$, while the theta funcition decomposes as

$$\Theta_Q(z) = E_0(z) + E_1(z) + \cdots + E_{r-1}(z) + f(z)$$

[20] Q is **totally positive definite** when Q is positive definite at all infinite places. If Q is totally definite, one may arrange for this by multiplying Q by some appropriate integer.

where $f(z)$ is a cusp form. Since we have no additional information about them, the presence of the forms $E_i(z)$ with $1 \leq i \leq r-1$ give rise to a fairly crude bound on the size of the error term. To make any substantial progress, one must somehow understand these more complicated Eisenstein series.

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