AN EXACT MASS FORMULA FOR
QUADRATIC FORMS OVER NUMBER FIELDS

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ABSTRACT. In this paper we give an explicit formula for the mass of a quadratic form in $n \geq 3$ variables with respect to a maximal lattice over an arbitrary number field k, and use this to find the mass of many $\mathfrak{a}$-maximal lattices. We make the minor technical assumption that locally the determinant of the form is a unit up to a square if n is odd. The corresponding formula for k totally real was recently computed by Shimura [Shi].

§0 SUMMARY

Our goal is to give an exact formula for the mass of the genus of a quadratic form φ on a maximal lattice defined over an arbitrary number field k. In §2 we explain how knowledge of the Tamagawa number of the special orthogonal group G^φ gives rise to a mass formula. Such a formula expresses the mass as a product of local factors over all places v of k, so our problem is reduced to computing each of these. For the non-archimedian places, these factors were recently computed by Shimura [Shi]. We state his result in §3 and for completeness include a translation between our language and his. In §4 we compute the archimedian factors, treating separately the 3 cases: v real, φ definite; v real, φ indefinite; and v complex. To define the factors in the last two cases, we choose a symmetric space $\mathfrak{Z}_v$ on which G^φ_v acts and a non-zero G^φ_v-invariant volume form $\omega_{\mathfrak{Z}}$. Finally, in §5 we compute the mass of φ with respect to a maximal lattice. We note that this formula agrees with Shimura's when k is totally real. In §6 we conclude by using the local similitude groups to show that this agrees with the mass of many genera of $\mathfrak{a}$-maximal lattices. Our results depend on several technical lemmas which we include as an appendix.

§1 INTRODUCTION

We begin with a quadratic space (V, φ) over an algebraic number field k. By this we mean a k-vector space V together with a non-degenerate quadratic form $\varphi : V \longrightarrow k$. Let O_k denote the ring of integers of k and let O_v denote the local ring of integers at each place v of k. We consider (V, φ) as well as its localizations (V_v, φ_v) given by linear extension of scalars to k_v. Given a lattice $\Lambda \subset (V, \varphi)$, we have the associated local lattice $\Lambda_{\mathfrak{p}} = \Lambda \otimes_{O_k} O_{\mathfrak{p}} \subset (V_{\mathfrak{p}}, \varphi_{\mathfrak{p}})$ at each non-archimedian

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place $\mathfrak{p}$ of k. We occasionally write (Λ, φ) for the restriction of the form φ to Λ, and $(\Lambda_{\mathfrak{p}}, \varphi_{\mathfrak{p}})$ for the restriction of $\varphi_{\mathfrak{p}}$ to $\Lambda_{\mathfrak{p}}$.

With (V, φ) as above, we let $G^{\varphi} = G(\varphi)$ be the special orthogonal group of (V, φ) by which we mean the group of determinant 1 invertible linear transformations of V preserving φ. We also define G_v^{φ} to be the special orthogonal group of (V_v, φ_v). Then we have a natural G^{φ}-action on (V, φ), and a natural G_v^{φ}-action on (V_v, φ_v). We say that two lattices $\Lambda, \Lambda' \subseteq (V, \varphi)$ are **globally equivalent** if there exists $g \in G^{\varphi}$ such that $\Lambda' = g\Lambda$, and **locally equivalent** if for each non-archimedian place v, there exists $g_v \in G_v^{\varphi}$ such that $\Lambda_v' = g_v \Lambda_v$. We define the **genus** of (Λ, φ) to be the set of all lattices locally equivalent to (Λ, φ), and say that the **classes** of (Λ, φ) are the global equivalence classes of (Λ, φ) in its genus.

Let $G_{\mathbf{A}}^{\varphi}$ be the adelization of G^{φ}, and let $G_{\mathbf{a}}^{\varphi}$ and $G_{\mathbf{h}}^{\varphi}$ be the product of G_v^{φ} over the archimedian and non-archimedian places respectively. Then there is a natural $G_{\mathbf{A}}^{\varphi}$-action on the space of lattices $\Lambda \subseteq (V, \varphi)$. To see this, take $g = (g_v) \in G_{\mathbf{A}}^{\varphi}$ and define $g\Lambda$ to be the lattice $\Lambda'' \subseteq (V, \varphi)$ such that $\Lambda_v'' = g_v \Lambda_v$ for all non-archimedian places v. The stabilizer of a lattice (Λ, φ) defines a subgroup $D \in G_{\mathbf{A}}^{\varphi}$ such that $D \subset G_{\mathbf{a}}^{\varphi}$ and $D \cap G_{\mathbf{h}}^{\varphi}$ is open and compact, and by fixing a lattice (Λ, φ) we may parametrize the classes $\mathfrak{Cl}$ of Λ by the elements of $G^{\varphi} \backslash G_{\mathbf{A}}^{\varphi} / D$ using $a \mapsto \Lambda^a := a\Lambda$. We denote by Γ^a the group of **automorphisms** of (Λ^a, φ), defined as those $g \in G^{\varphi}$ leaving Λ^a invariant. From an adelic perspective, we see that $\Gamma^a = G^{\varphi} \cap aDa^{-1}$.

We say that a lattice $\Lambda \subseteq (V, \varphi)$ is **maximal** if $\varphi(\Lambda) \subseteq O_k$ and Λ is not properly contained in some lattice Λ' with $\varphi(\Lambda') \subseteq O_k$. There is a similar notion of an $\mathfrak{a}$-**maximal lattice** for any ideal $\mathfrak{a}$, given by replacing O_k by $\mathfrak{a}$. It turns out that for any ideal $\mathfrak{a}$, all of the $\mathfrak{a}$-maximal lattices in (V, φ) are locally equivalent (see [Shi2, Lemma 5.9]), so it makes sense to speak about the genus of $\mathfrak{a}$-maximal lattices.

If (Λ, φ) is a totally definite lattice over a totally real number field k, then we define the mass of its genus to be

$$\text{Mass}(\Lambda, \varphi) = \sum_{a \in \mathfrak{Cl}} [\Gamma^a : 1]^{-1}.$$

If (Λ, φ) is not totally definite (e.g. when k is not totally real) then Γ^a will be an infinite group, but we would still like to somehow keep track of its size. To do this, we allow Γ^a to act on some symmetric space $\mathfrak{Z}$ and choose a measure on $\mathfrak{Z}$ invariant under this action. We then define the mass in terms of the measures of the quotients $\Gamma^a \backslash \mathfrak{Z}$. So in general, we define the **mass** of (Λ, φ) to be

$$(1.1) \qquad\qquad \text{Mass}(\Lambda, \varphi) = \sum_{a \in \mathfrak{Cl}} \nu(\Gamma^a),$$

where

$$\nu(\Gamma^a) = \begin{cases} [\Gamma^a : 1]^{-1} & \text{if } G_{\mathbf{a}} \text{ is compact,} \\ [\Gamma^a \cap \{\pm 1\} : 1]^{-1} \text{vol}(\Gamma^a \backslash \mathfrak{Z}) & \text{otherwise.} \end{cases}$$

Our main interest in this paper will be to find an exact formula for the mass of the genus of maximal lattices over an arbitrary number field k when $n \geq 3$. Our approach is to use the Tamagawa number of G^{φ} to extend Shimura's computation of the mass of a maximal lattice over a totally real number field to a general number

field k. Then by interpreting the mass in terms of the volume of the non-archimedian stabilizer of (Λ, φ), we use the local group of similitudes of φ to show that the mass is unchanged as we vary over certain genera of $\mathfrak{a}$-maximal lattices.

This exact mass formula essentially expresses the mass as a product of even integer values of the Dedekind zeta function of k, a power of the index of Λ in its dual lattice, and some gamma function factors. If $\dim_k(V)$ is even, a special value of the L-function of a certain quadratic extension of k also appears.

SUMMARY OF NOTATION

Throughout this paper we take k to be a number field, O_k its ring of integers, and D_k the discriminant of $k/\mathbb{Q}$. We denote by v a valuation (or place) of k. We also let $\mathbf{a}$ and $\mathbf{h}$ denote the archimedian and non-archimedian places of k respectively. Suppose $\mathfrak{p}$ is a prime ideal in O_k lying over the prime p in $\mathbb{Z}$, and $x \in k$. We let $|x|_{\mathfrak{p}}$ denote the usual $\mathfrak{p}$-adic absolute value of x defined by $|x|_{\mathfrak{p}} = q^{-\mathrm{ord}_{\mathfrak{p}}(x)}$, where we take $q = q_{\mathfrak{p}} = [O_{\mathfrak{p}} : \mathfrak{p}]$.

We follow the convention that if we have an object R defined at a certain valuation v, we denote it by R_v. If R_v is defined at each of the archimedian valuations, we also write
$$R_{\mathbf{a}} = \prod_{v \in \mathbf{a}} R_v.$$

For an algebraic group G defined over k, we denote the adelization of G by $G_{\mathbf{A}}$.

If R is an arbitrary set, we denote by R_n^m the $m \times n$ matrices with coefficients in R. We write the transpose of a matrix A as ^tA. If x is a matrix, then we let x_{ij} denote the entry of x in the ith row and j^{th} column. Conversely given numbers x_{ij}, we let (x_{ij}) denote the matrix whose entries satisfy $(x_{ij})_{ij} = x_{ij}$. We abbreviate the diagonal matrix
$$\begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & a_{nn} \end{pmatrix}$$
by $\mathrm{diag}[a_{11}, \cdots, a_{nn}]$, and denote the $n \times n$ identity matrix by 1_n. Given an arbitrary $n \times n$ matrix A and an integer l with $1 \leq l \leq n$, we define $\det_l(A)$ to be the determinant of the upper left $l \times l$ submatrix of A. If A is a matrix of functions, we define the matrix of 1-forms $dA = (dA_{ij})$. Given two $n \times n$ matrices A and B over $\mathbb{R}$, we say that $A > B$ if the matrix $A - B$ is positive definite, and we set
$$S_+^n = \{A \in \mathbb{R}_n^n \mid {}^tA = A > 0\}.$$

We let (V, φ) denote a non-degenerate quadratic space of dimension n over k, and take $V_v, \Lambda_{\mathfrak{p}}, G^{\varphi}, G_v^{\varphi}, G_{\mathbf{A}}^{\varphi}$ as defined in the introduction. If we choose a basis $\{v_1, \cdots, v_n\}$ for V, we may express the bilinear form $\varphi(v, w)$ associated to φ as the matrix $\psi = [\varphi(v_i, v_j)]_{1 \leq i,j \leq n}$. We also let $G^-(\varphi)$ denote the set of invertible linear transformations of V which preserve the form φ and have determinant -1.

For convenience, we define the symbols
$$X = k_n^n, \quad T = \{\text{Symmetric } n \times n \text{ matrices with coefficients in } k\},$$
and their local counterparts T_v, and X_v at a valuation v by replacing k by k_v in the above definition.

We set $i = \sqrt{-1} \in \mathbb{C}$. For $x \in \mathbb{R}$ we let $\lfloor x \rfloor$ be the greatest integer $\leq x$. Also, when there is no danger of confusion, we freely use the letters i, j, k, l as indices. Our equations and statements are numbered first by section, then by order within each section, with the appendix labeled by A (e.g. Lemma A2).

§2 The Tamagawa Number and Local Factors

The main fact that we use in what follows is that the Tamagawa number τ of the special orthogonal group $G = G^{\varphi}$ over any number field k is given by

$$(2.1) \qquad \tau(G) = 2 \qquad \text{if } n \geq 3,$$

where $n = \dim_k(V)$. To define this, we first choose a measure $(dx)_{\mathbf{A}}$ on $k_{\mathbf{A}}$ normalized so that

$$(2.2) \qquad \int_{k \backslash k_{\mathbf{A}}} (dx)_{\mathbf{A}} = 1.$$

We then define the **Tamagawa number** of G to be

$$(2.3) \qquad \tau(G) = \int_{G \backslash G_{\mathbf{A}}} |\omega_G|_{\mathbf{A}},$$

where ω_G is a non-zero left G-invariant differential form on G of highest degree and $|\omega_G|_{\mathbf{A}}$ is the volume element defined with respect to $(dx)_{\mathbf{A}}$. By the product formula we see $|c\omega_G|_{\mathbf{A}} = |\omega_G|_{\mathbf{A}}$ for $c \in k^{\times}$, and since ω_G is chosen from a 1 dimensional space, this specifies a left G-invariant measure on $G_{\mathbf{A}}$ which is independent of our choice of ω_G. We call the measure associated to ω_G the **Tamagawa measure** on $G_{\mathbf{A}}$. (For a more detailed introduction, see [Tam], [Vos], or [Weil].)

From now on when speaking of an invariant object, we always understand this to mean it is left invariant. For clarity we also define a **volume form** to be a nowhere zero diferential form of highest degree.

For our computations, it will be useful to define another measure $(d'x)_{\mathbf{A}}$ by the restricted product $(d'x)_{\mathbf{A}} = \prod'_v (d'x)_v$ with local measures

$$(d'x)_v = \begin{cases} \text{Haar measure on } k_v \text{ normalized by } \int_{O_{\mathfrak{p}}} (d'x)_v = 1 & \text{if } k_v = k_{\mathfrak{p}}, \\ \text{Lebesgue measure on } \mathbb{R} & \text{if } k_v = \mathbb{R}, \\ i\,dz \wedge d\bar{z} = 2 \times \text{Lebesgue measure on } \mathbb{R}^2 & \text{if } k_v = \mathbb{C}. \end{cases}$$

This gives $\int_{k \backslash k_{\mathbf{A}}} (d'x)_{\mathbf{A}} = |D_k|^{1/2}$, so in terms of $(d'x)_{\mathbf{A}}$ we have

$$
\begin{aligned}
\tau(G) &= |D_k|^{\frac{-\dim_k(G)}{2}} \int_{G \backslash G_{\mathbf{A}}} |\omega_G|'_{\mathbf{A}} \\
&= |D_k|^{\frac{-n(n-1)}{4}} \int_{G \backslash G_{\mathbf{A}}} |\omega_G|'_{\mathbf{A}},
\end{aligned}
$$
(2.4)

where $|\omega_G|'_{\mathbf{A}}$ is the volume element derived from ω_G using $(d'x)_{\mathbf{A}}$ instead of $(dx)_{\mathbf{A}}$.

We now give a general procedure for constructing a suitable invariant volume form $\widetilde{\omega_G}$ on G. By choosing a global basis $\{v_1, \cdots, v_n\}$ for (V, φ) we can represent the bilinear form associated to φ as a matrix ψ. This gives a natural map

$$(2.5) \qquad X = (k)_n^n \xrightarrow{\ \mathcal{F}\ } T$$
$$x \longmapsto {}^t x \psi x,$$

whose fibre over the matrix $\psi \in T$ is the full orthogonal group of φ. Given the volume forms

$$(2.6) \qquad \omega_X = \bigwedge_{i,j} dx_{ij}, \qquad \omega_T = \bigwedge_{i \leq j} dt_{ij}$$

on X and T respectively, we can find a differential form ω on X such that

$$(2.7) \qquad \omega_X = \mathcal{F}^*(\omega_T) \wedge \omega.$$

By pulling ω back to the fibre and then restricting to the identity component, we get a form $\widetilde{\omega_G}$ on G. From Lemma A6, we see that $\widetilde{\omega_G}$ is a non-zero G-invariant volume form, and is independent of our choice of ω. We will use this construction many times in our calculation, and consistently identify $G = G^\varphi = G^\psi$ as well as the image of Λ under this identification.

For each place v of k, we define the local representation density

$$(2.8) \qquad \beta_v(\psi) = \beta_v(\Lambda, \psi) = \frac{1}{2} \lim_{U \to \psi_v} \frac{\int_{U'} dX}{\int_U dT},$$

where $dX = \prod_{i,j}(dx_{ij})_v$ and $dT = \prod_{i \leq j}(dt_{ij})_v$ are the measures associated to ω_X and ω_T in these coordinates,

$$U' = \begin{cases} \mathcal{F}^{-1}(U) & \text{if } v \in \mathbf{a}, \\ \mathcal{F}^{-1}(U) \cap \{x \in X_v \mid x\Lambda_v = \Lambda_v\} & \text{if } v \in \mathbf{h}, \end{cases}$$

and U is an open neighborhood of ψ_v in T_v. From the construction of $\widetilde{\omega_G}$ above, we can easily see that $\int_{D_v} \widetilde{\omega_G} = \beta_v(\Lambda, \psi)$ where $D \subset G_{\mathbf{A}}$ is the stabilizer of Λ (see [Tam, §6, pp119–120]). In our calculations the lattice Λ will be fixed, so we will often supress Λ and write $\beta_v(\psi)$.

Remark. Notice that both the volume form $\widetilde{\omega_G}$ and the local densities $\beta_v(\psi)$ depend not only on (V, φ) and v, but also on our given choice of basis for (V, φ).

Any choice of volume form ω_G can be used to define an archimedian measure $\tau_{\mathbf{a}}$ on $G_{\mathbf{a}}$ by $\prod_{v \in \mathbf{a}} |\omega_G|'_v$. By choosing $\omega_G = \widetilde{\omega_G}$ as above and expressing (2.4) in terms of local measures, one can prove:

Theorem 2.1. *Let Λ be a lattice in (V, φ), and suppose ψ a matrix representing φ in some global basis for V. Then*

$$\sum_{a \in \mathfrak{Cl}} \tau_{\mathbf{a}}(\Gamma^a \backslash G_{\mathbf{a}}) = \tau(G) \, |D_k|^{\frac{n(n-1)}{4}} \prod_{v \in \mathbf{h}} \beta_v(\Lambda, \psi)^{-1},$$

with $\tau_{\mathbf{a}}$ and $\beta_v(\Lambda, \psi)$ as above, and Γ^a as defined in §1.

Proof. This is proved in [Cas, pp380-382] when $k = \mathbb{Q}$, but the argument there works for any number field k. In his notation, $\beta_v(\Lambda, \psi) = \lambda_v = \tau_v(O^+(\Lambda_v))$ and (due to a typographical error) the right side of (4.19) on p382 should read $2\lambda_\infty^{-1} \prod_{p \neq \infty} \lambda_p^{-1}$. See also [Tam, §6, pp119–120] and [Vos, §15, pp87–88].   □

To simplify our calculations, we use the invertible matrix $\sigma_v \in (k_v)_n^n$ to change basis locally at every place v, so that ψ_v has the standard form

$$(2.9) \qquad \phi_v = {}^t\sigma_v \psi_v \sigma_v = \begin{cases} \begin{bmatrix} 0 & 0 & 2^{-1}1_r \\ 0 & \theta_{\mathfrak{p}} & 0 \\ 2^{-1}1_r & 0 & 0 \end{bmatrix} & \text{if } k_v = k_{\mathfrak{p}}, \\[2em] \begin{bmatrix} 1_q & 0 \\ 0 & -1_r \end{bmatrix} & \text{if } k_v = \mathbb{R}, \\[1em] 1_n & \text{if } k_v = \mathbb{C}, \end{cases}$$

with $q, r \in \mathbb{N}$ satisfying either $q + r = n$ and $q \geq r$, or $\dim(\theta_{\mathfrak{p}}) + 2r = n$ and $\theta_{\mathfrak{p}}$ is some anisotropic symmetric matrix with $\dim(\theta_{\mathfrak{p}}) \leq 4$. Since we take Λ to be a maximal lattice, by [Shi2, Lemma 5.6], we can locally choose a free $O_{\mathfrak{p}}$-basis for $\Lambda_{\mathfrak{p}}$ so that $(\Lambda_{\mathfrak{p}}, \varphi_{\mathfrak{p}})$ is represented by the matrix $\phi_{\mathfrak{p}}$ above, and we choose the matrices $\sigma_{\mathfrak{p}}$ so this is true. The following lemma describes how the local factors change under such a change of basis.

Lemma 2.2. *Let v be a place of k and suppose that ψ and ψ' in $(k_v)_n^n$ are related by $\psi' = {}^tA\psi A$ for some invertible $n \times n$ matrix A. Then*

$$\beta_v(\Lambda, \psi') = |\det(A)|_v^{n+1} \, \beta_v(A\Lambda, \psi).$$

Proof. For $n \times n$ matrices $A \in X$ and $t \in T$ we let $[A] : T \longrightarrow T$ denote the map $[A](t) = {}^tAtA$, which corresponds to change of basis by A for the quadratic form associated to t.

Fix an open set U about ψ' in T, and let $V = [A^{-1}](U)$ be the corresponding neighborhood of ψ. Then one can easily check

$$\frac{\text{vol}_X(\mathcal{F}_{\psi'}^{-1}(U))}{\text{vol}_T(U)} \cdot \frac{\text{vol}_T(U)}{\text{vol}_T([A^{-1}](U))} = \frac{\text{vol}_X(A^{-1}\mathcal{F}_\psi^{-1}(V)A)}{\text{vol}_T(V)} = \frac{\text{vol}_X(\mathcal{F}_\psi^{-1}(V))}{\text{vol}_T(V)},$$

where $\mathcal{F}$ is as in (2.5) and the last equality follows from both parts of Lemma A2.

By passing to the limit as $U \to \psi'$, we have

$$\beta_v(\Lambda, \psi') = \lim_{U \to \psi'} \frac{\text{vol}_T([A^{-1}](U))}{\text{vol}_T(U)} \beta_v(A\Lambda, \psi).$$

This ratio of volumes is given by computing the pull-back of the volume form ω_T under the map $[A]$. We claim that

$$[A]^*(\omega_T) = \det(A)^{n+1}\omega_T,$$

which is to say

$$(2.10) \qquad \bigwedge_{i \leq j} d({}^t A t A)_{ij} = \det(A)^{n+1} \bigwedge_{i \leq j} dt_{ij}.$$

Since $[AB] = [B][A]$, we already know (2.10) is true if we replace $\det(A)^{n+1}$ by some multiplicative character on $GL_n(k_v)$. By construction $c(A)$ is a polynomial in the entries of A, and since the only continuous characters on GL_n are powers of the determinant, we easily verify (2.10) by checking the scalar matrices $A = \lambda \cdot 1_n$.

With this we have

$$\lim_{U \to \psi'} \frac{\mathrm{vol}_T([A]^{-1}(U))}{\mathrm{vol}_T(U)} = |\det(A)|_v^{n+1},$$

which proves our lemma. $\square$

§3 THE NON-ARCHIMEDIAN LOCAL FACTORS

The non-archmedian local factors appearing in the mass formula for a maximal lattice Λ have been calculated by Shimura in [Shi], under the condition that locally the determinant of φ is a unit up to a square if n is odd. We now show how his local factors relate to the local factors $\beta_{\mathfrak{p}}(\Lambda_{\mathfrak{p}}, \phi_{\mathfrak{p}})$ appearing in our mass formula.

Fix a basis $\{v_1, \cdots, v_n\}$ for $V_{\mathfrak{p}}$, let ϕ be the invertible $n \times n$ matrix defined over $k_{\mathfrak{p}}$ which represents $(V_{\mathfrak{p}}, \varphi_{\mathfrak{p}})$ in this basis, and let $\Lambda_{\mathfrak{p}}$ be a lattice in $(V_{\mathfrak{p}}, \varphi_{\mathfrak{p}})$. We define $\beta_{\mathfrak{p}}(\phi)$ as in §2 to be the limit of the ratio of volumes

$$(3.1) \qquad \beta_{\mathfrak{p}}(\phi) = \beta_{\mathfrak{p}}(\Lambda_{\mathfrak{p}}, \phi) = \frac{1}{2} \lim_{U \to \phi} \frac{\int_{U'} dX}{\int_U dT},$$

where U' is a neighborhood in $X_{\mathfrak{p}}$ determined by $\Lambda_{\mathfrak{p}}$ and an open neighborhood U of ϕ in $T_{\mathfrak{p}}$. We may also write U' as $U'(\phi)$ to emphasize its dependence on the matrix ϕ. Since we are working over a $\mathfrak{p}$-adic field, we have a natural choice of neighborhoods U_i to use for this limit, namely $U_i = \phi + P_i$ where $P_i = (\mathfrak{p}^i)_n^n \cap T_{\mathfrak{p}}$.

Lemma 3.1. *Let $\Lambda_{\mathfrak{p}}$ and ϕ be as above, and let $c \in k_{\mathfrak{p}}^\times$. Then we have*

$$\beta_{\mathfrak{p}}(\Lambda_{\mathfrak{p}}, \phi) = |c|_{\mathfrak{p}}^{\frac{n(n+1)}{2}} \beta_{\mathfrak{p}}(\Lambda_{\mathfrak{p}}, c\phi) = |\det(c \cdot 1_n)|_{\mathfrak{p}}^{\frac{(n+1)}{2}} \beta_{\mathfrak{p}}(\Lambda_{\mathfrak{p}}, c\phi).$$

Proof. Since $U \to \phi$, it suffices to compute the limit (3.1) for $U = U_i$. Consider the pre-images

$$U_i'(\phi) = \{x \in X_{\mathfrak{p}} \mid {}^t x \phi x \in \phi + P_i \text{ and } x\Lambda_{\mathfrak{p}} = \Lambda_{\mathfrak{p}}\},$$

and notice $U_i'(\phi) = U_{i+\mathrm{ord}_{\mathfrak{p}}(c)}'(c\phi)$. Using this we have

$$\begin{aligned}
\beta_{\mathfrak{p}}(\phi) &= \frac{1}{2} \lim_{i \to \infty} \frac{\int_{U_i'(\phi)} dX}{\int_{U_i} dT} \\
&= \frac{1}{2} \lim_{i \to \infty} \frac{\int_{U_{i+\mathrm{ord}_{\mathfrak{p}}(c)}'(c\phi)} dX}{\int_{U_i} dT} \\
&= |c|_{\mathfrak{p}}^{\frac{n(n+1)}{2}} \frac{1}{2} \lim_{i \to \infty} \frac{\int_{U_{i+\mathrm{ord}_{\mathfrak{p}}(c)}'(c\phi)} dX}{\int_{U_{i+\mathrm{ord}_{\mathfrak{p}}(c)}} dT} \\
&= |c|_{\mathfrak{p}}^{\frac{n(n+1)}{2}} \beta_{\mathfrak{p}}(c\phi),
\end{aligned}$$

which completes the proof. $\square$

Lemma 3.2. *Let $\Lambda_{\mathfrak{p}}$ and ϕ be as above, and suppose that for our choice of basis we have $\Lambda_{\mathfrak{p}} = \sum_{i=1}^{n} O_{\mathfrak{p}} v_i$. Then $\beta_{\mathfrak{p}}(\Lambda_{\mathfrak{p}}, \phi) = \frac{1}{2} e_{\mathfrak{p}}(\phi)$, where $e_{\mathfrak{p}}(\phi)$ is as in [Shi, §8].*

Proof. In [Shi, §8] $e_{\mathfrak{p}}(\phi)$ is given by

$$e_{\mathfrak{p}}(\phi) = \lim_{i \to \infty} q^{\frac{-n(n-1)}{2}} N_i',$$

where $N_i' = \#\{x \in (O_{\mathfrak{p}}/\mathfrak{p}^i O_{\mathfrak{p}})_n^n \mid {}^t x \phi x \equiv \phi \mod P_i\}$. However, U_i is a sum of cosets mod P_i and one can check that U_i' is a sum of cosets mod $(\mathfrak{p})_n^n$, so by counting them we have

$$\beta_{\mathfrak{p}}(\psi) = \frac{1}{2} \lim_{i \to \infty} \frac{\int_{U_i'} dX}{\int_{U_i} dT} = \frac{1}{2} \lim_{i \to \infty} \frac{\left(\frac{1}{q^i}\right)^{n^2} N_i'}{\left(\frac{1}{q^i}\right)^{\frac{n(n+1)}{2}}} = \frac{1}{2} e_{\mathfrak{p}}(\psi),$$

which proves the lemma. $\square$

We are interested in computing $\beta_{\mathfrak{p}}(\phi_{\mathfrak{p}})$ with respect to a maximal lattice $\Lambda_{\mathfrak{p}}$ in $(V_{\mathfrak{p}}, \varphi_{\mathfrak{p}})$, with $\phi_{\mathfrak{p}}$ as in §2. By Lemmas 3.1 and 3.2 we know

$$(3.2) \qquad\qquad \beta_{\mathfrak{p}}(\phi_{\mathfrak{p}}) = |\det(2 \cdot 1_n)|_{\mathfrak{p}}^{\frac{n+1}{2}} \frac{e_{\mathfrak{p}}(2\phi_{\mathfrak{p}})}{2},$$

and by combining this with [Shi; Theorem 8.6(3), Prop. 3.9, (3.1.9)], we obtain

$$(3.3) \qquad\qquad \beta_{\mathfrak{p}}(\phi_{\mathfrak{p}}) = |\det(2 \cdot 1_n)|_{\mathfrak{p}}^{\frac{n+1}{2}} q^{\kappa_{\mathfrak{p}} n} [\widetilde{\Lambda_{\mathfrak{p}}} : \Lambda_{\mathfrak{p}}] \xi,$$

where $q = \#(O_{\mathfrak{p}}/\mathfrak{p} O_{\mathfrak{p}})$, κ is defined by $2O_{\mathfrak{p}} = \mathfrak{p}^{\kappa}$,

$$\xi = \begin{cases} (1 - q^{-m}) \prod_{i=1}^{m-1}(1 - q^{-2i}) & \text{if } t = 0, \\ \prod_{i=1}^{m}(1 - q^{-2i}) & \text{if } t = 1, \\ (1 + q^{-m}) \prod_{i=1}^{m-1}(1 - q^{-2i}) & \text{if } t = 2, \mathfrak{p} \text{ is unramified in } K, \\ & \qquad \text{and } \widetilde{\Lambda_{\mathfrak{p}}} = \Lambda_{\mathfrak{p}}, \\ 2(1 + q)(1 + q^{1-m})^{-1} \prod_{i=1}^{m-1}(1 - q^{-2i}) & \text{if } t = 2, \mathfrak{p} \text{ is unramified in } K, \\ & \qquad \text{and } \widetilde{\Lambda_{\mathfrak{p}}} \neq \Lambda_{\mathfrak{p}}, \\ 2 \prod_{i=1}^{m-1}(1 - q^{-2i}) & \text{if } t = 2, \text{and } \mathfrak{p} \text{ is ramified in } K, \\ 2(1 + q) \prod_{i=1}^{m-1}(1 - q^{-2i}) & \text{if } t = 3, \\ 2(1 + q)(1 - q^{1-m})^{-1} \prod_{i=1}^{m-1}(1 - q^{-2i}) & \text{if } t = 4, \end{cases}$$

with $t = \dim(\theta_{\mathfrak{p}})$, $m = \lfloor n/2 \rfloor$, $K = k(\sqrt{(-1)^{n/2} \det(\varphi)})$, and $\widetilde{\Lambda_{\mathfrak{p}}} = \{x \in V_{\mathfrak{p}} \mid 2\varphi_{\mathfrak{p}}(x, \Lambda_{\mathfrak{p}}) \in O_{\mathfrak{p}}\}$. For future reference we explicitly state [Shi, (3.1.9)], which says

$$(3.4) \qquad\qquad [\widetilde{\Lambda_{\mathfrak{p}}} : \Lambda_{\mathfrak{p}}] = |\det(2\phi_{\mathfrak{p}})|_{\mathfrak{p}}^{-1}.$$

§4 Archimedian Local Factors

In this section we explicitly compute the volume form ω_G on $G_v = G_v^{\phi_v}$ described in §2 when v is archimedian, and relate ω_G to a natural volume form $\omega_{\mathfrak{Z}}$ on the symmetric space $\mathfrak{Z}_v$. The relationship between ω_G and $\omega_{\mathfrak{Z}}$ is established by constructing a non-zero C_v-invariant volume form ω_C on the fibre C_v of G_v over some chosen point $p_v \in \mathfrak{Z}_v$, and then evaluating $\int_{C_v} \omega_C$. This allows us to connect the associated measures on G_v and $\mathfrak{Z}_v$. We note that when v is real and φ is definite, the situation is much simpler since $\mathfrak{Z}_v = \{1_n\}$ and $C_v = G_v$.

For our calculations we would like to write down ω_G in some set of coordinates on G, and we choose the coordinates given by the strictly lower triangular matrix entries of the natural embedding $G \hookrightarrow (k_v)_n^n$. These give coordinates on an open subset of G whose compliment has measure zero, and the associated coordinate 1-forms give a basis for the cotangent space. The matrix $g^{-1}dg$ is a G-invariant matrix of 1-forms under left multiplication, and so the form

$$(4.1) \qquad \gamma_n = \bigwedge_{i>k} (g^{-1}dg)_{ik}$$

gives a G-invariant volume form on G. Since the space of such forms is 1 dimensional, any G-invariant volume form will be a constant multiple of γ_n.

Calculation 4.1. *Suppose v is archimedian. Then in the coordinates given by $G_v \hookrightarrow (k_v)_n^n$, the volume form ω_G described in §2 can we written as*

$$\omega_G = \pm \frac{1}{2^n} \gamma_n = \pm \frac{1}{2^n} \prod_{l=1}^n \det{}_l(x)^{-1} \bigwedge_{i>k} dx_{ik}.$$

Proof. To compute ω_G it suffices to compute any non-zero monomial Θ in $\mathcal{F}^*(\omega_T)$, since if $\Theta = f(x) \bigwedge_{(i,k)\in I} dx_{ik}$ for some indexing set I and $\omega = f(x)^{-1} \bigwedge_{(i,k)\notin I} dx_{ik}$ is its complimentary monomial, then $\mathcal{F}^*(\omega_T) \wedge \omega = \Theta \wedge \omega = \omega_X$. We choose to calculate the monomial $\Theta = f(x) \bigwedge_{i\leq k} dx_{ik}$. Since we are only interested in finding ω_G up to sign, it will be enough to compute ω_G for $\phi_v = 1_n$.

From (2.5) we have $t = \mathcal{F}(x) = {}^txx$ and so $\mathcal{F}^*(dt) = {}^t(dx)x + {}^tx(dx)$. Therefore

$$(4.2) \qquad \mathcal{F}^*(\omega_T) = \bigwedge_{i\leq k} \left(\sum_j dx_{ji}x_{jk} + x_{ji}dx_{jk} \right)$$
$$= \Theta + \text{other terms}.$$

We compute Θ by induction on the column bound k_0, showing that

$$(4.3) \qquad \bigwedge_{i\leq k\leq k_0} \left(\sum_j dx_{ji}x_{jk} + x_{ji}dx_{jk} \right) = 2^{k_0} \bigwedge_{i\leq k\leq k_0} \sum_j x_{ji}dx_{jk} + \Psi$$

where Ψ is a sum of terms each of which has some dx_{ik} factor with $i > k$.

The case $k_0 = 1$ is clear since the left side is just $2x_{11}dx_{11}$. If $k_0 > 1$ we have

(4.4)

$$\bigwedge_{i \leq k \leq k_0} \left(\sum_j dx_{ji}x_{jk} + x_{ji}dx_{jk} \right)$$

$$= \bigwedge_{i \leq k \leq k_0-1} \left(\sum_j dx_{ji}x_{jk} + x_{ji}dx_{jk} \right) \wedge \bigwedge_{i \leq k = k_0} \left(\sum_j dx_{ji}x_{jk_0} + x_{ji}dx_{jk_0} \right)$$

$$= \left(2^{k_0-1} \bigwedge_{i \leq k \leq k_0-1} \sum_j x_{ji}dx_{jk} + \Psi \right) \wedge \bigwedge_{i \leq k = k_0} \left(\sum_j dx_{ji}x_{jk_0} + x_{ji}dx_{jk_0} \right)$$

We now analyze the term $\Xi = \bigwedge_{i \leq k_0} \left(\sum_j dx_{ji}x_{jk_0} + x_{ji}dx_{jk_0} \right)$ appearing at the end of (4.4). The only terms of Ξ contributing non-zero terms to Θ come from the column k_0. This is because all of the dx_{jk} terms with $k \leq k_0 - 1$ already appear in each term of $\bigwedge_{i \leq k \leq k_0-1} \sum_j x_{ji}dx_{jk}$ contributing to Θ, and so the wedge product of the two is zero. Also, since the entries of dx are linearly independent, such factors dx_{jk_0} must satisfy $j \leq k_0$ to contribute to Θ. So Ξ in (4.4) can be replaced by

(4.5)
$$\bigwedge_{i < k_0} \left(\sum_j x_{ji}dx_{jk_0} \right) \wedge \left(\sum_j dx_{jk_0}x_{jk_0} + x_{jk_0}dx_{jk_0} \right)$$

$$= 2 \bigwedge_{i \leq k_0} \left(\sum_j x_{ji}dx_{jk_0} \right),$$

which proves (4.3).

By combining (4.3) with $k_0 = n$ and Lemma A3, we see that

(4.6) $$\Theta = 2^n \bigwedge_{i \leq k} (^t x dx)_{ik} = 2^n \prod_{l=1}^n \det_l(x) \bigwedge_{i \leq k} dx_{ik} +\ \text{other terms,}$$

which shows that

(4.7) $$\omega_G = \frac{1}{2^n} \prod_{l=1}^n \det_l(x)^{-1} \bigwedge_{i > k} dx_{ik}$$

satisfies (2.7). $\square$

§4.1 COMPUTATION FOR $k_v = \mathbb{R}$ WITH φ DEFINITE

If v is real and φ_v is definite, then $G_v = SO_n(\mathbb{R})$. Since $SO_n(\mathbb{R})$ is compact, $\tau_v(G_v)$ is finite. We now find the measure $\tau_{\mathbb{R}}$ of $SO_n(\mathbb{R})$ with respect to ω_G. From Calculation 4.1 and Lemma A3 we see that (up to sign) on G_v

$$\omega_G \sim \bigwedge_{i > k} (^t g dg)_{ik} \sim \bigwedge_{i > k} (g^{-1}dg)_{ik},$$

and this together with the volume computation in [Vos, (14.6), p85] relative to volume form $\bigwedge_{i > k}(g^{-1}dg)_{ik}$ gives

(4.1.1) $$\tau_{\mathbb{R}}(G_v) = \frac{1}{2} \pi^{\frac{n(n+1)}{4}} \left(\prod_{l=1}^n \Gamma(l/2) \right)^{-1}.$$

§4.2 COMPUTATION FOR $k_v = \mathbb{R}$ WITH φ INDEFINITE

If v is real and φ_v is indefinite, then we take $\phi_v = \text{diag}[1_q, -1_r]$ as in (2.9), $G_\mathbb{R} = SO(q, r)$, and define the (symmetric) space $\mathfrak{Z}_\mathbb{R}$ by

$$\mathfrak{Z}_\mathbb{R} = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}_r^q \,\middle|\, x \in \mathbb{R}_r^r, y \in \mathbb{R}_r^t, {}^t x + x > {}^t y y \right\}.$$

To define a $G_\mathbb{R}$-action on $\mathfrak{Z}_\mathbb{R}$, let

$$B(z) = \begin{bmatrix} {}^t x & {}^t y & x \\ 0 & 1_t & y \\ -1_r & 0 & 1_r \end{bmatrix}, \qquad \gamma = \begin{bmatrix} \frac{-1}{\sqrt{2}}r & 0 & \frac{1}{\sqrt{2}}r \\ 0 & 1_t & 0 \\ \frac{1}{\sqrt{2}}r & 0 & \frac{1}{\sqrt{2}}r \end{bmatrix},$$

$$\mathfrak{Y} = \{Y \in GL_n(\mathbb{R}) \mid {}^t Y \phi_v^{-1} Y = \text{diag}[A, -B] \text{ with } A \in S_+^q, B \in S_+^r\},$$

and induce a $G_\mathbb{R}$-action on $\mathfrak{Z}_\mathbb{R}$ from the bijection

$$\mathfrak{Z}_\mathbb{R} \times GL_q(\mathbb{R}) \times GL_r(\mathbb{R}) \xrightarrow{\ \sim\ } \mathfrak{Y}$$

(4.2.1)
$$(z, \lambda, \mu) \longmapsto B(z) \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix},$$

by allowing $\alpha \in G_\mathbb{R}$ to act on $\mathfrak{Y}$ by left multiplication. (See [Shi2, §6] for details.) Explicitly, (4.2.1) gives the action $z \mapsto \alpha z$ on $\mathfrak{Z}_\mathbb{R}$ by

(4.2.2)
$$\alpha B(z) = B(\alpha z) \begin{bmatrix} \lambda_\alpha(z) & 0 \\ 0 & \mu_\alpha(z) \end{bmatrix},$$

for some matrices $\lambda_\alpha(z)$ and $\mu_\alpha(z)$.

By choosing a distinguished point $p_\mathbb{R} = \begin{bmatrix} 1_r \\ 0_r^t \end{bmatrix} \in \mathfrak{Z}_\mathbb{R}$, we define a map

$$F_\mathbb{R} : G_\mathbb{R} \longrightarrow \mathfrak{Z}_\mathbb{R}$$

(4.2.3)
$$\alpha \longmapsto \alpha p_\mathbb{R}.$$

If we write $\alpha \in G_\mathbb{R}$ as

(4.2.4)
$$\alpha = \begin{bmatrix} a & b & c \\ g & e & f \\ h & l & w \end{bmatrix}$$

with $a, d \in \mathbb{R}_r^r$ and $e \in \mathbb{R}_t^t$, then the map $F_\mathbb{R}$ sends

(4.2.5)
$$\alpha \longmapsto \alpha p_\mathbb{R} = \begin{bmatrix} (w - c)(w + c)^{-1} \\ (\sqrt{2})_t f(w + c)^{-1} \end{bmatrix}.$$

In these coordinates the stabilizer of $p_\mathbb{R}$ is given by

(4.2.6)
$$C_\mathbb{R} = \{\alpha \in G_\mathbb{R} \mid f = 0_t^r, c = 0_r^r\},$$

and the relation ${}^t x \phi_v x = \phi_v$ implies that l and h are also zero. Thus $C_{\mathbb{R}}$ decomposes as

$$C_{\mathbb{R}} \cong \left[G_{\mathbb{R}}(1_q) \times G_{\mathbb{R}}(1_r) \right] \cup \left[G_{\mathbb{R}}^-(1_q) \times G_{\mathbb{R}}^-(1_r) \right]$$

(4.2.7)

$$\alpha \mapsto \left(\begin{bmatrix} a & b \\ g & e \end{bmatrix}, w \right).$$

We will be working with the $G_{\mathbb{R}}$-invariant volume form $\omega_{\mathfrak{z}}$ on $\mathfrak{z}_{\mathbb{R}}$ constructed in [Shi, §4.2], given by the expression

(4.2.8) $$\omega_{\mathfrak{z}} = \delta(z)^{-n/2} \bigwedge_{i,k} dz_{ik},$$

where $\delta(z) = \det(\frac{1}{2}({}^t x + x - {}^t y y))$.

Computation of ω_C and $\int_C \omega_C$

We now compute the expression for ω_C on $C_{\mathbb{R}} = \mathrm{Stab}(p_{\mathbb{R}})$ described in §4. For this it is enough, by the last part of Lemma A6, for us to consider forms whose restrictions to the fibre $C_{\mathbb{R}}$ are equal up to sign. We write this equivalence as $\approx$.

From (4.2.5) we have

$$F_{\mathbb{R}}^*(dx) = -(1_r + (w-c)(w+c)^{-1}) dc (w+c)^{-1}$$
$$+ (1_r - (w-c)(w+c)^{-1}) dw (w+c)^{-1}$$
$$\approx -2_r \, dc \, w^{-1},$$
$$F_{\mathbb{R}}^*(dy) = -(\sqrt{2})_r df (w+c)^{-1} - (\sqrt{2})_r f (w+c)^{-1} d(w+c)(w+c)^{-1}$$
$$\approx (\sqrt{2})_r df \, w^{-1}.$$

Applying Lemma A2 and $\det(w) \approx 1$ to these gives

$$\bigwedge_{i,k} F_{\mathbb{R}}^*(dx)_{ik} \approx 2^{r^2} \bigwedge_{i,k} dc_{ik},$$
$$\bigwedge_{i,k} F_{\mathbb{R}}^*(dy)_{ik} \approx 2^{\frac{rt}{2}} \bigwedge_{i,k} df_{ik},$$

which together with the observation $\delta(p_{\mathbb{R}}) = 1$ yields

$$F_{\mathfrak{z}}^*(\omega_{\mathbb{R}}) \approx 2^{\frac{rn}{2}} \bigwedge_{i,k} dc_{ik} \bigwedge_{i,k} df_{ik}.$$

We recall from Calculation 4.1,

$$\omega_G \approx 2^{-n} \prod_{l=1}^n \det_l(\alpha)^{-1} \bigwedge_{i>k} d\alpha_{ik}.$$

By the construction of ω_G in §2 and $F_{\mathbb{R}}^*(\omega_{\mathbb{R}})$ as above, and since the matrix $g^{-1} dg$ of §4 is skew symmetric. we see that the volume form ω_C on the fibre is

$$\omega_C \approx 2^{\frac{-rn}{2}} 2^{-n} \prod_{l=1}^n \det_l(\alpha)^{-1} \bigwedge_{i>k} da_{ik} \bigwedge_{i>k} de_{ik} \bigwedge_{i,k} dg_{ik} \bigwedge_{i>k} dw_{ik}$$
$$\approx 2^{\frac{-rn}{2}} \omega_{SO_q(\mathbb{R})} \wedge \omega_{SO_r(\mathbb{R})}.$$

By comparison with ω_G in §4.1 and the isomorphism (4.2.7), we find that

$$
\begin{aligned}
\operatorname{vol}_C(C_{\mathbb{R}}) &= \int_{C_{\mathbb{R}}} |\omega_C| \\
&= 2 \cdot 2^{\frac{-rn}{2}} \left[\int_{SO_q(\mathbb{R})} \omega_{SO_q(\mathbb{R})} \right] \left[\int_{SO_r(\mathbb{R})} \omega_{SO_r(\mathbb{R})} \right] \\
&= 2 \cdot 2^{\frac{-rn}{2}} \frac{1}{2} \pi^{\frac{q(q+1)}{4}} \left(\prod_{k=1}^{q} \Gamma(k/2) \right)^{-1} \frac{1}{2} \pi^{\frac{r(r+1)}{4}} \left(\prod_{k=1}^{r} \Gamma(k/2) \right)^{-1},
\end{aligned}
$$

which completes our calculation.

$$\S4.3 \text{ COMPUTATION FOR } k_v = \mathbb{C}$$

If v is complex, then $G_{\mathbb{C}} = SO_n(\mathbb{C})$ and we define the (symmetric) space $\mathfrak{Z}_{\mathbb{C}}$ by

$$\mathfrak{Z}_{\mathbb{C}} = \{ z \in \mathbb{R}_n^n \mid {}^t z = -z, {}^t z z < 1 \}.$$

To define a $G_{\mathbb{C}}$-action on $\mathfrak{Z}_{\mathbb{C}}$, we first let

$$
B(z) = \begin{bmatrix} 1_n & z \\ -z & 1_n \end{bmatrix}, \qquad I = \begin{bmatrix} 1_n & 0 \\ 0 & -1_n \end{bmatrix},
$$

$$
\mathfrak{X} = \left\{ X \in GL_{2n}(\mathbb{R}) \;\middle|\; {}^t X I X = \begin{bmatrix} A & 0 \\ 0 & -B \end{bmatrix} \text{ with } A, B \in S_+^n \right\}.
$$

One can check that this gives an injection

$$\mathfrak{Z}_{\mathbb{C}} \times GL_n(\mathbb{R}) \times GL_n(\mathbb{R}) \longrightarrow \mathfrak{X}$$

(4.3.1)

$$(z, \lambda, \mu) \longmapsto B(z) \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}.$$

Writing $\alpha = a + bi \in G_{\mathbb{C}}$ with $a, b \in \mathbb{R}_n^n$, we define $\iota(\alpha) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ and allow α to act on $x \in \mathfrak{X}$ by left multiplication by $\iota(\alpha)$

$$\alpha x = \iota(\alpha) x.$$

By a direct calculation we see that this gives a well-defined action on the image of (4.3.1) and can be used to define a $G_{\mathbb{C}}$-action on $\mathfrak{Z}_{\mathbb{C}}$ by

(4.3.2)
$$\alpha B(z) = \iota(\alpha) B(z) = B(\alpha z) \begin{bmatrix} \lambda_\alpha(z) & 0 \\ 0 & \mu_\alpha(z) \end{bmatrix},$$

the key observation being that ${}^t \iota(\alpha) I \iota(\alpha) = I$ for $\alpha \in G_{\mathbb{C}}$. The same calculation shows that

$$\lambda_\alpha(z) = \mu_\alpha(z) = (a + bz),$$

which we henceforth denote by $\mu_\alpha(z)$.

By choosing a distinguished point $p_{\mathbb{C}} = 0_n^n \in \mathfrak{Z}_{\mathbb{C}}$, we define a map

$$F_{\mathbb{C}} : G_{\mathbb{C}} \longrightarrow \mathfrak{Z}_{\mathbb{C}}$$

(4.3.3)

$$\alpha \longmapsto \alpha p_{\mathbb{C}}.$$

Writing this map out in real coordinates we see

(4.3.4)
$$\alpha = a + bi \longmapsto -ba^{-1},$$

where $a, b \in \mathbb{R}_n^n$. In these coordinates the stabilizer of $p_{\mathbb{C}}$ is given by

(4.3.5)
$$C_{\mathbb{C}} = \mathrm{Stab}(p_{\mathbb{C}}) = \{\alpha = a + bi \in G_{\mathbb{C}} \mid b = 0_n^n\} \cong SO_n(\mathbb{R}).$$

We now construct a $G_{\mathbb{C}}$-invariant volume form on $\mathfrak{Z}_{\mathbb{C}}$. To do this we need to know how the differentials transform under the map $F_{\mathbb{C}}$. We begin with a few definitions. For any two points $w, z \in \mathfrak{Z}_{\mathbb{C}}$ we let

(4.3.6)
$$\xi(w, z) = 1_n - {}^t w z, \qquad \xi(z) = \xi(z, z),$$

(4.3.7)
$$\delta(w, z) = \det(\xi(w, z)), \qquad \delta(z) = \delta(z, z),$$

which satisfies the relation

(4.3.8)
$$^t B(w) I B(z) = \begin{bmatrix} \xi(w, z) & z + {}^t w \\ z + {}^t w & -\xi(w, z) \end{bmatrix}.$$

By combining (4.3.8), $^t\iota(\alpha) I \iota(\alpha) = I$, and (4.3.2), we have

$$^t\mu_\alpha(w)(\alpha z - \alpha w)\mu_\alpha(z) = z - w,$$

$$^t\mu_\alpha(w)\xi(\alpha w, \alpha z)\mu_\alpha(z) = \xi(w, z).$$

Fixing $w \in \mathfrak{Z}_{\mathbb{C}}$, we differentiate these with respect to z and evaluate at $z = w$ to obtain

$$d(\alpha z) = {}^t\mu_\alpha(z)^{-1} \, dz \, \mu_\alpha(z)^{-1},$$

$$\delta(\alpha z) = \det(\mu_\alpha(z))^{-2}\delta(z).$$

By applying Lemma A4 to these two equations, we see that the expression

(4.3.9)
$$\omega_{\mathfrak{Z}} = \delta(z)^{\frac{1-n}{2}} \bigwedge_{i > k} dz_{ik}$$

gives a non-zero $G_{\mathbb{C}}$-invariant volume form on $\mathfrak{Z}_{\mathbb{C}}$.

Computation of ω_C and $\int_C \omega_C$

We now compute the form ω_C on $C_{\mathbb{C}} = \mathrm{Stab}(p_{\mathbb{C}})$ described in §4. By the last part of Lemma A6, it is enough to consider forms whose restrictions to the fibre $C_{\mathbb{C}}$ are equal up to sign. We write this equivalence as $\approx$.

First we compute $F_{\mathbb{C}}^*(\omega_{\mathfrak{Z}})$. From (4.3.4) we have

$$F_{\mathbb{C}}^*(dz) = -db \, a^{-1} - b \, d(a^{-1})$$

$$\approx db \, a^{-1},$$

and so

$$\bigwedge_{i>k} F_{\mathbb{C}}^*(dz)_{ik} \approx \bigwedge_{i>k} \left(db\, a^{-1}\right)_{ik}.$$

From the relations defining $G_{\mathbb{C}}$, we know that ${}^t a \approx a^{-1}$ and the restriction of ${}^t a\, db$ to $C_{\mathbb{C}}$ is skew symmetric, therefore so is $a({}^t a\, db)a^{-1} = db\, a^{-1}$. Applying Lemma A5 to this gives

$$\bigwedge_{i>k} db_{ik} = \prod_{l=1}^{n-1} \det_l(a) \bigwedge_{i>k} (db\, a^{-1})_{ik}$$

and so

$$F_{\mathbb{C}}^*(\omega_3) = \prod_{l=1}^{n-1} \det_l(a)^{-1} \bigwedge_{i>k} db_{ik}$$

since $\delta(p_{\mathbb{C}}) = 1$.

From our choice of local measures in §2, the real volume form $\widetilde{\omega}$ associated to the complex volume form ω is given by $\omega \wedge \overline{\omega}$. Combining this with Calculation 4.1 we have

$$\widetilde{\omega_G} = 2^{-2n} \prod_{l=1}^{n} \det_l(z)^{-1} \det_l(\bar{z})^{-1} \bigwedge_{i>k} (i dz_{ik} \wedge d\bar{z}_{ik})$$

$$= 2^{\frac{n(n-5)}{2}} \prod_{l=1}^{n} \det_l(z)^{-1} \det_l(\bar{z})^{-1} \bigwedge_{i>k} (da_{ik} \wedge db_{ik})$$

$$\approx 2^{\frac{n(n-5)}{2}} \prod_{l=1}^{n-1} \det_l(a)^{-2} \bigwedge_{i>k} (da_{ik} \wedge db_{ik}).$$

By applying the procedure in §2 to $\widetilde{\omega_G}$ and $F_{\mathbb{C}}^*(\omega_3)$ above, we see that the (real) volume form ω_C on the fibre is given by

$$\omega_C = 2^{\frac{n(n-5)}{2}} \prod_{l=1}^{n-1} \det_l(a)^{-1} \bigwedge_{i>k} da_{ik}.$$

From §4.1, we know

$$\int_{SO_n(\mathbb{R})} \omega_G = \frac{1}{2} \pi^{\frac{n(n+1)}{4}} \prod_{j=1}^{n} \Gamma(j/2)^{-1},$$

so we have

$$\text{vol}_C(C_{\mathbb{C}}) = \int_{C_{\mathbb{C}}} \omega_C = 2^{\frac{n(n-3)}{2}} \int_{SO_n(\mathbb{R})} \omega_G = 2^{\frac{n(n-3)}{2}} \left(\frac{1}{2} \pi^{\frac{n(n+1)}{4}} \prod_{j=1}^{n} \Gamma(j/2)^{-1} \right),$$

which completes our calculation.

§5 Mass Formula for Maximal Lattices

In this section we compute an exact mass formula for the genus of maximal lattices $\Lambda \subset (V, \varphi)$. We call a lattice $\Lambda \subset (V, \varphi)$ a **maximal lattice** if $\varphi(\Lambda) \subseteq O_k$ and Λ is maximal with this property.

To define the mass of a genus of lattices over an arbitrary number field k, we need to define symmetric spaces $\mathfrak{Z}_v$ for all $v \in \mathbf{a}$. If v is real and φ_v is definite, then we take $\mathfrak{Z}_v$ to be a single point with measure one. If v is real and φ_v is indefinite or v is complex, then we take $\mathfrak{Z}_v$ as in §4.2 or §4.3 respectively. The spaces $\mathfrak{Z}_v$ come equipped with a transitive G_v-action, an invariant volume form $\omega_{\mathfrak{Z}}$, and a distinguished point p_v. For each $v \in \mathbf{a}$, we define a surjective map

$$F_v : G_v \longrightarrow \mathfrak{Z}_v$$

(5.1)

$$\alpha \longmapsto \alpha p_v$$

and denote by C_v the fibre of F_v over p_v. We let

$$(5.2) \qquad \mathfrak{Z} = \prod_{v \in \mathbf{a}} \mathfrak{Z}_v, \qquad C = \prod_{v \in \mathbf{a}} C_v, \qquad p = (p_v)_{v \in \mathbf{a}},$$

and let F denote the product map

$$(5.3) \qquad\qquad\qquad F : G_{\mathbf{a}} \longrightarrow \mathfrak{Z}.$$

We observe that the $C = F^{-1}(p)$ is the fibre of F over p.

We define the **mass** of a quadratic form (V, φ) with respect to a lattice Λ to be

$$(5.4) \qquad\qquad \mathrm{Mass}(\Lambda, \varphi) = \sum_{a \in \mathfrak{Cl}} \nu(\Gamma^a)$$

where

$$(5.5) \qquad \nu(\Gamma^a) = \begin{cases} [\Gamma^a : 1]^{-1} & \text{if } G_{\mathbf{a}} \text{ is compact,} \\ [\Gamma^a \cap \{\pm 1\} : 1]^{-1} \mathrm{vol}(\Gamma^a \backslash \mathfrak{Z}) & \text{otherwise.} \end{cases}$$

Theorem 5.1. *Let (V, φ) be a non-degenerate quadratic space of dimension $n \geq 3$ defined over a number field k of degree d over $\mathbb{Q}$. Then the mass of (V, φ) with respect to a maximal lattice $\Lambda \subset (V, \varphi)$ is given by*

$$\mathrm{Mass}(\Lambda, \varphi) = 2|D_k|^{\lfloor \frac{(n-1)^2}{4} \rfloor} \left[\prod_{j=1}^{\lfloor \frac{n-1}{2} \rfloor} |D_k|^{\frac{1}{2}} \left(\frac{(2j-1)!}{(2\pi)^{2j}} \right)^d \zeta_k(2j) \right] [\widetilde{\Lambda} : \Lambda]^{\frac{n-1}{2}} \prod_{v | \mathfrak{e}} \lambda_v$$

$$\prod_{v \in \mathbf{a}} b_v^{\varphi} \prod_{v \text{ complex}} \left(2^{-\frac{(n-1)(n-2)}{2}} \pi^{\frac{n(n+1)}{4}} \prod_{j=1}^{n} \Gamma(j/2)^{-1} \right)$$

$$\begin{cases} 2^{-\left(\frac{n-1}{2}\right)d} & \text{if } 2 \nmid n, \\ |D_k|^{\frac{1}{2}} \left[(\frac{n}{2} - 1)! (2\pi)^{-\frac{n}{2}} \right]^d L_k(\frac{n}{2}, \chi) & \text{if } 2 | n, \end{cases}$$

where r_v and $t_v = \dim(\theta_v)$ are defined by the normalization of φ_v in §2,

$$\Gamma_i(s) = \pi^{\frac{i(i-1)}{4}} \prod_{j=0}^{i-1} \Gamma(s - (j/2)),$$

$$\widetilde{\Lambda} = \{x \in V \mid 2\varphi(x, \Lambda) \in O_k\},$$

$$b_v^\varphi = 2^{\frac{r_v n}{2}} \pi^{\frac{(n-r_v)r_v}{2}} \Gamma_{r_v}(r_v/2) \Gamma_{r_v}(n/2)^{-1},$$

$\mathfrak{e}$ *is the product of all prime ideals for which $\widetilde{\Lambda}_v \neq \Lambda_v$, $\zeta_k(s)$ and $L_k(s, \chi)$ are zeta and L-functions over k, χ is the non-trivial Hecke character associated to the extension K/k where $K = k(\sqrt{(-1)^{n/2}\det(\varphi)})$, and λ_v is defined by*

$$\lambda_v = \begin{cases} 1 & \text{if } t_v = 1, \\ 2^{-1}(1+q)^{-1}(1+q^{1-m})(1+q^{-m}) & \text{if } t_v = 2, \mathfrak{p} \text{ is unramified in } K, \\ & \qquad\qquad\qquad\qquad \text{and } \widetilde{\Lambda_\mathfrak{p}} \neq \Lambda_\mathfrak{p}, \\ 2^{-1} & \text{if } t_v = 2, \text{ and } \mathfrak{p} \text{ is ramified in } K, \\ 2^{-1}(1+q)^{-1}(1-q^{-2m}) & \text{if } t_v = 3, \\ 2^{-1}(1+q)^{-1}(1-q^{1-m})(1-q^{-m}) & \text{if } t_v = 4, \end{cases}$$

where q is the norm of the prime ideal at $v \in \mathbf{h}$ and $m = \lfloor \frac{n}{2} \rfloor$.

Proof. By Lemma A7 applied to F, for each class $a \in \mathfrak{Cl}$ we have

$$\tau_\mathbf{a}(\Gamma^a \backslash G_\mathbf{a}) = \mathrm{vol}_C((\Gamma^a \cap S)\backslash C_\mathbf{a})\mathrm{vol}_{\mathfrak{Z}}(\Gamma^a\backslash\mathfrak{Z}),$$

where $S = \{g \in G_\mathbf{a} \mid gz = z \text{ for every } z \in \mathfrak{Z}\}$. By Lemma A1, $S = \{(\pm 1)_{v, v \in \mathbf{a}}\}$ so this becomes

$$(5.6) \qquad \tau_\mathbf{a}(\Gamma^a \backslash G_\mathbf{a})\mathrm{vol}_C(C_\mathbf{a})^{-1} = [\Gamma^a \cap \{\pm 1\} : 1]^{-1}\mathrm{vol}_{\mathfrak{Z}}(\Gamma^a\backslash\mathfrak{Z}).$$

This together with Theorem 2.1 and (2.1) gives

$$(5.7) \quad \begin{aligned} \mathrm{Mass}(\Lambda, \varphi) &= 2|D_k|^{\frac{n(n-1)}{4}}\mathrm{vol}_C(C_\mathbf{a})^{-1}\prod_{v\in\mathbf{h}}\beta_v(\Lambda, \psi)^{-1} \\ &= 2|D_k|^{\frac{n(n-1)}{4}}\prod_{v\in\mathbf{a}}\mathrm{vol}_C(C_v)^{-1}\prod_{v\in\mathbf{h}}\beta_v(\Lambda, \psi)^{-1}. \end{aligned}$$

From here, we complete the proof in 3 parts:

Part 1: First we prove the case where φ_v is positive definite at all $v \in \mathbf{a}$. In this case $C_v = G_v$ for all $v \in \mathbf{a}$, so by (5.7) we have

$$\mathrm{Mass}(\Lambda, \varphi) = 2|D_k|^{\frac{n(n-1)}{4}}\prod_{v\in\mathbf{a}}\beta_v(\psi)^{-1}\prod_{v\in\mathbf{h}}\beta_v(\Lambda, \psi)^{-1}.$$

By (2.9), $\phi_v = {}^t\sigma_v\psi\sigma_v$ and $|\det(\sigma_v)|_v = \left(\frac{|\det(\phi_v)|_v}{|\det(\psi)|_v}\right)^{\frac{1}{2}}$. Combining this with Lemma 3.2 we have

$$2|D_k|^{\frac{n(n-1)}{4}}\prod_{v\in\mathbf{a}}\left(|\det(\psi)|_v^{\frac{-(n+1)}{2}}|\det(\phi_v)|_v^{\frac{n+1}{2}}\beta_v(\phi_v)^{-1}\right)$$

$$\prod_{v\in\mathbf{h}}\left(|\det(\psi)|_v^{\frac{-(n+1)}{2}}|\det(\phi_v)|_v^{\frac{n+1}{2}}\beta_v(\Lambda_v, \phi_v)^{-1}\right),$$

which by the product formula and $\det(\phi_v) = \pm 1$ for all $v \in \mathbf{a}$, gives

$$2|D_k|^{\frac{n(n-1)}{4}} \prod_{v \in \mathbf{a}} \beta_v(\phi_v)^{-1} \prod_{v \in \mathbf{h}} \left(|\det(\phi_v)|_v^{\frac{n+1}{2}} \beta_v(\Lambda_v, \phi_v)^{-1} \right).$$

Substituting (3.3) and (4.1.1), using (3.4), and noticing $\prod_{v|2} 2^{\kappa_v} = 2^n$, we get

$$2|D_k|^{\frac{n(n-1)}{4}} \left(2\pi^{\frac{-n(n+1)}{4}} \prod_{j=1}^{n} \Gamma(j/2) \right)^d [\widetilde{\Lambda} : \Lambda]^{\frac{n-1}{2}}$$

$$\left(2^{-nd} \prod_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} \zeta_k(2i) \prod_{v|\mathfrak{e}} \lambda_v \right) \begin{cases} 1 & \text{if } 2 \nmid n, \\ L_k(\frac{n}{2}, \chi) & \text{if } 2|n. \end{cases}$$

By rearranging terms we obtain

$$2|D_k|^{\frac{n(n-1)}{4}} \left(2^{-(n-1)d} \right) \left[\left(\pi^{\frac{-n(n+1)}{4}} \prod_{j=1}^{n} \Gamma(j/2) \right)^d \prod_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} \zeta_k(2i) \right]$$

$$[\widetilde{\Lambda} : \Lambda]^{\frac{n-1}{2}} \prod_{v|\mathfrak{e}} \lambda_v \begin{cases} 1 & \text{if } 2 \nmid n, \\ L_k(\frac{n}{2}, \chi) & \text{if } 2|n, \end{cases}$$

$$= 2|D_k|^{\frac{n(n-1)}{4}} \left(2^{-(n-1)d} \right) \left[\prod_{j=1}^{\lfloor \frac{n-1}{2} \rfloor} \left(\frac{(2j-1)!}{(2\pi)^{2j}} \right)^d \zeta_k(2j) \right]$$

$$[\widetilde{\Lambda} : \Lambda]^{\frac{n-1}{2}} \prod_{v|\mathfrak{e}} \lambda_v \begin{cases} 2^{\frac{n-1}{2}d} & \text{if } 2 \nmid n, \\ \left[2^{n-1}(\frac{n}{2}-1)!(2\pi)^{-\frac{n}{2}} \right]^d L_k(\frac{n}{2}, \chi) & \text{if } 2|n, \end{cases}$$

$$= 2|D_k|^{\frac{n(n-1)}{4}} \left[\prod_{j=1}^{\lfloor \frac{n-1}{2} \rfloor} \left(\frac{(2j-1)!}{(2\pi)^{2j}} \right)^d \zeta_k(2j) \right] [\widetilde{\Lambda} : \Lambda]^{\frac{n-1}{2}} \prod_{v|\mathfrak{e}} \lambda_v$$

$$\begin{cases} 2^{-(\frac{n-1}{2})d} & \text{if } 2 \nmid n, \\ \left[(\frac{n}{2}-1)!(2\pi)^{-\frac{n}{2}} \right]^d L_k(\frac{n}{2}, \chi) & \text{if } 2|n, \end{cases}$$

$$= 2|D_k|^{\lfloor \frac{(n-1)^2}{4} \rfloor} \left[\prod_{j=1}^{\lfloor \frac{n-1}{2} \rfloor} D_k^{\frac{1}{2}} \left(\frac{(2j-1)!}{(2\pi)^{2j}} \right)^d \zeta_k(2j) \right] [\widetilde{\Lambda} : \Lambda]^{\frac{n-1}{2}} \prod_{v|\mathfrak{e}} \lambda_v$$

$$\begin{cases} 2^{-(\frac{n-1}{2})d} & \text{if } 2 \nmid n, \\ D_k^{\frac{1}{2}} \left[(\frac{n}{2}-1)!(2\pi)^{-\frac{n}{2}} \right]^d L_k(\frac{n}{2}, \chi) & \text{if } 2|n. \end{cases}$$

Part 2: Now suppose that all $v \in \mathbf{a}$ are real, but perhaps φ_v is indefinite at some v. Take

$$b_v^\varphi = 2^{\frac{r_v n}{2}} \pi^{\frac{(n-r_v)r_v}{2}} \Gamma_{r_v}(r_v/2) \Gamma_{r_v}(n/2)^{-1}$$

as above where r_v is defined by the normalization of φ_v in (2.9). For each indefinite v, we add an additional factor of b_v^φ from the formula in part 1, which is seen by observing

$$\operatorname{vol}_C(C_v)^{-1} = \left(2\pi^{\frac{-n(n+1)}{4}} \prod_{j=1}^n \Gamma(j/2) \right) b_v^\varphi$$

and that $b_v^\varphi = 1$ if v is definite. Combined with the previous formula this proves the case where all $v \in \mathbf{a}$ are real.

Part 3: Finally, consider arbitrary $v \in \mathbf{a}$. We define $r_v = 0$ for v complex, and so for such v we have $b_v^\varphi = 1$. Since each complex place replaces two real places in the totally real formula, we again have a correction factor. The relevant calculation to check for v complex is

$$\operatorname{vol}_C(C_v)^{-1} = \left(2\pi^{\frac{-n(n+1)}{4}} \prod_{j=1}^n \Gamma(j/2) \right)^2 \left(2^{-\frac{(n-1)(n-2)}{2}} \pi^{\frac{n(n+1)}{4}} \prod_{j=1}^n \Gamma(j/2)^{-1} \right) b_v^\varphi.$$

This together with Part 2 proves the theorem. $\square$

One interesting application of Theorem 5.1 is to the case of a maximal indefinite quadratic form (Λ, φ) in $n \geq 3$ variables. In this case our formula explicitly computes the volume of the quotient $\Gamma^a \backslash \mathfrak{Z}$.

Corollary 5.2. *Let (Λ, φ) be a maximal indefinite quadratic form in $n \geq 3$ variables and let D the subgroup of $G_\mathbf{A}$ stabilizing Λ. Then*

$$\operatorname{vol}(\Gamma^a \backslash \mathfrak{Z}) = \varepsilon \left[k_\mathbf{A}^\times : k^\times \sigma(D) \right] \operatorname{Mass}(\Lambda, \varphi)$$

where σ is the spinor norm map $G_\mathbf{A}^\varphi \longrightarrow k_\mathbf{A}^\times / (k_\mathbf{A}^\times)^2$ (see [Shi, (2.1.1)]) and ε is either 1 or 2 depending on whether $\dim(V)$ is odd or even. If k has class number one, then

$$\operatorname{vol}(\Gamma^a \backslash \mathfrak{Z}) = \varepsilon \operatorname{Mass}(\Lambda, \varphi).$$

Proof. Since $n \geq 3$ and φ is indefinite, the classes and the spinor genera in the genus of Λ coincide. From this and [Shi, Lemma 2.3(4)] we know that the number of classes is $[k_\mathbf{A}^\times : k^\times \sigma(D)]$. We also know that $\nu(\Gamma^a)$ is independent of the class a by [Shi, Thrm 5.10(1)]. Finally, $-1 \in \Gamma^a$ exactly when $\det(-1_n) = 1$ which happens exactly when $2 | \dim(V)$. This proves the first assertion.

For the second part, from [Shi, Lemma 2.5] we know that $k_\mathbf{A}^\times / k^\times \sigma(D)$ is a quotient of the ideal class group of k. Thus if the class number of k is one, then $[k_\mathbf{A}^\times : k^\times \sigma(D)] = 1$. $\square$

§6 Mass formula for $\mathfrak{a}$-maximal lattices

In this section we use the local similitude groups $\widetilde{G}_\mathfrak{p}^\varphi$ to show that the mass of the genus of $\mathfrak{a}$-maximal lattices is the same for many ideals $\mathfrak{a}$. We do this by noticing that $\operatorname{Mass}(\Lambda, \varphi)$ depends only on the volume of non-archimedean stabilizer $D_\mathbf{h}$ of (Λ, φ), and then showing that the action of $\widetilde{G}_\mathfrak{p}^\varphi$ preserves these volumes.

Let $\widetilde{G}_{\mathfrak{p}}^{\varphi} = \{\widetilde{g} \in GL_n(k_{\mathfrak{p}}) \mid {}^t\widetilde{g}\,\varphi_{\mathfrak{p}}\widetilde{g} = \xi(\widetilde{g})\varphi_{\mathfrak{p}} \text{ for some } \xi(\widetilde{g}) \in k_{\mathfrak{p}}^{\times}\}$ be the local group of similitudes of φ, and let $\Xi_{\mathfrak{p}}(\varphi) = \{\xi(\widetilde{g}) \mid \widetilde{g} \in G_{\mathfrak{p}}^{\varphi}\}$ be the set of similitude multipliers of $G_{\mathfrak{p}}^{\varphi}$. We also recall the local decomposition

$$(V_{\mathfrak{p}}, \varphi_{\mathfrak{p}}) = (H_{2r}, \eta_{2r}) \bigoplus (W_{\mathfrak{p}}, \theta_{\mathfrak{p}})$$

where $(H_{2r}, \eta_{2r}) \cong \bigoplus_{i=1}^{r}(F_{\mathfrak{p}}^2, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix})$ and $(W_{\mathfrak{p}}, \theta_{\mathfrak{p}})$ is anisotropic of dimension $t_{\mathfrak{p}}$. By [OM, 63:19, p170] we know that $t_{\mathfrak{p}} \leq 4$. One can easily compute $\Xi_{\mathfrak{p}}(\eta_{2r}) = F_{\mathfrak{p}}^{\times}$, and so $\Xi_{\mathfrak{p}}(\varphi) = \Xi_{\mathfrak{p}}(\eta_{2r}) \cap \Xi_{\mathfrak{p}}(\theta_{\mathfrak{p}}) = \Xi_{\mathfrak{p}}(\theta_{\mathfrak{p}})$.

Lemma 6.1. *Suppose Λ and Λ' are two lattices in the quadratic space (V, φ) over k with stabilizers $D, D' \subset G_{\mathbf{A}}^{\varphi}$ respectively. Then*

$$\frac{\mathrm{Mass}(\Lambda, \varphi)}{\mathrm{Mass}(\Lambda', \varphi)} = \frac{\mathrm{vol}(D_{\mathbf{h}}')}{\mathrm{vol}(D_{\mathbf{h}})},$$

where the local volumes are defined by $\mathrm{vol}(D_v) = \int_{D_v} \widetilde{\omega_G}$.

Proof. This follows by combining (5.7) with the remarks after (2.8). □

Lemma 6.2. *Suppose $D_{\mathfrak{p}}$ is an open compact subgroup of $G_{\mathfrak{p}}^{\varphi}$ and $\alpha \in \widetilde{G}_{\mathfrak{p}}^{\varphi}$, then* $\mathrm{vol}(D_{\mathfrak{p}}) = \mathrm{vol}(\alpha^{-1}D_{\mathfrak{p}}\,\alpha)$.

Proof. This is equivalent to showing that the volume form ω_G on G_v is invariant under conjugation by α. To see this holds, following the procedure of §2 we can realize G as a fibre of the map $\mathcal{F} : \widetilde{G}_v \to k_v^{\times}$ given by $\mathcal{F}(\widetilde{g}) = \xi(\widetilde{g})$, which gives $\omega_{\widetilde{G}} = \omega_G \wedge \mathcal{F}^*(\frac{d\xi}{\xi})$. Since $\widetilde{G}_v$ is unimodular and $\frac{d\xi}{\xi}$ is clearly invariant under conjugation, we see that ω_G is also invariant. □

Theorem 6.3. *Suppose (Λ', φ) is a non-degenerate $\mathfrak{a}$-maximal lattice of dimension $n \geq 3$ defined over a number field k. Then*

$$\mathrm{Mass}(\Lambda', \varphi) = \mathrm{Mass}(\Lambda, \varphi)$$

where (Λ, φ) is a maximal lattice, and $\mathfrak{a}_{\mathfrak{p}}$ satisfies the following conditions:

If n is odd, then $\mathfrak{a}_{\mathfrak{p}}$ is a square,

If n is even and $t_{\mathfrak{p}} = 2$, then $\mathfrak{a}_{\mathfrak{p}}$ is a norm from $K_{\mathfrak{p}} = k_{\mathfrak{p}}\left(\sqrt{(-1)^{\frac{n}{2}}d_{\mathfrak{p}}}\right)$.

This mass is explicitly given in Theorem 5.1.

Proof. By Lemmas 6.1 and 6.2, it suffices to show for all primes $\mathfrak{p}$ that $\Lambda_{\mathfrak{p}}' = \widetilde{g}\Lambda_{\mathfrak{p}}$ for some $\widetilde{g} \in \widetilde{G}_{\mathfrak{p}}^{\varphi}$, and by [Shi2, Lemma 5.9, p33] we know this is true for any two $\mathfrak{a}$-maximal lattices. By comparing their values under φ we see that

$$\Lambda_{\mathfrak{p}} \text{ is } O_{\mathfrak{p}}\text{-maximal} \iff \widetilde{g}\Lambda_{\mathfrak{p}} \text{ is } \xi(\widetilde{g})O_{\mathfrak{p}}\text{-maximal},$$

so the proof reduces to characterizing the set $\mathrm{ord}_{\mathfrak{p}}(\Xi_{\mathfrak{p}}(\theta_{\mathfrak{p}}))$. We do this by using the local models for $(W_{\mathfrak{p}}, \theta_{\mathfrak{p}})$ in [Shi, §3.2].

If $t_{\mathfrak{p}}$ is odd then we can never find a similitude of odd valuation, since if $ord_{\mathfrak{p}}(\xi(\widetilde{g}))$ is odd then taking determinants gives $ord_{\mathfrak{p}}(\det(\widetilde{g})^2) = ord_{\mathfrak{p}}(\xi(\widetilde{g})^{t_{\mathfrak{p}}})$ which is odd. Conversely, if $\pi_{\mathfrak{p}}$ is a uniformizer in $k_{\mathfrak{p}}$ then we can construct $\pi_{\mathfrak{p}}^2$ in $\Xi(\Theta_p)$ by using $\widetilde{g} = \mathrm{diag}[\pi_p, \cdots, \pi_p]$.

If $t_{\mathfrak{p}} = 0$, then $(V_{\mathfrak{p}}, \varphi_{\mathfrak{p}})$ is a direct sum of hyperbolic planes and $\Xi_{\mathfrak{p}}(\theta_{\mathfrak{p}}) = k_{\mathfrak{p}}^{\times}$.

If $t_{\mathfrak{p}} = 2$, then $(W_{\mathfrak{p}}, \theta_{\mathfrak{p}}) \cong (K_{\mathfrak{p}}, cN_{K_{\mathfrak{p}}/k_{\mathfrak{p}}}(x))$ where $K_{\mathfrak{p}} = k_{\mathfrak{p}}(\sqrt{-\det(\varphi)})$ and $c \in k^{\times}$. Therefore $K_{\mathfrak{p}}^{\times} \subseteq \widetilde{G}_{\mathfrak{p}}^{\theta}$ and so $\Xi_{\mathfrak{p}}(\theta_{\mathfrak{p}}) \supseteq N_{K_{\mathfrak{p}}/F_{\mathfrak{p}}}(K_{\mathfrak{p}}^{\times})$.

If $t_{\mathfrak{p}} = 4$, then $(W_{\mathfrak{p}}, \theta_{\mathfrak{p}}) \cong (B_{\mathfrak{p}}, N_{B_{\mathfrak{p}}/k_{\mathfrak{p}}}(x))$ where $B_{\mathfrak{p}}$ is a division quaternion algebra over $k_{\mathfrak{p}}$. Since $B^{\times} \subseteq \widetilde{G}_{\mathfrak{p}}^{\theta}$ and $N_{B_{\mathfrak{p}}/k_{\mathfrak{p}}}(B_{\mathfrak{p}}^{\times}) = k_{\mathfrak{p}}^{\times}$, we have $\Xi_{\mathfrak{p}}(\theta_{\mathfrak{p}}) = k_{\mathfrak{p}}^{\times}$. $\square$

Remark. In terms of the invariants $(n_{\mathfrak{p}}, d_{\mathfrak{p}}, c_{\mathfrak{p}})$ for the local quadratic space $(V_{\mathfrak{p}}, \varphi_{\mathfrak{p}})$, the condition $t_v = 1$ is equivalent to $c_{\mathfrak{p}} = \left(\frac{(-1)^{n/2}, (-1)^{n/2}d_{\mathfrak{p}}}{\mathfrak{p}} \right)$ when n is odd, and $t_v = 2$ is equivalent to $[K_{\mathfrak{p}} : k_{\mathfrak{p}}] = 2$ when n is even and $K_{\mathfrak{p}}$ is as above.

APPENDIX

It will be convenient to know a few lemmas about matrices of differentials. If we take $x = (x_{ij})$ to be a matrix of functions, then we define the matrix dx to be the matrix (dx_{ij}) of differentials of x.

Lemma A1. *Let $\mathfrak{Z}_v$ be a symmetric space of the type described in §4.2 or §4.3. Then $\{g \in G_v \mid gz = z \text{ for every } z \in \mathfrak{Z}_v\} = \{\pm 1_n\}$.*

Proof. This is the analogous statement of [Shi2, Prop. 6.4(5)] for orthogonal groups, and has the same proof with obvious modifications. $\square$

Lemma A2. *Let dx and dx' be two $r \times t$ matrices of linearly independent differentials, and suppose $dx' = a(dx)$ for some $r \times r$ constant matrix a. Then*

$$\bigwedge_{i,k} dx'_{ik} = \det(a)^t \bigwedge_{i,k} dx_{ik}.$$

Similarly, if $dx' = (dx)a'$ for some $t \times t$ constant matrix a', then

$$\bigwedge_{i,k} dx'_{ik} = \det(a')^r \bigwedge_{i,k} dx_{ik}.$$

Proof. This is well known, and follows from the action of a (resp. a') on a column (resp. row) vector. $\square$

Lemma A3. *Let dx and dx' be two $n \times n$ matrices of linearly independent differentials and suppose $dx' = a(dx)$ for some $n \times n$ constant matrix a. Then*

$$\bigwedge_{i \leq k} dx'_{ik} = \prod_{l=1}^{n} \det_l(a) \bigwedge_{i \leq k} dx_{ik} + \sum \left(\begin{array}{c} \text{terms containing at least one} \\ \text{factor } dx_{ik} \text{ with } i > k \end{array} \right).$$

Proof. It will be enough to analyze the columns $k \geq k_0$, proving inductively that for each $1 \leq k_0 \leq n$ we have

$$(\text{A3.1}) \qquad \bigwedge_{\substack{i \leq k \\ k \geq k_0}} dx'_{ik} = \prod_{l=k_0}^{n-1} \det_l(a) \bigwedge_{\substack{i \leq k \\ k \geq k_0}} dx_{ik} + \Omega,$$

where Ω is a sum of terms each containing at least one factor dx_{ik} with $i > k$.

If $k_0 = n$ then

$$\bigwedge_{i \leq k_0} dx'_{ik_0} = \bigwedge_{i \leq k_0} \sum_j a_{ij} dx_{jk_0}$$

$$= \bigwedge_{i \leq k_0} \sum_{\sigma \in S_n} a_{i\sigma(i)} dx'_{\sigma(i)k_0}$$

$$= \det(a) \bigwedge_{i \leq k_0} dx_{ik_0}$$

since the only non-zero terms in the wedge product come from permutations of the row index i.

Proceeding inductively, we consider the row k_0 and assume (A3.1) holds for all $k > k_0$. Then

(A3.2)

$$\bigwedge_{\substack{i \leq k \\ k \geq k_0}} dx'_{ik} = \bigwedge_{i \leq k_0} dx'_{ik_0} \wedge \bigwedge_{\substack{i \leq k \\ k \geq k_0+1}} dx'_{ik}$$

$$= \left(\bigwedge_{i \leq k_0} \sum_j a_{ij} dx_{jk_0} \right) \wedge \left(\prod_{l=k_0+1}^{n-1} \det_l(a) \bigwedge_{\substack{i \leq k \\ k \geq k_0+1}} dx_{ik} + \Omega \right).$$

The terms dx_{jk_0} of $\bigwedge_{i \leq k_0} \sum_j a_{ij} dx_{jk_0}$ with $j > k_0$ cannot contribute to the term $\bigwedge_{i \leq k, k \geq k_0} dx_{ik}$ since the entries of dx are linearly independent. Therefore the only terms which contribute to it are the dx_{jk_0} with $j \leq k_0$ and these can be written as the following sum over permutations on the row index i:

$$\bigwedge_{i \leq k_0} \sum_{j \leq k_0} a_{ij} dx_{jk_0} = \bigwedge_{i \leq k_0} \sum_{\sigma \in S_{k_0}} a_{i\sigma(i)} dx'_{\sigma(i)k_0}$$

$$= \det_{k_0}(a) \bigwedge_{i \leq k_0} dx_{ik_0}.$$

Combining this with (A3.2), we prove (A3.1). Our lemma then follows from (A3.1) by taking $k_0 = 1$. $\square$

Lemma A4. *Let dx and dx' be two skew-symmetric $n \times n$ matrices of differentials whose upper triangular coordinates are linearly independent, and suppose $dx' = {}^t a (dx) a$ for some $n \times n$ constant matrix a. Then*

$$\bigwedge_{i > k} dx'_{ik} = \det(a)^{n-1} \bigwedge_{i > k} dx_{ik}.$$

Proof. This is proved in the same way as Lemma 3.2, the only difference being that the computation for scalar matrices here gives $\det(a)^{n-1}$. $\square$

Lemma A5. *Let dx and dx' be two skew-symmetric $n \times n$ matrices of differentials whose upper triangular coordinates are linearly independent, and suppose $dx' = (dx)a$ for some $n \times n$ constant matrix a. Then*

$$\bigwedge_{i>k} dx'_{ik} = \prod_{l=1}^{n-1} \det_l(a) \bigwedge_{i>k} dx_{ik}.$$

Proof. We prove by induction that

$$(A5.1) \qquad \bigwedge_{\substack{k<i \\ i\geq i_0}} dx'_{ik} = \prod_{l=i_0}^{n-1} \det_l(a) \bigwedge_{\substack{k<i \\ i\geq i_0}} dx_{ik}$$

for all $1 \leq i_0 \leq n$.

In the case $i_0 = n$, the non-zero terms of $\bigwedge_{k<n} \sum_j dx_{nj} a_{jk}$ come from choosing one term $dx_{nj} a_{jk}$ for each k with no repetition among the j indices. Thus the j index is a permutation of the k index, and we have

$$\bigwedge_{k<n} \sum_{\sigma \in S_{n-1}} dx_{n\sigma(k)} a_{\sigma(k)k} = \det_{n-1}(a) \bigwedge_{k<n} \sum_j dx_{nk}.$$

Now suppose $i_0 < n$. By induction we have

$$\bigwedge_{\substack{k<i \\ i\geq i_0}} \sum_j dx_{ij} a_{jk} = \left(\bigwedge_{k<i_0} \sum_j dx_{i_0 j} a_{jk} \right) \wedge \left(\bigwedge_{\substack{k<i \\ i\geq i_0+1}} \sum_j dx_{ij} a_{jk} \right)$$

$$= \left(\bigwedge_{k<i_0} \sum_j dx_{i_0 j} a_{jk} \right) \wedge \left(\prod_{l=i_0+1}^{n-1} \det_l(a) \bigwedge_{\substack{k<i \\ i\geq i_0+1}} dx_{ik} \right).$$

By skew-symmetry of dx, we see that all of the terms in $\bigwedge_{k<i_0} \sum_j dx_{i_0 j} a_{jk}$ with $j \geq i_0$ would give zero when wedged together with $\bigwedge_{k<i, i\geq i_0+1} dx_{ik}$. Thus the only terms that contribute have the form

$$\sum_{\sigma \in S_{i_0}} dx_{i_0 \sigma(k)} a_{jk} = \det_{i_0-1}(a) \bigwedge_{k<i_0} dx_{i_0 k},$$

which together with the above proves (A5.1). Our result follows from (A5.1) by taking $i_0 = 1$. $\square$

We now state two basic lemmas about volume forms on manifolds.

Lemma A6. *Let $F : X \to Y$ be a map of C^∞-manifolds of dimensions n and m respectively, with $\operatorname{rank}(F) = m$. Suppose that X is a group acting on Y and the map F commutes with this action. Choose $p \in Y$ and let $C = F^{-1}(p)$ be the fibre over p. Given X-invariant volume forms ω_X and ω_Y on X and Y respectively, we can define a unique volume form ω_C on C by choosing $\omega \in (\bigwedge^{n-m})^*(X)$ such that*

$$(A6.1) \qquad \omega \wedge F^*(\omega_Y) = \omega_X$$

and taking ω_C to be the restriction $\omega|_C$ of ω to C. Further, ω_C is C-invariant and when computing ω_C it suffices to take forms on X with coefficients in the fibre C over p.

Proof. In this situation, the forms on X are determined by their definition on any neighborhood, so it is sufficient to check locally on X.

Choose a point $q \in F^{-1}(p) \subset X$. Taking $y_1, \cdots, y_m$ to be a set of coordinates on Y in some neighborhood of p, we can pull these back to give coordinates $x_1, \cdots, x_m$ on some neighborhood of q in X. Since $F^{-1}(p)$ is a regular submanifold of X, we can extend these to give a complete set of coordinates $x_1, \cdots, x_n$ on a possibly smaller neighborhood of q. In these coordinates we have

$$(A6.2) \qquad\qquad \omega_X = f(x) \bigwedge_{i=1}^{n} dx_i,$$

$$(A6.3) \qquad\qquad F^*(\omega_Y) = f_1(x) \bigwedge_{i=1}^{m} dx_i.$$

From this we see that any ω on X satisfying (A6.1) must have the form

$$(A6.4) \qquad \omega = \frac{f(x)}{f_1(x)} \bigwedge_{i=m+1}^{n} dx_i + \sum \left(\begin{array}{c} \text{terms containing at least one} \\ \text{factor from } \{dx_1, \cdots, dx_m\} \end{array} \right).$$

Such an ω exists and is a volume form since both ω_X and ω_Y are nowhere vanishing. Uniqueness of ω_C follows since $x_1, \cdots, x_m$ are constant on C, so all terms of (A6.4) except the first term vanish on C.

To see the C-invariance of ω_C, let $c_0 \in C$ act on (A6.1). This gives

$$c_0^* \wedge F^*(\omega_Y) = \omega_X.$$

But by uniqueness of ω_C we have the second part of

$$c_0^*(\omega_C) = c_0^*(\omega)|_C = \omega_C,$$

so ω_C is C-invariant.

The final assertion is easy, and can be checked in the coordinates $x_1, \cdots, x_n$ above. We write $f_1(x) = f_2(x) + f_2'(x)$ where $f_2'(x)$ has coefficients all of which are zero on C, and observe that the $f_2'(x)$ term disappears whether we restrict coefficients before or after choosing ω. $\square$

Lemma A7. *Suppose we are in the setting of Lemma A6, and take some Fuchsian subgroup $\Gamma \subseteq X$. We let $\mu_C, \mu_X,$ and μ_Y denote the measures associated to $\omega_C, \omega_X,$ and ω_Y respectively. Then*

$$\mu_X(\Gamma \backslash X) = \mu_Y(\Gamma \backslash Y) \mu_C((\Gamma \cap S) \backslash C),$$

where $S = \{x \in X \mid xy = y \text{ for every } y \in Y\}$.

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