CHAPTER IV

THE OTHER CLASSICAL GROUPS

4.1. Classification and general theorems.

We consider only the algebraic groups, over a groundfield k, which, over the universal domain, are isogenous to products of simple groups of the three "classical" types : "special" linear, orthogonal and symplectic. Excluding the case of characteristic 2 (which has not been fully investigated) and certain "exceptional" forms of the orthogonal group in 8 variables (depending upon the principle of triality), such groups, up to isogeny, can be reduced to the following types, which will be called "classical" (the letter indicates the type over the universal domain, and K denotes any separably algebraic extension of k) :

L1. Special linear group (or projective group) over a division algebra D_K over K.

L2. (a) Hermitian (i.e., "special" unitary) group for a hermitian form over a quadratic extension K' of K.
(b) Id. for a non-commutative central division algebra $D_{K'}$ over K', with an involution inducing on K' the non-trivial automorphism of K' over K.

O1. Orthogonal group for a quadratic form over K.

O2. Antihermitian group for an antihermitian (or "skew-hermitian") form over a quaternion algebra over K, with its usual involution.

S1. Symplectic group over K.

S2. Hermitian group for a hermitian form over a quaternion algebra over K, with its usual involution.

Our problem was to solve, if possible, the questions (I), (II), (III) listed in 2.4, for these various groups, their products and the groups isogenous to such products; only a fraction of this program will be fulfilled. In Chapter III, (I) has been solved (affirmatively) for $D^{(1)}$, the special linear group of a division algebra (cf. Remark following Lemma 3.1.1); it will be shown presently that the answer to (1) is negative for $R^{(1)}$ when $R = M_m(D)$, $m \geq 2$; and the problem will be solved for all remaining types except L2(b), for which only a partial answer will be given. Problem (II) has been solved affirmatively, in Chapter III for the types L1, S1; an affirmative answer will be given to it, in this Chapter, for the types L2(a), O1, S2; perhaps the same method could be applied also to the types L2(b), O2, but this would require computations which have not been carried out. Problem (III) has been solved, in Chapter III, for the types L1, S1; it will be solved in this Chapter, for the types L2(a), O1, S2, by a method depending upon the construction of zeta-functions for quadratic forms (the same idea can be applied to the exceptional group G_2, as shown by Demazure in the Appendix); there is at present no obvious way in which one could hope to extend this method to the types L2(b) and O2. In many special cases, once the problems (I), (II), (III) have been solved for a group G, it is not too difficult to obtain a solution for a group G' isogenous to G; but general theorems by which this could be effected are still lacking. Perhaps the method described in 3.6 could be generalized.

From now on, we consider exclusively the types other than Li; it is enough to consider the case $K = k$ (since the case of an arbitrary K can be reduced to this by the operation $R_{K/k}$). The groups in question can all be described as follows :

Put $k' = k$ in the cases O1, O2, S1, S2; in the case L2, call k' a quadratic extension of k. Let D' be an algebra variety over k', such that $D'_{k'}$ is a division algebra with the center k'; let n^2 be the dimension of D'; put $D = R_{k'/k}(D')$, this being the same as D' for $k' = k$. In $D'_{k'}$, let $x \to \bar{x}$ be an involution (i.e., an involutory antiautomorphism) inducing the identity on k, and the non-trivial automorphism of k' over k if $k' \neq k$; this can be extended in an obvious manner to an involution in D, defined over k. Take $R' = M_m(D')$, $R = R_{k'/k}(R') = M_m(D)$; then $X \to {}^t\bar{X}$ is an involution in R, defined over k; so is the mapping $X \to S^{-1} \cdot {}^t\bar{X} \cdot S$ if S is an element of R^*_k such that ${}^t\bar{S} = \pm S$. The "classical group" defined by these data is the one given by

$$(1) \qquad G = \{X \in R^* \mid {}^t\bar{X}SX = S,\ N(X) = 1\}$$

where N is the reduced norm, mapping R^* into its center $Z^* = R_{k'/k}(G_m)$. For the symplectic group (type S1), we have to take $k' = k$, $n=1$, ${}^tS = -S$, and the relation ${}^t\bar{X}SX = S$ implies $N(X) = 1$ (as shown by the consideration of the pfaffian of S and of ${}^t\bar{X}SX$); therefore the same is true for the type S2, since these two types are one and the same over the universal domain. Similarly, in the case O1, we have $k' = k$, $n = 1$, ${}^tS = S$; and ${}^t\bar{X} \cdot S \cdot X = S$ implies $N(X) = \pm 1$; the additional condition $N(X) = 1$ serves to single out the component of the identity in the group ${}^t\bar{X} \cdot S \cdot X = S$; therefore the same holds for the type O2. On the other hand, in the case L2, the mapping $X \to N(X)$ maps the group ${}^t\bar{X} \cdot S \cdot X = S$ onto the subgroup U of $R_{k'/k}(G_m)$ determined by $\bar{z}z = 1$, and G is the kernel of that homomorphism.

Now we put $F(x) = {}^t\bar{x}Sx$, with $x \in D^m$; as the characteristic is not 2, S is uniquely determined by the values of F on D^m_k, except in

the case S1, when $F = 0$; F is called "quadratic" in the case O1, "hermitian" in cases L2, S2, "antihermitian" in the case O2; it is a scalar polynomial function in cases L2(a), O1, S2; in the case O2, it takes its values in the (3-dimensional) subspace D^- of odd elements of the quaternion algebra D (the elements such that $\bar{x} = -x$); in the case L2(b), it takes its values in the n^2-dimensional space D^+ of even elements of D (the elements of D such that $\bar{x} = x$).

Writing T for any one of the symbols L2(a), L2(b), O1, O2, S1, S2, we say that the group G defined by (1) is of type T_m. With this notation, we have the lemma :

<u>Lemma</u> 4.1.1. <u>Let</u> G <u>be the group of type</u> T_m <u>defined by</u> (1); <u>let</u> ξ <u>be a vector in</u> D^m_k, <u>other than</u> 0; <u>let</u> g <u>be the subgroup of</u> G <u>leaving</u> ξ <u>fixed</u>. <u>Then</u> : (a) <u>if</u> $m = 1$, $g = \{e\}$; (b) <u>if</u> $m \geq 2$ <u>and</u> $F(\xi) \neq 0$, g <u>is isomorphic to a group of type</u> T_{m-1}; (c) <u>if</u> $m = 2$ <u>and</u> $F(\xi) = 0$, g <u>is</u> $\{e\}$ <u>or isomorphic to a group</u> $(G_a)^r$; (d) <u>if</u> $m \geq 3$ <u>and</u> $F(\xi) = 0$, g <u>is the semidirect product of a group</u> g' <u>and of a group</u> G'' <u>of type</u> T_{m-2}, <u>where</u> g' <u>is either isomorphic to a group</u> $(G_a)^r$ <u>or to the semidirect product of two groups of that type</u>.

For $m = 1$, we must have $F(\xi) \neq 0$; and (a) is trivial. In case (b), g is isomorphic to the group of type T, acting on the vector-space ${}^t\bar{\xi}Sy = 0$ and leaving invariant the form induced on it by F. In the cases (c), (d), one can, by a suitable change of coordinates over k, transform S and ξ into matrices

$$S = \begin{pmatrix} 0 & 1 & 0 \\ \pm 1 & 0 & 0 \\ 0 & 0 & S'' \end{pmatrix}, \qquad \xi = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Then g consists of the matrices

$$\begin{pmatrix} 1 & x & -v \\ 0 & 1 & 0 \\ 0 & u & X'' \end{pmatrix}$$

with

$$\bar{x} \pm x + {}^t\bar{u}S''u = 0,\ v = \pm {}^t\bar{u}S''X'',\ {}^t\bar{X}''S''X'' = S'',\ N(X'') = 1 .$$

Call G'' the group consisting of the X'', which is of type T_{m-2}; call g' the subgroup of g for which $X'' = 1_{m-2}$; call g'' the subgroup of g' for which $u = 0$; if $m = 2$, $g = g' = g''$. Then g is the semidirect product of g' and $G'' = g/g'$; g' is the semidirect product of g'' and g'/g''; $g'' = \{e\}$ in the case 01; otherwise g'' is isomorphic to G_a, or to D^-, as the case may be; g'/g'' is isomorphic to D^{m-2}. This proves the lemma.

Now we apply the lemma, and Witt's theorem, to the consideration of "spheres"; by <u>the sphere of radius</u> ρ, we understand the variety $\Sigma = \Sigma(\rho)$ defined in D^m by the equation $F(x) = \rho$ (in the cases 01, L2(a), S2, this is actually, in the classical terminology, the sphere of radius $\sqrt{\rho}$). To begin with, Witt's theorem says that, if ξ, ξ' are two vectors, other than 0, in Σ_k, there is $M \in R_k^*$ such that ${}^t\bar{M}SM = S$ and $\xi' = M\xi$; for our purposes, however, we need more. Let H, as in 3.4, be the orbit of the vector e (and therefore also of every vector ξ, other than 0, in D_k^m) under R^*; put $\Sigma^* = \Sigma \cap H$; this is a Zariski-open subset of Σ. With these notations :

<u>Lemma</u> 4.1.2. <u>If</u> $\rho \in k$ (<u>case</u> L2(a), 01, S2), $\rho \in D_k^+$ (<u>case</u> L2(b)), $\rho \in D_k^-$ (<u>case</u> 02), <u>and</u> $\rho \neq 0$, <u>and</u> Σ <u>is the sphere of radius</u> ρ, <u>we have</u> $\Sigma \subset H$, <u>and consequently</u> $\Sigma^* = \Sigma$.

Let K be a field containing k; then $D_K = D_k \otimes K$ is either isomorphic to a matrix algebra $M_r(D'')$ over a central division algebra

D'' over K (cases 0, S, and L2 for $K \not\supset k'$) or to a direct sum $M_r(D'') \oplus M_r(D'')$ of two such algebras (case L2 for $K \supset k'$). In the former case, H is as described in 3.4; call σ an isomorphism of D_K onto $M_r(D'')$; we have to show that, if $x \in D_K^m$ and $F(x) = \rho \neq 0$, the (mr,r)-matrix $\sigma(x)$ is of rank r; this follows at once from the relation

$$\sigma({}^t\bar{x}) \cdot \sigma(S) \cdot \sigma(x) = \sigma(\rho)$$

since ρ is invertible in D_k, so that $\sigma(\rho)$ must be of rank r. Now consider the case L2, with $K \supset k'$; if σ is any isomorphism of D_K onto $M_r(D'') \oplus M_r(D'')$, the involution $x \to \bar{x}$ of D, transported to the latter algebra by means of σ, must exchange the two components, since it induces the non-trivial automorphism on the center $K \oplus K$; therefore we can choose σ so that this involution is $(Y,Z) \to ({}^tZ, {}^tY)$. Put $\sigma = (\sigma_1, \sigma_2)$, where σ_1, σ_2 are two homomorphism of D_K onto $M_r(D'')$, and extend σ, σ_1, σ_2 in the obvious manner to D_K^m and $R_K = M_m(D_K)$. Then H_K consists of the elements x of D_K^m such that the two (mr,r)-matrices $\sigma_1(x)$, $\sigma_2(x)$ over D'' are both of rank r; and one sees, just as above, that $F(x) = \rho \neq 0$ implies $x \in H_K$.

We can now generalize Witt's theorem as follows :

<u>Lemma</u> 4.1.3. (i) <u>Let</u> a, a' <u>be in</u> Σ_K^*; <u>then there is</u> $X \in R_K^*$ <u>such that</u> $a' = Xa$, ${}^t\bar{X} \cdot S \cdot X = S$. (ii) <u>Let</u> G <u>be of any type except</u> L2(b); <u>also, assume, if</u> G <u>is of type</u> 01 <u>or</u> L2(a), <u>that</u> $m \geq 3$, <u>or that</u> $\rho \neq 0$, $m \geq 2$; <u>and let</u> a, a' <u>be in</u> Σ_K^*; <u>then there is</u> $X \in G_K$ <u>such that</u> $a' = Xa$.

In the cases 01, S1, (i) is nothing else than Witt's theorem; in the case S1, (ii) is the same as (i); in case 01, take $X_1 \in R_K^*$ such that $a' = X_1 a$, ${}^tX_1 S X_1 = S$; if $N(X_1) = 1$, take $X = X_1$; if $N(X_1) = -1$, take $X_2 \in R_K^*$ such that $X_2 a = a$, ${}^tX_2 S X_2 = S$, $N(X_2) = -1$ (this can be constructed by reasoning just as in the proof of Lemma 4.1.1), and $X = X_1 X_2$. In the case S2, (ii) is the same as (i); (i) is Witt's theorem if D_K

is a quaternion algebra over K; if not, there is an isomorphism σ of D_K onto $M_2(K)$; $\sigma(a)$ is a $(2m,2)$-matrix, of rank 2 since $a \in H_K$, which can be written as $(b_1\ b_2)$, where b_1, b_2 are in K^{2m}, and we can similarly write $\sigma(a') = (b_1'\ b_2')$. The extension of σ to $M_m(D_K)$ maps $M_m(D_K)$ onto $M_{2m}(K)$, and G_K onto the symplectic group of an alternating bilinear form $\Phi(y_1,y_2)$ on $K^{2m} \times K^{2m}$; and it is easily seen that the assumption $F(a) = F(a')$ is then equivalent to $\Phi(b_1,b_2) = \Phi(b_1',b_2')$; our assumption is then a special case of Witt's theorem, applied to Φ and to the subspaces of K^{2m} respectively spanned by b_1, b_2 and by b_1', b_2'. The proof of (i) for the case 02 is quite similar; in order to deduce (ii) from this, we observe, if D_K is a quaternion algebra over K, that $N(X) = 1$ for all $X \in G_K$ (in fact, this is so for $m = 1$, as one finds by direct calculation; in the general case, it follows then immediately from the fact that G_K is generated by the "quasi-symmetries", i.e. by the elements which leave invariant a nonisotropic hyperplane; cf. Dieudonné, Géom. des gr. class. p.24 and 41); if there is an isomorphism σ of D_K onto $M_2(K)$, this transforms G_K into a group of type $(01)_{2m}$, and one has merely to apply what has been said above for the type 01, observing that the exceptional cases for that type cannot occur here since $2m \geq 2$, and since ρ cannot be 0 if $m = 1$ (for all types, Σ^* is empty if $m = 1$, $\rho = 0$). In the case L2(a), (i) is Witt's theorem if $K \not\supset k'$, and (ii) can be deduced from (i) just as in the case 01; if $K \supset k'$, we have $D_K = D_k \otimes K = k' \otimes K \cong K \oplus K$; let σ be an isomorphism of D_K onto $K \oplus K$; this transforms the involution $x \to \bar{x}$ of D_K into $(y,z) \to (z,y)$; if it maps S into (A,B), where A, B are in $M_m(K)$, we have $B = {}^tA$; it if maps a, a' into (b,c), (b',c'), b, c, b', c' must be vectors, other than 0, in K^m; and the assumption $F(a) = F(a')$ becomes ${}^tc \cdot A \cdot b = {}^tc' \cdot A \cdot b'$; the statement (i) amounts to saying that there is Y in $M_m(K)^*$ such that $b' = Yb$ and ${}^tA \cdot c' = {}^tY^{-1} \cdot {}^tA \cdot c$; (ii) says

that we can take Y such that $\det(Y) = 1$; these statements are easily verified. It seems to be an open question whether (ii) is still valid for the type L2(b).

Combining the above lemmas with Theorem 2.4.2, we have now :

Theorem 4.1.1. *Let ξ be a vector other than 0 in D_K^m; let g be the subgroup of G which leaves ξ fixed; put $\rho = F(\xi)$, and let Σ be the sphere of radius ρ in D^m. Then (except for the case $m = 2$, $\rho = 0$ and the case $m = 1$ of types 01, L2(a), and possibly for the type L2(b)), we have $\Sigma^* = G/g$, $\Sigma_K^* = G_K/g_K$ for every field $K \supset k$, and $\Sigma_A^* = G_A/g_A$; the same is true in the case L2(b) if G is replaced by the group $G^* = \{X \in R^* \mid {}^t\bar{X}SX = S\}$, and g by the subgroup of that group leaving ξ fixed.*

We now consider problems (I), (II) for the groups in question. All available evidence goes to show that (II) is to be answered affirmatively (i.e., that G_A/G_k has finite measure) for all semisimple groups, and that the answer to (I) is given by "Godement's conjecture" : *if G is semisimple, G_A/G_k is compact if and only if G_k contains no unipotent element.*

(Added in December 1960 : these statements have now been proved by Borel and Harishchandra *for all semisimple groups over number-fields*).

In the direction of Godement's conjecture, we prove :

Lemma 4.1.4. *Let Γ be any locally compact group; let γ be a discrete subgroup of Γ, such that Γ/γ is compact. Then the orbit of any $\xi \in \gamma$ under the group of inner automorphisms of Γ is closed.*

In fact, this orbit is the union of the compact sets $\{k^{-1}\xi' k \mid k \in K\}$, where K is a compact set such that $\Gamma = \gamma K$, and where one takes for ξ' all the distinct transforms of ξ under inner automorphisms of γ; and the family consisting of these compact sets is locally finite (i.e., only finitely many of them can meet a given compact subset of Γ). (This lemma is, in substance, due to Selberg; cf. Bombay Colloquium on Function-Theory, 1960, p. 148-149).

In view of this, the necessity of the condition in Godement's criterion will be proved if one shows the following : if G is a semi-simple algebraic group, and ξ is a unipotent element of G_k, the closure of the orbit of ξ under the inner automorphisms of G_A contains the neutral element of G. We do not discuss this in general. For the classical groups, it can be verified easily :

Type L1 : Consider the group $R^{(1)}$, $R = M_m(D)$, $m \geq 2$; as $M_2(D)^{(1)}$ is a subgroup of $M_m(D)^{(1)}$ for $m > 2$, it is enough to discuss the case $m = 2$. Consider the orbit of $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ under the inner automorphisms induced by elements $\begin{pmatrix} x^{-1} & 0 \\ 0 & x \end{pmatrix}$, $x \in I_k$, for a sequence of values of x tending to 0 in A_k.

Type $(01)_m$, $m \geq 3$, F not of index 0 : Take coordinates so that S is as in Lemma 4.1.1; consider the elements $X^{-1}AX$, with

$$A = \begin{pmatrix} 1 & b & c \\ 0 & 1 & 0 \\ 0 & a & 1_{m-2} \end{pmatrix} \qquad X = \begin{pmatrix} x^{-1} & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & 1_{m-2} \end{pmatrix}$$

where $a \in k^{m-2}$, $a \neq 0$, $b = -\frac{1}{2}{}^t aS''a$, $c = -{}^t aS''$, $x \in I_k$; and, as above, take a sequence of values of x tending to 0 in A_k.

Other types, F not of index 0 : As the matrix S must be invertible, the latter condition implies $m \geq 2$; take coordinates so that S is as in Lemma 4.1.1; then, as for the type L1, it is enough to discuss the case $m = 2$. Consider the unipotent element $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ in G_k, where $a \pm \bar{a} = 0$, $a \in D_k$, $a \neq 0$ (notations of Lemma 4.1.1; take $a \in k'$, $a + \bar{a} = 0$, $a \neq 0$, in the cases L2, S2, and $a \in k$, $a \neq 0$, in the cases 02, S1); take its transforms under the inner automorphisms induced by $\begin{pmatrix} x^{-1} & 0 \\ 0 & x \end{pmatrix}$, with $x \in I_k$, x tending to 0 in A_k.

We now seek to prove that G_A/G_k is compact, for the types other than L1, whenever F is of index 0. This will be done for all types except L2(b); for the type L2(b), we only do it for the group G^* defined in Theorem 4.1.1.

We begin by considerations which are valid, whether F is of index 0 or not. For the time being, however, we exclude the type L2(b), and also, for the types 01, L2(a), the cases $m = 2$, F not of index 0, and $m = 1$. Let C be a compact subset of D_A^m, of measure > 1; as in the proof of Lemma 3.1.1, put $C' = C + (-C)$. For $X \in G_A$, the automorphism $x \rightarrow X^{-1}x$ of D_A^m has the module 1; therefore it maps C onto a set $X^{-1}C$ of measure > 1, which cannot be mapped in a one-to-one manner onto its image in D_A^m/D_k^m; this means that $X^{-1}C' \cap D_k^m$ must contain an element $\xi \neq 0$, so that $\xi = X^{-1}c$ with $c \in C'$. Then $F(c) = F(\xi)$, and, if we put $\rho = F(\xi)$, ρ is in $F(C') \cap D_k$, which is a finite set since $F(C')$ is compact and D_k discrete in D_A; write that set as $\{\rho_0 = 0, \rho_1, \ldots, \rho_H\}$. For each i, let Σ_i be the sphere of radius ρ_i (the variety $F(x) = \rho_i$); choose a vector ξ_i in $(\Sigma_i^*)_k$, if there is one (i.e. a vector $\neq 0$ in D_k^m, such that $F(\xi_i) = \rho_i$; for $i = 0$, there is such a vector if and only if F is not of index 0); let g_i be the subgroup of G leaving ξ_i fixed, and call ϕ_i the mapping $X \rightarrow X\xi_i$ of G into Σ_i^*; by Theorem 4.1.1, we can identify Σ_i^* with G/g_i, and then ϕ_i becomes the canonical mapping of G onto G/g_i. Now put $E_i = (\Sigma_i^*)_A \cap C'$, and $B_i = \phi_i^{-1}(E_i)$. Our proof shows that, if X is any element of G_A, there is an i, and an element c of C', such that $\xi = X^{-1}c$ is in $(\Sigma_i^*)_k$, i.e., by Lemma 4.1.3, of the form $M^{-1}\xi_i$ with $M \in G_k$; then $X\xi = XM^{-1}\xi_i$ is in E_i, so that $XM^{-1} \in B_i$, $X \in B_iG_k$. We formulate this as a lemma :

Lemma 4.1.5. There are finitely many factors $\xi_i \neq 0$ in D_k^m, and a compact subset C' of D_A^m, with the following properties : (a) the $\rho_i = F(\xi_i)$ are distinct elements of D_k; (b) let Σ_i be the sphere

<u>of radius</u> ρ_i, g_i <u>the subgroup of</u> G <u>leaving</u> ξ_i <u>fixed</u>, ϕ_i <u>the mapping</u> $X \to X\xi_i$ <u>of</u> G <u>into</u> Σ_i^*; <u>put</u> $E_i = (\Sigma_i^*)_A \cap C'$, $B_i = \phi_i^{-1}(E_i)$; <u>then</u> $G_A = \bigcup_i B_i G_k$.

As indicated above, we choose notations so that $\rho_i \neq 0$ for $i \neq 0$; in particular, if F is of index 0, we have $\rho_i \neq 0$, i.e., in this notation, $i \neq 0$, for all i. By Lemma 4.1.2, for $i \neq 0$, we have $\Sigma_i^* = \Sigma_i$; as $(\Sigma_i)_A$ is a closed subset of D_A^m, $(\Sigma_i)_A \cap C'$, is a compact subset of $(\Sigma_i)_A$. Therefore, for every $i \neq 0$, there is a compact subset K_i of G_A such that $\phi_i(K_i) \supset E_i$; then $B_i \subset K_i(g_i)_A$.

Now assume that F is of index 0, and that $(g_i)_A/(g_i)_k$ is compact for every i, so that there is, for every i, a compact subset K_i' of $(g_i)_A$ such that $(g_i)_A = K_i'(g_i)_k$; then $B_i \subset K_i K_i'(g_i)_k$, and, in view of Lemma 4.1.5, we have $G_A = (\bigcup_i K_i K_i')G_k$, so that G_A/G_k is compact. But for each i, by Lemma 4.1.1, g_i is the group, of the same type as G but with $m-1$ substituted for m, acting on the space ${}^t\bar{\xi}_i S x = 0$ and leaving invariant the form induced on that space by F; obviously, the latter form is of index 0 if F is of index 0. Now, by induction on m, we can prove :

<u>Theorem</u> 4.1.2. <u>For all types other than</u> L2(b), <u>the group</u> G <u>determined by</u> (1) <u>is such that</u> G_A/G_k <u>is compact whenever</u> ${}^t\bar{x}Sx \neq 0$ <u>for</u> $x \neq 0$ <u>in</u> D_k^m.

The theorem is trivially true for $m = 1$ in the cases L2(a), 01 (for then G is reduced to the neutral element), and vacuously true for $m = 0$ in the cases 02, S2; the induction proof is valid for $m \geq 2$ for L2(a), 01, and for $m \geq 1$ for 02, S2 (one could also deduce the cases $m = 2$ of L2(a), 01, and $m = 1$ of 02, S2, directly from Lemma 3.1.1).

In the case L2(b), we consider, instead of G, the group G^* defined in Theorem 4.1.1. One proves then, exactly in the same manner, that G_A^*/G_k^* is compact if F is of index 0. We observe that G^* is

isogenous to $G \times U$, where U is, as before, the commutative subgroup of $R_{k'/k}(G_m)$ determined by $z\bar{z} = 1$. We shall not proceed further with the investigation of the type L2(b), which, in all respects, is the most difficult of all.

Now we apply Lemma 4.1.5 to proving that, for all types except possibly L2(b) and 02, G_A/G_k is of finite measure. Apply Lemma 2.4.1 to G_A, $(g_i)_A$, $(g_i)_k$, and to the characteristic function $f_i(w)$ of E_i on $(\Sigma_i^*)_A$; this shows that the image of B_i in $G_A/(g_i)_k$ is of finite measure if and only if $(g_i)_A/(g_i)_k$ and E_i are so (of course we are using invariant measures on G_A, $(g_i)_A$ and $(\Sigma_i^*)_A$). For $i \neq 0$, we have seen that E_i is compact, hence of finite measure. Proceeding by induction on m, and using Lemma 4.1.1, we may assume that $(g_i)_A/(g_i)_k$ is of finite measure; so the image of B_i in $G_A/(g_i)_k$ is of finite measure; as the obvious mapping from $G_A/(g_i)_k$ onto G_A/G_k is locally a measure-preserving isomorphism, this implies that the image of B_i, which is also the image of $B_i G_k$, in G_A/G_k, is of finite measure. In view of Lemma 4.1.5, the induction part of our proof will be complete if we show that E_o and $(g_o)_A/(g_o)_k$ are of finite measure. The latter fact, in view of Lemma 4.1.1, is also a consequence of the induction assumption. Thus it only remains to show that $E_o = \Sigma_A^* \cap C'$ is of finite measure (for the invariant measure on Σ_A^*) when C' is compact and Σ is the sphere of radius 0; this will be done in 4.2 for the types L2(a) $(m > 3)$, 01$(m > 5)$, S2$(m > 2)$; the case S1 has been treated in 3.5. The case L2(a), $m = 1$, is trivial, and the case L2(a), $m = 2$, has been treated in 3.7; the cases 01, $m = 3$ and 4, have been treated in 3.7; the case S2, $m = 1$, is included in Theorem 3.3.1; therefore this will prove :

<u>Theorem</u> 4.1.3. <u>If</u> G <u>is defined by</u> (1), G_A/G_k <u>is of finite measure for the types</u> L2(a), 01, S1, S2, <u>except only for the case</u> 01, $m = 2$, F <u>not of index</u> 0.

The same would be proved for the type 02 if we could show, also in that case, that E_o is of finite measure. As to the type L2(b), our method could be applied to the group G^*, and would show that G^*_A/G^*_k is of finite measure, again under the assumption that E_o is so. These cases will not be considered any further.

4.2. End of proof of Theorem 4.1.3 (types 01, L2(a), S2).

In the remainder of this chapter, we shall consider only the cases 01 (quadratic case), L2(a) (hermitian case) and S2 (quaternionic case); we put $\delta = [D_k : k]$; in the quadratic case, $D_k = k$ and $\delta = 1$; in the hermitian case, $D_k = k'$ and $\delta = 2$; int the quaternionic case, D_k is a field of quaternions with the center k, and $\delta = 4$. In all cases, D_k^m is a vector-space of dimension δm over k, $F(x) = {}^t\bar{x}Sx$ is a k-valued quadratic form in that space, D^m is an affine space of dimension δm in the sense of algebraic geometry, and the sphere of radius ρ is the hypersurface defined by $F(x) = \rho$.

In this section, we assume that F is not of index 0, and Σ will denote the sphere of radius 0, i.e. the hypersurface $F(x) = 0$ in D^m; as before, we put $\Sigma^* = \Sigma \cap H$, where H is the orbit of the vector $e = (1, 0, \dots, 0)$ in D^m under the group $R^* = M_m(D)^*$.

Lemma 4.2.1. For the variety Σ^*, (1) is a set of convergence factors, provided $\delta m > 4$.

This is done by computing the number of points of Σ^* modulo p for almost all p (the formulas for this are well known) and applying Theorem 2.2.5. We exclude all p for which S is not in $M_m(D_{\underline{o}_p})$, all p which divide $2N(S)$, and all p which are ramified in k' (resp. in D_k) in the hermitian (resp. quaternionic) case. Then, in the quadratic case, the number of solutions, other than 0, of $F(x) = 0$ in the field F_q with $q = N(p)$ elements is $q^{m-1} - 1$ if $m \equiv 1$ mod. 2, and $(q^{m'} - \varepsilon)(q^{m'-1} + \varepsilon)$ with $m' = m/2$ and $\varepsilon = \pm 1$ if $m \equiv 0$ mod. 2($\varepsilon = +1$ or -1 according as $(-1)^{m'}$ det (S) is or is not a square in F_q). In view of Theorem 2.2.5, this proves the lemma in that case. In the hermitian case, consider first the case when p does not split in k', i.e. when it can be extended in only one way to k', so that k'_p is a quadratic extension of k_p, and $\underline{o}'_p/p$ is a quadratic extension of $F_q = \underline{o}_p/p$; then $F(x)$ determines a quadratic form in $(\underline{o}'_p/p)^m$ considered as a vector-space of dimension $2m$ over F_q, so that the number of solutions, other than 0, of $F(x) = 0$ modulo p is given by $(q^m - \varepsilon)(q^{m-1} + \varepsilon)$ with a suitable $\varepsilon = \pm 1$. If p "splits" in k', i.e. if it can be extended to two distinct valuations p', p'' of k', then, reasoning as in the latter part of the proof of Lemma 4.1.3, we see that the number of points of $\Sigma^*_{\underline{o}_p}$ modulo p is the number of pairs of vectors x, y in F_q^m, other than 0, satisfying a relation ${}^tyS'x = 0$, where S' is an invertible matrix in $M_m(F_q)$; this is equal to $(q^m - 1)(q^{m-1} - 1)$. The conclusion is the same as before. In the quaternionic case, reasoning as in the first part of the proof of Lemma 4.1.3, we see that the number of points of $\Sigma^*_{\underline{o}_p}$ is the number of $(2m,2)$-matrices $(x_1\ x_2)$ of rank 2 over F_q such that $\Phi(x_1,x_2) = 0$, where Φ is a non-degenerate alternating bilinear form on $F_q^{2m} \times F_q^{2m}$; this has the value $(q^{2m} - 1)(q^{2m-1} - q)$. The conclusion is again the same.

Now let (ε_ν), for $0 \leq \nu \leq \delta - 1$, be a basis of D_k over k, with $\varepsilon_o = 1$; call E_ν the mapping $x \to x\varepsilon_\nu$ of D^m, considered as an automorphism of the (δm)-dimensional affine space. Let L be the group of all such automorphisms, which is isomorphic, over k, to $M^*_{\delta m}$, the full linear group in δm variables; then $R^* = M_m(D)^*$ is the subgroup of the elements of L which commute with E_ν for $1 \leq \nu \leq \delta - 1$ (in particular, $R^* = L$ for $\delta = 1$). Call H_o the orbit of the vector $e = (1,0,\dots,0)$ of D^m under the group L; this is a Zariski-open subset of D^m, and the orbit H of e under R^* is a Zariski-open subset of H_o. Let G_o be the group of type 01, consisting of the elements of L of determinant 1, leaving the quadratic form F invariant; the group G defined by 4.1 (1) is then a subgroup of G. As before, put $\Sigma^* = \Sigma \cap H$; and put $\Sigma^*_o = \Sigma \cap H_o$; then Σ^* is a Zariski-open subset of Σ^*_o. Assume, from now on, that $\delta m > 4$; in view of Lemma 4.2.1, and of Lemma 3.4.1, the Tamagawa measure on Σ^*_o, derived from any gauge-form dw, induces on Σ^* the Tamagawa measure derived from dw on Σ^*, and $\Sigma^*_o - \Sigma^*$ is of measure 0. Now take any $\xi \in \Sigma^*_k$ (F is not of index 0), and call g, g_o the subgroups of G and G_o leaving ξ fixed; by Theorem 4.1.1, we can identify Σ^*_o with G_o/g_o, and Σ^* with G/g. Choose the gauge-form dw on Σ^*_o so that it matches algebraically with invariant gauge-forms on G_o and g_o (in the sense of 2.4); then the Tamagawa measure dw_A derived from it is invariant under $(G_o)_A$, hence also under G_A; therefore it induces on Σ^*_A the (uniquely determined) Tamagawa measure on Σ^*_A which is invariant under G_A. In order to complete the proof of Theorem 4.1.3, we have to show that $\Sigma^*_A \cap C'$ is of finite measure, for that measure, whenever C' is compact in D^m_A; it is now clear that it is enough to prove this for Σ^*_o instead of Σ^*; in other words, it is enough to consider the quadratic case (with $m > 4$).

Actually, we shall prove a stronger result (needed in the next sections) :

Theorem 4.2.1. Assume that $\delta m > 4$, and call dw_A the Tamagawa measure on Σ^*_A, invariant under G_A. Then $\int_{\Sigma^*_A} \Phi(w) dw_A < +\infty$ for every function Φ in D^m_A, defined for $x = (x_v) \in D^m_A$ by $\Phi(x) = \prod_v \phi_v(x_v)$, where the ϕ_v are as follows : (i) for almost all p, ϕ_p is the characteristic function of $D^m_{\mathfrak{o}_p}$; (ii) for all p, ϕ_p is continuous with compact support; (iii) for any infinite place v_λ, $|\phi_\lambda(x)| \leq C_\lambda \exp(-Q_\lambda(x))$, where C_λ is a constant and Q_λ is a positive-definite quadratic form on $D^m_{k_\lambda}$ considered as a vector-space over $\mathbb{R}$.

We first show how this implies the finiteness of the measure of $\Sigma^*_A \cap C'$ for compact C'. Take for ϕ^o_p the characteristic function of $D^m_{\mathfrak{o}_p}$ for every p; take for ϕ^o_λ the characteristic function of $Q_\lambda(x) < 1$, where Q_λ is as in (iii); then $\Phi_o(x) = \prod_v \phi_v(x_v)$ is the characteristic function of an open neighborhood of 0 in D^m_A, so that C' can be covered by finitely many translates of this neighborhood; therefore the characteristic function of C' is majorized by a finite sum $\Sigma_i \Phi_o(x+a_i)$, with $a_i \in D^m_A$; as every term of that sum satisfies the conditions for Φ in Theorem 4.2.1, this proves our assertion.

Now we prove our theorem. It is clearly enough to consider the quadratic case (with $m > 4$); then $\Sigma^* = \Sigma - \{0\}$. Proceeding as in similar calculations in Chapter III, we see that it is enough to show that all the factors in the product

$$\prod_v \int_{\Sigma^*_{k_v}} \phi_v(w)(dw)_v$$

are finite and that the product is absolutely convergent. As to the first point, let Q be the quadratic defined in the projective space of dimension $m-1$ by the homogeneous equation $F(x) = 0$; let f be the obvious mapping of Σ^* onto Q; for each v, this determines a mapping of $\Sigma^*_{k_v}$ onto the compact space Q_{k_v}; and $\Sigma^*_{k_v}$ is fibered over Q_{k_v} by that mapping, with the fibre k^*_v. As Q_{k_v} is compact, the finiteness of our integral will be proved provided we show that each point of Q_{k_v} has a neighborhood Ω such that the same integral, taken over $f^{-1}(\Omega)$, is finite. Assume (as we may, since the characteristic is not 2) that coordinates have been taken so that $F(x) = \Sigma_i a_i x_i^2$; if Ω is a suitable neighborhood of a point of Q where $w_1 \neq 0$, we can write, in $f^{-1}(\Omega)$, $w_1 = t$, $w_i/w_1 = u_i$ for $2 \leq i \leq m$, so that $\Sigma_{i \neq 1} a_i u_i^2 = -a_1$. It is easily seen that the invariant gauge-form on Σ^* can be taken to be

$dw = dw_2 \dots dw_m / a_1 w_1$; this is equal to $t^{m-3}\omega(u)dt$, where $\omega(u)$ is a gauge-form on Ω. This gives :

$$\int_{f^{-1}(\Omega)} \phi_v(w)\cdot(dw)_v = \int_\Omega \omega_v \int_{k_v^*} \phi_v(w)\cdot|t|_v^{m-3}(dt)_v \,.$$

Because of the conditions on ϕ_v, the last integral is absolutely convergent and is a continuous function of u in Ω, provided $m \geq 3$.

Now take any p such that all the a_i and 2 are units in $\underline{o}_p$ and that ϕ_p is the characteristic function of $(\underline{o}_p)^m$; let π be a generator of the maximal ideal in $\underline{o}_p$. For every point w of $\Sigma^*_{k_p} \cap (\underline{o}_p)^m$, we can write $w = \pi^\nu w'$, where $\nu \geq 0$, $w' \in (\underline{o}_p)^m$, $w' \not\equiv 0 (\text{mod.} p)$, $F(w') = 0$; the latter conditions aumount to $w' \in \Sigma^*_{\underline{o}_p}$, since $\Sigma^* = \Sigma \cap H$ and we have seen in 3.4 that $H_{\underline{o}_p}$ consists of the points $x \in (\underline{o}_p)^m$ such that $x \not\equiv 0$ (mod. p). As the formula given above for dw shows that the mapping $w \to \pi^\nu w$ changes $(dw)_p$ into $q^{\nu(2-m)}(dw)_p$, with $q = N(p)$, this gives

$$\int_{\Sigma^*_{k_p}} \phi_p(w)\cdot(dw)_p = \sum_{\Sigma=o}^{\infty} q^{\nu(2-m)}\mu_p = (1-q^{2-m})^{-1}\mu_p$$

where μ_p is the $(dw)_p$-measure of $\Sigma^*_{\underline{o}_p}$. By Lemma 4.2.1, $\prod_p \mu_p$ is absolutely convergent (for $m > 4$). As the same is true of $\prod_p (1-q^{2-m})$, this completes our proof.

4.3. <u>The local zeta-functions for a quadratic form.</u>

Notations remain the same as above.

<u>Lemma</u> 4.3.1. <u>Let</u> V <u>be the Zariski-open set defined by</u> $F(x) \neq 0$ <u>in the affine space</u> D^m <u>of dimension</u> $\delta m \geq 4$. <u>Then</u> $(1-q^{-1})$, $q = N(p)$, <u>is a set of convergence factors for</u> V.

In fact, by the same formula which was used in the proof of Lemma 4.2.1, the number of points of V modulo p, for almost all p,

is $q^{\delta m - 1}(q-1)$ if δm is odd, and $q^{m'-1}(q^{m'}-\varepsilon)(q-1)$ if $\delta m = 2m'$. In view of Theorem 2.2.5, this proves the lemma.

As V is isomorphic to the variety, in the affine space of dimension $\delta m + 1$, with the generic point $(x, 1/F(x))$, V_A is the set of the points $x \in D_A^m$ such that $F(x) \in I_k$.

We now wish to calculate, for almost all p, the following "local zeta-function" :

(1) $$Z_p(s) = \int_{V_{k_p}} |F(x)|_p^s \phi_p(x) d'_p x \;;$$

here $\phi_p(x)$ is the characteristic function of $D^m_{\underline{o}_p}$; $d'_p x$ is the local measure for V_{k_p}, derived from the gauge-form $dx_1 \dots dx_{\delta m}$ for V (the x_i are the coordinates, for any basis of D_k^m over k) and from the convergence factor $1 - q^{-1}$. Clearly this can also be written :

(2) $$Z_p(s) = (1-q^{-1})^{-1} \int_{(\underline{o}_p)^{\delta m}} |F(x)|_p^s \, (dx)_p$$

where $(dx)_p$ is the additive measure in $(\underline{o}_p)^{\delta m}$, normalized so that the group has the measure 1; we may assume that the quadratic form $F(x)$ has been written as $F(x) = \Sigma_i a_i x_i^2$. Then :

<u>Lemma</u> 4.3.2. <u>For almost all</u> p, <u>we have</u>

$$Z_p(s) = (1-q^{-s-m})(1-q^{-s-1})^{-1}(1-q^{-2s-m})^{-1} \quad \underline{\text{for}} \quad \delta = 1,\ m \ \underline{\text{odd}};$$

$$Z_p(s) = (1-\varepsilon q^{-m'})(1-q^{-s-1})^{-1}(1-\varepsilon q^{-s-m'})^{-1} \quad \underline{\text{for}} \quad \delta m = 2m' ,$$

<u>where in the latter case</u> ε <u>is</u> $+1$ <u>or</u> -1 <u>according as</u> $(-1)^{m'} a_1 a_2 \dots a_{\delta m}$ <u>is or is not a quadratic residue in</u> $\underline{o}_p$ <u>molulo</u> p.

In fact, it will be shown that this is so whenever all a_i and 2 are units in $\underline{o}_p$. Let π be a prime element in $\underline{o}_p$; for $x \in (\underline{o}_p)^{\delta m}$, we can write $x = \pi^{\nu} x'$ with $\nu \geq 0$, $x' \in (\underline{o}_p)^{\delta m}$, $x' \not\equiv 0$ mod. p, and get

$$Z_p(s) = (1-q^{-1})^{-1}(1-q^{-2s-\delta m})^{-1} Z'_p(s) ,$$

with $Z'_p(s) = \int |F(x)|_p^s \cdot (dx)_p$, the integral being taken over $x \in (\underline{o}_p)^{\delta m}$, $x \not\equiv 0$ mod. p. Now let N_ν, for $\nu \geq 1$, be the number of those solutions in $(\underline{o}_p)^{\delta m}$ mod. p^ν for the congruence $F(x) \not\equiv 0$ mod. p^ν which are $\not\equiv 0$ mod.p; we know that N_1 is $q^{\delta m-1}-1$ if δm is odd, and $(q^{m'}-\varepsilon)(q^{m'-1}+\varepsilon)$ if $\delta m = 2m'$; and it is easily seen that $N_{\nu+1} = q^{\delta m-1} N_\nu$ for $\nu \geq 1$. In the set $x \in (\underline{o}_p)^{\delta m}$, $x \not\equiv 0$ mod. p, the measure of the subset where $F(x) \equiv 0$ mod. p^ν, i.e. where $|F(x)|_p \leq q^{-\nu}$, is equal to $q^{-\delta m \nu} N_\nu$ for $\nu \geq 1$; and the measure of the subset where $F(x) \not\equiv 0$ mod. p, i.e. where $|F(x)|_p = 1$, is $q^{-\delta m}(q^{\delta m}-1-N_1)$. This gives :

$$Z'_p(s) = q^{-\delta m}(q^{\delta m}-1-N_1) + \sum_{\nu=1}^{\infty} q^{-\nu s}(q^{-\delta m \nu} N_\nu - q^{-\delta m(\nu+1)} N_{\nu+1}) .$$

A trivial calculation gives the result in the lemma.

As to the value of ε, we remark the following :

(a) Quadratic case $(\delta = 1)$, m even : then $\varepsilon = (\Delta/p)$ (quadratic residue character of Δ mod.p), where $\Delta = (-1)^{m/2} \det(S)$ is the discriminant of F;

(b) Hermitian case $(\delta = 2)$: then, write $F = \Sigma_1^m \bar{x}_i a_i x_i$, $k' = k(\alpha)$ with $\alpha^2 = a \in k$, $x_i = y_i + \alpha z_i$, $\bar{x}_i = y_i - \alpha z_i$; then, in terms of the δm variables y_i, z_i, F has the coefficients a_i, $-a_i a$; therefore $\varepsilon = (a^m/p)$, i.e. $\varepsilon = 1$ for m even, and $\varepsilon = (a/p)$ for m odd.

(c) Quaternionic case $(\delta = 4)$: we can take a base 1,i,j,ij for D_k over k, with $i^2 = a \in k$, $j^2 = b \in k$, $ij = -ji$; if we put $F = \Sigma_i \bar{x}_i a_i x_i$, $x_i = t_i + iu_i + jv_i + ijw_i$, then, in terms of the δm variables t_i, u_i, v_i, w_i, F has the coefficients $a_i, -a_i a, -a_i b, a_i ab$, so that $\varepsilon = 1$.

4.4. The Tamagawa number (hermitian and quaternionic cases).

From now on (in this section) we assume that $\delta = 2$ (hermitian case) or $\delta = 4$ (quaternionic case); we use "resp." to refer to these two cases (in that order). In both cases, we shall denote by $z \to \nu(z) = \bar{z}z$ the norm-mapping of D^* into G_m; its kernel is U (in the notation of 3.7) resp. $D^{(1)}$. In both cases, ν maps D_A^* onto an open subgroup of I_k. In the hermitian case, by class-field theory, $\nu(D_A^*) \cdot k^*$ is an open subgroup of I_k of index 2; in the quaternionic case (cf. Lemma 3.3.2), $\nu(D_A^*)$ contains all elements of I_k whose components at the infinite places of k are all >0, so that $\nu(D_A^*) \cdot k^* = I_k$. In both cases, we define a character λ of I_k by putting $\lambda = 1$ on $\nu(D_A^*) \cdot k^*$ and $\lambda = -1$ on the complement of that group in I_k. In the hermitian case, λ is the character of I_k of order 2 belonging to the quadratic extension k'/k in the sense of class-field theory; in the quaternionic case, λ is the trivial character of I_k. By $|\ |$, we always denote the idele-module taken in I_k.

In the quaternionic case, we shall construct Fourier transforms of functions in D_A^m by means of the character $\chi_D({}^t\bar{x}Sy)$, where χ_D is the character of D_A introduced in 3.1. In the hermitian case, we have $D_A = A_{k'}$, and we do the same by means of $\chi'({}^t\bar{x}Sy)$, where χ' is the character of $A_{k'}$ defined by Theorem 2.1.1; in both cases, we simplify notations by writing χ instead of χ' resp. χ_D. If $\Psi(y)$ is the Fourier transform of $\Phi(x)$ in D_A^m, defined by

$$\Psi(y) = \int_{D_A^m} \Phi(x)\chi({}^t\bar{x}Sy)dx$$

(dx = Tamagawa measure in D_A^m), and if $X \in M_m(D_A)$ is such that ${}^t\bar{X}SX = S$, then the Fourier transform of $\Phi(Xx)$ is $\Psi(X'y)$ with $X' = S^{-1} \cdot {}^t\bar{X}^{-1} \cdot S$, as one sees by replacing x, y by Xx, X'y and observing that, for

${}^t\bar{X}SX = S$, the module of the automorphism $x \to Xx$ of D_A^m is 1. Similarly, if $z \in D_A^*$, the Fourier transform of $\Phi(xz)$ is $|\bar{z}z|^{-\delta m/2}\Psi(y\bar{z}^{-1})$.

Our method will depend upon the construction of a zeta-function (whose residue, as usual, gives the Tamagawa number) by means of a function $\Phi(x)$ in D_A^m of which we assume that it is "of standard type" in a sense similar to that defined in 3.1, and also that Theorem 4.2.1 is valid both for Φ and for its Fourier transform Ψ; such functions can be obtained by the procedure described in 3.1 (following the definition of the "standard type").

We are concerned with the group

$$G = \{X \in M_m(D) \mid {}^t\bar{X}SX = S,\ N(X) = I\}$$

(we know that $N(X) = 1$ is a consequence of ${}^t\bar{X}SX = S$ in the quaternionic case, but not in the hermitiant case). By 3.7(b) resp. Theorem 3.3.1, we know that $\tau(G) = 1$ for $\delta m = 4$. From now on, we assume $\delta m > 4$.

If V is as in Lemma 4.3.1, we denote by V_A^+ the open subset of V_A given by $\lambda(F(x)) = 1$. With this notation, we introduce the function

$$(1) \qquad Z^\Phi(s) = \int_{V_A^+} |F(x)|^s \Phi(x)\,d'x ,$$

where $d'x$ is the Tamagawa measure on V_A derived from the gauge-form $dx_1 \dots dx_{\delta m}$ (if the x_i are the coordinates of x for any choice of a basis of D_k^m over k) and from the convergence factors $(1-q^{-1})$, $q = N(p)$

We put, for $\nu = 0, 1$ (hermitian case) and for $\nu = 0$ (quaternionic case) :

$$I_\nu = \int_{V_A} |F(x)|^s \lambda(F(x))^\nu \Phi(x)\,d'x ;$$

then we have $Z^\Phi(s) = \frac{1}{2}(I_o + I_1)$ resp. $= I_o$. We give now a multiplicative calculation for I_o, I_1 resp. for I_o; this is similar to the corresponding calculations in Chapter III; I_ν is the product of a "finite part" (i.e.,of an integral over a finite product $\prod_{v \in S} V_{k_v}$) and of a product of "local zeta-functions"

$$\int_{V_{k_p}} |F(x)|_p^s \lambda_p(F(x))^\nu \phi_p(x)\,d_p'x .$$

For $\nu = 0$, this is given by Lemma 4.3.2; for $\nu = 1$, λ_p is the local character induced by λ on k_p^* considered (in the obvious manner) as a subgroup of I_k; if p is not ramified in k', this is given by $\lambda_p(t) = \lambda(p)^r$ if $|t|_p = q^r$, with $\lambda(p) = +1$ or -1 according as p "splits" or not in k' (i.e. according as there are two valuations p', p'' of k' extending p, with $k'_{p'} \cong k'_{p''} \cong k_p$, or there is only one such valuation p', with $k'_{p'}$ quadratic and non-ramified over k_p). But then we have

$$|F(x)|_p^s \lambda_p(F(x)) = |F(x)|_p^{s'} , \quad s' = s + \frac{\log\lambda(p)}{\log q} ,$$

so that Lemma 4.3.2 gives the value of the local zeta-function also in this case. In the quaternionic case, we find (in view of the remarks following Lemma 4.3.2) that the infinite product for I_o coincides, for almost all p, with that for

$$\zeta_k(s+1)\zeta_k(s+2m)\zeta_k(2m)^{-1} ,$$

which shows that it converges absolutely for $\mathrm{Re}(s) > 0$, and that

$$[sI_o]_{s=o} = \rho_k \int_{D_A^m} \Phi(x)\,dx .$$

Similarly, in the hermitian case, the infinite product for I_o is the same (for almost all p) as that for

$$\zeta_k(s+1)\zeta_k(s+m)\zeta_k(m)^{-1} \qquad (m \text{ even}),$$
$$\zeta_k(s+1)L_{k'/k}(s+m)L_{k'/k}(m)^{-1} \qquad (m \text{ odd}),$$

where $L_{k'/k} = \zeta_{k'}/\zeta_k$ is the L-function belonging to the quadratic extension k' of k, i.e. to the character λ. The infinite product for I_1 is the same as that for

$$L_{k'/k}(s+1)L_{k'/k}(s+m)\zeta_k(m)^{-1} \qquad (m \text{ even}),$$
$$L_{k'/k}(s+1)\zeta_k(s+m)L_{k'/k}(m)^{-1} \qquad (m \text{ odd}).$$

As $m > 2$ in this case, this proves the absolute convergence for $Re(s) > 0$. Furthermore, we find that

$$[sI_o]_{s=o} = \rho_k \int_{D_A^m} \Phi(x)dx \ , \ [sI_1]_{s=o} = 0 \ .$$

Thus, in all cases, the integral for $Z^\Phi(s)$ is absolutely convergent for $Re(s) > 0$, and

(2) $$[s \cdot Z^\Phi(s)]_{s=o} = \tfrac{1}{4}\delta\rho_k \int_{D_A^m} \Phi(x)dx \ .$$

Now we give the additive calculation. Take any $x \in V_A^+$; this means that $x \in D_A^m$ and that $F(x)$ is of the form $\bar{z}\rho z$ with $\rho \in k^*$, $z \in D_A^*$, i.e. that $F(xz^{-1}) = \rho$. By Hasse's fundamental theorem on quadratic forms, the fact that the equation $F(x') = \rho$ has a solution x' in D_A^m implies that it has a solution ξ in D_k^m; then we have $F(x) = F(\xi z)$. On V_k, which is the set of the vectors $\xi \in D_k^m$ such that $F(\xi) \neq 0$, consider the equivalence relation $F(\xi')/F(\xi) = \bar{\zeta}\zeta$ with $\zeta \in D_k^*$; by Lemma 4.1.3,

two vectors ξ, ξ' are in the same equivalence class for this if and only if $\xi' = M\xi\zeta$ with $M \in G_k$, $\zeta \in D_k^*$. Let (ξ_μ) be a complete set of representatives for the equivalence classes on V_k under this relation; put $\rho_\mu = F(\xi_\mu)$; for $\mu \neq \nu$, ρ_ν/ρ_μ cannot be of the form $\bar{\zeta}\zeta$ with $\zeta \in D_k^*$; therefore (by the norm theorem for cyclic extensions, applied to k'/k in the hermitian case, and by Eichler's norm theorem in the quaternionic case) ρ_ν/ρ_μ cannot be of the form $\bar{z}z$ with $z \in D_A^*$. From this, one concludes at once that, for every $x \in V_A^+$, there is one and only one μ such that $F(x) = \bar{z}\rho_\mu z$ with $z \in D_A^*$; let Ω_μ be the subset of V_A^+ where this is so for a given μ; this is an open subset of V_A, and we get :

(3) $$Z^\Phi(s) = \sum_\mu \int_{\Omega_\mu} |F(x)|^s \Phi(s)d'x \ .$$

Put $\Gamma = G \times D^*$, and make it act on D^m by $((X,z),x) \to Xxz$. By Lemma 4.1.3, Ω_μ is the same as the orbit of ξ_μ under Γ_A; call $\Gamma^{(\mu)}$ the subgroup of Γ leaving ξ_μ fixed; this consists of the (X,z) such that $X\xi_\mu = \xi_\mu z^{-1}$; when that is so, we have $F(\xi_\mu) = F(\xi_\mu z^{-1})$, and therefore $\bar{z}z = 1$. In order to determine the structure of $\Gamma^{(\mu)}$, change coordinates so that S, ξ_μ appear as matrices

$$S = \begin{pmatrix} a & 0 \\ 0 & S' \end{pmatrix}, \qquad \xi_\mu = \begin{pmatrix} 1 \\ 0 \end{pmatrix} ;$$

then, for $(X,z) \in \xi_\mu$, X must be a matrix $\begin{pmatrix} z^{-1} & 0 \\ 0 & X' \end{pmatrix}$, with ${}^t\bar{X}'S'X' = S'$, and, in the hermitian case, $\det(X') = z$. This shows that, in the quaternionic case, $\Gamma^{(\mu)}$ is isomorphic to $G' \times D^{(1)}$, where G' is the group of type $(S2)_{m-1}$ belonging to the matrix S'; in the hermitian case, $\Gamma^{(\mu)}$ is isomorphic to the group $G'^* = \{X' \mid {}^t\bar{X}'S'X' = S'\}$. Algebraically, the orbit of ξ_μ under Γ is V, so that we can identify V, algebraically (i.e., over the universal domain) with $\Gamma/\Gamma^{(\mu)}$; moreover, it is easily seen that the gauge-form $F(x)^{-\delta m/2}dx$ on V is invariant under Γ, so that we can find invariant gauge-forms on Γ and $\Gamma^{(\mu)}$ which match algebraically with that form on V.

We now proceed with the proof in the quaternionic case, and will then indicate the changes required to adapt it to the hermitian case. By the induction assumption, (1) is a set of convergence factors for G', and $\tau(G')=1$; therefore (1) is a set of convergence factors for $\Gamma^{(\mu)}$, and $\tau(\Gamma^{(\mu)})=1$ (since $\tau(D^{(1)})=1$); reasoning as in Theorems 2.4.3 and 2.4.4, we conclude, since the orbit Ω_μ of ξ_μ under Γ_A is open in V_A, and since $(1-q^{-1})$ is a set of convergence factors for V, that $(1-q^{-1})$ is a set of convergence factors for Γ (and therefore, since it is such a set for D^*, that (1) is a set of convergence factors for G), and also that the Tamagawa measure on Γ_A, $\Gamma_A^{(\mu)}$ and V_A, derived from matching gauge-forms and from these convergence factors, match together topologically when one identifies Ω_μ with $\Gamma_A/\Gamma_A^{(\mu)}$. We can now apply Lemma 2.4.2 to Γ_A, $\Gamma_A^{(\mu)}$, Ω_μ and to the discrete groups Γ_k, $\Gamma_k^{(\mu)}$; this gives

$$\int_{\Omega_\mu} |F(x)|^{s+\frac{1}{2}\delta m}\Phi(x)\cdot|F(x)|^{-\frac{1}{2}\delta m}d'x$$

$$= \int_{\Gamma_A/\Gamma_k} \Big(\sum_{\xi} |F(X\xi z)|^{s+\frac{1}{2}\delta m}\Phi(X\xi z)\Big)d'(X,z) ,$$

where the sum in the right-hand side is extended to all $\xi\in\Gamma_k/\Gamma_k^{(\mu)}$, i.e. to the orbit of ξ_μ under Γ_k; this is nothing else than the equivalence class of ξ_μ in V_k for the equivalence relation defined above; therefore, when we take the sum of both sides over all μ, the sum in the right-hand side gets extended to all $\xi\in V_k$. At the same time, we

have $F(X\xi z)=\bar{z}F(\xi)z$, with $F(\xi)\in k^*$, hence $|F(\xi)|=1$; also, we can write $d'(X,z)=dX\cdot d'z$, where dX, $d'z$ are the Tamagawa measures, on G and on D^*, derived from the convergence factors (1) and $(1-q^{-1})$, respectively. Therefore :

$$(4)\qquad Z^\Phi(s) = \int_{\Gamma_A/\Gamma_k} |\bar{z}z|^{s+\frac{1}{2}\delta m}\cdot\Big(\sum_{\xi\in V_k}\Phi(X\xi z)\Big)dX\cdot d'z ,$$

and the multiplicative calculation shows that this is absolutely convergent for $Re(s)>0$. Just as in the similar calculation in 3.1, we split this into two parts, $Z_+^\Phi(s)$ and $Z_-^\Phi(s)$, by introducing into the integral a factor $1=f_+(z)+f_-(z)$, where f_+ is 0, $\frac{1}{2}$ or 1 according as $|\bar{z}z|$ is <1, $=1$ or >1, and $f_-(z)=f_+(z^{-1})$; then $Z_+^\Phi(s)$ is an entire function. In $Z_-^\Phi(s)$, we apply Poisson summation, observing that $V_k=D_k^m-\Sigma_k$, where Σ is the sphere of radius 0, i.e. the variety $F(x)=0$ in D^m. If Ψ is the Fourier transform of Φ, the Fourier transform of $\Phi(Xxz)$, considered as a function of $x\in D_A^m$ for a given $(X,z)\in\Gamma_A$, is $|\bar{z}z|^{-\delta m/2}\Psi(X'y\bar{z}^{-1})$ with $X'=S^{-1}.{}^t\bar{X}^{-1}.S$; this gives

$$\sum_{\xi\in D_k^m}\Phi(X\xi z) = |\bar{z}z|^{-\frac{1}{2}\delta m}\sum_{\eta\in D_k^m}\Psi(X'\eta\bar{z}^{-1}) .$$

On the other hand, $(X,z)\to(X',\bar{z}^{-1})$ is a measure-preserving automorphism of Γ_A, which maps Γ_k onto itself; if we make that substitution in the integral for $Z_+^\Psi(-s-\frac{1}{2}\delta m)$, and compare it with the expression for $Z_-^\Phi(s)$ obtained by Poisson summation as we have just said, we get :

$$Z_-^\Phi(s) - Z_+^\Psi(s+\tfrac{1}{2}\delta m)$$

$$= \int_{\Gamma_A/\Gamma_k}\Big(|\bar{z}z|^{s}\sum_{\eta\in\Sigma_k}\Psi(X'\eta\bar{z}^{-1}) - |\bar{z}z|^{s+\frac{1}{2}\delta m}\sum_{\xi\in\Sigma_k}\Phi(X\xi z)\Big)f_-(z)dXd'z.$$

If F is of index 0, then $\Sigma_k=\{0\}$, and the calculation can be completed immediately by applying Theorem 3.1.1 (iii) to D^*; it shows that $\tau(G)$ is finite, since the right-hand side must be absolutely convergent for $\mathrm{Re}(s) > 0$, and it gives the value of the right-hand side, showing in particular that it has the residue $\tau(G)\Psi(0)\rho_k$ for $s = 0$; comparing this with (2), we get $\tau(G) = 1$. In the general case, we apply Lemma 2.4.2 to Σ_A^*, which, by Lemma 4.1.4, we may identify with G_A/g_A, where g is the subgroup of G leaving some vector $\xi_o \in \Sigma_k^*$ fixed; by Lemma 4.1.1 and the induction assumption, we have $\tau(g) = 1$. This gives :

$$\int_{\Sigma_A^*} \Phi(s)dw_A = \int_{G_A/G_k} \Big(\sum_{\xi \in \Sigma_k^*} \Phi(X\xi)\Big)\, dX \;;$$

replacing here $\Phi(w)$ by $\Phi(wz)$, and applying the same formula to $\Psi(w\bar{z}^{-1})$, we get (since the automorphism $w \to wz$ of Σ_A^*, for $z \in D_A^*$, changes dw_A into $|\bar{z}z|^{\frac{1}{2}\delta m - 1} dw_A$) :

$$\text{(5)} \qquad Z^\Phi(s) = Z_+^\Phi(s) + Z_+^\Psi(-s-\tfrac{1}{2}\delta m) + \rho_k\tau(G)\Big(\frac{\Psi(0)}{s} - \frac{\Phi(0)}{s+\frac{1}{2}\delta m}\Big)$$

$$+ \rho_k\Big(\frac{1}{s+\frac{1}{2}\delta m-1} \int_{\Sigma_A^*} \Psi(w)dw_A - \frac{1}{s+1} \int_{\Sigma_A^*} \Phi(w)dw_A\Big)$$

in the number-field case, and a similar formula, which we omit, in the function-field case. Since (2) shows that the residue at $s = 0$ must be $\rho_k\Psi(0)$, we get $\tau(G) = 1$. One may observe that, when G_A/G_k is compact, i.e. when F is of index 0, the zeta-function has no other residue than $s = 0$, $s = \frac{1}{2}\delta m$ (and these residues gives the value of the Tamagawa number) while otherwise it also has poles at $s = -1$, $s = 1 - \frac{1}{2}\delta m$; this should be compared with similar results in 3.8 for the zeta-functions of simple algebras.

In the hermitian case, let again G^* be the group $\{{}^t\bar{X}SX = S\}$; if we take coordinates so that S appears as $\begin{pmatrix} a & 0 \\ 0 & S' \end{pmatrix}$, with $a \in k^*$

and $S' \in M_{m-1}(k')$, we see that G^* is the semidirect product of G and of the group $\left\{\begin{pmatrix} z & 0 \\ 0 & 1_{m-1} \end{pmatrix} \,\middle|\, \bar{z}z = 1\right\}$, which is isomorphic to U. Now we have seen that $\Gamma^{(\mu)}$ is isomorphic to a group $G'^* = \{{}^t\bar{X}'S'X' = S'\}$; if G' is the subgroup of G'^* determined by $\det(X') = 1$, we see that $\Gamma^{(\mu)}$ is isomorphic to a semidirect product of G' and U; and, by the induction assumption, (1) is a set of convergence factors for G', and $\tau(G') = 1$. As we have seen in 3.7(c) that $(1-\lambda(p)q^{-1})$ is a set of convergence factors for U, it is also such for $\Gamma^{(\mu)}$; and the measure of $\Gamma_A^{(\mu)}/\Gamma_k^{(\mu)}$, for the Tamagawa measure derived from those convergence factors, is $\tau' = 2\rho_{k'}/\rho_k$, since it has been shown in 3.7(c) that this is so for U. As we have seen that $(1-q^{-1})$ is a set of convergence factors for V, we conclude from this, as above, that the factors

$$(1-q^{-1})(1-\lambda(p)q^{-1})$$

are convergence factors for $\Gamma = G \times D^*$; as they are such for D^*, this shows that (1) is a set of convergence factors for G. Using these sets of convergence factors, we can again apply Lemma 2.4.2 to Γ_A, $\Gamma_A^{(\mu)}$, Ω_μ, Γ_k, $\Gamma_k^{(\mu)}$, and get a formula similar to (4), except that $Z^\Phi(s)$ has now to be replaced by $\tau' Z^\Phi(s)$. The continuation of the calculation is just as before, except that the application of Theorem 3.1.1(iii) to D^* introduces now the constant $\rho_{k'}$ instead of ρ_k. Thus (5), or the corresponding formula in the function-field case, will be valid provided we replace Z^Φ, Z_+^Φ, Z_-^Φ by $\tau' Z^\Phi$, $\tau' Z_+^\Phi$, $\tau' Z_-^\Phi$, and ρ_k by $\rho_{k'}$; since $\tau' = 2\rho_{k'}/\rho_k$, this means that (5) <u>is valid in the hermitian case if</u> ρ_k <u>is replaced by</u> $\frac{1}{2}\rho_k$. Comparing this with (2), we get $\tau(G) = 1$ as before, and the result is the same in the function-field case.

<u>Theorem</u> 4.4.1. <u>We have</u> $\tau(G) = 1$ <u>for the group</u> G <u>defined by</u> (1) <u>in</u> 4.1, <u>in the cases</u> L2(a) (<u>hermitian case</u>) <u>and</u> S2 (<u>quaternionic case</u>).

Remark. The group Γ, acting on D^m by $x \to Xxz$, is not effective; in fact, (X,z) induces the identity on D^m if z is in Z^*, where Z is the center of D, and $X = z^{-1} \cdot 1_m$; the condition $N(X) = 1$ gives then $z^{\delta m/2} = 1$. We have found it more convenient to use Γ, rather than the effective group which could be derived from it. On the other hand, for even $m \geq 4$ in the hermitian case, our methods can also be applied to the following group :

$$\Gamma' = \{(X,\mu) \in M_m(D)^* \times G_m \mid {}^t\bar{X}SX = \mu S,\ \det(X) = \mu^{m/2}\} ;$$

if $(X,\mu) \in \Gamma'$, X is called a similitude of multiplicator μ; if M_k, M_A are the sets of the multiplicators belonging respectively to the elements of Γ'_k and Γ'_A, it follows from Dieudonné's theorems on similitudes that $M_k = M_A \cap k^*$, and that M_A consits of the $\mu \in I_k$ such that $\mu_v > 0$ for every real infinite place v of k for which (i) $k_v = \underline{\underline{R}}$, $k'_w = \underline{\underline{C}}$ for $w|v$, and (ii) F and $-F$ are note equivalent as hermitian forms over k_v. Using this, it can be shown that, when we write $2Z^{\Phi}(s) = I_o + I_1$ as above, I_1 is an entire function and consequently I_o is a meromorphic function, and that both of them satisfy "functional equations" similar to (5). The method fails for odd m.

4.5. The Tamagawa number of the orthogonal group.

From now on, we consider exclusively the quadratic case $\delta = 1$. Our purpose is to prove, by induction on m, the Siegel-Tamagawa theorem $\tau(G) = 2$ (this can be shown, by purely formal calculations, to be equivalent to Siegel's main theorem on the number of representations of a quadratic form by a genus of quadratic forms). The reduction from m to $m-1$ can be effected, for even m, by the consideration of the following zeta-function

$$\int_{V_A^+} |F(x)|^s \Phi(x) d'x ,$$

where V, Φ, $d'x$ are as explained in 4.4, and V_A^+ is defined, as in 4.4, by $\lambda(F(x)) = 1$, except that here λ is the character of the quadratic extension k' of k given $k' = k(\Delta^{1/2})$, $\Delta = (-1)^{m/2} \det(S)$. The method used in 4.4 can be applied with small changes. This fails, however, for odd m (because the group of similarity transformations has not the same structure for odd m as for even m). The following treatment is valid for all values of m.

We change our notation by writing F for the matrix S, so that we have now $F(x) = {}^t xFx$. Whenever convenient, we may assume that F has been put into diagonal form, $F(x) = \Sigma_i a_i x_i^2$. For m even, we denote by Δ the discriminant of F, $\Delta = (-1)^{m/2} \det(F)$; for m odd, we write $D = (-1)^{(m-1)/2} \det(F)$.

By Hasse's theorem, if $\rho \in k^*$, and Σ is the sphere of radius ρ, Σ_k is empty if and only if Σ_A is empty, or also if and only if there is v such that Σ_{k_v} is empty.

Lemma 4.5.1. If $\rho \in k^*$ and Σ is the sphere of radius ρ, and if Σ_A is not empty, (1) is a set of convergence factors for Σ, provided $m \geq 4$.

From the formula (cf. proof of Lemma 4.2.1) for the number of solutions of a homogeneous quadratic equation mod. p, one deduces the number of solutions of $F(x) \equiv \rho$ mod. p in the field F_q with $q = N(p)$ elements; assuming p to be such that $F \in M_m(\underline{o}_p)$, $\rho \in \underline{o}_p$, and that $2\rho \det(F)$ is a unit in $\underline{o}_p$, this number is $q^{m'-1}(q^{m'} - \varepsilon)$ for $m = 2m'$, with $\varepsilon = (\Delta/p)$ (quadratic residue character of the discriminant Δ modulo p); it is $q^{m'}(q^{m'} + \eta)$ if $m = 2m'+1$, with $\eta = (D\rho/p)$, $D = (-1)^{m'} \det(F)$. The conclusion follows now from Theorem 2.2.5.

For $\rho \in k^*$, we consider the variety $F(x) = \rho y^2$ in the affine space of dimension $m+1$, and, on that variety, the Zariski-open subset $T(\rho)$ defined by $y \neq 0$; this is isomorphic to $\Sigma \times G_m$, if Σ is the sphere of radius ρ (the mapping $(x,y)\to(xy,y)$ is an isomorphism of $\Sigma\times G_m$ onto $T(\rho)$); in view of Lemma 4.5.1, $(1-q^{-1})$ is therefore a set of convergence factors for $T(\rho)$, for $m\geq 4$. As $dx = dx_1dx_2\dots dx_m$ is a gauge-form on the (non-singular) variety $T(\rho)$, we can take on $T(\rho)$ the Tamagawa measure $d'x$ derived from this and the factors $(1-q^{-1})$. we introduce, for each $\rho\in k^*$, the "spherical zeta-function"

$$(1)\qquad Z(s,\rho) = \int_{T(\rho)_A} |F(x)|^s\Phi(x)d'x = \int_{T(\rho)_A} |y|^{2s}\Phi(x)d'x ,$$

where Φ is a function in A_k^m, of the type described in 4.4. It should be understood that this is 0 if Σ_A, and consequently $T(\rho)_A$, are empty. For every $\lambda\in k^*$, we have $Z(s,\rho\lambda^2) = Z(s,\rho)$, as we see by making the change of variables $(x,y)\to(x,\lambda^{-1}y)$, which maps $T(\rho)$ onto $T(\rho\lambda^2)$ and leaves the integrand in $Z(s,\rho)$ invariant. The zeta-function for the quadratic form F will be defined as

$$(2)\qquad Z(s) = \sum_{\rho\in k^*/k^{*2}} Z(s,\rho) ,$$

where the sum is taken over a full set of representatives of k^* modulo $(k^*)^2$. As usual, we start with a multiplicative calculation (which will give the proof for convergence, and the principal residue), and this depends upon the calculation of the local zeta-functions for almost all p.

Take a finite set S of valuations of k, containing as usual all the infinite places, and such that, for $p\notin S$: (i) all coefficients of F are in $\underline{o}_p$; (ii) $2\det(F)$ is a unit in $\underline{o}_p$; (iii) the factor ϕ_p occurring in the definition of Φ is the characteristic function of $(\underline{o}_p)^m$ (there will be further conditions on S later on). For any $u\in k_p^*$, consider the integral

$$(3)\qquad Z_p(s,u) = (1-q^{-1})^{-1} \int_{T(u)} |F(x)|_p^s\phi_p(x)\cdot(dx)_p .$$

As before, we see that $Z_p(s,uv^2) = Z_p(s,u)$ for $v\in k_p^*$; therefore, if we write $u=\pi^\alpha v$, where v is a unit in $\underline{o}_p$, $Z_p(s,u)$ can depend only upon α mod. 2 and upon the quadratic residue character (v/p) of v mod. p (in fact, since 2 is a unit in $\underline{o}_p$, every element of $\underline{o}_p$ which is $\equiv 1$ mod. p is in $(k_p^*)^2$). The value of $Z_p(s,u)$ is given by :

Lemma 4.5.2. Put $u=\pi^\alpha v$, where v is a unit in $\underline{o}_p$. Then :

(i) for odd m, $m=2m'+1$, we have

$$(1-q^{-2s-2})(1-q^{-2s-m})Z_p(s,u) = \begin{cases} q^{-s-1}(1-q^{1-m}) & \text{for } \alpha \text{ odd}, \\ (1+\eta q^{-m'})(1-\eta q^{-2s-2-m'}) & \text{for } \alpha \text{ even}, \end{cases}$$

where $\eta = (Dv/p)$, $D = (-1)^{m'}\det(F)$.

(ii) for even m, $m=2m'$, we have

$$(1-q^{-2s-2})(1-q^{-2s-m})Z_p(s,u) = \begin{cases} q^{-s-1}(1-\varepsilon q^{-m'})(1+\varepsilon q^{1-m'}) & \text{for } \alpha \text{ odd} \\ (1-\varepsilon q^{-m'})(1+\varepsilon q^{-2s-m'-1}) & \text{for } \alpha \text{ even}, \end{cases}$$

where $\varepsilon = (\Delta/p)$, $\Delta = (-1)^{m'}\det(F)$.

Assume that we have put F into diagonal form, $F(x) = \Sigma_i a_i x_i^2$. We have to take the integral (1) over the set of points (x,y) of k_p^{m+1} for which $F(x) = uy^2$, $x\in(\underline{o}_p)^m$, $y\neq 0$; for each such point, we can write, in one and only one way, $y^{-1}x$ and y in the form $y^{-1}x = \pi^\lambda z$, $y=\pi^\mu t$, where $z\in(\underline{o}_p)^m$, $t\in\underline{o}_p$, $z\not\equiv 0$ mod. p and $t\not\equiv 0$ mod. p; then we have $\lambda+\mu\geq 0$ (since x must be in $(\underline{o}_p)^m$), and $F(z) = \pi^{\alpha-2\lambda}v$ and therefore $2\lambda\leq\alpha$. We split up our domain of integration into the open subsets such that, on each of these sets, λ,μ have given values, and z,t have given values $\bar{z},\bar{t}$ modulo p; the latter must then be such that $F(\bar{z})\equiv v$ mod. p if $2\lambda=\alpha$, and $F(\bar{z})\equiv 0$ mod. p if $2\lambda<\alpha$. Conversely, assume that $\lambda,\mu,\bar{z},\bar{t}$ are so given, and e.g. $\bar{z}_1\not\equiv 0$ mod. p; then, for every choice of $z_2,\dots,z_m$ such that $z_i \equiv \bar{z}_i$ mod. p for $2 \leq i \leq m$, the equation $F(z) = \pi^{\alpha-2\lambda}v$ has exactly one solution z_1 such that $z_1 \equiv \bar{z}_1$ mod. p, and this is an analytic function of $z_2,\dots,z_m$; therefore, on the subset of the domain of integration determined by those values of $\lambda,\mu,\bar{z},\bar{t}$, we can use $z_2,\dots,z_m$, t as local coordinates, and the mapping $(x,y) \to (z_2,\dots,z_m,t)$ is an analytic homeomorphism of that set onto the subset of $(\underline{o}_p)^m$ given by $z_i \equiv \bar{z}_i$, $t \equiv \bar{t}$ mod. p. On that set, and in terms of those local coordinates, the gauge-form $dx_1\dots dx_m$ is given by

$$dx_1\dots dx_m = \pi^{m(\lambda+\mu)}F(z)dtdz_2\dots dz_m/a_1z_1 ,$$

which gives, on that set, since a_1z_1 on that set is a unit in $\underline{o}_p$:

$$(dx)_p = q^{-m(\lambda+\mu)}\cdot q^{2\lambda-\alpha}(dtdz_2\dots dz_m)_p ;$$

therefore, on that set, our integral has the value q^N, with

$$N = (s+1)(2\lambda-\alpha) - m(\lambda+\mu+1) .$$

For given values of λ, μ, this gives to our integral a contribution $(q-1)\nu(v)q^N$ if $2\lambda=\alpha$, where $\nu(v)$ is the number of solutions of $F(\bar{z}) \equiv v$ mod. p, and $(q-1)\nu(0)q^N$ if $2\lambda<\alpha$, where $\nu(0)$ is the number of solutions, other than 0, of $F(\bar{z}) \equiv 0$ mod. p; we have to take the sum of these contributions over all values of λ, μ satisfying $2\lambda \leq \alpha$, $\lambda+\mu \geq 0$. Using the formulas given above (in the proofs of Lemmas 4.2.1 and 4.5.1) for $\nu(v)$, $\nu(0)$, we get our result.

Now, for $p \notin S$, denote by U_p the multiplicative group of the units in $\underline{o}_p$, and put $H_S = (I_k)^2 \prod_{p\notin S} U_p$; clearly H_S is an open subgroup of I_k. Also, write $|S|$ for the number of elements of S.

Lemma 4.5.3. The group $H_S k^*$ is of finite index in I_k. Moreover, if S is large enough, this index is $[I_k : H_S k^*] = 2^{|S|}$, and we have $k^* \cap H_S = k^{*2}$.

The characters of $I_k/H_S k^*$ are those characters of order 2 of I_k which are 1 on k^*, and on U_p for every $p \in S$; by class-field theory, they are the characters attached to those quadractic extensions $k(d^{\frac{1}{2}})$ of k which are unramified outside S. In the number-field case, take S such that it contains all the primes dividing 2, and that the primes in S generate the group C of ideal-classes of k. If $k(d^{\frac{1}{2}})$ is unramified outside S, every prime $p \notin S$ must occur in d with an even exponent, so that we can write the principal ideal (d) as a product of primes $p \in S$ and of the square m^2 of an ideal m; by our assumption on S, we can write m as $\lambda m'$ with $\lambda \in k^*$, where m' has no prime factor $p \notin S$; therefore every quadratic extension of k, unramified outside S, can be written as $k(d^{\frac{1}{2}})$ with $d \in E_S$, where E_S is the group of the S-units of k (elements of k which are p-units for every $p \notin S$). The index we have to compute is then $[E_S : E_S^2]$. It is well-known (Dirichlet-Chevalley) that E_S is the product of a finite cyclic group of even order, and of a free abelian group with $|S|-1$ generators. The proof is even simpler in the function-field case. This proves the first part (note that, if the index is finite for large S, it must be finite for all S). Now take $d \in k^* \cap H_S$; then, in $k' = k(d^{\frac{1}{2}})$, each $p \in S$ splits (i.e. can be extended to two distinct valuations of k') and each $p \notin S$ is unramified. As there are only finitely many non-ramified quadratic extensions of k, this implies $d^{\frac{1}{2}} \in k^*$ provided S has been so chosen that, to every non-ramified quadratic extension k' of k, there is a least one $p \in S$ which does not split in k'.

Now take any S satisfying the conditions in Lemma 4.5.3 as well as the earlier ones. Also, let V be, as in 4.3, the open set $F(x) \neq 0$ in the affine m-space; and call Ω_S the open subset of V_A determined by the condition $F(x) \in H_S k^*$; if $\gamma(S)$ is the group of order

$2^{|S|}$ consisting of the characters of I_k which are 1 on $H_S k^*$, Ω_S can also be defined as the subset of V_A where $\lambda(F(x)) = 1$ for every $\lambda \in \gamma(S)$. Now we introduce the function

$$Z_S(s) = \int_{\Omega_S} |F(x)|^s \Phi(x)d'x = 2^{-|S|} \sum_{\lambda\in\gamma(S)} \int_{V_A} |F(x)|^s \lambda(F(x))\Phi(x)d'x .$$

The multiplicative calculation for the $2^{|S|}$ integrals in the last sum is the same as the one given in 4.4 for I_o, I_1 (this depends only upon Lemma 4.3.2); it shows that these integrals are absolutely convergent for $Re(s) > 0$, and that only the one corresponding to $\lambda = 1$ gives a residue for $s = 0$, this residue being $\rho_k \int \Phi(x)dx$. Therefore the integral for Z_S is absolutely convergent for $Re(s) > 0$, and we have :

$$(4) \qquad [sZ_S(s)]_{s=o} = 2^{-|S|}\rho_k \int_{A_k^m} \Phi(x)dx .$$

Now we can also write $H_S k^*$ as the disjoint union of the sets $H_S\rho$ when we take for ρ a complete set of representatives of k^* modulo $k^* \cap H_S$, i.e., under our assumptions on S, modulo k^{*2}; therefore, if we put

$$(5) \qquad Z_S(s,\rho) = \int_{F(x)\in H_S\rho} |F(x)|^s \Phi(x)d'x ,$$

these integrals are absolutely convergent, and we have

$$(6) \qquad Z_S(s) = \sum_{\rho\in k^*/k^{*2}} Z_S(s,\rho) ,$$

this series being also absolutely convergent for $Re(s) > 0$.

Now consider $Z(s,\rho)Z_S(s,\rho)^{-1}$; the multiplicative calculation shows that this is the product, extended over all valuations v of k, of the factors

$$(7) \qquad \left(\int_{T(\rho)} |F(x)|_v^s \phi_v(x)dx_v\right) \left(\int_{F(x)\in H_v\rho} |F(x)|_v^s \phi_v(x)dx_v\right)^{-1}$$

with $H_v = k_v^{*2}$ if $v \in S$, $H_p = k_p^{*2}U_p$ if $p \notin S$. For $v \in S$, $T(\rho)$, which is the subset of the variety $F(x) = \rho y^2$ determined by $y \neq 0$, covers twice the subset of A_k^m determined by $F(x) \in \rho k_v^{*2}$; therefore the above factor has then the value 2. For $p \notin S$, take $u_p \in U_p - U_p^2$, so that $U_p = U_p^2 \cup U_p^2 u_p$ and $H_p = k_p^{*2} \cup k_p^{*2}u_p$; then the second integral in (7) is the sum of the same integrals taken under the restrictions $F(x) \in k_p^2\rho$, $F(x) \in k_p^2 u_p \rho$, respectively; and these, for the reasons just explained, differ from the same integrals, taken over $T(\rho)$ and $T(u_p\rho)$, only by the factor $\frac{1}{2}$. Thus the factor (7), for $p \notin S$, has the value

$$\frac{2Z_p(s,\rho)}{Z_p(s,\rho) + Z_p(s,u_p\rho)}$$

By Lemma 4.5.2, <u>this is equal to</u> 1 <u>for</u> m <u>even</u>; in this case, therefore, the integral in (1) and the series in (2) are absolutely convergent for $Re(s) > 0$; and we have :

$$Z(s,\rho) = 2^{|S|}Z_S(s,\rho), \quad Z(s) = 2^{|S|}Z_S(s), \quad [sZ(s)]_{s=o} = \rho_k \int_{A_k^m} \Phi(x)dx .$$

<u>For</u> m <u>odd</u>, Lemma 4.5.2 shows that the factor (7), for $p \in S$, has the value 1 whenever ρ contains an odd power of p; when that is not so, this same factor has the value

$$\theta(p) = \frac{(1+\eta q^{-m'})(1-\eta q^{-2s-2-m'})}{1-q^{-2s-m-1}}$$

with $\eta = \pm 1$; in all cases, therefore, we have

$$1 - q^{-m'} < \theta(p) < 1 + q^{-m'} .$$

As $m > 4$, we have here $m' \geq 2$. This shows that the infinite product for $Z(s,\rho)$ is absolutely convergent together with that for $Z_S(s,\rho)$; also, if we put

$$\mu(S) = \prod_{p \notin S} (1-q^{-m'}) \ , \ \mu'(S) = \prod_{p \notin S} (1+q^{-m'}) \ ,$$

we see that $2^{-|S|}Z(s,\rho)Z_S(s,\rho)^{-1}$ is $> \mu(S)$ and $< \mu'(S)$ for $Re(s) > 0$; this implies that the series (2) for $Z(s)$ is absolutely convergent, and that $2^{-|S|}Z(s)Z_S(s)^{-1}$ remains between the same constants. In view of (4), this shows that lim. inf. $sZ(s)$ and lim. sup. $sZ(s)$, for $s=0$, are between $\mu(S)$ and $\mu'(S)$; since we may here take S as large as we please, we have proved :

$$[sZ(s)]_{s=0} = \rho_k \int_{A_k^m} \Phi(x)dx \ , \tag{8}$$

so that this formula holds for all $m > 4$. This completes the "multiplicative calculation".

Now we take up the additive calculation. Take any ρ such that $T(\rho)$ is not empty, i.e. (by Hasse's theorem) such that there is $\xi_o \in k^m$ for which $F(\xi_o) = \rho$. Put $\Gamma = G \times G_m$, and let Γ act on $T(\rho)$ by

$$((X,t),(x,y)) \to (Xxt, yt) \ ;$$

by Witt's theorem (Lemma 4.1.3), Γ acts transitively on $T(\rho)$, and the subgroup leaving $(\xi_o, 1)$ fixed has a cross-section; this subgroup is $\Gamma' = G' \times \{1\}$, where G' is the subgroup of G leaving ξ_o fixed. By Lemma 4.1.1 and the induction assumption, (1) is a set of convergence factors for G', and $\tau(G') = 2$. Let dX, dX' be invariant gauge-forms for G, G'; then $dX \cdot (dt/t)$ is such a form for Γ. Clearly $y^{-m}dx_1 \ldots dx_m$ is a gauge-form on $T(\rho)$, invariant under Γ. Therefore,

by Theorem 2.4.3, Γ has the same set of convergence factors as $T(\rho)$, viz. $(1-q^{-1})$, so that G has (1) as a set of convergence factors; also, the Tamagawa measures for $\Gamma, \Gamma', T(\rho)$, derived from these convergence factors and the gauge-forms $dX \cdot (dt/t)$, dX', $y^{-m}dx$, match together topologically. This gives, by Lemma 2.4.2 :

$$Z(s,\rho) = \tfrac{1}{2} \int_{T(\rho)_A} |y|^{2s+m} \Phi(x) |y|^{-m} d'x =$$

$$= \tfrac{1}{2} \int_{\Gamma_A/\Gamma_k} \Big(\sum_{(M,\tau)} |\tau t|^{2s+m} \Phi(XM\xi_o \tau t) \Big) dX (dt/t)' \ ,$$

where the sum is taken over all $(M,\tau) \in \Gamma_k/\Gamma'_k$, i.e. over all $M \in G_k/G'_k$ and all $\tau \in k^*$; but then, by Witt's theorem, the vector $\xi = M\xi_o\tau$ runs <u>twice</u> through the set of all vectors in k^m such that $F(\xi) \in \rho k^{*2}$; if then we let ρ run through a full set of representatives of k^* modulo k^{*2}, ξ runs twice through the set of vectors ξ in k^m such that $F(\xi) \neq 0$. This gives :

$$Z(s) = \int_{\Gamma_A/\Gamma_k} |t|^{2s+m} \Big(\sum_{F(\xi) \neq 0} \Phi(X\xi t) \Big) dX \cdot (dt/t)' \ .$$

From here on, the calculation is exactly the same as in 4.4, beginning with formula (4) of that section. The conclusions are the same; in particular, the residue of $Z(s)$ at $s=0$ turns out to be $\rho_k \tau(G)/2$; comparing this with (8), we get :

<u>Theorem</u> 4.5.1. <u>The Tamagawa number of the orthogonal group in</u> $m \geq 3$ <u>variables is</u> 2.

<u>Remark</u>. For <u>indefinite</u> quadratic forms, Siegel has defined Zeta-functions for individual <u>classes</u> of such forms. This can perhaps be explained by the fact that, for indefinite forms, classes and "spinor-genera" are the same (Eichler-Kneser). If G is the orthogonal group,

and $\mathcal{G}$ is the corresponding spin-group, the spinor-norm, for an element of G_K (K = any field) is an element of K^*/K^{*2} which gives the obstruction against lifting that element from G_K to $\mathcal{G}_K$. In particular, this defines a homomorphism of G_A into $I_k/I_k^2k^*$, with the value 1 on G_k; therefore, to every character of $I_k/I_k^2k^*$, i.e. to every character of I_k belonging to a quadratic extension of k, one can assign a character of order 2 of G_A, with the value 1 on G_k; it is not unlikely that, by introducing such characters into our zeta-functions, one might get Siegel's zeta-functions for indefinite forms.

APPENDIX 1.

THE CASE OF THE GROUP G_2

by M. Demazure

The method used in the case of orthogonal groups can also give the Tamagawa number of the groups of type G_2, which turns out to be 1 as expected.

We first recall some results on Cayley Algebras, after JACOBSON, Composition Algebras and Their Automorphisms, Rend. Palermo, 1958. For the time being, k is any field of characteristic not 2.

A Cayley algebra over k is a vector space Ω_k of dimension 8 over k, together with a k-linear map $\Omega_k \times \Omega_k \to \Omega_k$ denoted $(x,y) \to x \cdot y$ and a non-degenerate quadratic form called the norm $N : \Omega_k \to k$ subject to the following axioms :

(i) there exists in Ω_k a unit-element, i.e. $1 \in \Omega_k$ with $x \cdot 1 = 1 \cdot x = x$;

(ii) for any $x, y \in \Omega_k$, $N(x \cdot y) = N(x)N(y)$.

One can easily show that the form N is uniquely determined by the structure of (non-associative) algebra of Ω_k.

Let Ω be the algebra-variety defined by Ω_k. We denote by Ω_o the orthogonal space, for N, of the one-dimensional line $k \cdot 1$. For $x \in \Omega_o$, $N(x)1 = -x \cdot x$. An automorphism of Ω is a linear mapping $g : \Omega \to \Omega$ such that $g(x \cdot y) = g(x) \cdot g(y)$. Then $g(1) = 1$ and $g(\Omega_o) = \Omega_o$. If $x \in \Omega_o$, then $N(g(x)) = N(x)$. Moreover, one proves (Jacobson, Theorem 2) that g is a rotation, i.e. of determinant 1. Hence the group G of all automorphisms of Ω is imbedded in $SO(\Omega_o, N)$. It is a semi-simple algebraic group defined over k which becomes isomorphic over $\bar{k}$ to the group G_2 of the Cartan-Killing classification.

A Witt-type theorem is true for G (Jacobson § 3) :

Let (Ω_k, N) and (Ω'_k, N') be two Cayley algebras with equivalent norms (in the sense of quadratic forms). Let B (resp. B') be a non-isotropic subalgebra of Ω_k (resp. Ω'_k). Let there be given an algebra-isomorphism $f : B \to B'$. Then f can be extended to an isomorphism of Ω_k onto Ω'_k.

Corollary. If $x, y \in \Omega_k$, $x, y \neq 0$, $N(x) = N(y)$, then there exists $g \in G_k$ mapping x on y.

Proof : 1) If $N(x) = N(y) \neq 0$, then $K(x)$ and $K(y)$ are two isomorphic quadratic fields.

2) If $N(x) = N(y) = 0$, then x and y can be imbedded in two quaternion algebras isomorphic under a map carrying x into y.

We finally have the two following results :

(i) Let $a \in \Omega_o$, $a \cdot a = b \cdot 1 \neq 0$. Let $K = k(a)$. Then the orthogonal L of K is a 3-dimensional vector-space over K. The subgroup of G leaving K point-wise fixed is isomorphic to the unimodular unitary group of L as a vector space over K relative to the form $(x,y) + b^{-1}a(ax,y)$. (Jacobson, Theorem 3.).

(ii) Let B be a quaternion subalgebra of Ω. The subgroup of G leaving B point-wise fixed is isomorphic to the multiplicative group of elements of norm 1 in B.

From now on, is a number field. (In the case of a function field of characteristic not 2, everything is valid, provided that we prove that G_k is Zariski-dense in G).

A careful analysis of 4.5 shows that what was proved there amounts to the following :

Let Ω_o be a finite-dimensional vector-space (of dimension ≥ 5) defined over k, with a quadratic form N. Let G be an algebraic group of rotations of N defined over k. Suppose that G verifies a Witt-type theorem (i.e. if $x, y \in \Omega_o$, $N(x) = N(y)$, there exists $g \in G$ carrying x into y; if x and y are rational over k, then g may be taken rational over k). For $a \in \Omega_o$, let $G(a)$ be the isotropy group of a in G. Assume the two following properties :

(i) There exists a finite number τ such that for non-isotropic $a \in \Omega_k$, the Tamagawa number of $G(a)$ is finite and equal to τ, independently of a.

(ii) For any isotropic $a \in \Omega_k$, the Tamagawa number of $G(a)$ is finite. Then the Tamagawa number of G is finite and equal to τ.

In our case, G satisfies a Witt-type theorem. We have only to verify properties (i) and (ii).

(i) By property (i) recalled above, for non-isotropic $a \in \Omega_k$, $G(a)$ is a unimodular unitary group and $\tau(G(a)) = 1$.

(ii) Let now a be isotropic. The subgroup of G leaving the line $k \cdot a$ fixed has at most rank two. It contains a multiplicative factor not contained in $G(a)$. Hence $G(a)$ has at most rank one. On the other hand, let B be a quaternion algebra containing a. Then the subgroup of G leaving B pointwise fixed is semi-simple of rank one and contained in $G(a)$; its Tamagawa number is one (by property (ii) recalled above). By the general properties of algebraic groups, it must be normal in $G(a)$. The quotient being of rank zero is unipotent, hence has Tamagawa number one. On the other hand, the quotient being unipotent, the fibration admits local cross-sections (Rosenlicht). By Theorem 2.4.4, the Tamagawa number of $G(a)$ is finite and equal to 1. This proves :

Theorem. The Tamagawa number of a group of type G_2 is 1.

APPENDIX 2.

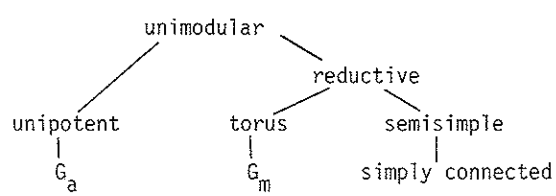
A SHORT SURVEY OF SUBSEQUENT RESEARCH ON TAMAGAWA NUMBERS

by Takashi Ono

The following is a short survey of works on Tamagawa numbers which have appeared since the original publication of Adeles and Algebraic groups, The Institute for Advanced Study, Princeton, N.J., 1961 (Notes by M; Demazure and T. Ono). In the sequel, we shall quote these notes as [AAG]. In the bibliography at the end of this survey we have also included some earlier work, chiefly by Siegel and Weil.

§ 1. Definition of $\tau(G)$ for unimodular groups ([AAG, Ch.I, Ch.II, Ch.III, 3.6])

Let k be an algebraic number field, G be a connected linear algebraic group defined over k and ω be a left invariant differential form of highest degree on G defined over k. The group G is called unimodular if the form ω is also right invariant. We have the following chains of containment :



where G_a, G_m mean the additive group and the multiplicative group,

respectively. We now define the Tamagawa number $\tau(G)$. Let $\hat{G}$ be the group of rational characters of G, $\hat{G} = \mathrm{Hom}(G, G_m)$, and $\hat{G}_k$ be the subgroup of $\hat{G}$ of characters defined over k. Let G_A be the adele group and let $G_A^1 = \{x \in G_A, |\xi(x)|_A = 1 \text{ for all } \xi \in \hat{G}_k\}$. Then G_A/G_A^1 is isomorphic to the vector group $\mathbb{R}^r$, $r = \mathrm{rank}\, \hat{G}_k$. As a measure on G_A/G_A^1 we take the usual measure of $\mathbb{R}^r$ which we denote by $d(G_A/G_A^1)$. Since G_k is discrete in G_A, we define dG_k to be the canonical discrete measure. Thanks to the fundamental result of Borel and Harish-Chandra [1] (see also Borel [1]) the space G_A^1/G_k has a finite measure. We can then define $\tau(G)$ as the measure one has to give to the space G_A^1/G_k in order that

$$dG_A = d(G_A/G_A^1)d(G_A^1/G_k)dG_k,$$

where dG_A is some canonical measure on G_A to be determined. Now, take a finite Galois extension K/k so that $\hat{G} = \hat{G}_K$. Then $\hat{G}$ becomes a $\mathbb{Z}$-free $\mathrm{Gal}(K/k)$-module and we denote by χ_G the character of the integral representation of $\mathrm{Gal}(K/k)$. The Artin L-function $L(s, \chi_G)$ has a pole of order r at $s = 1$. We put

$$dG_A = \rho_G^{-1}|\Delta_k|^{-\dim G/2} \prod_{v/\infty}\omega_v \prod_p L_p(1, \chi_G)\omega_p ,$$

where $\rho_G = \lim_{s\to 1} (s-1)^r L(s, \chi_G)$, Δ_k = the discriminant of k. (As for the convergence of dG_A, see the beginning of Ono [7]). It can be shown that dG_A is instrinsic, i.e. it is independent of the choice of K/k and ω. Thus, the definition of the Tamagawa number

$$\tau(G) = \tau_k(G) = \int_{G_A^1/G_k} d(G_A^1/G_k) , \quad G : \text{unimodular},$$

is settled. Note the properties : $\tau(G \times G') = \tau(G)\tau(G')$, $\tau_k(G) = \tau_k(R_{K/k}G)$

for any finite extension K/k. The definition of the number $\tau(G)$ is chosen so that $\tau(G_a) = \tau(G_m) = 1$. The latter equality is equivalent to the classical formula for the residue at $s = 1$ of $\zeta_k(s)$.

§ 2. The weil conjecture.

The statement

(W) $\tau(G) = 1$ when G is simply connected,

is known as Weil's conjecture. One cannot find (W) explicitly in [AAG]; however, it is stated in Weil [4]. The conjecture (W) has been settled for almost all simply connected groups. Among absolutely almost simple groups, (W) is not yet settled for groups of type ${}^3D_4$, ${}^6D_4$, E_6, E_7 and E_8. On the other hand, (W) is settled for all classical groups (Weil [9] and Mars [2]), for the split groups (Langlands [1]) and even for the quasi-split groups (Lai [1], [2]). (W) was also settled for groups of type G_2 by Demazure ([AAG, Appendix]), for groups of type F_4, and some groups of type ${}^1E_6$ (Mars [1]). We shall later discuss the situation for classical groups and quasi-split groups in a little more detail. There are many survey papers refering to $\tau(G)$ and/or (W) (Borel [2], Cassels [1], Iyanaga [1], Kneser [1], Mackey [1], [2], Mars [3], Ono [7]). Near the bottom of p.311 of Weil [3] one finds the following passage in allusion to (W) after a description of Siegel's theorem on quadratic forms :

............"Est-il possible d'en donner un énoncé général, qui permette d'un seul coup d'obtenir tous les résultats de cette nature, de même que la découverte du théorème des résidus a permis de calculer par une méthode uniforme tant d'intégrales et de séries qu'on ne traitait auparavant que par des procédés disparates ?"

As is well known, it was Tamagawa's discovery that $\tau(SO(f)) = 2$ (Siegel's theorem on the quadratic form f) and that $\tau(spin(f)) = 1$ in the late fifties, which stimulated the work discussed in [AAG].

§3. Tamagawa number of tori and relative Tamagawa numbers.

In [AAG, Ch.III, 3.6], Weil employed a trick to compare $\tau(G)$ and $\tau(G')$ when G, G' are isogenous. A typical case is : G = the projective group of a division algebra of dimension n^2 over its center, G' = the special linear group of the division algebra. One first proves that $\tau(G) = n$. Then, the trick implies that $\tau(G') = 1$, a solution of (W) for G'. One can describe the trick in a more general setting as follows : let G be semisimple and let (G', π) be the universal covering group of G over k. Then $\ker \pi$ (the fundamental group of G) of the isogeny π is endowed with a structure of a $Gal(\bar{k}/k)$-module, which can be imbedded in a torus $T = (R_{K/k} G_m)^r$ for some finite extension K/k. One has a commutative diagram with exact row and column :

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & & \downarrow & & & & \\ & & & & T & \searrow & & & \\ & & & & \downarrow & & & & \\ 1 & \to & G' & \to & G^* & \to & T' & \to & 1 \\ & & & \pi\searrow & \downarrow & & & & \\ & & & & G & & & & \\ & & & & \downarrow & & & & \\ & & & & 1 & & & & \end{array}$$

where $G^* = (G' \times T)/\mathrm{Ker}\,\pi$ with the diagonal imbedding of $\mathrm{Ker}\,\pi$. Using G^* as a dummy, one reduces the computation of the ratio $\tau(G)/\tau(G')$ to the isogeny of tori : $T \to T'$. As for a torus T, the Tamagawa number can be written in terms of the Galois cohomology (Ono [3]) :

$$\tau(T) = \frac{\#\,\mathrm{Pic}\,T}{\#\,Ш(T)}$$

where, for any algebraic group A, Pic A = Picard group and $Ш(A) = \mathrm{Ker}(H^1(k,A) \to \prod_v H^1(k_v,A))$. Note that $\mathrm{Pic}\,T = H^1(k,\hat{T})$ for a torus T. Finally, we arrive at a formula which expresses the ratio $\tau(G)/\tau(G')$ in terms of the $Gal(\bar{k}/k)$-module $F = \mathrm{Ker}\,\pi$ (Ono [4], [7]) :

$$\tau(G)/\tau(G') = \frac{\# H^0(k, \hat{F})}{\# Ш(\hat{F})}$$

where $\hat{F} = \mathrm{Hom}(F, G_m)$. Therefore, if (W) holds for G', we have

(W)* $$\tau(G) = \frac{\# H^0(k, \hat{F})}{\# Ш(\hat{F})},$$

a conjectural formula for any connected semisimple group G.

§ 4. Tamagawa number of classical groups ([AAG Ch.III, IV])

The key idea of the use of the Poisson summation formula for the calculation of the Tamagawa number of classical groups goes back to the paper Weil [2] where the formula is applied to determine the volume of $SL(n,\mathbb{R})/SL(n,\mathbb{Z})$. When Siegel [7] improved an inequality of Hlawka on $SL(n,\mathbb{R})$, he obtained without Poisson summation the value $\zeta(2)\zeta(3)\ldots\zeta(n)$ for that volume, when measured in the nice fashion. Weil [2] gives a simplification of Siegel [7] from the point of view of the book Weil [1].

In 1964-65, two important papers appeared in the Acta Mathematica (Weil [8], [9]). Pushing his way in the direction of Poisson summation, Weil obtained there a formula in the framework of adeles and algebraic groups which is a wide generalization of Siegel's work on indefinite quadratic forms (Siegel [8], [9]). As an application of this Siegel-Weil formula the Tamagawa number of all classical groups except the groups of type L2(b) ([AAG, p.76]) were determined systematically. Later Mars [2] took care of this last case and (W) was settled for all classical groups.

§5. Tamagawa number of quasi-split groups.

Let us begin with the case of the Chevalley group. Let G be the Chevalley group, i.e. the identity component of the group of automorphisms of a complex semisimple Lie algebra g (Chevalley [1]). With respect to a Chevalley basis of g, G becomes an algebraic matrix group defined over $\mathbb{Q}$. Let ω be a left invariant form on G of the highest degree defined over $\mathbb{Q}$ which is obtained by taking the wedge product of 1-forms dual to vectors in the Chevalley basis. Let $G_{\mathbb{R}}$, $G_{\mathbb{Z}}$ be the subgroups of G whose coordinates are in $\mathbb{R}$, $\mathbb{Z}$, respectively, and let $\omega_{\mathbb{R}}$ be the Haar measure on $G_{\mathbb{R}}$ derived from ω. As an application of his theory of Eisenstein series (Langlands [2]), Langlands [1] proved the remarkable equality :

$$\int_{G_{\mathbb{R}}/G_{\mathbb{Z}}} \omega_{\mathbb{R}} = \# F \prod_{i=1}^{\ell} \zeta(a_i)$$

where F = the fundamental group of $G = \mathrm{Ker}\,\pi$ (in §3) and the integers a_i appear in the Poincaré polynomial $\prod_{i=1}^{\ell} (t^{2a_i-1}+1)$ of the maximal compact subgroup of G. Obviously, this is a generalization of the Siegel's result for the volume of $SL(n,\mathbb{R})/SL(n,\mathbb{Z})$ mentioned in §4. Combining Langlands' result with a computation of the volume of $G_{\mathbb{Z}_p}$ for any prime p (Ono [5]) and the formula of the relative Tamagawa number in §3, one settles (W) for the universal covering group of G. Later in 1974, Lai [1] verified that the method of Langlands works for any quasi-split group over k, as well. Namely, let G be a connected semisimple quasi-split group over k, let B be a Borel subgroup of G defined over k, and T be a maximal torus in B defined over k. The basic observation of Langlands is $\tau(G) = (1,1) = (f,1)(1,g)/(Pf,Pg)$ for $f,g \in L^2(G_A/G_k)$ where P means the orthogonal projection onto the space of constant functions in $L^2(G_A/G_k)$. Using the theory of Eisenstein series, Lai computes the terms on the right hand side with suitable f,g and obtains $\tau(G) = c\tau(T)$ where c is the index of the lattice of k-rational weights of G in the corresponding lattice for the universal covering group of G. Combining this with the formula of $\tau(T)$ in §3, one settles (W) for all quasi-split groups. As far as we know, this is the most general result about $\tau(G)$ obtained without appealing to the classification theory.

§6. Remarks

(1) Unlike [AAG], we have totally ignored the function field case in this survey, because most of subsequent papers have treated the number field case only, and also because the function field case has not matured to the level of the number field case (See, however, Harder [2]).

(2) The use of the Gauss-Bonnet theorem for the computation of the volumes of the various fundamental domains goes back to Siegel (e.g. Siegel [5]). There are interesting relations among Tamagawa numbers, Bernoulli numbers, Euler numbers and special values of L-functions. (See, Borel [4], Harder [1], Harder and Narasimhan [1], Ono [5], [6], Satake [1], Serre [2]).

(3) It is desirable to extend the notion of the Tamagawa number to a more general category of algebraic varieties defined over k. Birch and Swinnerton-Dyer considered the case of elliptic curves and produced a very plausible conjecture (See, Cassels [1], Swinnerton-Dyer [1], Tate [1]). Recently, Bloch [1] obtained a purely volume-theoretic interpretation of the Birch and Swinnerton-Dyer conjecture. It is interesting to note that the generalized Tamagawa number still has the form

$$\tau(X) = \frac{\# \mathrm{Pic}(X)_{tors}}{\# Ш(X)}$$

for some commutative group variety. (See also Sansuc [1]).

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