101 Illustrated Analysis Bedtime Stories

Special *Bounded* Edition

AS TOLD BY:

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AND

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Dedication

For Dr. E. R. Bishop, who inspired this work and whose classes provided ample time to ponder the connection between real analysis and fairy tales.

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Chapter 1

ε-Red Riding Hood and the Big Bad Bolzano-Weierstrass Theorem

Once upon a time[1] a long long time ago back when Fermat's Last Theorem would still fit in a margin, $\exists$ a little[2] girl named ε-Red Riding Hood (see Figure 1.1). ε-Red

Figure 1.1: ε-Red Riding Hood with her basket of lemmas and π.

Riding Hood was trying to find the shortest path through the forest $\mathbb{F}$, a subfield of $\mathbb{X}$, to Γ's domain. She was carrying a basket full of lemmas[3] and π, to give to Γ (see Figure 1.2) who had a degenerate case of discontinuity.

Meanwhile, independently, the Big Bad Bolzano-Weierstrass Theorem (see Figure 1.3) was on a random walk through $\mathbb{F}$. As t approached T_0, T-time, the paths of ε-Red Riding Hood and the Big Bad Bolzano-Weierstrass Theorem converged.

"Hello ε-Red Riding Hood, may I ask you a question?", asked the Big Bad Bolzano-Weierstrass Theorem.

"You may indeed provided it is well-posed", stated ε-Red Riding Hood.

"Then what is your limit?" queried the Big Bad Bolzano-Weierstrass Theorem.

"Why, I'm uniformly bound to Γ's domain" replied ε-Red Riding Hood.

[1] $\exists$ a day $\in \mathbb{T}$ime...

[2] That is, given $\epsilon > 0$, she was within ϵ of 0

[3] And you know what they say about lemmas; "...when life hands you lemmas, make lemmanade."

Figure 1.2: ε-Red Riding Hood's Γ.

"Well in that case Q.E.D." concluded the Big Bad Bolzano-Weierstrass Theorem, and with that he commuted off into the forest. The Big Bad Bolzano-Weierstrass Theorem was able to map himself into $\mathbb{C}^n$ and thus approached Γ's domain in a way such that

$$\frac{\partial}{\partial t}(\text{Big Bad Bolzano-Weierstrass Theorem}) > \frac{\partial}{\partial t}(\varepsilon\text{-Red Riding Hood}) \tag{1.1}$$

held. Soon afterwards, at $t = T_1$, the Big Bad Bolzano-Weierstrass Theorem reached the boundary at Γ's domain. However Γ's domain was compact and thus by the Heine-Borel theorem it was closed and bounded.[4]

Figure 1.3: The Big Bad Bolzano-Weierstrass Theorem.

"Who approaches $\partial(\text{dom}(\Gamma))$?" asked Γ.

"It is I, ε-Red Riding Hood" replied the Big Bad Bolzano-Weierstrass Theorem, "I'm bringing you lemmas and π."

"In that case, I will find a convergent sequence, x_k, such that,

$$\lim_{k\to\infty} x_k \notin \text{dom}(\Gamma),$$

[4]provided of course that The Big Bad Bolzano-Weierstrass Theorem knocked on the door, $\mathbb{D} \subset \partial(\text{dom}(\Gamma))$.

thus creating an opening in $\partial(\mathrm{dom}(\Gamma))$." And having stated thus, she unlocked and opened $\mathbb{D}$.

"You're not ε-Red Riding Hood, you're the Big Bad Bolzano-Weierstrass Theorem!", gasped Γ.

"Be that as it may, I'm still going to eat you!", exclaimed the Big Bad Bolzano-Weierstrass Theorem.

"No, that proposition is false!" argued Γ.

But the Big Bad Bolzano-Weierstrass Theorem made the assumption that her argument was in fact an integer and thus, by the identity

$$\Gamma(n+1) = n!, \qquad n \in \mathbb{Z},$$

turned Γ in to a factorial. It follows that this made her more appetizing and he thus proceeded to gobble her up. Next, the Big Bad Bolzano-Weierstrass Theorem cleverly disguised himself as

$$\int_0^\infty t^{x-1}e^{-t}dt, \qquad x \in (0,\infty).$$

He climbed into Γ's bed where he found a collection of blankets, $\mathcal{V} = \{V_\alpha\}_{\alpha\in\Lambda}$. The Big Bad Bolzano-Weierstrass Theorem arranged the blankets such that

$$\text{Big Bad Bolzano-Weierstrass Theorem} \subseteq \bigcup_{\alpha\in\Lambda} V_\alpha.$$

That is, such that he was *covered* by them.

At $t = T_2$ such that $T_2 > T_1$, ε-Red Riding Hood approached $\mathrm{dom}(\Gamma)$. She knocked on $\mathbb{D}$ and was greeted by the cleverly disguised voice of the Big Bad Bolzano-Weierstrass Theorem: "Come in Dear and let us both choose $n, m \geq N \in \mathbb{N}$ such that our sequences of positions will be within $\epsilon > 0$ of each other."

"Oh Γ, then our sequences will be Cauchy and therefore we will converge!" exclaimed ε-Red Riding Hood.[5] As ε-Red Riding Hood was climbing[6] into bed next to the cleverly disguised Big Bad Bolzano-Weierstrass Theorem, she dislodged one of the blankets in $\mathcal{V}$. Therefore, she noted that part of him was starting to diverge.[7] She found this rather suspicious and observed "What discrete points you have, Γ."

"All the better to disconnect you with, my Dear!" replied the cleverly disguised Big Bad Bolzano-Weierstrass Theorem.

"And what big reals you have, Γ," exclaimed ε-Red Riding Hood.

"All the better to bound you with, my Dear!" shouted the cleverly disguised Big Bad Bolzano-Weierstrass Theorem as he leapt out from under $\mathcal{V}$.

[5] We note that $\mathrm{dom}(\Gamma)$ is a *complete* topological space and thus by Cauchy's theorem, a sequence converges if and only if it is Cauchy.

[6] That is, she was *monotonically increasing.*

[7] The Big Bad Bolzano-Weierstrass Theorem failed to notice that the gamma function is defined by $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$, for $x \in (0,\infty)$ *only when* this integral converges.

It just so happens that a friendly boundary cutter (who incidentally enjoyed surfing in his spare time) was walking by in $\mathbb{X} \setminus \mathrm{dom}(\Gamma)$ on his way to work with his trusty axe of extended reals (see Figure 1.4). He heard ε-Red Riding Hood's cries for

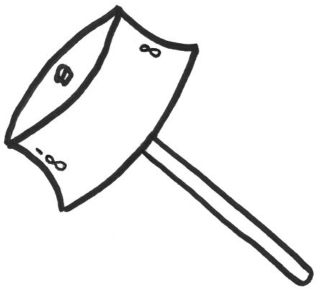

Figure 1.4: The axe of extended reals.

help and rushed through the open $\mathbb{D}$. With one swipe of his axe of extended reals, he unbounded the Big Bad Bolzano-Weierstrass Theorem limb from limb. "Dude! that's like *so* totally disconnected!" exclaimed the boundary cutter.

They then noticed the dismembered pieces of Γ were in the interior of the Big Bad Bolzano-Weierstrass Theorem. "Oh my poor poor Γ" cried ε-Red Riding Hood.

"Don't be sad dude, I think we could, like, fix her and stuff given necessary and sufficient conditions" comforted the friendly boundary cutter.

"As a matter of fact, I just happen to have the Pasting Lemma right here in my basket!" replied ε-Red Riding Hood in excitement and handed it to him.

With a flourish of activity, he Pasted Γ back together and announced "There now she's totally bounded dude!"

Both ε-Red Riding Hood and Γ were very thankful and offered the friendly boundary cutter some π from ε-Red Riding Hood's basket. They all sat down and shared some π. Unfortunately, ε-Red Riding Hood found the π to be somewhat *coarse*[8].

After the π, the friendly boundary cutter went merrily on his way to work.

And $\forall x \in \{\varepsilon\text{-Red Riding Hood}, \Gamma, \text{friendly boundary cutter}\}$, x lived happily $\forall t$ as $t \to \infty$. ■

Q.E.D.

[8]This is probably because $\pi \subseteq$ (ε-Red Riding Hood) and ε-Red Riding Hood was mighty fine!

Chapter 2

Chapters 2–101 are left as an exercise for the reader ...

Bibliography

[1] William R. Wade, *An Introduction To Analysis*, Addison-Wesley Publishing Company, (1995).

[2] James Munkres, *Topology*, 2nd ed, Prentice Hall, (2000).

[3] *The Little Red Riding Hood Project: Text and Image Archive Inventory*, `http://www-dept.usm.edu/~engdept/lrrh/inventt.htm`, accessed: 30/03/2001.