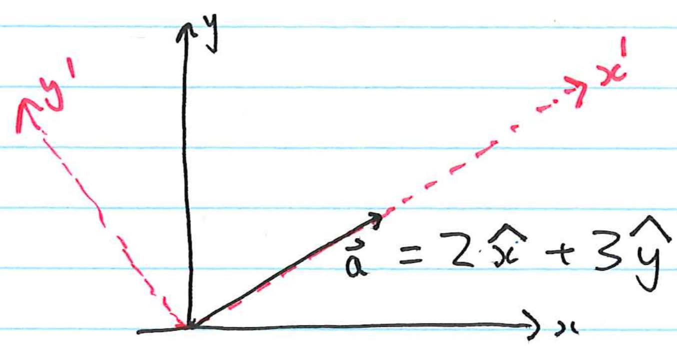
Math 152 Lecture 2

Last day: vectors, today: more vectors :)

Often we fix the coordinate system; then it is "safe" to write $\vec{a} = (2,3)$

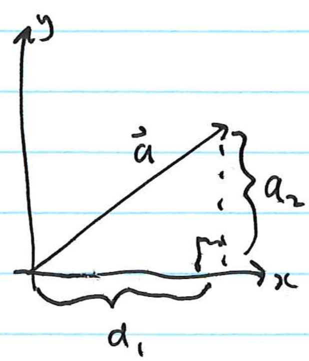


But if we have more than one coor sys then better to say $\vec{a} = 2\hat{x} + 3\hat{y}$

also $\vec{a} = \sqrt{13}\,\hat{x'} + 0\,\hat{y'}$

- Same vector, multiple representations
- Should think of the vector as independent of coordinate system

Length of a vector: notation is $\|\vec{a}\|$ a.k.a. "norm($\vec{a}$)"



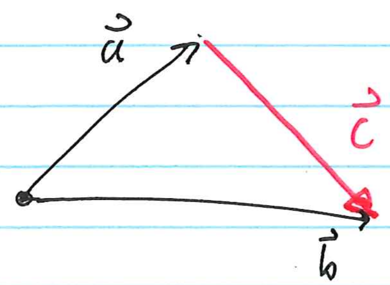
Pythagoras: $\|\vec{a}\| = \sqrt{a_1^2 + a_2^2}$

Note: formula involves the coor sys but length of the vector must be independent of that choice.

Ex $\vec{a} = (2,3)$ then $\|\vec{a}\| = \sqrt{2^2+3^2} = \sqrt{13}$

Ex (from above) $\vec{a} = \sqrt{13}\,\hat{x}$ then $\|\vec{a}\| = \sqrt{(\sqrt{13})^2 + 0^2}$
$= \sqrt{13}$

__Distance__ b/w two vectors $\vec{a}$ and $\vec{b}$...



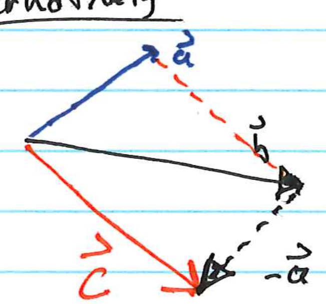
... is the length of $\vec{c}$.

__Note!__ $\vec{a} + \vec{c} = \vec{b}$

so $\vec{c} = \vec{b} - \vec{a}$

Distance b/w $\vec{a}$ and $\vec{b}$
is $\|\vec{b} - \vec{a}\|$
or $\|\vec{a} - \vec{b}\|$

alternatively



recall, vector does not depend on location ← dir & length.

__Ex__ $\vec{a} = (2,3)$ and $\vec{b} = (2,1)$, find distance d b/w them.

$$d = \|\vec{a} - \vec{b}\| = \|(2,3) - (2,1)\| = \|(0,2)\| = 2$$

Unit vectors : sometimes we want a vector in same direction as $\vec{a}$ but has length 1.

↳ $\hat{a} = s\vec{a}$ for some scalar s.

↳ scale by 1 over the length of $\vec{a}$.

$s = \dfrac{1}{\|\vec{a}\|}$

$$\boxed{\hat{a} = \frac{1}{\|\vec{a}\|}\vec{a} = \frac{\vec{a}}{\|\vec{a}\|}}$$

↳ alt: $\hat{a}_1^2 + \hat{a}_2^2$

$= (sa_1)^2 + (sa_2)^2$

$= s^2(a_1^2 + a_2^2) = \underbrace{1}_{\text{want}}$

$\Rightarrow s^2 = \dfrac{1}{a_1^2 + a_2^2}$

Ex Find unit vector in the same dir as $\vec{a} = (2, 3)$.

$\|\vec{a}\| = \sqrt{13}$ so $\hat{a} = \left(\dfrac{2}{\sqrt{13}}, \dfrac{3}{\sqrt{13}}\right)$

The dot product b/w two vectors
result is a scalar.

Def'n: let $\vec{a} = (a_1, a_2)$ and $\vec{b} = (b_1, b_2)$

then $\vec{a} \bullet \vec{b} = a_1 b_1 + a_2 b_2$ $\quad$ ($+ a_3 b_3 + \dots$ in 3D, 4D, etc)

dot product
a.k.a. the inner product

The sum of the component-wise multiplication

In general, $\vec{a}, \vec{b} \in \mathbb{R}^n$

Properties of dot:

$$\underbrace{\vec{a} \circ \vec{a}}_{a_1 a_1 + a_2 a_2} = \underbrace{\|\vec{a}\|^2}_{a_1^2 + a_2^2} \qquad (1)$$

$$\vec{a} \circ \vec{b} = \vec{b} \circ \vec{a} \qquad (2)$$

$$\vec{a} \circ (\vec{b} + \vec{c}) = \vec{a} \circ \vec{b} + \vec{a} \circ \vec{c} \qquad (3)$$

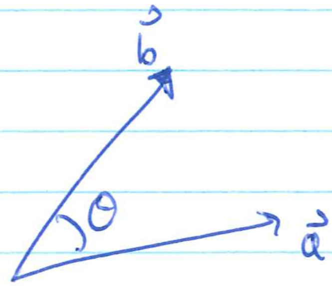
$$s\underbrace{(\vec{a} \circ \vec{b})}_{\text{scalar}} = (s\vec{a}) \bullet \vec{b} \qquad (4)$$

$$\vec{0} \bullet \vec{a} = 0 \quad \text{for all } \vec{a} \qquad (5)$$

Rules seem "natural": just keep track of type: vec vs scalars.

(Geom. interpretation of dot)

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \, \|\vec{b}\| \cos \underbrace{\theta}_{\text{angle b/w } \vec{a} \text{ and } \vec{b}} \qquad (6)$$



Note: this ~~defines~~ defines angles in a vector space. angle b/w $\vec{a}$, $\vec{b}$ $\in \mathbb{R}^{25}$ (even though traditional "angle" hard to visualize).

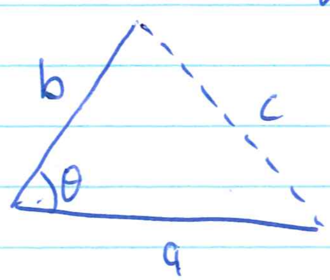
Note: also shows that the dot product independent of coor sys

Note: abstract vector space (eg. polynomials) ... $\Rightarrow$ angle b/w functions (?)

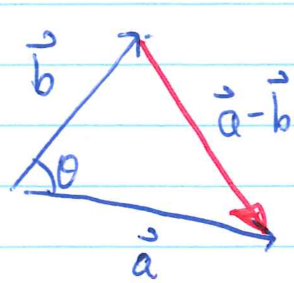
Proof:

Law of cosines

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$



$$\|\vec{a} - \vec{b}\|^2 = \|\vec{a}\|^2 + \|\vec{b}\|^2 - 2\|\vec{a}\|\|\vec{b}\|\cos\theta \qquad (*)$$



③ twice

Algebra: $$\|\vec{a}-\vec{b}\|^2 \overset{①}{=} (\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b}) = \vec{a}\cdot\vec{a} - \vec{b}\cdot\vec{a} - \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{b}$$

② 6

$\cancel{+\vec{a}\cdot\vec{b}}$

$$= \vec{a}\cdot\vec{a} + \vec{b}\cdot\vec{b} - 2\vec{a}\cdot\vec{b}$$
$$= \|\vec{a}\|^2 + \|\vec{b}\|^2 - 2\vec{a}\cdot\vec{b} \qquad (**)$$

Subtract (**) from (*)

$$0 = -2\|\vec{a}\|\,\|\vec{b}\|\cos\theta + 2\vec{a}\cdot\vec{b}$$

$\boxtimes$

Ex find angle b/w $\vec{a} = (1,1,1)$
$\vec{b} = (1,2,3)$

$$\cos\theta = \frac{(1,1,1)\cdot(1,2,3)}{\sqrt{3}\quad\sqrt{14}} = \frac{6}{\sqrt{3}\sqrt{14}}$$

$$\theta = \arccos\left(\frac{6}{\sqrt{3}\sqrt{14}}\right) \approx 0.388$$ radians

$\cancel{\approx 22.2^\circ}$

$\approx 22.2^\circ$

Notes: if $\vec{a} \cdot \vec{b} = 0$ then $\|\vec{a}\|\|\vec{b}\| \cos\theta = 0$

$\Rightarrow$ either $\cancel{\|\vec{a}\| = 0}$ $\vec{a} = \vec{0}$ or $\cancel{\|\vec{b}\| = 0}$ $\vec{b} = \vec{0}$

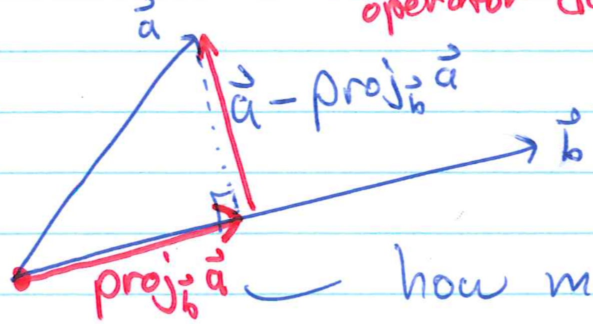
or $\cos\theta = 0$

☆ if $\vec{a} \cdot \vec{b} = 0$ but $\vec{a} \neq \vec{0}$ and $\vec{b} \neq \vec{0}$

then $\theta = \pi/2$ $(90°)$

$\Rightarrow$ $\vec{a}$ and $\vec{b}$ are orthogonal (perpendicular).

Projections projection of vector $\vec{a}$ in the direction of $\vec{b}$ is denoted $\underbrace{\text{proj}_{\vec{b}}}_{\text{operator acting on } \vec{a}} \vec{a}$:



— how much of $\vec{a}$ is in dir $\vec{b}$?

Derive formula: $\text{proj}_{\vec{b}}\vec{a} = s\vec{b}$ scalar; same dir as b.

Also: $(\vec{a} - \text{proj}_{\vec{b}}\vec{a}) \cdot \vec{b} = 0$ rejection is orthog to $\vec{b}$.

$$\Rightarrow (\vec{a} - s\vec{b}) \cdot \vec{b} = 0$$

$$\Rightarrow \vec{a} \cdot \vec{b} - s\,\vec{b} \cdot \vec{b} = 0$$

$$s = \frac{\vec{a} \cdot \vec{b}}{\vec{b} \cdot \vec{b}}$$

$$\boxed{\text{proj}_{\vec{b}}\vec{a} = \frac{\vec{a} \cdot \vec{b}}{\|\vec{b}\|^2}\,\vec{b}}$$

scalar

Note: if $\hat{b}$ is a unit vector, $\text{proj}_{\hat{b}}\vec{a} = \vec{a} \cdot \hat{b}\,\hat{b}$

Ex find the projection of $(2,3) = \vec{a}$ onto direction $(2,1) = \vec{b}$

$$\|\vec{b}\| = \sqrt{5} \quad \& \quad \vec{a} \cdot \vec{b} = 4 + 3 = 7$$

$$\text{proj}_{\vec{b}}\vec{a} = \frac{7}{5}(2,1) = \left(\frac{14}{5}, \frac{7}{5}\right)$$

↳ draw picture!