Lecture 3

2018-01-11

- office hours — on website.

Determinant §2.4 (in 2D/3D for now)

The determinant is a scalar number associated with a square matrix

array of numbers e.g. $\begin{bmatrix} 2 & 4 & 8 \\ 10 & 11 & 13 \end{bmatrix}$

defined to be

In 2D: $\det \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} := a_1 b_2 - b_1 a_2 \in \mathbb{R}$

3D: $\det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} := a_1 \det \begin{bmatrix} b_2 & b_3 \\ c_2 & c_3 \end{bmatrix}$

$- a_2 \det \begin{bmatrix} b_1 & b_3 \\ c_1 & c_3 \end{bmatrix}$

$+ a_3 \det \begin{bmatrix} b_1 & b_2 \\ c_1 & c_2 \end{bmatrix} \in \mathbb{R}$

For now, think of these as

$$\det\begin{bmatrix} - \vec{a} - \\ - \vec{b} - \end{bmatrix} = \det\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$$

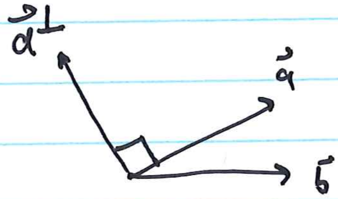
$\to$ $\vec{a}$ ~~then~~ and $\vec{b} \in \mathbb{R}^2$

$$\det\begin{bmatrix} - \vec{a} - \\ - \vec{b} - \\ - \vec{c} - \end{bmatrix}$$

Geometry in 2D: note: $\det\begin{bmatrix} - \vec{a} - \\ - \vec{b} - \end{bmatrix} = a_1 b_2 - a_2 b_1 = [-a_2, a_1] \bullet \overset{\vec{b}}{\cancel{[b_1, b_2]}}$

$= \vec{a}^{\perp} \bullet \vec{b}$

ccw rotation of $\vec{a}$



Q1: $\boxed{\det\begin{bmatrix} - \vec{a} - \\ - \vec{b} - \end{bmatrix} = 0 \iff \vec{a} \parallel \vec{b} \text{ or } \vec{a} = \vec{0} \text{ or } \vec{b} = \vec{0}}$

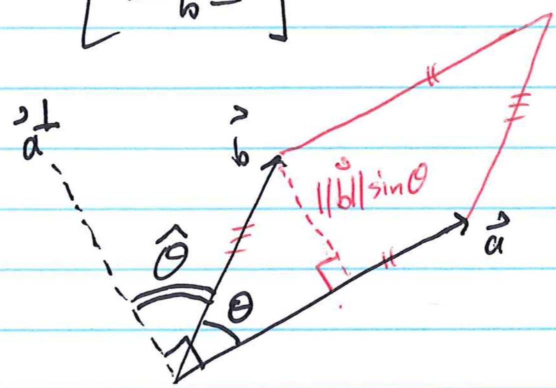
Ex ~~Ex~~ Say $\vec{b} = s\vec{a}$. $\det\begin{bmatrix} a_1 & a_2 \\ sa_1 & sa_2 \end{bmatrix} = sa_1 a_2 - a_2 s a_1$
$= 0$

last day

$$\det\begin{bmatrix} - \vec{a} - \\ - \vec{b} - \end{bmatrix} = \vec{a}^{\perp} \cdot \vec{b} = \|\vec{a}^{\perp}\| \, \|\vec{b}\| \cos\hat{\theta} = \|\vec{a}\| \, \|\vec{b}\| \cos\left(\frac{\pi}{2} - \theta\right)$$

$$= \|\vec{a}\| \, \|\vec{b}\| \sin(\theta)$$

$$= \pm \text{ area of parallelogram with sides spanned by } \vec{a} \text{ and } \vec{b}$$



here θ is the signed angle from $\vec{a}$ to $\vec{b}$

+ for c.c.w.

− for c.w.

$$\boxed{\left| \det\begin{bmatrix} - \vec{a} - \\ - \vec{b} - \end{bmatrix} \right| = \text{area of the parallelogram}}$$

(3D later!)

only defined in 3D

Cross Product — take two 3D vectors, get another vector

$\vec{a} = \langle a_1\ a_2\ a_3 \rangle, \quad \vec{b} = \langle b_1\ b_2\ b_3 \rangle$

$$\vec{a} \times \vec{b} := \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$$

"Flakey but useful" Wetton.

↓

Mnemonic

$$\text{"det"} \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} = \hat{\imath} \det \begin{bmatrix} a_2 & a_3 \\ b_2 & b_3 \end{bmatrix} - \hat{\jmath} \det \begin{bmatrix} a_1 & a_3 \\ b_1 & b_3 \end{bmatrix} + \hat{k} \det \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$$

not a matrix.

$$= \vec{a} \times \vec{b}$$

Ex $\langle 1, 2, 3 \rangle \times \langle 0, 1, -1 \rangle = \text{"det"} \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & 3 \\ 0 & 1 & -1 \end{bmatrix}$

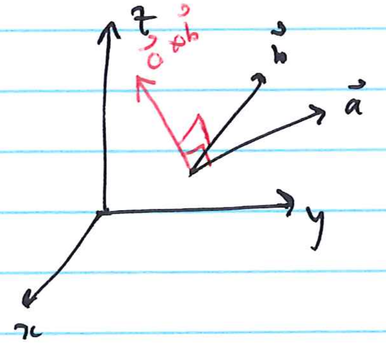
$$= \hat{\imath} \det \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix} - \hat{\jmath} \det \begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix} + \hat{k} \det \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$= \hat{\imath}(-5) \qquad - \hat{\jmath}(-1 - 0) \qquad + \hat{k}(1 - 0)$$

$$= \begin{bmatrix} -5 \\ 1 \\ 1 \end{bmatrix}$$

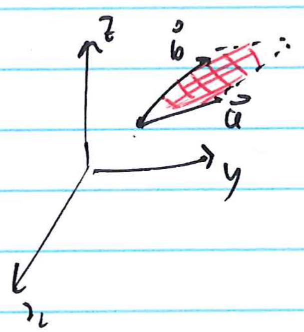
Geometry of $\vec{a} \times \vec{b}$

- $\vec{a} \times \vec{b}$ orthogonal to both $\vec{a}$ and $\vec{b}$



(proof later)

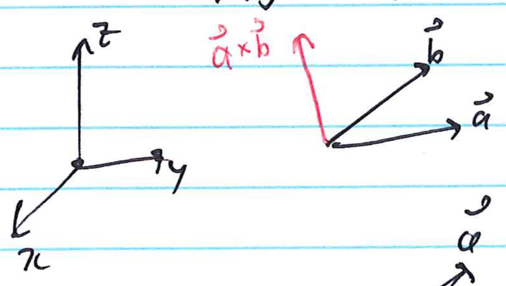
- $\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \, \|\vec{b}\| \sin\theta$ where $0 \le \theta \le \pi$

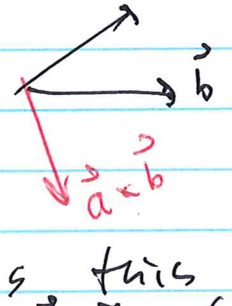


Ex parallelogram ~~spanned~~ spanned by $\begin{bmatrix}1\\2\\3\end{bmatrix}$ and $\begin{bmatrix}0\\1\\-1\end{bmatrix}$

the area is $\|\langle -5, 1, 1\rangle\| = \sqrt{27}$

- $\vec{a}$, $\vec{b}$ and $\vec{a}\times\vec{b}$ obey the "right-hand rule". $\rightarrow$ curl ~~fingers~~ fingers (of right hand) from $\vec{a}$ to $\vec{b}$... thumb points in $\vec{a}\times\vec{b}$ direction





<u>Consequence</u> $\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$

Aside

Q: why does this look like the area formula for $\det\begin{bmatrix}-\vec{a}-\\-\vec{b}-\end{bmatrix}$ (2D vectors).

A: Consider $\vec{a}$ and $\vec{b} \in \mathbb{R}^2$

$$[a_1 \; a_2 \; 0] \times [b_1 \; b_2 \; 0] = \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & 0 \\ b_1 & b_2 & 0 \end{bmatrix}$$

$$= \dots = \hat{k} \det \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$$

$$\| \hat{k} \det(\quad) \| = | \det(\quad) |$$

Properties of Cross Product

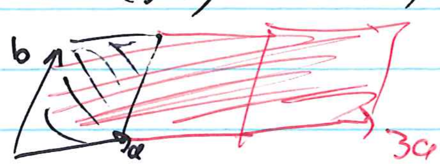
$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a} \qquad (1)$$

$$\vec{a} \times (\vec{b} \times \vec{c}) = \underbrace{(\vec{c} \cdot \vec{a})}_{\text{Scalar}} \vec{b} - \underbrace{(\vec{b} \cdot \vec{a})}_{\text{Scalar}} \vec{c} \qquad (2)$$

Lagrange's formula

no cross prod on RHS

$$s(\vec{a} \times \vec{b}) = (s\vec{a}) \times \vec{b} = \vec{a} \times (s\vec{b}) \qquad (3)$$



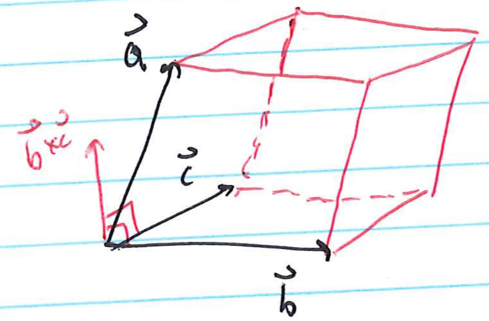
$$\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c} \qquad (4)$$

Triple product

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c} \qquad (5)$$

$$= b \cdot (\vec{c} \times \vec{a})$$

Triple product: volume of a "parallelopiped"



Connection to det: $\vec{a}\cdot(\vec{b}\times\vec{c}) = \vec{a}\cdot\det\begin{bmatrix}\hat{i} & \hat{j} & \hat{k}\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{bmatrix}$

$$= \begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix}\cdot\begin{bmatrix}\det[\quad]\\ -\det[\quad]\\ \det[\quad]\end{bmatrix} = a_1\det[\quad] - a_2\det[\quad] + a_3\det[\quad]$$

= formula from page 1

$$= \det\begin{bmatrix}- & \vec{a} & -\\ - & \vec{b} & -\\ - & \vec{c} & -\end{bmatrix}$$